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A constant rank theorem for spacetime convex solutions of heat equation

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Abstract. We prove a constant rank theorem for the spacetime Hessian of the spacetime convex solutions of standard heat equation. Moreover, we apply this technique to get a constant rank theorem for the spacetime hessian of a spacetime convex solution of a nonlinear heat equation.

1. Introduction

Heat equation is a basic partial differential equation. It appears in index theorem [2], Brownian motion [11] and geometry analysis [30], etc. It is important to understand the geometric and analysis properties of the solutions for heat equation. In this paper we concentrate on the spacetime convexity of the solutions of heat equation. We first state some related developments on this subject.

In 1976, Brascamp and Lieb [8] used the Brownian motion to study the heat equation and proved the log-concavity of the first eigenfunction and the Brunn–Minkowski inequality for the first eigenvalue of Laplacian equation in convex domains. In 1986, Li and Yau [26] proved a gradient estimate and a Harnack inequality for parabolic equations with Schrödinger potential on Riemannian manifolds. And Hamilton [16] gave a matrix version of Li and Yau Harnack inequality, in which he proved that the Hessian matrix of the logarithmic function of solutions to heat equation is positive definite under certain curvature condition.

Now we turn into the spacetime convexity for the solutions of heat equation. In a series of papers by Borell [5–7], he studied certain spacetime convexities of the solution or the level sets of the solution for heat equation with Schrödinger potential. As a consequence, he gave a new proof of the Brascamp and Lieb's [8] theorem and the Brownian motion proof of the classical Brunn–Minkowski inequality.

In this paper we establish a *constant rank theorem* for spacetime convex solutions of heat equation. Using similar calculations, we obtain a constant rank theorem

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for solutions of a nonlinear parabolic equation, whose spacetime convexity was obtained by Borell [6].

First, we give the definition of the spacetime convexity of a function $u(x,t)$. We denote by Du and D^2u the spatial gradient and the spatial Hessian matrix respectively.

Definition 1.1. Suppose $u \in C^{2,1}(\Omega \times (0,T))$, where Ω is a domain in $\mathbb{R}^n$, then u is spacetime convex if u is convex for every $(x,t) \in (\Omega \times (0,T))$, i.e

$$\widehat{D^2u} = \begin{pmatrix} D^2u & (Du_t)^T \\ Du_t & u_{tt} \end{pmatrix} \geq 0. \tag{1.1}$$

Our main results is the following theorem.

Theorem 1.2 *Let Ω be a domain in $\mathbb{R}^n$, and $u \in C^{3,1}(\Omega \times (0,T))$ be a spacetime convex solution of the heat equation*

$$\frac{\partial u}{\partial t} = \Delta u, \quad (x,t) \in \Omega \times (0,T). \tag{1.2}$$

If $\widehat{D^2u}$ attains its minimum rank l $(0 \leq l \leq n)$ at some point $(x_o, t_o) \in \Omega \times (0,T)$, then the rank of $\widehat{D^2u}$ is constant on $\Omega \times (0, t_o]$. Moreover, let $l(t)$ be the minimum rank of $\widehat{D^2u}$ in Ω for fixed t, then $l(t_1) \leqslant l(t_2)$ for all $0 < t_1 \leq t_2 < T$.

Now we give an application of Theorem 1.2 to the result by Borell [6]. Let

$$\Phi(x) = \int_{-\infty}^{x} exp(-\lambda^2/2)d\lambda/\sqrt{2\pi}, \quad x \in (-\infty, +\infty)$$

be the distribution function of a $\mathcal{N}(0;1)$-distributed random variable and $\Phi^{-1}: [0,1] \to (-\infty, +\infty)$ be its inverse function. Moreover, we define a bijection $(y,s) = \psi(x,t)$ of $\mathbb{R}^n \times \mathbb{R}_+$ onto $\mathbb{R}^n \times \mathbb{R}_+$ by setting

$$\begin{cases} y = \dfrac{x}{\sqrt{t}}, \\ s = \dfrac{1}{\sqrt{t}}, \end{cases}$$

then $v(y,s) := -\Phi^{-1} \circ u \circ \psi^{-1}(y,s)$ satisfies the equation

$$sv_s + 2\Delta v + y \cdot \bigtriangledown v - 2v|\bigtriangledown v|^2 = 0, \tag{1.3}$$

where u satisfies the Eq. 1.2.

For the spacetime convex solutions of the Eq. 1.3, we also have the following constant rank theorem.

Theorem 1.3 *Let Ω be a domain in $\mathbb{R}^n$, and $v \in C^{3,1}(\Omega \times (0,T))$ be a spacetime convex solution of the Eq. 1.3. If $\widehat{D^2v}$ attains the minimum rank l $(0 \leq l \leq n)$ at some point $(y_0, s_0) \in \Omega \times (0,T)$, then the rank of $\widehat{D^2v}$ takes the constant l on $\Omega \times (0, s_0]$. Moreover, let $l(s)$ be the minimum rank of $\widehat{D^2v}$ in Ω for fixed s, then $l(s_1) \leq l(s_2)$ for all $0 < s_1 \leq s_2 < T$.*

Remark 1.4. In 1996, Borell [6] proved that if $D^+ = D \times \{t > 0\}$ and $A \subset \mathbb{R}^n$ satisfies some convexity conditions, and

$$u(\zeta) = \int_A p(\zeta, (x, 0))dx, \quad \zeta \in D^+,$$

where $D \subset \mathbb{R}^n \times \mathbb{R}$, and $p : D \times D \longrightarrow [0, +\infty)$ is the Green function of the heat operator in D equipped with zero Dirichlet boundary condition (see Watson [32]), then $-\Phi^{-1} \circ u \circ \psi^{-1}$ is spacetime convex in $\psi(D^+)$.

Combining this result of Borell [6] and Theorem 1.3, we conclude that $-\Phi^{-1} \circ u \circ \psi^{-1}$ not only be a spacetime convex solution of the Eq. 1.3, but also has constant rank property.

We shall give a short review on the history of the convexity for the solution of elliptic partial differential equations. So far as we know, there are two important methods to approach this problem, which are the macroscopic and microscopic methods. The former is based on a weak maximum principle, while the latter uses the strong maximum principle coupled with a continuity argument. For the macroscopic convexity argument, Korevaar made breakthroughs in [24], in which he introduced a concavity maximum principles for a class of quasilinear elliptic equations. Later it was improved by Kennington [22] and Kawhol [21]. The theory was further developed to its great generality by Alvarez et al. [1].

The key of the study of microscopic convexity is a method called *constant rank theorem* which was discovered in 2 dimension by Caffarelli and Friedman [9] (a similar result was also discovered by Singer et al. [31] at the same time). Later the result in [9] was generalized to $\mathbb{R}^n$ by Korevaar and Lewis [25]. Recently the *constant rank theorem* was generalized to fully nonlinear elliptic and parabolic equations in [10] and [3,4], where the elliptic result in [3] is the microscopic version of the macroscopic convexity principle in [1]. For parabolic equations, the *constant rank theorem* in [10,3] has been proved with respect to the space variable only.

Constant rank theorem is a very useful tool to produce convex solutions in geometric analysis. By the corresponding homotopic deformation, the existence of convex solution comes from the *constant rank theorem*. For the geometric application of the *constant rank theorem*, the Christoffel-Minkowski problem and the related prescribing Weingarten curvature problems were studied in [15,14,18]. The preservation of convexity for the general geometric flows of hypersurfaces was given in [3]. The Brunn–Minkowski inequality for the first eigenvalue of some elliptic operators are studied in [27].

There are some related results on the spacetime convexity of the solutions of parabolic equation in [19,20,23,29] using the similar elliptic macroscopic convexity technique in Kennington [22] and Kawhol [21]. In fact Ishige and Salani [19,20] studied the spacetime convexity of the level sets of the solution for a class parabolic equation.

The rest of the paper is organized as follows. In Sect. 2, we give some preliminaries. In Sect. 3, we prove the constant rank theorem of heat equation. Then in Sect. 4, we prove Theorem 1.3.

2. Notations and preliminaries

In this section, we shall collect some basic properties of elementary symmetric functions, which could be found in [28].

Definition 2.1. For any $k = 1, 2, \dots, n$, we set

$$S_k(\lambda) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_k}, \qquad \text{for any} \quad \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n.$$

We also set $S_0 = 1$ and $S_k = 0$ for $k > n$.

For $1 \le k \le n$, Γ_k is a cone in $\mathbb{R}^n$ determined by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \quad S_1(\lambda) > 0, \dots, and\ S_k(\lambda) > 0\},$$

and Γ_k contains the positive cone $\Gamma_n = \{\lambda \in \mathbb{R}^n; \lambda_i > 0, 1 \le i \le n\}$. Γ_k is symmetric, namely, if $\lambda \in \Gamma_k$, then any permutation of λ lies in Γ_k too. In fact, Γ_k is a convex cone.

Definition 2.2. Let $W = (W_{ij})$ be an $n \times n$ symmetry matrix, for $1 \le k \le n$ we define

$$S_k(W) = S_k(\lambda(W_{ij})) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1}\lambda_{i_2}\dots\lambda_{i_k},$$

where $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ denote the eigenvalues of the matrix (W_{ij}). Equivalently, $S_k(W)$ can be defined as the sum of the $k \times k$ principal minors of W.

We define $S_k(W|i) = S_k(\lambda(W|i))$ where $(W|i)$ means that the matrix W exclude the i-column and i-row, and $S_k(W|ij) = S_k(\lambda(W|ij))$ where $(W|ij)$ means that the matrix W exclude the i, j columns and i, j rows. Then we have the following well-known identities.

Proposition 2.3 *Suppose $n \times n$ matrix $W = (W_{ij})$ is diagonal and $1 \le k \le n$, then*

$$\frac{\partial S_k(W)}{\partial W_{ij}} = \begin{cases} S_{k-1}(W\,|i), & if\, i = j, \\ 0, & if\, i \ne j, \end{cases}$$

and

$$\frac{\partial^2 S_k(W)}{\partial W_{ij}\partial W_{rs}} = \begin{cases} S_{k-2}(W\,|ir), & if\, i = j, r = s, i \ne r, \\ -S_{k-2}(W\,|ir), & if\, i = s, j = r, i \ne j, \\ 0, & otherwise\ . \end{cases}$$

Let's denote by $S_k(\lambda|i)$ the sum of the terms in $S_k(\lambda)$ not containing the factor λ_i, and denote by $S_k(\lambda|ij)$ the sum of the terms in $S_k(\lambda)$ not containing the factors λ_i, λ_j. Now we state some basic formulas of the elementary symmetric functions.

Proposition 2.4 *Let* $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ *and* $k = 0, 1, \dots, n$, *then*

$$S_k(\lambda) = S_k(\lambda|i) + \lambda_i S_{k-1}(\lambda|i), \quad \forall\, 1 \le i \le n, \tag{2.1}$$

$$\sum_i \lambda_i S_{k-1}(\lambda|i) = k S_k(\lambda), \tag{2.2}$$

$$\sum_i S_k(\lambda|i) = (n-k) S_k(\lambda). \tag{2.3}$$

From the definition of $S_k(W)$, it follows that for

$$\widehat{D^2u} = \begin{pmatrix} D^2u & (Du_t)^T \\ Du_t & u_{tt} \end{pmatrix},$$

we have the following relation.

Lemma 2.5 $\forall\, 0 \le l \le n$,

$$\begin{aligned} S_{l+1}(\widehat{D^2u}) = {} & S_{l+1}(D^2u) + u_{tt} S_l(D^2u) - \sum_i S_{l-1}(D^2u|i) u_{ti}^2 \\ & + \sum_{i \ne j} S_{l-2}(D^2u|ij) u_{ti} u_{tj} u_{ij} - \sum_{i \ne j, i \ne k, j \ne k} s S_{l-3}(D^2u|ijk) u_{ti} u_{tj} u_{ki} u_{kj} \\ & + T, \end{aligned} \tag{2.4}$$

where the terms which contains at least three u_{ij} $(i \ne j)$ *consist of* T.

3. The proof of Theorem 1.2

In order to prove the Theorem 1.2, we first prove the following differential inequality locally on spacetime.

Lemma 3.1 *Let* Ω *be a domain in* $\mathbb{R}^n$, *and* $u \in C^{3,1}(\Omega \times (0, T))$ *be a spacetime convex solution of the heat equation*

$$\frac{\partial u}{\partial t} = \Delta u, \quad (x, t) \in \Omega \times (0, T). \tag{3.1}$$

Suppose there are a point $(x_0, t_0) \in \Omega \times (0, T)$ *and a positive constant* C_0, *such that for a fixed integer* $n \ge l \ge 1$, $S_l(D^2u)(x_0, t_0) \ge C_0$. *Then there are positive constants* C_1 *and* C_2 *depending only on* $||u||_{C^{3,2}}$, n *and* C_0, $\mathcal{O} \subset \Omega$ *which is a small neighborhood of* x_0 *and* δ *is a small positive constant, such that*

$$\Delta\phi(x,t) - \phi_t(x,t) \le C_1 \phi(x,t) + C_2 |\nabla \phi|(x,t), \quad \forall (x,t) \in \mathcal{O} \times (t_0 - \delta, t_0 + \delta), \tag{3.2}$$

where $\phi = S_{l+1}(\widehat{D^2u})$.

Proof. We divide into two steps to prove this lemma.

Step 1: First we give the routine calculations for

$$\Delta\phi - \phi_t$$

at some fixed point (x, t).

For each fixed $(x, t) \in \Omega \times (0, T)$, we choose a local orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ so that D^2u is diagonal. All the calculations will be at the point (x, t).

At (x, t), by Lemma 2.5

$$S_{l+1}(\widehat{D^2u}) = S_{l+1}(D^2u) + u_{tt}S_l(D^2u) - \sum_i S_{l-1}(D^2u|i)u_{ti}^2, \tag{3.3}$$

and T satisfy

$$T = 0, \quad \frac{\partial T}{\partial x_i} = 0, \quad \frac{\partial^2 T}{\partial x_i \partial x_j} = 0, \quad \forall 1 \le i, j \le n.$$

Taking the first derivatives of ϕ with respect to t, we have

$$\begin{aligned}\phi_t = &\sum_i S_l(D^2u|i)u_{tii} + u_{ttt}S_l(D^2u) + u_{tt}\sum_i S_{l-1}(D^2u|i)u_{tii}\\ &-2\sum_i S_{l-1}(D^2u|i)u_{ti}u_{tti} - \sum_{i\neq j} S_{l-2}(D^2u|ij)u_{tj}^2u_{tii}\\ &+\sum_{i\neq j} S_{l-2}(D^2u|ij)u_{ti}u_{tj}u_{tij}.\end{aligned} \tag{3.4}$$

Computing the first and second derivatives of ϕ in the direction e_p, we have

$$\begin{aligned}\phi_p = &\sum_i S_l(D^2u|i)u_{iip} + u_{ttp}S_l(D^2u) + u_{tt}\sum_i S_{l-1}(D^2u|i)u_{iip}\\ &- 2\sum_i S_{l-1}(D^2u|i)u_{ti}u_{tip} - \sum_{i\neq j} S_{l-2}(D^2u|ij)u_{tj}^2u_{iip}\\ &+\sum_{i\neq j} S_{l-2}(D^2u|ij)u_{ti}u_{tj}u_{ijp},\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\phi_{pp} = &\sum_{i\neq j} S_{l-1}(D^2u|ij)u_{iip}u_{jjp} - \sum_{i\neq j} S_{l-1}(D^2u|ij)u_{ijp}^2 + \sum_i S_l(D^2u|i)u_{iipp}\\ &+S_l(D^2u)u_{ttpp} + 2\sum_i S_{l-1}(D^2u|i)u_{ttp}u_{iip} + u_{tt}\sum_{i\neq j} S_{l-2}(D^2u|ij)u_{iip}u_{jjp}\\ &-u_{tt}\sum_{i\neq j} S_{l-2}(D^2u|ij)u_{ijp}^2 + u_{tt}\sum_i S_{l-1}(D^2u|i)u_{iipp} - 2\sum_i S_{l-1}(D^2u|i)u_{tip}^2\\ &-2\sum_i S_{l-1}(D^2u|i)u_{ti}u_{tipp} - 4\sum_{i\neq j} S_{l-2}(D^2u|ij)u_{tj}u_{tjp}u_{iip}\\ &-\sum_{i\neq j, i\neq k, j\neq k} S_{l-3}(D^2u|ijk)u_{tk}^2u_{iip}u_{jjp} + \sum_{i\neq j, i\neq k, j\neq k} S_{l-3}(D^2u|ijk)u_{tk}^2u_{ijp}^2\end{aligned}$$

$$-\sum_{i\neq j} S_{l-2}(D^2u|ij)u_{tj}^2 u_{iipp} + 4\sum_{i\neq j} S_{l-2}(D^2u|ij)u_{ti}u_{tjp}u_{ijp}$$
$$+\sum_{i\neq j} S_{l-2}(D^2u|ij)u_{ti}u_{tj}u_{ijpp} + 2\sum_{i\neq j, i\neq k, j\neq k} S_{l-3}(D^2u|ijk)u_{ti}u_{tj}u_{ijp}u_{kkp}$$
$$-2\sum_{i\neq j, i\neq k, j\neq k} S_{l-3}(D^2u|ijk)u_{ti}u_{tj}u_{kip}u_{kjp}. \tag{3.6}$$

Combining (3.4) and (3.6) yield that

$$\Delta\phi - \phi_t = I + II + III + IV + V,$$

where

$$I = \sum_i S_l(D^2u|i)(\Delta u - u_t)_{ii} + S_l(D^2u)(\Delta u - u_t)_{tt}$$
$$+u_{tt}\sum_i S_{l-1}(D^2u|i)(\Delta u - u_t)_{ii} - 2\sum_i S_{l-1}(D^2u|i)u_{ti}(\Delta u - u_t)_{ti}$$
$$-\sum_{i\neq j} S_{l-2}(D^2u|ij)u_{tj}^2(\Delta u - u_t)_{ii} + \sum_{i\neq j} S_{l-2}(D^2u|ij)u_{ti}u_{tj}(\Delta u - u_t)_{ij}$$
$$= 0,$$

since u is the solution of heat equation $u_t = \Delta u$; the second term is

$$II = -\sum_p\sum_{i\neq j} S_{l-1}(D^2u|ij)u_{ijp}^2 - 2\sum_{i,p} S_{l-1}(D^2u|i)u_{tip}^2$$
$$+4\sum_p\sum_{i\neq j} S_{l-2}(D^2u|ij)u_{ti}u_{tjp}u_{ijp}$$
$$-2\sum_p\sum_{i\neq j, i\neq k, j\neq k} S_{l-3}(D^2u|ijk)u_{ti}u_{tj}u_{kip}u_{kjp}; \tag{3.7}$$

the third term is

$$III = \sum_p\sum_{i\neq j} S_{l-1}(D^2u|ij)u_{iip}u_{jjp} - 4\sum_p\sum_{i\neq j} S_{l-2}(D^2u|ij)u_{tj}u_{tjp}u_{iip}$$
$$+2\sum_p\sum_{i\neq j, i\neq k, j\neq k} S_{l-3}(D^2u|ijk)u_{ti}u_{tj}u_{ijp}u_{kkp}; \tag{3.8}$$

the fourth term is

$$IV = 2\sum_{i,p} S_{l-1}(D^2u|i)u_{ttp}u_{iip}; \tag{3.9}$$

and the fifth term is

$$V = u_{tt}\sum_p\sum_{i\neq j} S_{l-2}(D^2u|ij)u_{iip}u_{jjp} - u_{tt}\sum_p\sum_{i\neq j} S_{l-2}(D^2u|ij)u_{ijp}^2$$
$$-\sum_p\sum_{i\neq j, i\neq k, j\neq k} S_{l-3}(D^2u|ijk)u_{tk}^2u_{iip}u_{jjp}$$
$$+\sum_p\sum_{i\neq j, i\neq k, j\neq k} S_{l-3}(D^2u|ijk)u_{tk}^2u_{ijp}^2. \tag{3.10}$$

Step 2: Under the assumption $S_l(D^2u)(x_0, t_0) \geq C_0 > 0$, there exist a small neighborhood $\mathcal{O} \subset \Omega$ of x_0 and a small positive constant δ such that

$$S_l(D^2u)(x_0, t_0) \geq \frac{C_0}{2} > 0, \quad \forall (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0 + \delta).$$

In this step, we prove the following important inequality

$$\Delta\phi - \phi_t \lesssim 0, \quad \forall (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0 + \delta), \tag{3.11}$$

where we follow the notations in [9,15], let h and g are two function defined in $\mathcal{O} \times (t_0 - \delta, t_0 + \delta)$, we say $h \lesssim g$ if there are positive constants C_1 and C_2 depending only on $||u||_{C^{2,1}}, n$ (independent of (x, t)), such that $(h - g)(x, t) \leq (C_1|\nabla\phi| + C_2\phi)(x, t)$, $\forall (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0 + \delta)$. We also write

$$h \sim g \quad \text{if} \quad h \lesssim g, \quad g \lesssim h.$$

For each fixed $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0 + \delta)$, we choose a local orthonormal frame $e_1, \ldots, e_n$ so that D^2u is diagonal and let $u_{ii} = \lambda_i$, $i = 1, \ldots, n$. We arrange $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ are the eigenvalues of D^2u at (x, t).

Since $S_l(D^2u) \geq C_0/2$ and $u \in C^{3,1}$, for any $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0 + \delta)$, there is constant $C > 0$ depending only on $||u||_{C^{2,1}}$, n, and C_0, such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq C > 0$. Let $G = \{1, \ldots, l\}$ and $B = \{l + 1, \ldots, n\}$ be the "good" and "bad" sets of indices respectively. All the calculations will be at the point (x, t) in this step.

At (x, t), by Lemma 2.5 we have

$$\begin{aligned} 0 \sim \phi = S_{l+1}(\widehat{D^2u}) &= S_{l+1}(D^2u) + u_{tt}S_l(D^2u) - \sum_i S_{l-1}(D^2u|i)u_{ti}^2 \\ &\geq S_l(G)S_1(B) + S_l(G)(u_{tt} - \sum_{i \in G} \frac{u_{ti}^2}{\lambda_i}), \end{aligned} \tag{3.12}$$

so

$$\lambda_i \sim 0, \quad i \in B; \tag{3.13}$$

$$u_{tt} - \sum_{i \in G} \frac{u_{ti}^2}{\lambda_i} \sim 0, \tag{3.14}$$

where we used $\widehat{D^2u}$ is positive semi-definite, and we have chosen only the two $l + 1$ order principal minors of all in (3.12). This relation yield that, for $1 \leq k \leq l$

$$S_k(D^2u) \sim S_k(G), \quad S_k(D^2u|i) \sim \begin{cases} S_k(G|i), & \text{if } i \in G, \\ S_k(G), & \text{if } i \in B, \end{cases}$$

$$S_k(D^2u|ij) \sim \begin{cases} S_k(G|ij), & \text{if } i, j \in G, \ i \neq j, \\ S_k(G|i), & \text{if } i \in G, \ j \in B, \\ S_k(G), & \text{if } i, j \in B, \ i \neq j. \end{cases}$$

Actually, we also yield $S_{l+2}(\widehat{D^2u}) = O(\phi^2)$ from $S_{l+1}(\widehat{D^2u}) = \phi$. Expending $S_{l+2}(\widehat{D^2u})$ and reserving only three $l+2$ order principal minors of all, it follows that

$$\begin{aligned}
O(\phi^2) = S_{l+2}(\widehat{D^2u}) &= S_{l+2}(D^2u) + u_{tt}S_{l+1}(D^2u) - \sum_i S_l(D^2u|i)u_{ti}^2 \\
&= S_l(G)S_2(B) + u_{tt}S_l(G)S_1(B) - \sum_{i\in G} S_{l-1}(G|i)S_1(B)u_{ti}^2 \\
&\quad - S_l(G)\sum_{i\in B}u_{ti}^2 + u_{tt}S_{l-1}(G)S_2(B) - \sum_{i\in G}S_{l-2}(G|i)S_2(B)u_{ti}^2 \\
&\quad - S_{l-1}(G)\sum_{i\in B}S_1(B|i)u_{ti}^2 + O(\phi^2) \\
&= S_l(G)S_2(B) + S_l(G)S_1(B)(u_{tt} - \sum_{i\in G}\frac{u_{ti}^2}{\lambda_i}) \\
&\quad + S_2(B)[u_{tt}S_{l-1}(G) - \sum_{i\in G}S_{l-2}(G|i)u_{ti}^2] \\
&\quad - \sum_{i\in B}[S_l(G) + S_{l-1}(G)S_1(B|i)]u_{ti}^2 + O(\phi^2),
\end{aligned}$$

so we have

$$u_{ti} \sim 0, \quad i \in B. \tag{3.15}$$

For $1 \le p \le n$, we introduce the following notations:

$$C_p = \sum_{i\in B} u_{iip}\,; \quad B_{ip} = \sum_{j\in G}\frac{u_{tj}}{\lambda_j}u_{ijp}\,, \quad \text{for} \quad i \in G. \tag{3.16}$$

Inserting (3.13)–(3.15) into (3.5), we get

$$\begin{aligned}
0 \sim \phi_p \sim& S_l(G)\sum_{i\in B}u_{iip} + S_l(G)u_{ttp} + u_{tt}\sum_{i\in G}S_{l-1}(G|i)u_{iip} + u_{tt}S_{l-1}(G)\sum_{i\in B}u_{iip} \\
&- 2\sum_{i\in G}S_{l-1}(G|i)u_{ti}u_{tip} - \sum_{\substack{i,j\in G\\ i\neq j}}S_{l-2}(G|ij)u_{tj}^2u_{iip} \\
&- \sum_{j\in G}S_{l-2}(G|j)u_{tj}^2\sum_{i\in B}u_{iip} + \sum_{\substack{i,j\in G\\ i\neq j}}S_{l-2}(G|ij)u_{ti}u_{tj}u_{ijp} \\
\sim& S_l(G)\sum_{i\in B}u_{iip} + S_l(G)u_{ttp} + S_l(G)\sum_{i\in G}\frac{u_{ti}^2}{\lambda_i^2}u_{iip} + S_l(G)\sum_{j\in G}\frac{u_{tj}^2}{\lambda_j^2}\sum_{i\in B}u_{iip} \\
&- 2S_l(G)\sum_{i\in G}\frac{u_{ti}}{\lambda_i}u_{tip} + S_l(G)\sum_{\substack{i,j\in G\\ i\neq j}}\frac{u_{ti}}{\lambda_i}\frac{u_{tj}}{\lambda_j}u_{ijp},
\end{aligned}$$

using (3.16), it follows that

$$u_{ttp} \sim -\left(1+\sum_{j\in G}\frac{u_{tj}^2}{\lambda_j^2}\right)C_p-\sum_{i\in G}\frac{u_{ti}}{\lambda_i}B_{ip}+2\sum_{i\in G}\frac{u_{ti}}{\lambda_i}u_{tip}. \tag{3.17}$$

Now we will use the relations (3.13)–(3.17) to treat the terms II; III; IV; V in (3.7)–(3.10). We shall divide all the terms in $II + III + IV + V$ again into four classes. The first class is of the quadratic terms of C_p^2; the second class is of the terms who contain C_p; the third class consists of the terms like u_{tip} and u_{klp} ($\forall i, j, k, l \in G$); the other terms become the last class.

First we use (3.13) and (3.15) to simply II,

$$\begin{aligned}
II \sim & -\sum_p\left(\sum_{\substack{i\in G\\ j\in B}}+\sum_{\substack{i\in B\\ j\in G}}+\sum_{\substack{i,j\in G\\ i\neq j}}+\sum_{\substack{i,j\in B\\ i\neq j}}\right)S_{l-1}(D^2u|ij)u_{ijp}^2\\
&-2\sum_p\left(\sum_{i\in G}+\sum_{i\in B}\right)S_{l-1}(D^2u|i)u_{tip}^2\\
&+4\sum_p\sum_{i\in G}\left(\sum_{j\in B}+\sum_{\substack{j\in G\\ j\neq i}}\right)S_{l-2}(D^2u|ij)u_{ti}u_{tjp}u_{ijp}\\
&-2\sum_p\sum_{\substack{i,j\in G\\ i\neq j}}\left(\sum_{k\in B}+\sum_{\substack{k\in G\\ k\neq i,k\neq j}}\right)S_{l-3}(D^2u|ijk)u_{ti}u_{tj}u_{kip}u_{kjp}\\
\sim & \ II_1+II_2,
\end{aligned}$$

where

$$\begin{aligned}
II_1 = & -2\sum_p\sum_{i\in G}S_{l-1}(G|i)u_{tip}^2+4\sum_p\sum_{\substack{i,j\in G\\ i\neq j}}S_{l-2}(G|ij)u_{ti}u_{tjp}u_{ijp},\\
&-2\sum_p\sum_{\substack{i,j,k\in G\\ i\neq j,i\neq k,j\neq k}}S_{l-3}(G|ijk)u_{ti}u_{tj}u_{kip}u_{kjp},\\
II_2 = & -2\sum_p\sum_{i\in B}\sum_{j\in G}S_{l-1}(G|j)u_{ijp}^2-S_{l-1}(G)\sum_p\sum_{\substack{i,j\in B\\ i\neq j}}u_{ijp}^2\\
&-2S_{l-1}(G)\sum_p\sum_{i\in B}u_{tip}^2+4\sum_p\sum_{i\in B}\sum_{j\in G}S_{l-2}(G|j)u_{tj}u_{tip}u_{ijp}\\
&-2\sum_p\sum_{k\in B}\sum_{\substack{i,j\in G\\ i\neq j}}S_{l-3}(G|ij)u_{ti}u_{tj}u_{kip}u_{kjp}.
\end{aligned}$$

Also by (3.13) and (3.15), we yield that

$$
\begin{aligned}
III \sim & \sum_p \left(\sum_{\substack{i\in G\\ j\in B}} + \sum_{\substack{i\in B\\ j\in G}} + \sum_{\substack{i,j\in G\\ i\neq j}} + \sum_{\substack{i,j\in B\\ i\neq j}} \right) u_{iip}u_{jjp} \\
& -4\sum_p \sum_{j\in G} \left(\sum_{i\in B} + \sum_{\substack{i\in G\\ i\neq j}} \right) S_{l-2}(D^2u|ij)u_{tj}u_{tjp}u_{iip} \\
& +2\sum_p \sum_{\substack{i,j\in G\\ i\neq j}} \left(\sum_{k\in B} + \sum_{\substack{k\in G\\ k\neq i,k\neq j}} \right) S_{l-3}(D^2u|ijk)u_{ti}u_{tj}u_{ijp}u_{kkp} \\
\sim & \; S_{l-1}(G)\sum_p \sum_{\substack{i,j\in B\\ i\neq j}} u_{iip}u_{jjp} + 2\sum_p \sum_{j\in G} S_{l-1}(G|j)u_{jjp}\sum_{i\in B} u_{iip} \\
& -4\sum_p \sum_{\substack{i,j\in G\\ i\neq j}} S_{l-2}(G|ij)u_{tj}u_{tjp}u_{iip} - 4\sum_p \sum_{j\in G} S_{l-2}(G|j)u_{tj}u_{tjp}\sum_{i\in B} u_{iip} \\
& +2\sum_p \sum_{\substack{i,j\in G\\ i\neq j}} S_{l-3}(G|ij)u_{ti}u_{tj}u_{ijp}\sum_{k\in B} u_{kkp} \\
& +2\sum_p \sum_{\substack{i,j,k\in G\\ i\neq j,i\neq k,j\neq k}} S_{l-3}(G|ijk)u_{ti}u_{tj}u_{ijp}u_{kkp}.
\end{aligned}
$$

Since

$$
\sum_{\substack{i,j\in B\\ i\neq j}} u_{iip}u_{jjp} = \sum_{i\in B} u_{iip}\left(\sum_{j\in B} u_{jjp} - u_{iip} \right) = C_p^2 - \sum_{i\in B} u_{iip}^2, \tag{3.18}
$$

it follows that

$$
III \sim III_1 + III_2 + III_3 + III_4,
$$

where

$$
\begin{aligned}
III_1 &= S_{l-1}(G)\sum_p C_p^2, \\
III_2 &= 2\sum_p C_p[\sum_{j\in G} S_{l-1}(G|j)u_{jjp} - 2\sum_{j\in G} S_{l-2}(G|j)u_{tj}u_{tjp} \\
&\quad + \sum_{\substack{i,j\in G\\ i\neq j}} S_{l-3}(G|ij)u_{ti}u_{tj}u_{ijp}],
\end{aligned}
$$

$$
\begin{aligned}
III_3 =& -4\sum_p \sum_{\substack{i,j\in G\\ i\neq j}} S_{l-2}(G|ij)u_{tj}u_{tjp}u_{iip}\\
&+2\sum_p \sum_{\substack{i,j,k\in G\\ i\neq j, i\neq k, j\neq k}} S_{l-3}(G|ijk)u_{ti}u_{tj}u_{ijp}u_{kkp},\\
III_4 =& -S_{l-1}(G)\sum_p \sum_{i\in B} u_{iip}^2.
\end{aligned}
$$

We apply (3.16) and (3.17) to IV and obtain that,

$$
\begin{aligned}
IV =& 2\sum_p \left(\sum_{i\in G} + \sum_{i\in B}\right) S_{l-1}(D^2u|i)u_{ttp}u_{iip}\\
\sim& 2S_{l-1}(G)\sum_p \sum_{i\in B} u_{ttp}u_{iip} + 2\sum_p \sum_{i\in G} S_{l-1}(G|i)u_{ttp}u_{iip}\\
\sim& 2\sum_p [S_{l-1}(G)C_p + \sum_{i\in G} S_{l-1}(G|i)u_{iip}][-\left(1+\sum_{j\in G}\frac{u_{tj}^2}{\lambda_j^2}\right)C_p\\
&-\sum_{i\in G}\frac{u_{ti}}{\lambda_i}B_{ip} + 2\sum_{i\in G}\frac{u_{ti}}{\lambda_i}u_{tip}]\\
=& IV_1 + IV_2 + IV_3,
\end{aligned}
$$

where

$$
\begin{aligned}
IV_1 =& -2S_{l-1}(G)\left(1+\sum_{j\in G}\frac{u_{tj}^2}{\lambda_j^2}\right)\sum_p C_p^2,\\
IV_2 =& 2\sum_p C_p[2S_{l-1}(G)\sum_{i\in G}\frac{u_{ti}}{\lambda_i}u_{tip} - S_{l-1}(G)\sum_{i\in G}\frac{u_{ti}}{\lambda_i}B_{ip}\\
&-\left(1+\sum_{j\in G}\frac{u_{tj}^2}{\lambda_j^2}\right)\sum_{i\in G} S_{l-1}(G|i)u_{iip}],\\
IV_3 =& 4\sum_p \sum_{j\in G} S_{l-1}(G|j)u_{jjp}\sum_{i\in G}\frac{u_{ti}}{\lambda_i}u_{tip} - 2\sum_p \sum_{j\in G} S_{l-1}(G|j)u_{jjp}\sum_{i\in G}\frac{u_{ti}}{\lambda_i}B_{ip}.
\end{aligned}
$$

Again by (3.13) and (3.15), we have

$$
\begin{aligned}
V \sim& u_{tt}\sum_p \left(\sum_{\substack{i\in G\\ j\in B}} + \sum_{\substack{i\in B\\ j\in G}} + \sum_{\substack{i,j\in G\\ i\neq j}} + \sum_{\substack{i,j\in B\\ i\neq j}}\right) S_{l-2}(D^2u|ij)u_{iip}u_{jjp}\\
&-u_{tt}\sum_p \left(\sum_{\substack{i\in G\\ j\in B}} + \sum_{\substack{i\in B\\ j\in G}} + \sum_{\substack{i,j\in G\\ i\neq j}} + \sum_{\substack{i,j\in B\\ i\neq j}}\right) S_{l-2}(D^2u|ij)u_{ijp}^2
\end{aligned}
$$

$$-\sum_p\sum_{k\in G}\left(\sum_{\substack{i,j\in B\\ i\neq j}}+\sum_{\substack{i\in B\\ j\in G,j\neq k}}+\sum_{\substack{j\in B\\ i\in G,i\neq k}}+\sum_{\substack{i,j\in G\\ i\neq j,i\neq k,j\neq k}}\right)S_{l-3}(D^2u|ijk)u_{tk}^2u_{iip}u_{jjp}$$

$$+\sum_p\sum_{k\in G}\left(\sum_{\substack{i,j\in B\\ i\neq j}}+\sum_{\substack{i\in B\\ j\in G,j\neq k}}+\sum_{\substack{j\in B\\ i\in G,i\neq k}}+\sum_{\substack{i,j\in G\\ i\neq j,i\neq k,j\neq k}}\right)S_{l-3}(D^2u|ijk)u_{tk}^2u_{ijp}^2,$$

it follows that

$$\begin{aligned}V\sim\ & u_{tt}S_{l-2}(G)\sum_p\sum_{\substack{i,j\in B\\ i\neq j}}u_{iip}u_{jjp}+u_{tt}\sum_p\sum_{\substack{i,j\in G\\ i\neq j}}S_{l-2}(G|ij)u_{iip}u_{jjp}\\ &+2u_{tt}\sum_p\sum_{j\in G}S_{l-2}(G|j)u_{jjp}\sum_{i\in B}u_{iip}-u_{tt}S_{l-2}(G)\sum_p\sum_{\substack{i,j\in B\\ i\neq j}}u_{ijp}^2\\ &-u_{tt}\sum_p\sum_{\substack{i,j\in G\\ i\neq j}}S_{l-2}(G|ij)u_{ijp}^2-2u_{tt}\sum_p\sum_{i\in B}\sum_{j\in G}S_{l-2}(G|j)u_{ijp}^2\\ &-\sum_p\sum_{k\in G}\sum_{\substack{i,j\in B\\ i\neq j}}S_{l-3}(G|k)u_{tk}^2u_{iip}u_{jjp}-\sum_p\sum_{\substack{i,j,k\in G\\ i\neq j,i\neq k,j\neq k}}S_{l-3}(G|ijk)u_{tk}^2u_{iip}u_{jjp}\\ &-2\sum_p\sum_{\substack{j,k\in G\\ j\neq k}}S_{l-3}(G|jk)u_{tk}^2u_{jjp}\sum_{i\in B}u_{iip}+\sum_p\sum_{k\in G}\sum_{\substack{i,j\in B\\ i\neq j}}S_{l-3}(G|k)u_{tk}^2u_{ijp}^2\\ &+\sum_p\sum_{\substack{i,j,k\in G\\ i\neq j,i\neq k,j\neq k}}S_{l-3}(G|ijk)u_{tk}^2u_{ijp}^2+2\sum_p\sum_{i\in B}\sum_{\substack{j,k\in G\\ j\neq k}}S_{l-3}(G|jk)u_{tk}^2u_{ijp}^2.\end{aligned}\tag{3.19}$$

In order to simplify the above expression, we still need to use Proposition 2.4, (3.14) and (3.18). By the first term and seventh term in (3.19), we get

$$\begin{aligned}&u_{tt}S_{l-2}(G)\sum_p\sum_{i,j\in B,i\neq j}u_{iip}u_{jjp}-\sum_p\sum_{k\in G}\sum_{i,j\in B,i\neq j}S_{l-3}(G|k)u_{tk}^2u_{iip}u_{jjp}\\ &=S_{l-2}(G)\left[u_{tt}-\sum_k\frac{S_{l-3}(G|k)}{S_{l-2}(G)}u_{tk}^2\right]\sum_p\left(C_p^2-\sum_{i\in B}u_{iip}^2\right)\\ &\sim\sum_{k\in G}S_{l-2}(G|k)\frac{u_{tk}^2}{\lambda_k}\sum_p\left(C_p^2-\sum_{i\in B}u_{iip}^2\right).\end{aligned}$$

Coupling the second term and the eight term, we have

$$u_{tt}\sum_p\sum_{\substack{i,j\in G\\ i\neq j}} S_{l-2}(G|ij)u_{iip}u_{jjp}-\sum_p\sum_{\substack{i,j,k\in G\\ i\neq j,i\neq k,j\neq k}} S_{l-3}(G|ijk)u_{tk}^2u_{iip}u_{jjp}$$
$$\sim 2\sum_p\sum_{\substack{i,j\in G\\ i\neq j}} S_{l-2}(G|ij)\frac{u_{ti}^2}{\lambda_i}u_{iip}u_{jjp}.$$

Combining the third term and the ninth term, we obtain

$$2u_{tt}\sum_p\sum_{j\in G} S_{l-2}(G|j)u_{jjp}\sum_{i\in B}u_{iip}-2\sum_p\sum_{\substack{j,k\in G\\ j\neq k}} S_{l-3}(G|jk)u_{tk}^2u_{jjp}\sum_{i\in B}u_{iip}$$
$$=2\sum_p C_p[\sum_{j\in G} S_{l-2}(G|j)\frac{u_{tj}^2}{\lambda_j}u_{jjp}+\sum_{\substack{j,k\in G\\ j\neq k}} S_{l-2}(G|jk)\frac{u_{tk}^2}{\lambda_k}u_{jjp}].$$

By the fourth term and the tenth term, it follows that

$$-u_{tt}S_{l-2}(G)\sum_p\sum_{\substack{i,j\in B\\ i\neq j}} u_{ijp}^2+\sum_p\sum_{k\in G}\sum_{\substack{i,j\in B\\ i\neq j}} S_{l-3}(G|k)u_{tk}^2u_{ijp}^2$$
$$=-\sum_{k\in G} S_{l-2}(G|k)\frac{u_{tk}^2}{\lambda_k}\sum_p\sum_{\substack{i,j\in B\\ i\neq j}} u_{ijp}^2.$$

Using the fifth term and the eleventh term we get

$$-u_{tt}\sum_p\sum_{\substack{i,j\in G\\ i\neq j}} S_{l-2}(G|ij)u_{ijp}^2+\sum_p\sum_{\substack{i,j,k\in G\\ i\neq j,i\neq k,j\neq k}} S_{l-3}(G|ijk)u_{tk}^2u_{ijp}^2$$
$$=-2\sum_p\sum_{\substack{i,j\in G\\ i\neq j}} S_{l-2}(G|ij)\frac{u_{ti}^2}{\lambda_i}u_{ijp}^2.$$

At last, by the sixth term and the twelfth term, we have

$$-2u_{tt}\sum_p\sum_{i\in B}\sum_{j\in G} S_{l-2}(G|j)u_{ijp}^2+2\sum_p\sum_{i\in B}\sum_{\substack{j,k\in G\\ j\neq k}} S_{l-3}(G|jk)u_{tk}^2u_{ijp}^2$$
$$=-2\sum_p\sum_{i\in B}\sum_{j\in G} S_{l-2}(G|j)\frac{u_{tj}^2}{\lambda_j}u_{ijp}^2-2\sum_p\sum_{i\in B}\sum_{\substack{j,k\in G\\ j\neq k}} S_{l-2}(G|jk)\frac{u_{tk}^2}{\lambda_k}u_{ijp}^2.$$

So we can write

$$V \sim V_1 + V_2 + V_3 + V_4,$$

where

$$V_1 = \sum_p \sum_{j\in G} S_{l-2}(G|j)\frac{u_{tj}^2}{\lambda_j}C_p^2,$$

$$V_2 = 2\sum_p C_p\left[\sum_{j\in G} S_{l-2}(G|j)\frac{u_{tj}^2}{\lambda_j}u_{jjp} + \sum_{\substack{j,k\in G\\ j\neq k}} S_{l-2}(G|jk)\frac{u_{tk}^2}{\lambda_k}u_{jjp}\right],$$

$$V_3 = 2\sum_p \sum_{\substack{i,j\in G\\ i\neq j}} S_{l-2}(G|ij)\frac{u_{ti}^2}{\lambda_i}u_{iip}u_{jjp} - 2\sum_p \sum_{\substack{i,j\in G\\ i\neq j}} S_{l-2}(G|ij)\frac{u_{ti}^2}{\lambda_i}u_{ijp}^2,$$

$$V_4 = -\sum_{k\in G} S_{l-2}(G|k)\frac{u_{tk}^2}{\lambda_k}\sum_p \sum_{i,j\in B} u_{ijp}^2 - 2\sum_p \sum_{i\in B}\sum_{j\in G} S_{l-2}(G|j)\frac{u_{tj}^2}{\lambda_j}u_{ijp}^2$$
$$-2\sum_p \sum_{i\in B}\sum_{\substack{j,k\in G\\ j\neq k}} S_{l-2}(G|jk)\frac{u_{tk}^2}{\lambda_k}u_{ijp}^2.$$

Now we let $II + III + IV + V = T_1 + T_2 + T_3 + T_4$. Making use of some basic formulas of elementary symmetric function in Proposition 2.4, we have

$$\begin{aligned} T_1 &:= III_1 + IV_1 + V_1 \\ &= \sum_p C_p^2\left[-S_{l-1}(G) - 2S_{l-1}(G)\sum_{j\in G}\frac{u_{tj}^2}{\lambda_j^2} + \sum_{j\in G} S_{l-2}(G|j)\frac{u_{tj}^2}{\lambda_j}\right] \\ &= -\left[S_{l-1}(G) + \sum_{j\in G} S_{l-2}(G|j)\frac{u_{tj}^2}{\lambda_j}\right]\sum_p C_p^2 - 2\sum_{j\in G} S_{l-1}(G|j)\frac{u_{tj}^2}{\lambda_j^2}\sum_p C_p^2. \end{aligned} \tag{3.20}$$

Similarly, we have

$$\begin{aligned} T_2 &:= III_2 + IV_2 + V_2 \\ &= 2\sum_p C_p\left[\sum_{j\in G} S_{l-1}(G|j)u_{jjp} - 2\sum_{j\in G} S_{l-2}(G|j)u_{tj}u_{tjp}\right. \\ &\quad + \sum_{\substack{i,j\in G\\ i\neq j}} S_{l-3}(G|ij)u_{ti}u_{tj}u_{ijp} + 2S_{l-1}(G)\sum_{i\in G}\frac{u_{ti}}{\lambda_i}u_{tip} \end{aligned}$$

$$-S_{l-1}(G)\sum_{i\in G}\frac{u_{ti}}{\lambda_i}B_{ip}-\left(1+\sum_{j\in G}\frac{u_{tj}^2}{\lambda_j^2}\right)\sum_{i\in G}S_{l-1}(G|i)u_{iip}$$
$$\left.+\sum_{j\in G}S_{l-2}(G|j)\frac{u_{tj}^2}{\lambda_j}u_{jjp}+\sum_{\substack{j,k\in G\\ j\neq k}}S_{l-2}(G|jk)\frac{u_{tk}^2}{\lambda_k}u_{jjp}\right], \tag{3.21}$$

then we use Proposition 2.4 to simplify (3.21). First

$$-2\sum_{j\in G}S_{l-2}(G|j)u_{tj}u_{tjp}+2S_{l-1}(G)\sum_{i\in G}\frac{u_{ti}}{\lambda_i}u_{tip}=2\sum_{i\in G}S_{l-1}(G|i)\frac{u_{ti}}{\lambda_i}u_{tip}, \tag{3.22}$$

and

$$\sum_{j\in G}S_{l-1}(G|j)u_{jjp}-\left(1+\sum_{j\in G}\frac{u_{tj}^2}{\lambda_j^2}\right)\sum_{i\in G}S_{l-1}(G|i)u_{iip}$$
$$+\sum_{\substack{j,k\in G\\ j\neq k}}\frac{u_{tk}^2}{\lambda_k}S_{l-2}(G|jk)u_{jjp}=-S_l(G)\sum_{i\in G}\frac{u_{ti}^2}{\lambda_i^3}u_{iip}. \tag{3.23}$$

From (2.1), it is easy to check that

$$S_{l-1}(G)=S_{l-1}(G|i)+S_{l-1}(G|j)+\lambda_i\lambda_j S_{l-3}(G|ij),\quad \forall i,j\in G,\quad i\neq j.$$

Coupling the above relation and the definition of B_{ip}, we have

$$\sum_{\substack{i,j\in G\\ i\neq j}}S_{l-3}(G|ij)u_{ti}u_{tj}u_{ijp}-S_{l-1}(G)\sum_{i\in G}\frac{u_{ti}}{\lambda_i}B_{ip}$$
$$=-S_{l-1}(G)\sum_{i,j\in G}\frac{u_{ti}}{\lambda_i}\frac{u_{tj}}{\lambda_j}u_{ijp}+\sum_{\substack{i,j\in G\\ i\neq j}}S_{l-3}(G|ij)u_{ti}u_{tj}u_{ijp}$$
$$=-S_{l-1}(G)\sum_{i\in G}\frac{u_{ti}^2}{\lambda_i^2}u_{iip}+\sum_{\substack{i,j\in G\\ i\neq j}}\left[S_{l-3}(G|ij)-\frac{S_{l-1}(G)}{\lambda_i\lambda_j}\right]u_{ti}u_{tj}u_{ijp}$$
$$=-S_{l-1}(G)\sum_{i\in G}\frac{u_{ti}^2}{\lambda_i^2}u_{iip}-2S_l(G)\sum_{\substack{i,j\in G\\ i\neq j}}\frac{u_{ti}u_{tj}}{\lambda_i^2\lambda_j}u_{ijp},$$

it follows that

$$\sum_{\substack{i,j\in G\\ i\neq j}} S_{l-3}(G|ij)u_{ti}u_{tj}u_{ijp} - S_{l-1}(G)\sum_{i\in G}\frac{u_{ti}}{\lambda_i}B_{ip}$$
$$= -S_{l-1}(G)\sum_{i\in G}\frac{u_{ti}^2}{\lambda_i^2}u_{iip} - 2S_l(G)\sum_{i,j\in G}\frac{u_{ti}u_{tj}}{\lambda_i^2\lambda_j}u_{ijp} + 2S_l(G)\sum_{i\in G}\frac{u_{ti}^2}{\lambda_i^3}u_{iip}$$
$$= -2S_l(G)\sum_{i\in G}\frac{u_{ti}}{\lambda_i^2}B_{ip} + 2S_l(G)\sum_{i\in G}\frac{u_{ti}^2}{\lambda_i^3}u_{iip} - S_{l-1}(G)\sum_{i\in G}\frac{u_{ti}^2}{\lambda_i^2}u_{iip}. \tag{3.24}$$

Note that

$$S_l(G)\sum_{i\in G}\frac{u_{ti}^2}{\lambda_i^3}u_{iip} - S_{l-1}(G)\sum_{i\in G}\frac{u_{ti}^2}{\lambda_i^2}u_{iip} + \sum_{j\in G}S_{l-2}(G|j)\frac{u_{tj}^2}{\lambda_j}u_{jjp} = 0. \tag{3.25}$$

Inserting (3.22)–(3.25) to (3.21), we can reduce T_2 to

$$\begin{aligned} T_2 :=\ & III_2 + IV_2 + V_2 \\ =\ & 2\sum_p C_p\Big[2\sum_{i\in G}S_{l-1}(G|i)\frac{u_{ti}}{\lambda_i}u_{tip} - S_l(G)\sum_{i\in G}\frac{u_{ti}^2}{\lambda_i^3}u_{iip} \\ & -2S_l(G)\sum_{i\in G}\frac{u_{ti}}{\lambda_i^2}B_{ip} + 2S_l(G)\sum_{i\in G}\frac{u_{ti}^2}{\lambda_i^3}u_{iip} \\ & -S_{l-1}(G)\sum_{i\in G}\frac{u_{ti}^2}{\lambda_i^2}u_{iip} + \sum_{j\in G}\frac{u_{tj}^2}{\lambda_j}S_{l-2}(G|j)u_{jjp}\Big] \\ =\ & 4\sum_p C_p\sum_{i\in G}S_{l-1}(G|i)\frac{u_{ti}}{\lambda_i}(u_{tip} - B_{ip}). \end{aligned} \tag{3.26}$$

For T_3, we have

$$\begin{aligned} T_3 :=\ & II_1 + III_3 + IV_3 + V_3 \\ =\ & -2\sum_p\sum_{i\in G}S_{l-1}(G|i)u_{tip}^2 + 4\sum_p\sum_{\substack{i,j\in G\\ i\neq j}}S_{l-2}(G|ij)u_{ti}u_{tjp}u_{ijp} \\ & -2\sum_p\sum_{\substack{i,j,k\in G\\ i\neq j, i\neq k, j\neq k}}S_{l-3}(G|ijk)u_{ti}u_{tj}u_{kip}u_{kjp} \\ & -4\sum_p\sum_{\substack{i,j\in G\\ i\neq j}}S_{l-2}(G|ij)u_{tj}u_{tjp}u_{iip} \\ & +2\sum_p\sum_{\substack{i,j,k\in G\\ i\neq j, i\neq k, j\neq k}}S_{l-3}(G|ijk)u_{ti}u_{tj}u_{ijp}u_{kkp} \end{aligned}$$

$$+4\sum_p\sum_{j\in G}S_{l-1}(G|j)u_{jjp}\sum_{i\in G}\frac{u_{ti}}{\lambda_i}u_{tip}-2\sum_p\sum_{j\in G}S_{l-1}(G|j)u_{jjp}\sum_{i\in G}\frac{u_{ti}}{\lambda_i}B_{ip}$$

$$+2\sum_p\sum_{\substack{i,j\in G\\ i\neq j}}S_{l-2}(G|ij)\frac{u_{ti}^2}{\lambda_i}u_{iip}u_{jjp}-2\sum_p\sum_{\substack{i,j\in G\\ i\neq j}}S_{l-2}(G|ij)\frac{u_{ti}^2}{\lambda_i}u_{ijp}^2. \quad (3.27)$$

Combining the second term, the fourth term and the sixth term in (3.27), we get

$$\begin{aligned}
& 4\sum_p\sum_{\substack{i,j\in G\\ i\neq j}}S_{l-2}(G|ij)u_{ti}u_{tjp}u_{ijp}-4\sum_p\sum_{\substack{i,j\in G\\ i\neq j}}S_{l-2}(G|ij)u_{tj}u_{tjp}u_{iip} \\
& +4\sum_p\sum_{j\in G}S_{l-1}(G|j)u_{jjp}\sum_{i\in G}\frac{u_{ti}}{\lambda_i}u_{tip} \\
=\ & 4\sum_p\sum_{\substack{i,j\in G\\ i\neq j}}S_{l-2}(G|ij)u_{ti}u_{tjp}u_{ijp}+4S_l(G)\sum_p\sum_{i\in G}\frac{u_{ti}}{\lambda_i^2}u_{tip}u_{iip} \\
=\ & 4S_l(G)\sum_p\sum_{i,j\in G}\frac{u_{ti}}{\lambda_i\lambda_j}u_{tjp}u_{ijp} \\
=\ & 4\sum_p\sum_{i\in G}S_{l-1}(G|i)u_{tip}B_{ip}. \quad (3.28)
\end{aligned}$$

By using the relation

$$\sum_{i,j,k}=\sum_{i=j=k}+\sum_{i=j,i\neq k}+\sum_{i=k,i\neq j}+\sum_{j=k,i\neq j}+\sum_{i\neq j,i\neq k,j\neq k},$$

it follows that

$$\begin{aligned}
& -2\sum_p\sum_{\substack{i,j,k\in G\\ i\neq j,i\neq k,j\neq k}}S_{l-3}(G|ijk)u_{ti}u_{tj}(u_{kip}u_{kjp}-u_{ijp}u_{kkp}) \\
& +2\sum_p\sum_{\substack{i,j\in G\\ i\neq j}}S_{l-2}(G|ij)\frac{u_{ti}^2}{\lambda_i}(u_{iip}u_{jjp}-u_{ijp}^2) \\
=\ & 2S_l(G)\sum_p\sum_{\substack{i,j,k\in G\\ i\neq j,i\neq k,j\neq k}}\frac{u_{ti}u_{tj}}{\lambda_i\lambda_j\lambda_k}(u_{ijp}u_{kkp}-u_{kip}u_{kjp}) \\
& +2S_l(G)\sum_p\sum_{\substack{i,j\in G\\ i\neq j}}\frac{u_{ti}^2}{\lambda_i^2\lambda_j}(u_{iip}u_{jjp}-u_{ijp}^2) \\
=\ & 2S_l(G)\sum_p\sum_{i,j,k\in G}\frac{u_{ti}u_{tj}}{\lambda_i\lambda_j\lambda_k}(u_{ijp}u_{kkp}-u_{kip}u_{kjp}) \\
=\ & 2\sum_p\sum_{j\in G}S_{l-1}(G|j)u_{jjp}\sum_{i\in G}\frac{u_{ti}}{\lambda_i}B_{ip}-2\sum_p\sum_{i\in G}S_{l-1}(G|i)B_{ip}^2. \quad (3.29)
\end{aligned}$$

Inserting (3.28) and (3.29) to (3.27), we obtain

$$\begin{aligned} T_3 :=&\ II_1 + III_3 + IV_3 + V_3 \\ =&\ -2\sum_p\sum_{i\in G} S_{l-1}(G|i)u_{tip}^2 + 4\sum_p\sum_{i\in G} S_{l-1}(G|i)u_{tip}B_{ip} \\ &-2\sum_p\sum_{i\in G} S_{l-1}(G|i)B_{ip}^2 \\ =&\ -2\sum_p\sum_{i\in G} S_{l-1}(G|i)(u_{tip}-B_{ip})^2. \end{aligned} \tag{3.30}$$

The last class is

$$\begin{aligned} T_4 :=&\ II_2 + III_4 + V_4 \\ =&-2\sum_p\sum_{i\in B}\sum_{j\in G} S_{l-1}(G|j)u_{ijp}^2 - S_{l-1}(G)\sum_p\sum_{\substack{i,j\in B\\ i\neq j}} u_{ijp}^2 - 2S_{l-1}(G)\sum_p\sum_{i\in B} u_{tip}^2 \\ &+4\sum_p\sum_{i\in B}\sum_{j\in G} S_{l-2}(G|j)u_{tj}u_{tip}u_{ijp} - 2\sum_p\sum_{k\in B}\sum_{\substack{i,j\in G\\ i\neq j}} S_{l-3}(G|ij)u_{ti}u_{tj}u_{kip}u_{kjp} \\ &-S_{l-1}(G)\sum_p\sum_{i\in B} u_{iip}^2 - \sum_{k\in G} S_{l-2}(G|k)\frac{u_{tk}^2}{\lambda_k}\sum_p\sum_{i,j\in B} u_{ijp}^2 \\ &-2\sum_p\sum_{i\in B}\sum_{j\in G} S_{l-2}(G|j)\frac{u_{tj}^2}{\lambda_j}u_{ijp}^2 - 2\sum_p\sum_{i\in B}\sum_{\substack{j,k\in G\\ j\neq k}} S_{l-2}(G|jk)\frac{u_{tk}^2}{\lambda_k}u_{ijp}^2. \end{aligned} \tag{3.31}$$

Subsequently, we state the following two claims:
Claim 1: $T_1 + T_2 + T_3 \le 0$;
Claim 2: $T_4 \le 0$.
If **Claim 1** and **Claim 2** are true, it follows that

$$\Delta\phi - \phi_t = II + III + IV + V \sim T_1 + T_2 + T_3 + T_4 \le 0. \tag{3.32}$$

Then we complete the proof of the Lemma 3.1 follows from (3.32).

Now we prove **Claim 1** and **Claim 2**.

Proof of the Claim 1.

Taking the terms in (3.20), (3.26) and (3.30) together, we have

$$\begin{aligned} &T_1 + T_2 + T_3 \\ &= -\left[S_{l-1}(G) + \sum_{j\in G} S_{l-2}(G|j)\frac{u_{tj}^2}{\lambda_j}\right]\sum_p C_p^2 - 2\sum_{i\in G} S_{l-1}(G|i)\frac{u_{ti}^2}{\lambda_i^2}\sum_p C_p^2 \\ &+4\sum_p C_p\sum_{i\in G} S_{l-1}(G|i)\frac{u_{ti}}{\lambda_i}(u_{tip}-B_{ip}) - 2\sum_p\sum_{i\in G} S_{l-1}(G|i)(u_{tip}-B_{ip})^2 \end{aligned}$$

$$
\begin{aligned}
&= -\left[S_{l-1}(G) + \sum_{j\in G} S_{l-2}(G|j)\frac{u_{tj}^2}{\lambda_j}\right]\sum_p C_p^2 \\
&-2\sum_p\sum_{i\in G} S_{l-1}(G|i)\left(u_{tip} - \frac{u_{ti}}{\lambda_i}C_p - B_{ip}\right)^2 \\
&\leq 0.
\end{aligned}
$$

Then we finish the proof of **Claim 1**.

Proof of the Claim 2.

At first, we throw some useless negative terms yield that

$$
\begin{aligned}
T_4 \leq& -2S_{l-1}(G)\sum_p\sum_{i\in B} u_{tip}^2 + 4\sum_p\sum_{i\in B}\sum_{j\in G} S_{l-2}(G|j)u_{tj}u_{tip}u_{ijp} \\
&-2\sum_p\sum_{i\in B}\sum_{j\in G} S_{l-2}(G|j)\frac{u_{tj}^2}{\lambda_j}u_{ijp}^2 - 2\sum_p\sum_{i\in B}\sum_{\substack{j,k\in G\\ j\neq k}} S_{l-2}(G|jk)\frac{u_{tk}^2}{\lambda_k}u_{ijp}^2 \\
&-2\sum_p\sum_{k\in B}\sum_{\substack{i,j\in G\\ i\neq j}} S_{l-3}(G|ij)u_{ti}u_{tj}u_{kip}u_{kjp} \\
=& -2S_{l-1}(G)\sum_p\sum_{i\in B}\left[u_{tip} - \sum_{j\in G}\frac{S_{l-2}(G|j)}{S_{l-1}(G)}u_{tj}u_{ijp}\right]^2 \\
&+2\frac{1}{S_{l-1}(G)}\sum_p\sum_{i\in B}\left[\sum_{j\in G} S_{l-2}(G|j)u_{tj}u_{ijp}\right]^2 \\
&-2\sum_p\sum_{i\in B}\sum_{j\in G} S_{l-2}(G|j)\frac{u_{tj}^2}{\lambda_j}u_{ijp}^2 \\
&-2\sum_p\sum_{i\in B}\sum_{\substack{j,k\in G\\ j\neq k}} S_{l-2}(G|jk)\frac{u_{tk}^2}{\lambda_k}u_{ijp}^2 - 2\sum_p\sum_{k\in B}\sum_{\substack{i,j\in G\\ i\neq j}} S_{l-3}(G|ij)u_{ti}u_{tj}u_{kip}u_{kjp} \\
=& T_{41} + T_{42},
\end{aligned}
$$

where

$$
\begin{aligned}
T_{41} =& \frac{2}{S_{l-1}(G)}\sum_p\sum_{i\in B}\left[\sum_{j\in G} S_{l-2}(G|j)u_{tj}u_{ijp}\right]^2 \\
&-2\sum_p\sum_{i\in B}\sum_{j\in G} S_{l-2}(G|j)\frac{u_{tj}^2}{\lambda_j}u_{ijp}^2 \\
&-2\sum_p\sum_{k\in B}\sum_{\substack{i,j\in G\\ i\neq j}} S_{l-3}(G|ij)u_{ti}u_{tj}u_{kip}u_{kjp},
\end{aligned}
$$

$$T_{42} = -2\sum_p \sum_{i\in B} \sum_{\substack{j,k\in G\\ j\neq k}} S_{l-2}(G|jk)\frac{u_{tk}^2}{\lambda_k}u_{ijp}^2$$
$$= -2S_l(G)\sum_p \sum_{i\in B} \sum_{\substack{j,k\in G\\ j\neq k}} \frac{u_{tk}^2}{\lambda_j\lambda_k^2}u_{ijp}^2.$$

By Proposition 2.4, it is easy to check that for $i, j \in G, i \neq j$,

$$S_{l-2}(G|i)S_{l-2}(G|j) = S_{l-1}(G)S_{l-3}(G|ij) + S_{l-2}^2(G|ij),$$

it follows that

$$\frac{2}{S_{l-1}(G)}\sum_p \sum_{i\in B}\left[\sum_{j\in G} S_{l-2}(G|j)u_{tj}u_{ijp}\right]^2$$
$$= \frac{2}{S_{l-1}(G)}\sum_p \sum_{i\in B}\sum_{j\in G} S_{l-2}^2(G|j)u_{tj}^2u_{ijp}^2$$
$$+\frac{2}{S_{l-1}(G)}\sum_p \sum_{i\in B}\sum_{\substack{j,k\in G\\ j\neq k}} S_{l-2}(G|j)S_{l-2}(G|k)u_{tj}u_{tk}u_{ijp}u_{ikp}$$
$$= \frac{2}{S_{l-1}(G)}\sum_p \sum_{i\in B}\sum_{j\in G} S_{l-2}^2(G|j)u_{tj}^2u_{ijp}^2$$
$$+\frac{2}{S_{l-1}(G)}\sum_p \sum_{i\in B}\sum_{\substack{j,k\in G\\ j\neq k}} \left[S_{l-1}(G)S_{l-3}(G|kj) + S_{l-2}^2(G|kj)\right]u_{tj}u_{tk}u_{ijp}u_{ikp}.$$

So we get

$$T_{41} = \frac{2S_l(G)}{S_{l-1}(G)}\sum_p \sum_{i\in B} \sum_{\substack{j,k\in G\\ j\neq k}} S_{l-2}(G|jk)\frac{u_{tk}}{\lambda_k}\frac{u_{tj}}{\lambda_j}u_{ijp}u_{ikp}$$
$$-2\sum_p \sum_{i\in B}\sum_{j\in G}\left[\frac{S_{l-2}(G|j)}{\lambda_j} - \frac{S_{l-2}^2(G|j)}{S_{l-1}(G)}\right]u_{tj}^2u_{ijp}^2. \tag{3.33}$$

Since

$$\sum_{\substack{j,k\in G\\ j\neq k}} S_{l-2}(G|jk)\frac{u_{tk}}{\lambda_k}\frac{u_{tj}}{\lambda_j}u_{ijp}u_{ikp}$$
$$\leq \frac{1}{2}\sum_{\substack{j,k\in G\\ j\neq k}} S_{l-2}(G|jk)\left(\frac{u_{tk}^2}{\lambda_k^2}u_{ijp}^2 + \frac{u_{tj}^2}{\lambda_j^2}u_{ikp}^2\right)$$
$$= \sum_{\substack{j,k\in G\\ j\neq k}} S_{l-2}(G|jk)\frac{u_{tk}^2}{\lambda_k^2}u_{ijp}^2$$

$$= S_l(G) \sum_{\substack{j,k\in G\\ j\neq k}} \frac{u_{tk}^2}{\lambda_j \lambda_k^3} u_{ijp}^2$$

$$\leq S_{l-1}(G) \sum_{\substack{j,k\in G\\ j\neq k}} \frac{u_{tk}^2}{\lambda_j \lambda_k^2} u_{ijp}^2, \tag{3.34}$$

in last inequality we have used the observation $S_l(G) \leq \lambda_k S_{l-1}(G)$ for $k \in G$, we insert (3.34) to (3.33) obtain that

$$T_{41} \leq 2S_l(G) \sum_p \sum_{i\in B} \sum_{\substack{j,k\in G\\ j\neq k}} \frac{u_{tk}^2}{\lambda_j \lambda_k^2} u_{ijp}^2 - 2 \sum_p \sum_{i\in B} \sum_{j\in G} \frac{S_{l-1}(G|j) S_{l-2}(G|j)}{\lambda_j S_{l-1}(G)} u_{tj}^2 u_{ijp}^2. \tag{3.35}$$

That is to say

$$T_4 \leq T_{41} + T_{42} \leq 0.$$

The **Claim 2** is verified.

The proof of the lemma is complete. □

We give a remark on the proof of the above lemma.

Remark 3.2. In the proof of Step 2, we can use the Lemma 2.5 in Bian and Guan [3] to treat the terms $\sum_{i,j\in B} u_{ijp}^2$. In our case the calculations is simple, so we contain all these terms.

Now it is standard to give the proof of Theorem 1.2.

The proof of Theorem 1.2: Suppose $\widehat{D^2 u}$ attains the minimum rank l $(0 \leq l \leq n)$ at (x_0, t_0), since $\widehat{D^2 u}$ is positive semi-define, we get

$$S_l(\widehat{D^2 u})(x,t) > 0, \qquad \forall (x,t) \in \Omega \times (0,T), \tag{3.36}$$

$$S_{l+1}(\widehat{D^2 u})(x,t) \begin{cases} = 0, \ (x,t) = (x_0, t_0), \\ \geq 0, \ \Omega \times (0,T) \setminus (x_0, t_0). \end{cases} \tag{3.37}$$

Notice that the matrix $D^2 u$ is also positive semi-define, by the relation between $S_k(\widehat{D^2 u})$ and $S_k(D^2 u)$ $(1 \leq k \leq n)$, we claim that $S_{l-1}(D^2 u) > 0$ on $\Omega \times (0,T)$. Otherwise, if there is a point (x_1, t_1) such that $S_{l-1}(D^2 u)(x_1, t_1) = 0$, we choose a local orthonomal frame $\{e_1, e_2, \ldots, e_n\}$ at (x_1, t_1) so that the $D^2 u$ is diagonal. By Lemma 2.5, at (x_1, t_1)

$$S_l(\widehat{D^2 u}) = S_l(D^2 u) + u_{tt} S_{l-1}(D^2 u) - \sum_i S_{l-1}(D^2 u|i) u_{ti}^2$$

$$= -\sum_i S_{l-1}(D^2 u|i) u_{ti}^2$$

$$\leq 0,$$

this is contract to (3.36).

We will divide the proof into two cases.

Case1: $S_l(D^2u)(x_0,t_0)=0$. Since $S_{l-1}(D^2u)>0$ on $\Omega\times(0,T)$, the matrix D^2u must attain the minimum rank $l-1$ at (x_0,t_0). Let $\varphi=S_l(D^2u)$, then

$$\Delta\varphi-\varphi_t\le C_1|\bigtriangledown\varphi|+C_2\varphi$$

holds on $\mathcal{O}\times(t_0-\delta,t_0+\delta)$, where $\delta>0$ is a small positive constant, $\mathcal{O}\subset\Omega$ is a small neighborhood of x_0, such that $S_{l-1}(D^2u)\ge C_0$ on $\mathcal{O}\times(t_0-\delta,t_0+\delta)$, and C_1, C_2 depending only on $||u||_{C^{2,1}}$, and n, C_0 (see the proof of Lemma 4.6 in Sect. 4 or [15]).

By the maximum principle and continuity method, it follows that

$$\varphi(x,t)=S_l(D^2u)(x,t)\equiv 0,\quad\forall(x,t)\in\Omega\times(0,t_0],$$

so we have

$$S_{l+1}(\widehat{D^2u})(x,t)=0,\quad\forall(x,t)\in\Omega\times(0,t_0].\tag{3.38}$$

Coupling (3.36) and (3.38), we obtain that the rank of the matrix $\widehat{D^2u}$ takes the constant l on $\Omega\times(0,t_0]$.

Case2: $S_l(D^2u)(x_0,t_0)>0$.

As $S_l(D^2u)(x_0,t_0)>0$, there exist a small positive constant $\delta>0$ and a small neighborhood $\mathcal{O}\subset\Omega$ of x_0, such that $S_l(D^2u)(x,t)\ge C_0, \forall(x,t)\in\mathcal{O}\times(t_0-\delta,t_0+\delta)$. By the Lemma 3.1, combine the maximum principle and continuity method, we yield $S_{l+1}(\widehat{D^2u})\equiv 0$ on $\Omega\times(0,t_0]$ at once. So the rank of the matrix $\widehat{D^2u}$ always be l on the $\Omega\times(0,t_0]$. □

4. Constant rank theorem of nonlinear heat equation

In this section, we give an application on the calculus in the proof of Theorem 1.2 to the result by Borell [6]. From now on, the function

$$\Phi(x)=\int_{-\infty}^{x}exp(-\lambda^2/2)d\lambda/\sqrt{2\pi},\quad x\in(-\infty,+\infty)$$

denote the distribution function of a $\mathcal{N}(0;1)$-distributed random variable and we let $\Phi^{-1}:[0,1]\to(-\infty,+\infty)$ be its inverse function. We study the standard parabolic equation

$$u_t=\Delta u.$$

Define a new function v as

$$v(y,s):=-\Phi^{-1}\circ u\circ\psi^{-1}(y,s),$$

where $\psi:(x,t)\to(y,s)$ satisfy

$$\begin{cases} y=\dfrac{x}{\sqrt{t}},\\ s=\dfrac{1}{\sqrt{t}},\end{cases}$$

then v is the solution of the equation

$$sv_s + 2\Delta v + y \cdot \nabla v - 2v|\nabla v|^2 = 0. \tag{4.1}$$

Definition 4.1. [6] Let $H_n = \mathbb{R}^n \times \mathbb{R}_+$, and $E \subseteq H_n$, then we say the domain E is parabolically convex if $\psi(E)$ is convex.

Lemma 4.2 *[6] Let Ω be a domain in $\mathbb{R}^n$, then the set $\Omega \times \mathbb{R}_+$ is parabolically convex if and only if Ω is convex.*

Using the Ehrhard inequality [12], Borell proved the following theorem in [6].

Theorem 4.3 *[6] Let D be a domain in $\mathbb{R}^n \times \mathbb{R}$ such that the set $D^+ = D \cap \{t > 0\}$ is parabolically convex. Moreover, assume $A \subseteq \mathbb{R}^n$ is a non-empty domain such that $A \times \{0\} \subseteq D \times \{t = 0\}$ and define*

$$u(\zeta) = \int_A p(\zeta, (x, 0))dx, \quad \zeta \in D^+, \tag{4.2}$$

where $p : D \times D \longrightarrow [0, +\infty)$ is the Green function of the heat operator in D equipped with the Dirichlet boundary condition zero.

Then the function $\Phi^{-1} \circ u \circ \psi^{-1}$ is concave in $\psi(D^+)$, and especially the level sets $\{u > r\}$, $r \geq 0$, are parabolically convex.

Remark 4.4. Borell [6] gave the following example: if $D = \mathbb{R}^n \times \mathbb{R}$, $A = \{x_n \geq 0\}$, and

$$u(x, t) = \int_A exp(-|x - z|^2/2t)dz/\sqrt{2\pi t^n} = \Phi(x_n/\sqrt{t}), \quad (x, t) \in D^+,$$

thus the function $\Phi^{-1} \circ u \circ \psi^{-1}(y, s) = y_n$ is a linear in this particular case.

By Theorem 4.3 [6], we know that there exist a spacetime convex solution for the Eq. 4.1. In the rest of this section, We will prove a *constant rank theorem* for the spacetime hessian of the spacetime convex solution $v(y, s)$ of the Eq. 4.1. First we prove a similar conclusion of Lemma 3.1 is also true for the spacetime convex solution $v(y, s)$ of the Eq. 4.1.

Lemma 4.5 *Let Ω be a domain in $\mathbb{R}^n$, $v \in C^{3,1}(\Omega \times (0, T))$ be a spacetime convex solution of the nonlinear heat Eq. 4.1. Suppose there is a positive constant C_0, such that for a fixed integer $n \geq l \geq 1$, $S_l(D^2v)(y_0, s_0) \geq C_0$ for $(y_0, s_0) \in \Omega \times (0, T)$. Then there is a small neighborhood $\mathcal{O} \times (s_0 - \delta, s_0 + \delta)$ of (y_0, s_0), where $\mathcal{O} \subset \Omega$ be a small open subset, and a positive constant C (independent of ϕ) depending only on $||v||_{C^{3,2}}$, n, C_0, such that*

$$s\phi_s(y, s) + 2\Delta\phi(y, s) \leq C(\phi(y, s) + |\nabla\phi|(y, s)), \quad \forall (y, s) \in \mathcal{O} \times (s_0 - \delta, s_0 + \delta), \tag{4.3}$$

where $\phi = S_{l+1}(\widehat{D^2v})$.

Proof. We divide three steps to prove this lemma.

Step 1: We calculate the expression of $s\phi_s + 2\triangle\phi$. Under the conditions of the lemma, there is a small neighborhood $\mathcal{O} \times (s_0 - \delta, s_0 + \delta) \subset \Omega \times (0, T)$, such that the number of the eigenvalue of $D^2 v$ which more than $C_0/2$ at least l at each point. For each fixed $(y, s) \in \mathcal{O} \times (s_0 - \delta, s_0 + \delta)$, we choose a local orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ such that $D^2 v$ is diagonal, and let $v_{ii} = \lambda_i$ for $1 \le i \le n$. All the calculations will be at the point (y, s).

Similar to (3.4) and (3.6), if we take v and (y, s) instead of u and (x, t) in the II, III, IV, V (defined in (3.7)–(3.10)) respectively, and denote them by $\tilde{II}, \tilde{III}, \tilde{IV}, \tilde{V}$ correspondingly, then it is easy to check that

$$s\phi_s + 2\triangle\phi = J + 2(\tilde{II} + \tilde{III} + \tilde{IV} + \tilde{V}),$$

where

$$\begin{aligned} J = &\sum_i S_l(D^2v|i)(sv_s + 2\Delta v)_{ii} + S_l(D^2v)(sv_{sss} + 2\Delta v_{ss}) \\ &+v_{ss}\sum_i S_{l-1}(D^2v|i)(sv_s + 2\Delta v)_{ii} - 2\sum_i S_{l-1}(D^2v|i)v_{si}(sv_{ssi} + 2\Delta v_{si}) \\ &-\sum_{i\neq j} S_{l-2}(D^2v|ij)v_{sj}^2(sv_s+2\Delta v)_{ii} + \sum_{i\neq j} S_{l-2}(D^2v|ij)v_{si}u_{sj}(sv_s+2\Delta v)_{ij}. \end{aligned} \tag{4.4}$$

Step 2: In this step we shall prove the terms

$$\tilde{II} + \tilde{III} + \tilde{IV} + \tilde{V} \le 0.$$

Let $G = \{1, \ldots, l\}$ and $B = \{l + 1, \ldots, n\}$ be the "good" and "bad" sets of indices respectively. All the calculations will be at the point (y, s) using the notation "$\lesssim$". By step 2 in Lemma 3.1, we could obtain

$$\begin{aligned} v_{ii} \sim 0, \quad i \in B; \\ v_{ss} - \textstyle\sum_{i\in G} \frac{v_{si}^2}{v_{ii}} \sim 0 \\ v_{si} \sim 0, \quad i \in B. \end{aligned} \tag{4.5}$$

Similarly, we can obtain the following relation from $\phi_p \sim 0$,

$$v_{ttp} \sim -\left(1 + \sum_{j\in G} \frac{v_{sj}^2}{\lambda_j^2}\right) C_p - \sum_{i\in G} \frac{v_{si}}{\lambda_i} B_{ip} + 2\sum_{i\in G} \frac{v_{si}}{\lambda_i} v_{sip},$$

with

$$C_p = \sum_{i\in B} v_{iip}, \quad B_{ip} = \sum_{j\in G} \frac{v_{sj}}{\lambda_j} v_{ijp}, \quad \text{for} \quad i \in G.$$

By **Claim 1** and **Claim 2** in last section, we can conclude that

$$\tilde{II} + \tilde{III} + \tilde{IV} + \tilde{V} \lesssim 0.$$

Step 3: we shall prove $J \sim 0$.
Differentiating Eq. 4.1 yield that

$$(sv_s + 2\triangle v)_{ij} = 4vv_k v_{ijk} - y_k v_{ijk} - 2v_{ij} + 2v_{ij}|\nabla v|^2 + 4v_i v_k v_{kj} + 4v_j v_k v_{ki} + 4vv_{ki}v_{kj}, \quad (4.6)$$

$$sv_{ssi} + 2\Delta v_{si} = 4vv_k v_{sik} - y_k v_{sik} - 2v_{si} + 2v_{si}|\nabla v|^2 + 4v_i v_k v_{sk} + 4v_s v_k v_{ki} + 4vv_{sk}v_{ki}, \quad (4.7)$$

$$sv_{sss} + 2\Delta v_{ss} = 4vv_k v_{ssk} - y_k v_{ssk} - 2v_{ss} + 2v_{ss}|\nabla v|^2 + 8v_s v_k v_{sk} + 4vv_{sk}^2. \quad (4.8)$$

Inserting (4.6)–(4.8) into (4.4), it follows from directly computation that

$$J = J_1 + J_2,$$

where

$$J_1 = 4v\sum_i v_i\phi_i - \sum_i y_i\phi_i - 2(l+1)\phi + 2(l+1)|\nabla v|^2\phi \sim 0,$$

$$\begin{aligned} J_2 = {} & 4\sum_i S_l(D^2v|i)(2v_i^2 v_{ii} + vv_{ii}^2) + 4S_l(D^2v)(2v_s v_i v_{si} + vv_{si}^2) \\ & + 4v_{ss}\sum_i S_{l-1}(D^2v|i)(2v_i^2 v_{ii} + vv_{ii}^2) - 8\sum_{i,j} S_{l-1}(D^2v|i)v_i v_j v_{si} v_{sj} \\ & - 8\sum_i S_{l-1}(D^2v|i)(v_s v_i v_{ii} + vv_{si}v_{ii})v_{si} \\ & - 4\sum_{i\neq j} S_{l-2}(D^2v|ij)(2v_i^2 v_{ii} + vv_{ii}^2)v_{sj}^2 \\ & + 8\sum_{i\neq j} S_{l-2}(D^2v|ij)v_i v_j v_{si} v_{sj} v_{jj}. \end{aligned}$$

Using (4.5) to simplify the above expression, we obtain

$$\begin{aligned} J_2 \sim {} & 8v_s S_l(G)\sum_{i\in G} v_i v_{si} + 4vS_l(G)\sum_{i\in G} v_{si}^2 + 8v_{ss}S_l(G)\sum_{i\in G} v_i^2 \\ & + 4vv_{ss}S_l(G)\sum_{i\in G} v_{ii} - 8\sum_{i,j\in G} S_{l-1}(G|i)v_i v_j v_{si} v_{sj} - 8v_s S_l(G)\sum_{i\in G} v_i v_{si} \\ & - 8vS_l(G)\sum_{i\in G} v_{si}^2 - 8\sum_{\substack{i,j\in G\\ i\neq j}} S_{l-1}(G|j)v_i^2 v_{sj}^2 - 4v\sum_{\substack{i,j\in G\\ i\neq j}} S_{l-1}(G|j)v_{ii}v_{sj}^2 \\ & + 8\sum_{\substack{i,j\in G\\ i\neq j}} S_{l-1}(G|i)v_i v_j v_{si} v_{sj} \end{aligned}$$

$$\sim -4vS_l(G)\sum_{i\in G}v_{si}^2 + 8S_l(G)\sum_{i\in G}v_i^2\left(v_{ss}-\sum_{\substack{j\in G\\ j\neq i}}\frac{v_{sj}^2}{v_{jj}}\right)$$
$$+4vS_l(G)\sum_{i\in G}v_{ii}(v_{ss}-\sum_{\substack{j\in G\\ j\neq i}}\frac{v_{sj}^2}{v_{jj}}) - 8\sum_{i\in G}S_{l-1}(G|i)v_i^2v_{si}^2$$
$$\sim 0.$$

Then $s\phi_s + 2\Delta\phi \lesssim 0$, and the proof is completed. □

Now we prove another lemma, it comes from the usual calculations as in [15].

Lemma 4.6 *Let Ω be a domain in $\mathbb{R}^n$, $v \in C^{3,1}(\Omega\times(0,+\infty))$ be a space convex solution of the nonlinear heat Eq. 4.1. If $S_{l-1}(D^2v)(y_0,s_0)\geq C_0>0$ $(0\leq l-1\leq n-1)$for some point $(y_0,s_0)\in\Omega\times(0,T)$, then there is a neighborhood $\mathcal{O}\times(s_0-\delta,s_0+\delta)$ of (y_0,s_0), where $\mathcal{O}\subset\Omega$ be an open subset, and a positive constant C (independent of φ) depending only on $||v||_{C^{3,2}}$, n, C_0, such that*

$$s\varphi_s(y,s)+2\Delta\varphi(y,s)\leq C(\varphi(y,s)+|\nabla\varphi|(y,s)),\quad \forall (y,s)\in\mathcal{O}\times(s_0-\delta,s_0+\delta), \tag{4.9}$$

where $\varphi = S_l(D^2v)$.

Proof. Since $S_{l-1}(D^2v)(y_0,s_0)\geq C_0$, there exist a small positive constant $\delta>0$ and a small neighborhood $\mathcal{O}\subset\Omega$ of y_0, such that $S_{l-1}(D^2v)(y_0,s_0)>C_0/2$ for all $(y,s)\in\mathcal{O}\times(s_0-\delta,s_0+\delta)$.

Fix $(y,s)\in\mathcal{O}\times(s_0-\delta,s_0+\delta)$, we choose a local orthonormal frame $\{e_1,e_2,\ldots,e_n\}$ such that D^2v is diagonal. Let $G=\{1,\ldots,l-1\}$, $B=\{l,\ldots,n\}$ be the "good" and "bad" set of indices respectively. All the calculations will be at the point (y,s) using the relation "$\lesssim$". By (4.6)–(4.8) in Guan and Ma [15], we could obtain

$$0\sim\varphi\sim S_{l-1}(G)\sum_{i\in B}v_{ii}\sim v_{ii},\quad i\in B; \tag{4.10}$$
$$0\sim\varphi_\alpha\sim S_{l-1}(G)\sum_{i\in B}v_{ii\alpha}\sim\sum_{i\in B}v_{ii\alpha},\quad for\quad 1\leq\alpha\leq n. \tag{4.11}$$

As the formula (4.15) in Guan and Ma [15], we have

$$\Delta\varphi\sim S_{l-1}(G)\sum_{i\in B}(\Delta v)_{ii}-S_{l-2}(G)\sum_{\alpha=1}^n\sum_{i,j\in B}v_{ij\alpha}^2-2\sum_{\alpha=1}^n\sum_{i\in G,j\in B}S_{l-2}(G|i)v_{ij\alpha}^2.$$

Note that

$$\varphi_s\sim\sum_i S_{l-1}(G|i)v_{sii}\sim S_{l-1}(G)\sum_{i\in B}v_{sii},$$

it follows that

$$s\varphi_s + 2\Delta\varphi \sim S_{l-1}(G)\sum_{i\in B}(sv_s + 2\Delta v)_{ii} - 2S_{l-2}(G)\sum_{\alpha=1}^{n}\sum_{i,j\in B} v_{ij\alpha}^2$$
$$-4\sum_{\alpha=1}^{n}\sum_{i\in G, j\in B} S_{l-2}(G|i)v_{ij\alpha}^2.$$

Using (4.6) and the formulas (4.10) and (4.11), we have for $i \in B$

$$(sv_s + 2\Delta v)_{ii} = 4v\sum_j v_j v_{iij} - \sum_j y_j v_{iij} - 2v_{ii} + 2v_{ii}|\nabla v|^2 + 8v_i^2 v_{ii} + 4vv_{ii}^2$$
$$\sim 0,$$

then

$$\sum_{i\in B}(sv_s + 2\Delta v)_{ii} \sim 0.$$

That is to say

$$s\phi_s + 2\Delta\phi \lesssim 0.$$

Similarly in last section, the constant rank theorem of matrix $\widehat{D^2 v}$ will be as the directly corollary of the above Lemmas 4.5 and 4.6.

Proof of Theorem 1.3: Suppose $\widehat{D^2 v}$ attains the minimum rank $l(0 \le l \le n)$ at (y_0, s_0), then as in the proof of Theorem 1.2, we will divide the proof into two cases.

Case1: $S_l(D^2 v)(y_0, s_0) = 0$.

By using Lemma 4.6 and the maximum principle, we obtain that the rank of the matrix $\widehat{D^2 v}$ takes the constant l on $\Omega \times (0, s_0]$.

Case2: $S_l(D^2 v)(y_0, s_0) > 0$.

As $S_l(D^2 v)(y_0, s_0) > 0$, there exist $\delta > 0$ and $\mathcal{O} \subset \Omega$ such that $S_l(D^2 u)(y, s) \ge C_0, \forall (y, s) \in \mathcal{O} \times (s_0 - \delta, s_0 + \delta)$, where $\mathcal{O}$ is a neighborhood of y_0, C_0 depend on $||v||_{C^{2,1}}$, and n. We yield $S_{l+1}(\widehat{D^2 v}) \equiv 0$ on $\Omega \times (0, s_0]$ by the Lemma 4.5 and the maximum principle at once. So the rank of the matrix $\widehat{D^2 v}$ always be l on the $\Omega \times (0, s_0]$. □

Remark 4.7. We apply Theorem 1.3 to the function studied by Borell [6] , and yield that the spacetime concave function $\Phi^{-1} \circ u \circ \psi^{-1}$ u defined in (4.2) has constant rank property on $\psi(D^+)$. Moreover, for generally parabolically convex domain D, we can use the strong maximum principle (see in Friedman [13, p. 34]), and get a similar statement.

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