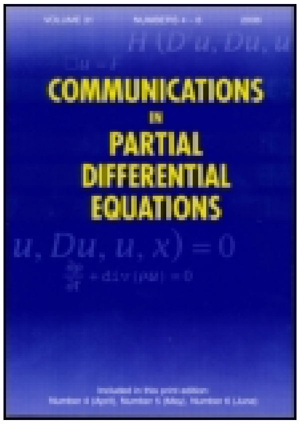

Communications in Partial Differential Equations
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lpde20

The Convexity Estimates for the Solutions of Two Elliptic Equations

Xi-Nan Ma [a], Shujun Shi [a b] & Yu Ye [a]

[a] School of Mathematical Sciences, University of Science and Technology of China, Hefei, China

[b] School of Mathematical Sciences, Harbin Normal University, Harbin, China

Accepted author version posted online: 18 Sep 2012.Version of record first published: 05 Nov 2012.

To cite this article: Xi-Nan Ma, Shujun Shi & Yu Ye (2012): The Convexity Estimates for the Solutions of Two Elliptic Equations, Communications in Partial Differential Equations, 37:12, 2116-2137

To link to this article: http://dx.doi.org/10.1080/03605302.2012.727129

Communications in Partial Differential Equations, 37: 2116–2137, 2012

ISSN 0360-5302 print/1532-4133 online
DOI: 10.1080/03605302.2012.727129

The Convexity Estimates for the Solutions of Two Elliptic Equations

XI-NAN MA[1], SHUJUN SHI[1,2], AND YU YE[1]

[1]School of Mathematical Sciences, University of Science and Technology of China, Hefei, China
[2]School of Mathematical Sciences, Harbin Normal University, Harbin, China

In this paper, for the solutions of two elliptic equations we find the auxiliary curvature functions which attain respective minimum on the boundary. These results are the generalization of the classical ones in Makar-Limanov [17] for the torsion equation and Acker et al. [1] for the first eigenfunction of the Laplacian in convex domains of dimension 2. Then we get the new proof of the specific convexity of the solutions of the above two elliptic equations. As a consequence, for the elliptic equation $v\Delta v = -(1+|\nabla v|^2)$ in a smooth, bounded and strictly convex domain Ω in $\mathbb{R}^n$ with homogeneous Dirichlet boundary value condition, we also get a sharply lower bound estimate of the Gaussian curvature for the solution surface by the curvature of the boundary of the domain.

Keywords Convex domain; Convexity estimates; First eigenvalue problem; Torsion problem.

Mathematics Subject Classification 35B30; 35J25; 35E10.

1. Introduction and Results

Let Ω be a bounded convex domain in $\mathbb{R}^n$. In this paper, we will consider the convexity estimates for the homogeneous Dirichlet problems of two elliptic partial differential equations. The first one is the torsion problem:

$$\begin{cases} \Delta u = -2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

And the second one is the first eigenvalue problem of the Laplace operator:

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Received October 12, 2011; Accepted August 22, 2012
Address correspondence to Shujun Shi, School of Mathematical Sciences, University of Harbin Normal University, Harbin 150025, China; E-mail: shjshi@mail.ustc.edu.cn

In 1971, Makar-Limanov [17] considered the boundary value problem (1.1) in a bounded plane convex domain Ω. He introduced the function

$$P_1 = 2u \det D^2 u + 2u_1 u_2 u_{12} - u_{11} u_2^2 - u_{22} u_1^2$$

and proved that P_1 is a superharmonic function. Then he could obtain that $u^{\frac{1}{2}}$ is strictly concave.

In 1976, Brascamp and Lieb [4] established the log-concavity of the first eigenfunction of the Laplace operator for the problem (1.2) in convex domains. For the case of dimension two, Acker et al. [1] utilized the idea of Makar-Limanov [17] to find the following function

$$P_2 = \frac{1}{u}\left[u \det D^2 u + 2u_1 u_2 u_{12} - u_{11} u_2^2 - u_{22} u_1^2\right],$$

which satisfies the following elliptic differential equality:

$$\Delta P_2 = 0, \quad \mathrm{mod}(\nabla P_2) \text{ in } \Omega \backslash \{x \in \Omega | \Theta(x) = 0\},$$

where we have suppressed the terms containing the gradient of P_2 with locally bounded coefficients, and $\Theta(x) = 4v_{12}^2 + (v_{11} - v_{22})^2$ for $v = -\log u$. Then they obtained a new proof for the Brascamp-Lieb's result in two dimensional case.

Now we state a brief history on the convexity of elliptic PDE. In 1983, Korevaar [11] introduced a very useful technique to study the convexity of the solution for a class of elliptic equations. The new proofs of the log-concavity of the first eigenfunction of convex domains were given respectively by Korevaar [12] and Caffarelli and Spruck [6]. In different extent, Kawohl [9] and Kennington [10] improved Korevaar's methods, which enabled them to give a higher dimensional generalization of the result of Makar-Limanov [17]. In particular, Kennington pointed out that the concavity number $\frac{1}{2}$ of u is sharp in the problem (1.1) in higher dimensional case. Singer et al. [18] and Caffarelli and Friedman [5] introduced a new deformation technique to deal with the convexity, and Caffarelli and Friedman [5] established the strict convexity of the solution for some equations in 2-dimensional convex domains. Korevaar and Lewis [13] generalized the deformation method to higher dimensions, and obtained the strict concavity of $u^{\frac{1}{2}}$ in (1.1) in higher dimensional case. A survey of this subject is given by Kawohl [8] and Guan and Ma [7].

In this paper, we generalize the technique of Makar-Limanov [17] and Acker et al. [1] to the higher dimensional case. We also find new corresponding auxiliary functions, and modulo the gradient terms we prove that they are superharmonic under the strict convexity assumption of the solutions. So from the minimum principle we get the convexity estimates for the solutions of (1.1) and (1.2) via boundary data. Combining the deformation methods we can give the new proof of the specific convexity of the solutions of the above two elliptic equations, and we obtain the Gaussian curvature estimate for the graph of $v = -\sqrt{u}$ in the problem (1.1) using the curvature of the boundary of domain.

In order to state our results, we need the standard curvature formula of the level sets of a function (see Trudinger [20]). Since the level sets of the solutions in the problems (1.1) and (1.2) are convex with respect to the normal direction ∇u, we get

$$K(x) = (-1)^{n-1} \sum_{i,j=1}^{n} \frac{\partial \det D^2 u}{\partial u_{ij}} u_i u_j |\nabla u|^{-(n+1)}, \tag{1.3}$$

where $K(x)$ is the Gaussian curvature of the level sets of the solution u at $x \in \Omega$.

Now we state our main results. (A function v is called strictly convex here if $D^2 v > 0$. Similarly a domain Ω is called strictly convex if at every point of $\partial\Omega$ all principal curvatures are strictly positive.)

Theorem 1.1. *Let Ω be a smooth, bounded convex domain in $\mathbb{R}^n$, $n \geq 2$, and u the solution for the problem* (1.1). *If $v = -\sqrt{u}$ is a strictly convex function, then the function*

$$\psi_1 = (-v)^{n+2} \det D^2 v = (-2)^{-n} u \det D^2 u + (-2)^{-n-1} \sum_{i,j=1}^{n} \frac{\partial \det D^2 u}{\partial u_{ij}} u_i u_j$$

satisfies the following elliptic differential inequality:

$$\Delta \psi_1 \leq 0 \quad mod(\nabla \psi_1) \ in \ \Omega, \tag{1.4}$$

where we have suppressed the terms containing the gradient of ψ_1 with locally bounded coefficients. Moreover, the function ψ_1 attains its minimum on the boundary. So from (1.4), *we have the following estimate for the solution of the problem* (1.1)

$$\psi_1 = (-v)^{n+2} \det D^2 v \geq 2^{-(n+1)} \min_{\partial\Omega} K \min_{\partial\Omega} |\nabla u|^{n+1}, \tag{1.5}$$

where K is the Gaussian curvature of $\partial\Omega$. Finally, the function ψ_1 attains its minimum in Ω if and only if Ω is an ellipsoid (*ellipse*).

For the eigenvalue problem (1.2), we give a similar result as Theorem 1.1, that is

Theorem 1.2. *Let Ω be a smooth, bounded convex domain in $\mathbb{R}^n$, $n \geq 2$, and $u > 0$ the first eigenfunction for the eigenvalue problem* (1.2). *If $v = -\log u$ is a strictly convex function, then the function*

$$\psi_2 = e^{-(n+1)v} \det D^2 v = (-1)^n u \det D^2 u + (-1)^{n-1} \sum_{i,j=1}^{n} \frac{\partial \det D^2 u}{\partial u_{ij}} u_i u_j$$

satisfies the following elliptic differential inequality:

$$\Delta \psi_2 \leq 0 \quad mod(\nabla \psi_2) \ in \ \Omega, \tag{1.6}$$

where we have suppressed the terms containing the gradient of ψ_2 with locally bounded coefficients. Moreover, the function ψ_2 attains its minimum on the boundary. So from (1.6), *we have the following estimate for the solution of the problem* (1.2)

$$\psi_2 = e^{-(n+1)v} \det D^2 v \geq \min_{\partial\Omega} K \min_{\partial\Omega} |\nabla u|^{n+1}, \tag{1.7}$$

where K is the Gaussian curvature of $\partial\Omega$.

From the results in Theorem 1.1, we further give the Gaussian curvature estimate for the graph of $v = -\sqrt{u}$ in the problem (1.1) via the boundary geometry of the domain.

Corollary 1.3. *Let Ω be a smooth, bounded and strictly convex domain in $\mathbb{R}^n$, κ_{min}, κ_{max} and K_{min} the minimal principal curvature, maximal principal curvature and the minimal Gaussian curvature of the boundary $\partial\Omega$ respectively. If u is the solution for* (1.1) *and $v = -\sqrt{u}$, then Gaussian curvature K_G for the graph of v satisfies the following sharp estimate*

$$K_G = \frac{\det D^2 v}{(1+|\nabla v|^2)^{\frac{n+2}{2}}} \geq \frac{K_{min}\kappa_{min}^{n+2}}{n^{\frac{n}{2}}\kappa_{max}^{n+1}} \quad in\ \Omega. \tag{1.8}$$

When we take Ω being the unit ball $B_1(0)$ of dimension n centering at 0, *the equality in* (1.8) *holds at the origin* 0.

For the Gauss curvature estimates of the solution surface to elliptic partial differential equations, Corollary 1.3 is few example which generalize the two dimensional result by Ma [14] for the problem (1.1).

In a different situation, Andrews and Clutterbuck [2] obtained another type convexity estimates for the eigenvalue problem (1.2) and found a beautiful application on the proof of the fundamental gap conjecture.

This paper is organized as follows. In Section 2, we prove Theorem 1.1 through establishing a differential inequality for the given function. Then, we give some estimates on the solution in (1.1) and its gradient on the boundary. Combining these estimates with Theorem 1.1, we obtain the proof of Corollary 1.3. In Section 3, using the same process as the proof of Theorem 1.1, we prove Theorem 1.2.

2. Convexity Estimates for the Torsion Problem

In this section, through establishing a differential inequality for the given function and applying minimum principle, we first prove Theorem 1.1. In order to prove Corollary 1.3, we give two lemmas which involve the estimates of the solution u and the modulus of its gradient $|\nabla u|$ on the boundary $\partial\Omega$. Then Corollary 1.3 is proved by applying Theorem 1.1.

Now, we begin to prove Theorem 1.1

Proof. Let u be the solution for the problem (1.1) and $v = -\sqrt{u}$. Then v is strictly convex from our assumption and satisfies the following problem

$$\begin{cases} \Delta v = -\frac{1+|\nabla v|^2}{v} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

For

$$\psi_1 = (-v)^{n+2}\det D^2 v = (-2)^{-n} u \det D^2 u + (-2)^{-n-1}\sum_{i,j=1}^{n}\frac{\partial \det D^2 u}{\partial u_{ij}} u_i u_j,$$

we shall show that

$$\varphi = \log \psi_1 = (n+2)\log(-v) + \log\det D^2 v$$

satisfies the following elliptic differential inequality:

$$\Delta\varphi \le 0 \quad \mathrm{mod}(\nabla\varphi) \text{ in } \Omega, \tag{2.2}$$

which implies the inequality (1.4). Moreover, we obtain that the function ψ_1 attains its minimum on the boundary by the standard minimum principle. Therefore, from (1.3), we have

$$\begin{aligned}\psi_1 &= (-v)^{n+2}\det D^2 v \ge \min_{\partial\Omega}\psi_1\\ &= 2^{-(n+1)}\min_{\partial\Omega}\{K|\nabla u|^{n+1}\}\\ &\ge 2^{-(n+1)}\min_{\partial\Omega}K\min_{\partial\Omega}|\nabla u|^{n+1},\end{aligned}$$

which is the estimate (1.5).

In order to prove (2.2) at an arbitrary point x_o, we can choose the coordinates at x_o such that the matrix $(v_{ij}(x_o))(1 \le i, j \le n)$ is diagonal. If we can establish (2.2) at x_o under the above coordinates assumption, then go back to the original coordinates we find that (2.2) remain valid with new locally bounded coefficients on $\nabla\varphi$ in (2.2), depending smoothly on the independent variables. Thus it remains to establish (2.2) under the above assumption that the matrix $(v_{ij}(x_o))(1 \le i, j \le n)$ is diagonal. From now on, all the calculation will be done at the fixed point x_o.

Because v is strictly convex, the Hessian matrix (v_{ij}) is positive definite. Let $\lambda_i = v_{ii}$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\sigma_1(\lambda|i) = \sum_{k=1,k\ne i}^n \lambda_k$, $i = 1, 2, \ldots, n$. Let (v^{ij}) be the inverse matrix of (v_{ij}). Taking the first derivative of φ, we have

$$\varphi_i = \frac{(n+2)v_i}{v} + \sum_{k,l=1}^n v^{kl}v_{kli}, \tag{2.3}$$

and it follows that

$$\sum_{k=1}^n v^{kk}v_{kki} = -\frac{(n+2)v_i}{v} + \varphi_i. \tag{2.4}$$

Taking the second derivative of φ, we get

$$\varphi_{ii} = \frac{(n+2)(vv_{ii} - v_i^2)}{v^2} + \sum_{k,l=1}^n v^{kl}v_{klii} - \sum_{k,l,p,q=1}^n v^{kq}v^{pl}v_{kli}v_{pqi}. \tag{2.5}$$

Then

$$\Delta\varphi = \frac{(n+2)(v\Delta v - |\nabla v|^2)}{v^2} + \sum_{k=1}^n v^{kk}\Delta(v_{kk}) - \sum_{k,l,i=1}^n v^{kk}v^{ll}v_{kli}^2 \triangleq A + B + C, \tag{2.6}$$

where

$$A = \frac{(n+2)(v\Delta v - |\nabla v|^2)}{v^2},$$
$$B = \sum_{k=1}^{n} v^{kk}\Delta(v_{kk}),$$
$$C = -\sum_{k,l,i=1}^{n} v^{kk}v^{ll}v_{kli}^2.$$

We will deal with the three terms above respectively. For the term A, we have

$$A = \frac{(n+2)\Delta v}{v} - \frac{(n+2)|\nabla v|^2}{v^2}. \tag{2.7}$$

For the term B, taking derivative of the equation in (2.1), it follows that

$$\begin{aligned}
(\Delta v)_k &= \frac{(1+|\nabla v|^2)v_k}{v^2} - 2\frac{\sum_{i=1}^n v_i v_{ik}}{v},\\
(\Delta v)_{kk} &= -2\frac{(1+|\nabla v|^2)v_k^2}{v^3} + \frac{(1+|\nabla v|^2)v_{kk} + 2\sum_{i=1}^n v_i v_{ik} v_k}{v^2}\\
&\quad + 2\frac{\sum_{i=1}^n v_i v_{ik} v_k}{v^2} - 2\frac{\sum_{i=1}^n (v_{ik}^2 + v_i v_{ikk})}{v}\\
&= -2\frac{(1+|\nabla v|^2)v_k^2}{v^3} + \frac{(1+|\nabla v|^2)v_{kk} + 4v_k^2 v_{kk}}{v^2} - 2\frac{v_{kk}^2 + \sum_{i=1}^n (v_i v_{ikk})}{v}.
\end{aligned}$$

Thus,

$$\begin{aligned}
B &= \sum_{k}^{n} v^{kk}\Delta(v_{kk}) = \sum_{k}^{n} v^{kk}(\Delta v)_{kk}\\
&= -2\sum_{k=1}^{n} \frac{(1+|\nabla v|^2)v_k^2}{\lambda_k v^3} + \frac{n + (n+4)|\nabla v|^2}{v^2} - 2\frac{\Delta v + \sum_{i=1}^n v_i(-\frac{(n+2)v_i}{v} + \varphi_i)}{v}\\
&= 2\frac{\Delta v}{v^2}\sum_{k=1}^{n}\frac{v_k^2}{\lambda_k} - \frac{(n+2)\Delta v}{v} + \frac{(2n+8)|\nabla v|^2}{v^2} - 2\sum_{i=1}^{n}\frac{v_i\varphi_i}{v},
\end{aligned} \tag{2.8}$$

where we have used (2.4) and (2.1) in the last two equalities. Therefore, combining (2.7) with (2.8), we yield that

$$\begin{aligned}
A + B &= \frac{(n+6)|\nabla v|^2}{v^2} + 2\frac{\Delta v}{v^2}\sum_{k=1}^{n}\frac{v_k^2}{\lambda_k} - 2\sum_{i=1}^{n}\frac{v_i\varphi_i}{v}\\
&= \sum_{k=1}^{n}\left[(n+8) + 2\frac{\sigma_1(\lambda|k)}{\lambda_k}\right]\frac{v_k^2}{v^2} - 2\sum_{i=1}^{n}\frac{v_i\varphi_i}{v}\\
&= \sum_{k=1}^{n}\left[(n+8) + 2\frac{\sigma_1(\lambda|k)}{\lambda_k}\right]\frac{v_k^2}{v^2} \quad \mathrm{mod}(\nabla\varphi).
\end{aligned} \tag{2.9}$$

Next, we treat the term C and have

$$\begin{aligned}
-C &= \sum_{k,l,i=1}^{n} \frac{v_{kli}^2}{\lambda_k\lambda_l} \\
&= \left[\sum_{k=l=i=1}^{n} + \sum_{\substack{k,l,i=1\\ k=l,k\neq i}}^{n} + \sum_{\substack{k,l,i=1\\ k=i,k\neq l}}^{n} + \sum_{\substack{k,l,i=1\\ l=i,k\neq l}}^{n} + \sum_{\substack{k,l,i=1\\ k\neq l,l\neq i,k\neq i}}^{n}\right]\frac{v_{kli}^2}{\lambda_k\lambda_l} \\
&= \sum_{i=1}^{n}\frac{v_{iii}^2}{\lambda_i^2} + \sum_{\substack{k,l=1\\ k\neq l}}^{n}\left(\frac{1}{\lambda_k^2}+\frac{2}{\lambda_k\lambda_l}\right)v_{kkl}^2 + \sum_{\substack{k,l,i=1\\ k\neq l,l\neq i,k\neq i}}^{n}\frac{v_{kli}^2}{\lambda_k\lambda_l} \\
&\geq \sum_{i=1}^{n}\frac{v_{iii}^2}{\lambda_i^2} + \sum_{\substack{k,l=1\\ k\neq l}}^{n}\left(\frac{1}{\lambda_k^2}+\frac{2}{\lambda_k\lambda_l}\right)v_{kkl}^2 \\
&= \sum_{l=1}^{n}\left(\frac{v_{lll}^2}{\lambda_l^2} + \sum_{k=1,k\neq l}^{n}\left(\frac{1}{\lambda_k^2}+\frac{2}{\lambda_k\lambda_l}\right)v_{kkl}^2\right) \\
&\triangleq \sum_{l=1}^{n} C_l,
\end{aligned}$$

where

$$C_l = \frac{v_{lll}^2}{\lambda_l^2} + \sum_{k=1,k\neq l}^{n}\left(\frac{1}{\lambda_k^2}+\frac{2}{\lambda_k\lambda_l}\right)v_{kkl}^2, \quad l = 1,\ldots,n.$$

We claim that

$$C_l \geq \left[(n+8) + 2\frac{\sigma_1(\lambda|l)}{\lambda_l}\right]\frac{v_l^2}{v^2} \quad \mathrm{mod}(\nabla\varphi), \;\; l = 1,2,\ldots,n. \tag{2.10}$$

From this claim, we have

$$C \leq -\sum_{l=1}^{n} C_l \leq -\sum_{l=1}^{n}\left[(n+8) + 2\frac{\sigma_1(\lambda|l)}{\lambda_l}\right]\frac{v_l^2}{v^2} \quad \mathrm{mod}(\nabla\varphi). \tag{2.11}$$

Consequently, (2.6), (2.9) and (2.11) yields

$$\Delta\varphi \leq 0 \quad \mathrm{mod}(\nabla\varphi),$$

which is exactly (2.2).

Now, we prove the claim for $l = 1$ and others are the same completely. Taking derivative of the equation in (2.1) with respect to x_1, we get the equality

$$\begin{aligned}
(\Delta v)_1 &= \frac{(1+|\nabla v|^2)v_1}{v^2} - 2\frac{\sum_{i=1}^{n} v_i v_{i1}}{v} \\
&= -(3\lambda_1 + \sigma_1(\lambda\,|\,1))\frac{v_1}{v},
\end{aligned}$$

that is

$$v_{111} = -\sum_{k=2}^{n} v_{kk1} - (3\lambda_1 + \sigma_1(\lambda \mid 1))\frac{v_1}{v}. \tag{2.12}$$

From (2.4), we have

$$\sum_{k=1}^{n} \frac{v_{kk1}}{\lambda_k} = -\frac{(n+2)v_1}{v} + \varphi_1. \tag{2.13}$$

That (2.13) subtracts (2.12) multiplied by $\frac{1}{\lambda_1}$ yields that

$$\sum_{k=2}^{n}\left(\frac{1}{\lambda_k} - \frac{1}{\lambda_1}\right)v_{kk1} = -\left(n - 1 - \frac{\sigma_1(\lambda \mid 1)}{\lambda_1}\right)\frac{v_1}{v} + \varphi_1. \tag{2.14}$$

Applying (2.12) to C_1, we have

$$\begin{aligned} C_1 &= \frac{1}{\lambda_1^2}\left[\sum_{k=2}^{n} v_{kk1} + (3\lambda_1 + \sigma_1(\lambda \mid 1))\frac{v_1}{v}\right]^2 + \sum_{k=2}^{n}\left(\frac{1}{\lambda_k^2} + \frac{2}{\lambda_k\lambda_1}\right)v_{kk1}^2 \\ &= \frac{1}{\lambda_1^2}\left(\sum_{k=2}^{n} v_{kk1}\right)^2 + \sum_{k=2}^{n}\left(\frac{1}{\lambda_k^2} + \frac{2}{\lambda_k\lambda_1}\right)v_{kk1}^2 \\ &\quad + \frac{2}{\lambda_1^2}(3\lambda_1 + \sigma_1(\lambda \mid 1))\frac{v_1}{v}\left(\sum_{k=2}^{n} v_{kk1}\right) + \frac{1}{\lambda_1^2}(3\lambda_1 + \sigma_1(\lambda \mid 1))^2\left(\frac{v_1}{v}\right)^2. \end{aligned} \tag{2.15}$$

Under the condition (2.14), we will compute the minimum of C_1. Let

$$\begin{aligned} f(y_2, y_3, \ldots, y_n) &= \frac{1}{\lambda_1^2}\left(\sum_{k=2}^{n} y_k\right)^2 + \sum_{k=2}^{n}\left(\frac{1}{\lambda_k^2} + \frac{2}{\lambda_k\lambda_1}\right)y_k^2 \\ &\quad + \frac{2}{\lambda_1^2}(3\lambda_1 + \sigma_1(\lambda \mid 1))\frac{v_1}{v}\left(\sum_{k=2}^{n} y_k\right) \\ &\quad + \frac{1}{\lambda_1^2}(3\lambda_1 + \sigma_1(\lambda \mid 1))^2\left(\frac{v_1}{v}\right)^2 \end{aligned}$$

and

$$g(y_2, y_3, \ldots, y_n) = \sum_{k=2}^{n}\left(\frac{1}{\lambda_k} - \frac{1}{\lambda_1}\right)y_k + \left(n - 1 - \frac{\sigma_1(\lambda \mid 1)}{\lambda_1}\right)\frac{v_1}{v},$$

where we have omitted the gradient term $\nabla\varphi$. We compute the least value of f under the condition $g = 0$.

Case 1: $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. In this case, $g \equiv 0$ and only need to compute the minimum of

$$f = \frac{1}{\lambda_1^2}\left(\sum_{k=2}^{n} y_k\right)^2 + \frac{3}{\lambda_1^2}\sum_{k=2}^{n} y_k^2 + \frac{2(n+2)}{\lambda_1}\frac{v_1}{v}\left(\sum_{k=2}^{n} y_k\right) + (n+2)^2\left(\frac{v_1}{v}\right)^2.$$

Because f is a quadratic polynomial and is strictly convex, f has a unique critical point and must take the least value at this point. We only need to compute critical value of f. Taking partial derivatives for f with respect to $y_k, k = 2, \ldots, n$, we obtain

$$\frac{\partial f}{\partial y_k} = \frac{2}{\lambda_1^2}\left(\sum_{i=2}^{n} y_i\right) + \frac{6}{\lambda_1^2} y_k + \frac{2(n+2)}{\lambda_1}\frac{v_1}{v}.$$

Let $\frac{\partial f}{\partial y_k} = 0, k = 2, \ldots, n$. Solving this system, we get

$$y_2 = y_3 = \cdots = y_n = -\frac{\lambda_1 v_1}{v}.$$

At this unique critical point, f takes the least value $(3n+6)\frac{v_1^2}{v^2}$. Therefore,

$$f(y_2, \ldots, y_n) \geq (3n+6)\frac{v_1^2}{v^2} = \left(n + 8 + \frac{2\sigma_1(\lambda \mid 1)}{\lambda_1}\right)\frac{v_1^2}{v^2}.$$

Case 2: There exists $2 \leq i \leq n$ such that $\lambda_i \neq \lambda_1$. In this case, we use the Lagrange method of multipliers to calculate the minimum. Let

$$\begin{aligned}
&L(y_2, y_3, \ldots, y_n; \mu)\\
&\quad = f(y_2, y_3, \ldots, y_n) + \mu g(y_2, y_3, \ldots, y_n)\\
&\quad = \frac{1}{\lambda_1^2}\left(\sum_{k=2}^{n} y_k\right)^2 + \sum_{k=2}^{n}\left(\frac{1}{\lambda_k^2} + \frac{2}{\lambda_k\lambda_1}\right)y_k^2 + \frac{2}{\lambda_1^2}(3\lambda_1 + \sigma_1(\lambda \mid 1))\frac{v_1}{v}\left(\sum_{k=2}^{n} y_k\right)\\
&\qquad + \frac{1}{\lambda_1^2}(3\lambda_1 + \sigma_1(\lambda \mid 1))^2\left(\frac{v_1}{v}\right)^2 + \mu\left[\sum_{k=2}^{n}\left(\frac{1}{\lambda_k} - \frac{1}{\lambda_1}\right)y_k + \left(n - 1 - \frac{\sigma_1(\lambda \mid 1)}{\lambda_1}\right)\frac{v_1}{v}\right].
\end{aligned}$$

Taking partial derivatives for L, we have, for $k = 2, \ldots, n$,

$$\begin{aligned}
\frac{\partial L}{\partial y_k} &= 2\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_k}\right)^2 y_k + \frac{2}{\lambda_1^2}\sum_{i=2, i\neq k}^{n} y_i + \frac{2}{\lambda_1^2}(3\lambda_1 + \sigma_1(\lambda \mid 1))\frac{v_1}{v} + \left(\frac{1}{\lambda_k} - \frac{1}{\lambda_1}\right)\mu,\\
\frac{\partial L}{\partial \mu} &= \sum_{i=2}^{n}\left(\frac{1}{\lambda_i} - \frac{1}{\lambda_1}\right)y_i + \left(n - 1 - \frac{\sigma_1(\lambda \mid 1)}{\lambda_1}\right)\frac{v_1}{v}.
\end{aligned}$$

Let

$$\frac{\partial L}{\partial y_k} = 0, \quad k = 2, \ldots, n, \quad \text{and} \quad \frac{\partial L}{\partial \mu} = 0,$$

we have

$$\underbrace{\begin{pmatrix} 2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)^2 & \frac{2}{\lambda_1^2} & \dots & \frac{2}{\lambda_1^2} & \frac{1}{\lambda_2}-\frac{1}{\lambda_1} \\ \frac{2}{\lambda_1^2} & 2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_3}\right)^2 & \dots & \frac{2}{\lambda_1^2} & \frac{1}{\lambda_3}-\frac{1}{\lambda_1} \\ \dots & \dots & \dots & \dots & \dots \\ \frac{2}{\lambda_1^2} & \frac{2}{\lambda_1^2} & \dots & 2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_n}\right)^2 & \frac{1}{\lambda_n}-\frac{1}{\lambda_1} \\ \frac{1}{\lambda_2}-\frac{1}{\lambda_1} & \frac{1}{\lambda_3}-\frac{1}{\lambda_1} & \dots & \frac{1}{\lambda_n}-\frac{1}{\lambda_1} & 0 \end{pmatrix}}_{U} \begin{pmatrix} y_2 \\ y_3 \\ \vdots \\ y_n \\ \mu \end{pmatrix} = \begin{pmatrix} -\frac{2}{\lambda_1^2}(3\lambda_1+\sigma_1(\lambda\,|\,1))\frac{v_1}{v} \\ -\frac{2}{\lambda_1^2}(3\lambda_1+\sigma_1(\lambda\,|\,1))\frac{v_1}{v} \\ \vdots \\ -\frac{2}{\lambda_1^2}(3\lambda_1+\sigma_1(\lambda\,|\,1))\frac{v_1}{v} \\ -\left(n-1-\frac{\sigma_1(\lambda\,|\,1)}{\lambda_1}\right)\frac{v_1}{v} \end{pmatrix}.$$

Because of the positive definiteness of the matrix

$$V = \begin{pmatrix} 2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)^2 & \frac{2}{\lambda_1^2} & \dots & \frac{2}{\lambda_1^2} \\ \frac{2}{\lambda_1^2} & 2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_3}\right)^2 & \dots & \frac{2}{\lambda_1^2} \\ \dots & \dots & \dots & \dots \\ \frac{2}{\lambda_1^2} & \frac{2}{\lambda_1^2} & \dots & 2\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_n}\right)^2 \end{pmatrix} \quad \text{and} \quad \mathbf{w} = \begin{pmatrix} \frac{1}{\lambda_2}-\frac{1}{\lambda_1} \\ \frac{1}{\lambda_3}-\frac{1}{\lambda_1} \\ \vdots \\ \frac{1}{\lambda_n}-\frac{1}{\lambda_1} \end{pmatrix} \neq \mathbf{0},$$

$\det(U) = (-\mathbf{w}^T V^{-1}\mathbf{w})\det V < 0$ and so the coefficient matrix U is nonsingular. We can solve the linear system above and obtain the unique minimal value point

$$(y_2, y_3, \dots, y_n) = -\left(\frac{\lambda_2 v_1}{v}, \frac{\lambda_3 v_1}{v}, \dots, \frac{\lambda_n v_1}{v}\right).$$

At this point,

$$f = \left(n + 8 + \frac{2\sigma_1(\lambda\,|\,1)}{\lambda_1}\right)\frac{v_1^2}{v^2}$$

must be the least value. Consequently,

$$f(y_2, \dots, y_n) \geq \left(n + 8 + \frac{2\sigma_1(\lambda\,|\,1)}{\lambda_1}\right)\frac{v_1^2}{v^2}$$

under the condition $g(y_2, \dots, y_n) = 0$.

From the two cases above, we get

$$C_1 \geq \left(n + 8 + \frac{2\sigma_1(\lambda\,|\,1)}{\lambda_1}\right)\frac{v_1^2}{v^2} \mod(\nabla\varphi)$$

and complete the claim (2.10).

Finally, we prove that the function ψ_1 attains its minimum in Ω if and only if Ω is an ellipsoid (ellipse). In fact, from the above process of the proof, we can see further that

$$
\begin{aligned}
-\Delta\varphi &= -(A+B+C) \\
&= \sum_{i=1}^{n} \frac{1}{\lambda_i^2}\left(v_{iii} + \frac{3\lambda_i v_i}{v}\right)^2 + \sum_{\substack{i,k=1 \\ i\neq k}}^{n} \left(\frac{1}{\lambda_i^2} + \frac{2}{\lambda_i\lambda_k}\right)\left(v_{iik} + \frac{\lambda_i v_k}{v}\right)^2 \\
&\quad + \sum_{\substack{i,j,k=1 \\ i\neq j, j\neq k, k\neq i}}^{n} \frac{v_{ijk}^2}{\lambda_j\lambda_k}.
\end{aligned}
$$

If ψ_1 attains its minimum in Ω, then ψ_1 is constant from the strong minimum principle. Therefore, $\Delta\varphi = \Delta \log\psi_1 = 0$ and

$$
v_{ijk} = \begin{cases} -\dfrac{3\lambda_i v_i}{v}, & i=j=k, \\ -\dfrac{\lambda_i v_k}{v}, & i=j, i\neq k, \\ 0, & i\neq j, j\neq k, k\neq i, \end{cases} \tag{2.16}
$$

where $i, j, k = 1, 2, \dots, n$. Because of $u = v^2$,

$$
u_{ijk} = 2(v_i v_{jk} + v_j v_{ki} + v_k v_{ij} + v v_{ijk}).
$$

Note that the matrix (v_{ij}) is diagonal by the choice of the coordinates, from (2.16), we have

$$
u_{ijk} = 0, \quad i, j, k = 1, 2, \dots, n,
$$

that is, all the third derivatives of u are vanish. Since $\Omega = \{x \in \mathbb{R}^n | u(x) > 0\}$ is convex, Ω must be an ellipsoid. On the contrary, if Ω is an ellipsoid, then, up to a translation,

$$
\Omega = \left\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \,\middle|\, \frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} < 1, a_i > 0, i = 1, 2, \dots, n\right\}.
$$

At this time, for the problem (1.1), the solution

$$
u = -b\left(\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_n^2}{a_n^2} - 1\right),
$$

where $b = \left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2}\right)^{-1}$, and the function $\psi_1 \equiv \frac{b^{n+1}}{a_1^2 a_2^2 \cdots a_n^2}$ is constant. Naturally, ψ_1 attains its minimum in Ω. □

Lemma 2.1 ([19]). *If u is a smooth positive solution of the problem* (1.1), *then the function $|\nabla u|^2 + \frac{4u}{n}$ attains its maximum on the boundary $\partial\Omega$ and satisfies*

$$
\max_{\Omega}\left\{|\nabla u|^2 + \frac{4u}{n}\right\} \leq \max_{\partial\Omega} |\nabla u|^2. \tag{2.17}
$$

Lemma 2.2 ([3]). *Let Ω be a smooth, bounded, and strictly convex domain in $\mathbb{R}^n$, $x \in \partial\Omega$, and $\kappa_i(x)$, $i = 1, 2, \ldots, n-1$, all principal curvatures of $\partial\Omega$ at x. Let*

$$\kappa_m(x) = \min\{\kappa_i(x) | i = 1, 2, \ldots, n-1\}, \kappa_M(x) = \max\{\kappa_i(x) | i = 1, 2, \ldots, n-1\},$$
$$\kappa_{min} = \min\{\kappa_m(x) | x \in \partial\Omega\} \quad and \quad \kappa_{max} = \max\{\kappa_M(x) | x \in \partial\Omega\}.$$

If u is a smooth positive solution of the problem (1.1), *then, on the boundary $\partial\Omega$, the modulus of its gradient $|\nabla u|_{\partial\Omega}$ satisfies the following estimate*

$$\frac{2}{n\kappa_{max}} \leq |\nabla u|_{\partial\Omega} \leq \frac{2}{n\kappa_{min}}. \tag{2.18}$$

Proof. For the completion of the paper, we give the proof here. For any boundary point x, let $\Omega \subseteq \Omega_0$ and $\Omega_1 \subseteq \Omega$ be two balls of radius $R = \kappa_{\min}^{-1}$ and $r = \kappa_{\max}^{-1}$ with the property that $x \in \bar{\Omega} \cap \bar{\Omega}_j$, $j = 0, 1$. Let u_{Ω_j}, $j = 0, 1$, be the solution to the problem

$$\begin{cases} \Delta u = -2 & \text{in } \Omega_j, \\ u = 0 & \text{on } \partial\Omega_j. \end{cases}$$

Since u vanishes on $\partial\Omega$, it follows immediately that

$$|\nabla u_{\Omega_1}(x)| \leq |\nabla u(x)| \leq |\nabla u_{\Omega_0}(x)|.$$

An explicit calculation yields

$$|\nabla u_{\Omega_1}(x)| = \frac{2r}{n}, \quad |\nabla u_{\Omega_0}(x)| = \frac{2R}{n}$$

and so

$$\frac{2r}{n} \leq |\nabla u(x)| \leq \frac{2R}{n}.$$

Therefore,

$$\frac{2}{n\kappa_{\max}} \leq |\nabla u|_{\partial\Omega} \leq \frac{2}{n\kappa_{\min}}.$$

□

Next, we start the proof of Corollary 1.3.

Proof. Since $v = -\sqrt{u}$, we have $|\nabla v|^2 = \frac{|\nabla u|^2}{4u}$ and

$$\det D^2 v = (-2)^{-n} u^{-\frac{n}{2}} \det D^2 u + (-2)^{-n-1} u^{-\frac{n+2}{2}} \sum_{i,j=1}^{n} \frac{\partial \det D^2 u}{\partial u_{ij}} u_i u_j.$$

So

$$K_G = \frac{\det D^2 v}{(1 + |\nabla v|^2)^{\frac{n+2}{2}}}$$

$$= \frac{2^{n+2}(-v)^{n+2}\det D^2 v}{2^{n+2}(-v)^{n+2}(1+|\nabla v|^2)^{\frac{n+2}{2}}}$$
$$= \frac{2^{n+2}\psi_1}{(4u+|\nabla u|^2)^{\frac{n+2}{2}}}. \tag{2.19}$$

By the estimate (1.5) in Theorem 1.1 and (2.18) in Lemma 2.2, we have

$$\min_{\bar{\Omega}} \psi_1 \ge 2^{-n-1}\Big(\min_{\partial\Omega} K\Big)\bigg(\frac{2}{n\kappa_{\max}}\bigg)^{n+1}$$
$$= \frac{K_{\min}}{n^{n+1}\kappa_{\max}^{n+1}}. \tag{2.20}$$

For the term $(4u+|\nabla u|^2)^{\frac{n+2}{2}}$ in Ω, by the (2.17) and (2.18), we have

$$4u+|\nabla u|^2 = n\bigg(\frac{|\nabla u|^2}{n}+\frac{4u}{n}\bigg)$$
$$\le n\max_{\Omega}\bigg\{|\nabla u|^2+\frac{4u}{n}\bigg\}$$
$$\le n\max_{\partial\Omega}|\nabla u|^2$$
$$\le \frac{4}{n\kappa_{\min}^2}. \tag{2.21}$$

Therefore, applying (2.20) and (2.21) to (2.19), we obtain

$$K_G(x) \ge \frac{2^{n+2}\frac{K_{\min}}{n^{n+1}\kappa_{\max}^{n+1}}}{\Big(\frac{4}{n\kappa_{\min}^2}\Big)^{\frac{n+2}{2}}} = \frac{K_{\min}\kappa_{\min}^{n+2}}{n^{\frac{n}{2}}\kappa_{\max}^{n+1}} \quad x\in\Omega, \tag{2.22}$$

which is exactly (1.8). If Ω is the unit ball $B_1(0)=\{x\in\mathbb{R}^n \,|\, |x|^2<1\}$, then $u=\frac{1-|x|^2}{n}$ is the solution to the problem (1.1), $v=-\sqrt{u}=-\sqrt{\frac{1-|x|^2}{n}}$ and $\kappa_{\max}=\kappa_{\min}=K_{\min}=1$. By the direct calculations, we get

$$K_G(x) = \frac{\det D^2 v(x)}{(1+|\nabla v(x)|^2)^{\frac{n+2}{2}}} = n^{-\frac{n}{2}}\bigg(1-|x|^2+\frac{|x|^2}{n}\bigg)^{-\frac{n+2}{2}},$$

$K_G(0)=n^{-\frac{n}{2}}$, and the equality holds at the origin 0 in (2.22). We have completed the proof of Corollary 1.3. □

Remark 2.3. When $n=2$, Corollary 1.3 is exactly Theorem 3.4 in [14].

Remark 2.4. From the Theorem 1.1, we can combine the deformation process to give a new proof of strict $\frac{1}{2}$-concavity of the solution u for (1.1), i.e., $v=-\sqrt{u}$ is strictly convex when Ω is a smooth, bounded and strictly convex domain. The deformation process is well-known and can be found in Ma and Xu [16] or Ma et al. [15].

Suppose $0 \in \Omega$. At the initial time, let the domain be the standard unit ball $B_1(0)$ and

$$\Omega_t = (1-t)B_1(0) + t\Omega, \quad 0 \le t \le 1,$$

where the sum is the Minkowski vector sum. So the domain Ω_t is a family of smooth and strictly convex domains. We assume the function v_t satisfies the problem (2.1) in the domain Ω_t, that is

$$\begin{cases} \Delta v_t = -\dfrac{1+|\nabla v_t|^2}{v_t} & \text{in } \Omega_t, \\ v_t = 0 & \text{on } \partial\Omega_t. \end{cases}$$

Now we prove the strict convexity of v. Because Ω_t, $0 \le t \le 1$, are all strictly convex, the Gauss curvature of $\partial\Omega_t$ have an uniformly positive lower bound. Suppose that v is not strictly convex in Ω. Then there exists the first time $0 < t_o \le 1$ such that the determinant $\det D^2 v_{t_o}$ for the Hessian of v_{t_o} becomes zero at some point $x_{t_o} \in \Omega_{t_o}$. Taking a sequence $\{t_k\}$ such that $0 < t_k < t_o$ and $t_k \to t_o$, $k \to \infty$, then from the estimate (1.5) in Theorem 1.1 and the Hopf's Lemma, we get an uniformly positive lower bound for the function sequence $(-v_{t_k})^{n+2} \det D^2 v_{t_k}$, $k = 1, 2, \ldots$. By the Schauder estimates and taking limit for k, we get a positive lower bound on the function $(-v_{t_o})^{n+2} \det D^2 v_{t_o}$. This is a contradiction. Then we complete the proof of the strict convexity of v on the strictly convex domain case.

For the general convex domain, we can first use the approximation with strictly convex domain to get the convexity of v and then obtain its strict convexity through using the constant rank theorem of the Hessian of v_t by Korevaar and Lewis [13].

3. Convexity Estimates for the Eigenvalue Problem

We now give the proof of Theorem 1.2.

Proof. The process of the proof is similar to that of Theorem 1.1.

Suppose that $u > 0$ is the first eigenfunction of Laplace operator, i.e., u is the solution for the eigenvalue problem (1.2) with $\lambda > 0$ being the first eigenvalue. From the assumption, $v = -\log u$ is strictly convex and satisfies the following Dirichlet problem

$$\begin{cases} \Delta v = \lambda + |\nabla v|^2 & \text{in } \Omega, \\ v(x) \to +\infty & \text{as } x \to \partial\Omega. \end{cases} \tag{3.1}$$

For

$$\psi_2 = e^{-(n+1)v} \det D^2 v = (-1)^n u \det D^2 u + (-1)^{n-1} \sum_{i,j=1}^{n} \frac{\partial \det D^2 u}{\partial u_{ij}} u_i u_j,$$

we shall show that

$$\varphi = \log \psi_2 = -(n+1)v + \log \det D^2 v$$

satisfies the following elliptic differential inequality:

$$\Delta \varphi \le 0 \quad \mathrm{mod}(\nabla \varphi) \text{ in } \Omega, \tag{3.2}$$

which implies the inequality (1.6). Moreover, we obtain that the function ψ_2 attains its minimum on the boundary by the standard minimum principle. Therefore, by (1.3), we have

$$\begin{aligned}\psi_2 &= e^{-(n+1)v}\det D^2 v\\ &\geq \min_{\partial\Omega}\psi_2 = \min_{\partial\Omega}\left\{K|\nabla u|^{n+1}\right\}\\ &\geq \min_{\partial\Omega}K\min_{\partial\Omega}|\nabla u|^{n+1},\end{aligned}$$

which is the estimate (1.7).

As in the proof of Theorem 1.1, in order to prove (3.2) at an arbitrary point x_o, we only need to establish (3.2) under the assumption that the matrix $(v_{ij}(x_o))(1 \leq i, j \leq n)$ is diagonal. From now on, all the calculation will be done at the fixed point x_o.

Because v is strictly convex, the Hessian matrix (v_{ij}) is positive definite. Let $\lambda_i = v_{ii} > 0$ and (v^{ij}) the inverse matrix of (v_{ij}). Taking the first derivative of φ, we have

$$\varphi_i = -(n+1)v_i + \sum_{k,l=1}^{n} v^{kl}v_{kli}, \tag{3.3}$$

and it follows that

$$\sum_{k=1}^{n} v^{kk}v_{kki} = (n+1)v_i + \varphi_i. \tag{3.4}$$

Taking the second derivative of φ, we get

$$\varphi_{ii} = -(n+1)v_{ii} + \sum_{k,l=1}^{n} v^{kl}v_{klii} - \sum_{k,l,p,q=1}^{n} v^{kq}v^{pl}v_{kli}v_{pqi}. \tag{3.5}$$

Then

$$\Delta\varphi = -(n+1)\Delta v + \sum_{k}^{n} v^{kk}\Delta(v_{kk}) - \sum_{k,l,i=1}^{n} v^{kk}v^{ll}v_{kli}^2 \triangleq E + F + G, \tag{3.6}$$

where

$$\begin{aligned}E &= -(n+1)\Delta v,\\ F &= \sum_{k=1}^{n} v^{kk}\Delta(v_{kk}),\\ G &= -\sum_{k,l,i=1}^{n} v^{kk}v^{ll}v_{kli}^2.\end{aligned}$$

For the term F, taking derivative of the equation in (3.1), it follows that

$$(\Delta v)_k = 2\sum_{i=1}^{n} v_i v_{ik},$$

$$(\Delta v)_{kk} = 2\sum_{i=1}^{n}(v_{ik}^2 + v_i v_{ikk})$$
$$= 2\left[v_{kk}^2 + \sum_{i=1}^{n} v_i v_{ikk}\right].$$

Thus,

$$F = \sum_{k}^{n} v^{kk}\Delta(v_{kk}) = \sum_{k}^{n} v^{kk}(\Delta v)_{kk}$$
$$= 2\Delta v + 2\sum_{i,k=1}^{n}(v_i v_{ikk} v^{kk})$$
$$= 2\Delta v + 2(n+1)|\nabla v|^2 + 2\sum_{i=1}^{n} v_i \varphi_i, \tag{3.7}$$

where we have used (3.4) in the last equality. Therefore, using (3.7), we yield that

$$E + F = (1-n)\Delta v + 2(n+1)|\nabla v|^2 + 2\sum_{i=1}^{n} v_i\varphi_i$$
$$= (1-n)\lambda + (n+3)|\nabla v|^2 + 2\sum_{i=1}^{n} v_i\varphi_i$$
$$< (n+3)|\nabla v|^2 \mod(\nabla\varphi), \tag{3.8}$$

where we have used the equation in (3.1) and $\lambda > 0$. Next, we treat the term G as C in the proof of Theorem 1.1, we have

$$-G \geq \sum_{l=1}^{n}\left(\frac{v_{lll}^2}{\lambda_l^2} + \sum_{k=1,k\neq l}^{n}\left(\frac{1}{\lambda_k^2} + \frac{2}{\lambda_k\lambda_l}\right)v_{kkl}^2\right) \triangleq \sum_{l=1}^{n} G_l,$$

where

$$G_l = \frac{v_{lll}^2}{\lambda_l^2} + \sum_{k=1,k\neq l}^{n}\left(\frac{1}{\lambda_k^2} + \frac{2}{\lambda_k\lambda_l}\right)v_{kkl}^2, \quad l = 1,\ldots,n.$$

We claim that

$$G_l \geq (n+3)v_l^2 \mod(\nabla\varphi), \;\; l = 1,2,\ldots,n. \tag{3.9}$$

From this claim, we have

$$G \leq -\sum_{l=1}^{n} G_l \leq -(n+3)|\nabla v|^2 \mod(\nabla\varphi). \tag{3.10}$$

Consequently, (3.6), (3.8) and (3.10) yield

$$\Delta\varphi \leq 0 \mod(\nabla\varphi),$$

which is exactly (3.2) and we have completed the proof.

Now, we prove the claim for $l = 1$ and others are the same completely. Taking derivative of the equation in (3.1) with respect to x_1, we get the equality

$$(\Delta v)_1 = 2\sum_{i=1}^{n} v_i v_{i1} = 2\lambda_1 v_1,$$

that is

$$v_{111} = -\sum_{k=2}^{n} v_{kk1} + 2\lambda_1 v_1. \tag{3.11}$$

From (3.4), we have

$$\sum_{k=1}^{n} \frac{v_{kk1}}{\lambda_k} = (n+1)v_1 + \varphi_1. \tag{3.12}$$

That (3.12) subtracts (3.11) multiplied by $\frac{1}{\lambda_1}$ yields that

$$\sum_{k=2}^{n}\left(\frac{1}{\lambda_k} - \frac{1}{\lambda_1}\right)v_{kk1} = (n-1)v_1 + \varphi_1 \tag{3.13}$$

and

$$v_1 = \frac{1}{n-1}\sum_{k=2}^{n} \frac{\lambda_1 - \lambda_k}{\lambda_1\lambda_k} v_{kk1} \quad \mathrm{mod}(\nabla\varphi). \tag{3.14}$$

Applying (3.11) to G_1, we have

$$\begin{aligned} G_1 &= \frac{1}{\lambda_1^2}\left(\sum_{k=2}^{n} v_{kk1} - 2\lambda_1 v_1\right)^2 + \sum_{k=2}^{n}\left(\frac{1}{\lambda_k^2} + \frac{2}{\lambda_k\lambda_1}\right)v_{kk1}^2 \\ &= \frac{1}{\lambda_1^2}\left(\sum_{k=2}^{n} v_{kk1}\right)^2 + \sum_{k=2}^{n}\left(\frac{1}{\lambda_k^2} + \frac{2}{\lambda_k\lambda_1}\right)v_{kk1}^2 - \frac{4}{\lambda_1}v_1\left(\sum_{k=2}^{n} v_{kk1}\right) + 4v_1^2. \end{aligned} \tag{3.15}$$

From (3.14) and (3.15), we get that

$$\begin{aligned} &G_1 - (n+3)v_1^2 \\ &= \frac{1}{\lambda_1^2}\left(\sum_{k=2}^{n} v_{kk1}\right)^2 + \sum_{k=2}^{n}\left(\frac{1}{\lambda_k^2} + \frac{2}{\lambda_k\lambda_1}\right)v_{kk1}^2 \\ &\quad - \frac{4}{(n-1)\lambda_1^2}\left(\sum_{k=2}^{n} \frac{\lambda_1 - \lambda_k}{\lambda_k} v_{kk1}\right)\left(\sum_{l=2}^{n} v_{ll1}\right) - \frac{1}{(n-1)\lambda_1^2}\left(\sum_{k=2}^{n} \frac{\lambda_1 - \lambda_k}{\lambda_k} v_{kk1}\right)^2 \\ &= \frac{1}{\lambda_1^2}\left(\sum_{k=2}^{n} v_{kk1}\right)^2 + \sum_{k=2}^{n}\left(\frac{1}{\lambda_k^2} + \frac{2}{\lambda_k\lambda_1}\right)v_{kk1}^2 \\ &\quad - \frac{1}{(n-1)\lambda_1^2}\left(\sum_{k=2}^{n} \frac{\lambda_1 - \lambda_k}{\lambda_k} v_{kk1}\right)\left(\sum_{l=2}^{n} \frac{\lambda_1 + 3\lambda_l}{\lambda_l} v_{ll1}\right) \end{aligned}$$

$$= \frac{1}{(n-1)\lambda_1^2}\Bigg[\sum_{k=2}^{n}\left(4+(n-2)\left(1+\frac{\lambda_1}{\lambda_k}\right)^2\right)v_{kk1}^2$$
$$+2\sum_{k,l=2,k<l}^{n}\left(n+3-\left(1+\frac{\lambda_1}{\lambda_k}\right)\left(1+\frac{\lambda_1}{\lambda_l}\right)\right)v_{kk1}v_{ll1}\Bigg] \mod(\nabla\varphi). \tag{3.16}$$

By the Lemma 3.1, the matrix

$$\begin{pmatrix} 4+(n-2)\left(1+\frac{\lambda_1}{\lambda_2}\right)^2 & n+3-\left(1+\frac{\lambda_1}{\lambda_2}\right)\left(1+\frac{\lambda_1}{\lambda_3}\right) & \dots & n+3-\left(1+\frac{\lambda_1}{\lambda_2}\right)\left(1+\frac{\lambda_1}{\lambda_n}\right) \\ n+3-\left(1+\frac{\lambda_1}{\lambda_3}\right)\left(1+\frac{\lambda_1}{\lambda_2}\right) & 4+(n-2)\left(1+\frac{\lambda_1}{\lambda_3}\right)^2 & \dots & n+3-\left(1+\frac{\lambda_1}{\lambda_3}\right)\left(1+\frac{\lambda_1}{\lambda_n}\right) \\ \cdots & \cdots & \cdots & \cdots \\ n+3-\left(1+\frac{\lambda_1}{\lambda_n}\right)\left(1+\frac{\lambda_1}{\lambda_2}\right) & n+3-\left(1+\frac{\lambda_1}{\lambda_n}\right)\left(1+\frac{\lambda_1}{\lambda_3}\right) & \dots & 4+(n-2)\left(1+\frac{\lambda_1}{\lambda_n}\right)^2 \end{pmatrix}$$

is semi-positive definite for $a_i = 1+\frac{\lambda_1}{\lambda_{i+1}}$ and $m = n-1$. Therefore, by (3.16), we obtain

$$G_1 - (n+3)v_1^2 \geq 0 \mod(\nabla\varphi).$$

The claim (3.9) is true. □

Now, we give the following lemma for the semi-positive definiteness of a class of matrices.

Lemma 3.1. *Let $m \geq 1$ be an integer, $a_i \geq 1$, $i = 1, \dots, m$, and*

$$M = \begin{pmatrix} 4+(m-1)a_1^2 & m+4-a_1a_2 & \dots & m+4-a_1a_m \\ m+4-a_1a_2 & 4+(m-1)a_2^2 & \dots & m+4-a_2a_m \\ \cdots & \cdots & \cdots & \cdots \\ m+4-a_ma_1 & m+4-a_ma_2 & \dots & 4+(m-1)a_m^2 \end{pmatrix}.$$

Then the matrix M is semi-positive definite.

Proof. The conclusion is trivial for $m = 1$ and so we only consider $m \geq 2$. Let

$$M_i^{(1)} = \begin{pmatrix} m+4-a_ia_1 \\ m+4-a_ia_2 \\ \vdots \\ m+4-a_ia_m \end{pmatrix}, \quad M_i^{(2)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m(a_i^2-1) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th row}.$$

Then,

$$M = \left(M_1^{(1)} + M_1^{(2)},\ M_2^{(1)} + M_2^{(2)},\ \ldots,\ M_m^{(1)} + M_m^{(2)}\right)$$

and we have

$$\det M = \sum_{i_1, i_2, \ldots, i_m = 1}^{2} \det(M_1^{(i_1)}, M_2^{(i_2)}, \ldots, M_m^{(i_m)}).$$

Because

$$(M_1^{(1)}, M_2^{(1)}, \ldots, M_m^{(1)}) = (m+4)(1)_{m\times m} - (a_i a_j)_{m\times m}$$

and the rank of both matrices $(m+4)(1)_{m\times m}$ and $(a_i a_j)_{m\times m}$ is 1, we know that the rank of $(M_1^{(1)}, M_2^{(1)}, \ldots, M_m^{(1)})$ is less than or equal to 2. Therefore,

$$\begin{aligned}
\det M &= \sum_{i_1, i_2, \ldots, i_m = 1}^{2} \det(M_1^{(i_1)}, M_2^{(i_2)}, \ldots, M_m^{(i_m)}) \\
&= \sum_{\substack{i,j=1 \\ i<j}}^{m} \det(M_1^{(2)}, \ldots, M_{i-1}^{(2)}, M_i^{(1)}, M_{i+1}^{(2)}, \ldots, M_{j-1}^{(2)}, M_j^{(1)}, M_{j+1}^{(2)}, \ldots, M_m^{(2)}) \\
&\quad + \sum_{i=1}^{m} \det(M_1^{(2)}, \ldots, M_{i-1}^{(2)}, M_i^{(1)}, M_{i+1}^{(2)}, \ldots, M_m^{(2)}) \\
&\quad + \det(M_1^{(2)}, \ldots, M_m^{(2)}).
\end{aligned} \tag{3.17}$$

We will compute the three terms respectively in the last equality above.

$$\begin{aligned}
I &\triangleq \sum_{\substack{i,j=1 \\ i<j}}^{m} \det(M_1^{(2)}, \ldots, M_{i-1}^{(2)}, M_i^{(1)}, M_{i+1}^{(2)}, \ldots, M_{j-1}^{(2)}, M_j^{(1)}, M_{j+1}^{(2)}, \ldots, M_m^{(2)}) \\
&= \sum_{\substack{i,j=1 \\ i<j}}^{m} ((m+4-a_i^2)(m+4-a_j^2) - (m+4-a_i a_j)^2) \prod_{\substack{k=1 \\ k\neq i, k\neq j}}^{m} m(a_k^2 - 1) \\
&= -(m+4)m^{m-2} \sum_{\substack{i,j=1 \\ i<j}}^{m} (a_i - a_j)^2 \prod_{\substack{k=1 \\ k\neq i, k\neq j}}^{m} (a_k^2 - 1),
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
II &\triangleq \sum_{i=1}^{m} \det(M_1^{(2)}, \ldots, M_{i-1}^{(2)}, M_i^{(1)}, M_{i+1}^{(2)}, \ldots, M_m^{(2)}) \\
&= \sum_{i=1}^{m} (m+4-a_i^2) \prod_{\substack{k=1 \\ k\neq i}}^{m} m(a_k^2 - 1) \\
&= m^{m-1} \sum_{i=1}^{m} (m+4-a_i^2) \prod_{\substack{k=1 \\ k\neq i}}^{m} (a_k^2 - 1),
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}III &\triangleq \det(M_1^{(2)}, \dots, M_m^{(2)})\\ &= \prod_{k=1}^m m(a_k^2 - 1) = m^{m-1} \sum_{i=1}^m (a_i^2 - 1) \prod_{\substack{k=1\\k\neq i}}^m (a_k^2 - 1).\end{aligned} \tag{3.20}$$

Combining (3.19) with (3.20), we get

$$\begin{aligned}II + III &= m^{m-1} \sum_{i=1}^m (m + 4 - a_i^2) \prod_{\substack{k=1\\k\neq i}}^m (a_k^2 - 1) + m^{m-1} \sum_{i=1}^m (a_i^2 - 1) \prod_{\substack{k=1\\k\neq i}}^m (a_k^2 - 1)\\ &= m^{m-1} \sum_{i=1}^m (m + 3) \prod_{\substack{k=1\\k\neq i}}^m (a_k^2 - 1)\\ &= \frac{(m+3)m^{m-1}}{m-1} \sum_{\substack{i,j=1\\i\neq j}}^m (a_j^2 - 1) \prod_{\substack{k=1\\k\neq i,k\neq j}}^m (a_k^2 - 1)\\ &= \frac{(m+3)m^{m-1}}{2(m-1)} \sum_{\substack{i,j=1\\i\neq j}}^m [(a_i^2 - 1) + (a_j^2 - 1)] \prod_{\substack{k=1\\k\neq i,k\neq j}}^m (a_k^2 - 1)\\ &= \frac{(m+3)m^{m-1}}{m-1} \sum_{\substack{i,j=1\\i< j}}^m (a_i^2 + a_j^2 - 2) \prod_{\substack{k=1\\k\neq i,k\neq j}}^m (a_k^2 - 1).\end{aligned} \tag{3.21}$$

Combining the above (3.21) with (3.18), we have

$$\begin{aligned}\det(M) &= I + II + III\\ &= -(m+4)m^{m-2} \sum_{\substack{i,j=1\\i< j}}^m (a_i - a_j)^2 \prod_{\substack{k=1\\k\neq i,k\neq j}}^m (a_k^2 - 1)\\ &\quad + \frac{(m+3)m^{m-1}}{m-1} \sum_{\substack{i,j=1\\i< j}}^m (a_i^2 + a_j^2 - 2) \prod_{\substack{k=1\\k\neq i,k\neq j}}^m (a_k^2 - 1)\\ &= \frac{m^{m-2}}{m-1} \sum_{\substack{i,j=1\\i< j}}^m \Big[m(m+3)(a_i^2 + a_j^2 - 2)\\ &\quad - (m+4)(m-1)(a_i - a_j)^2 \Big] \prod_{\substack{k=1\\k\neq i,k\neq j}}^m (a_k^2 - 1)\\ &= \frac{m^{m-2}}{m-1} \sum_{\substack{i,j=1\\i< j}}^m [4(a_i - a_j)^2 + 2m(m+3)(a_i a_j - 1)] \prod_{\substack{k=1\\k\neq i,k\neq j}}^m (a_k^2 - 1)\\ &\geq 0\end{aligned}$$

by $a_i \geq 1, i = 1, 2, \dots, m$.

For the kth principal minor $[M]_k$ of M, $k = 2, \ldots, m$, according to the same method above, we can get that

$$
\begin{aligned}
[M]_k &= \det\begin{pmatrix} 4+(m-1)a_1^2 & m+4-a_1a_2 & \ldots & m+4-a_1a_k \\ m+4-a_1a_2 & 4+(m-1)a_2^2 & \ldots & m+4-a_2a_k \\ \cdots & \cdots & \cdots & \cdots \\ m+4-a_ka_1 & m+4-a_ka_2 & \ldots & 4+(m-1)a_k^2 \end{pmatrix} \\
&= -(m+4)m^{k-2}\sum_{\substack{i,j=1\\ i<j}}^{k}(a_i-a_j)^2\prod_{\substack{l=1\\ l\neq i, l\neq j}}^{k}(a_l^2-1) \\
&\quad + m^{k-1}\sum_{i=1}^{k}(m+4-a_i^2)\prod_{\substack{l=1\\ l\neq i}}^{k}(a_l^2-1) + \frac{m^k}{k}\sum_{i=1}^{k}(a_i^2-1)\prod_{\substack{l=1\\ l\neq i}}^{k}(a_l^2-1) \\
&\geq -(m+4)m^{k-2}\sum_{\substack{i,j=1\\ i<j}}^{k}(a_i-a_j)^2\prod_{\substack{l=1\\ l\neq i, l\neq j}}^{k}(a_l^2-1) + (m+3)m^{k-1}\sum_{i=1}^{k}\prod_{\substack{l=1\\ l\neq i}}^{k}(a_l^2-1) \\
&\geq -(m+4)m^{k-2}\sum_{\substack{i,j=1\\ i<j}}^{k}(a_i-a_j)^2\prod_{\substack{l=1\\ l\neq i, l\neq j}}^{k}(a_l^2-1) \\
&\quad + \frac{(m+3)m^{k-1}}{m-1}\sum_{\substack{i,j=1\\ i<j}}^{k}(a_i^2+a_j^2-2)\prod_{\substack{l=1\\ l\neq i, l\neq j}}^{k}(a_l^2-1) \\
&= \frac{m^{k-2}}{m-1}\sum_{\substack{i,j=1\\ i<j}}^{k}[4(a_i-a_j)^2+2m(m+3)(a_ia_j-1)]\prod_{\substack{l=1\\ l\neq i, l\neq j}}^{k}(a_l^2-1) \\
&\geq 0,
\end{aligned}
$$

and $[M]_1 = 4 + (m-1)a_1^2 > 0$. Up to now, we obtain that all principal minors of M are nonnegative and so the matrix M is semi-positive definite. □

Remark 3.2. As the discussions in Remark 2.4, from the Theorem 1.2, we can combine the deformation process to give a new proof of log-concavity of the first eigenfunction u for the problem (1.2), i.e., $v = -\log u$ is strictly convex when Ω is a smooth, bounded and convex domain.

Acknowledgments

The first author and the third author were supported by the National Nature Science Foundation of China. The second author was partially supported by the Science Research Program from the Education Department of Heilongjiang Province (Grant No. 11551137). All authors were supported by Wu Wen-Tsun Key Laboratory of Mathematics.

References

[1] Acker, A., Payne, L.E., Philippin, G. (1981). On the convexity of level lines of the fundamental mode in the clamped membrane problem, and the existence of convex solutions in a related free boundary problem. *Z. Angew. Math. Phys.* 32:683–694.

[2] Andrews, B., Clutterbuck, J. (2011). Proof of the fundamental gap conjecture. *J. Amer. Math. Soc.* 24:899–916.

[3] Bandle, C. (1979). On the isoperimetric gradient bounds for Poisson problems and problems of torsional creep. *Z. Angew. Math. Phys.* 30:713–715.

[4] Brascamp, H.J., Lieb, E.H. (1976). On extensions of the Brunn-Minkowski and Prekopa-Leindler theorems, including inequalities for log-concave functions, with an application to the diffusion equation. *J. Funct. Anal.* 22:366–389.

[5] Caffarelli, L.A., Friedman, A. (1985). Convexity of solutions of some semilinear elliptic equations. *Duke Math. J.* 52:431–455.

[6] Caffarelli, L.A., Spruck, J. (1982). Convexity properties of solutions to some classical variational problems. *Commun. Part. Differ. Eqs.* 7:1337–1379.

[7] Guan, P., Ma, X.N. (2005). Convex solutions of fully nonlinear elliptic equations in classical differential geometry. Geometric evolution equations. In: *Contemporary Mathematics*. Vol. 367. Providence, RI: AMS, pp. 115–127.

[8] Kawohl, B. (1985). *Rearrangements and Convexity of Level Sets in PDE. Lecture Notes in Mathematics*. Vol. 1150. New York: Springer.

[9] Kawohl, B. (1986). A remark on N. Korevaar's concavity maximum principle and on the asymptotic uniqueness of solutions to the plasma problem. *Math. Methods Appl. Sci.* 8:93–101.

[10] Kennington, A.U. (1985). Power concavity and boundary value problems. *Indiana Univ. Math. J.* 34:687–704.

[11] Korevaar, N.J. (1983). Capillary surface convexity above convex domains. *Indiana Univ. Math. J.* 32:73–81.

[12] Korevaar, N.J. (1983). Convex solutions to nonlinear elliptic and parabolic boundary value problems. *Indiana Univ. Math. J.* 32:603–614.

[13] Korevaar, N.J., Lewis, J. (1987) Convex solutions of certain elliptic equations have constant rank hessians. *Arch. Rational Mech. Anal.* 91:19–32.

[14] Ma, X.N. (2002). Concavity estimates for a class of nonlinear elliptic equations in two dimensions. *Math. Zeit.* 240:1–11.

[15] Ma, X.N., Ou, Q.Z., Zhang, W. (2010). Gaussian curvature estimates for the convex level sets of p-harmonic functions. *Comm. Pure Appl. Math.* 63:935–971.

[16] Ma, X.N., Xu, L. (2008). The convexity of solution of a class Hessian equation in bounded convex domain in $\mathbb{R}^n$. *J. Funct. Anal.* 255:1713–1723.

[17] Makar-Limanov, L.G. (1971). Solution of Dirichlet's problem for the equation $\Delta u = -1$ on a convex region. *Math. Notes Acad. Sci. USSR* 9:52–53.

[18] Singer, I., Wong, B., Yau, S.T., Yau, S.S.T. (1985). An estimate of gap of the first two eigenvalues in the Schrödinger operator. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 12:319–333.

[19] Sperb, R.P. (1981). *Maximum Principle and Their Applications*. New York: Academic Press.

[20] Trudinger, N.S. (1997). On new isoperimetric inequalities and symmetrization. *J. Reine Angew. Math.* 488:203–220.