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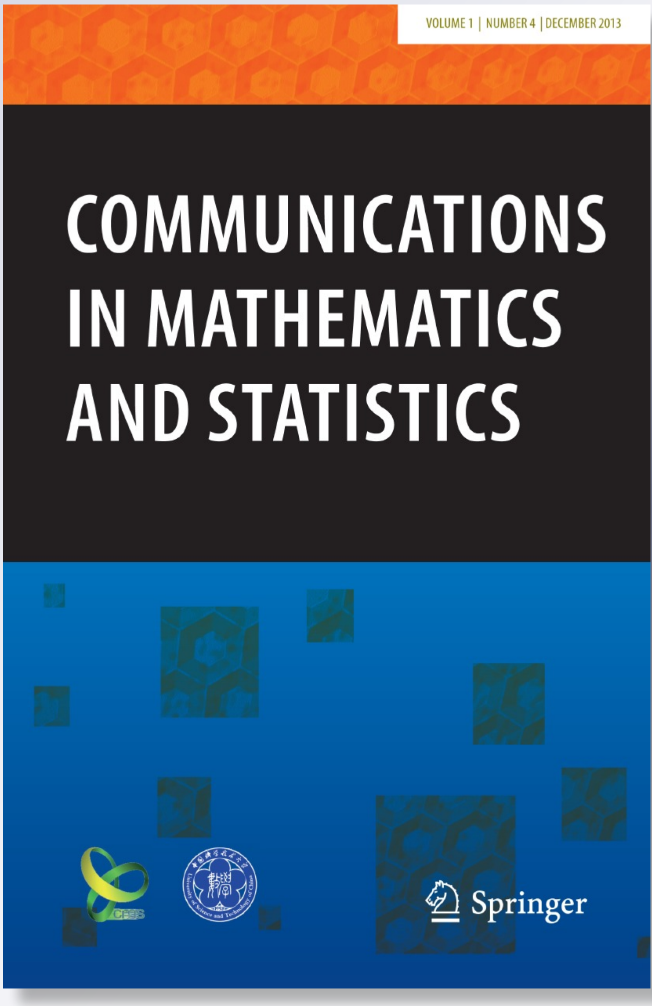

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The Concavity of the Gaussian Curvature of the Convex Level Sets of p-Harmonic Functions with Respect to the Height

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Abstract For the p-harmonic function with strictly convex level sets, we find an auxiliary function which comes from the combination of the norm of gradient of the p-harmonic function and the Gaussian curvature of the level sets of p-harmonic function. We prove that this curvature function is concave with respect to the height of the p-harmonic function. This auxiliary function is an affine function of the height when the p-harmonic function is the p-Green function on ball.

Keywords p-harmonic function · Level set · Gaussian curvature · Support function · Maximum prinicple

Mathematics Subject Classification(2010) 35B45 · 35J92 · 35B50 · 26A51

1 Introduction

In this paper, for the p-harmonic function with strictly convex level sets, we shall explore the relation between the Gaussian curvature of its level sets and the height of the function.

The convexity of the level sets of the solutions of elliptic partial differential equations has been studied for a long time. For instance, Ahlfors [1] contains the well-

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known result that level curves of Green function on convex domain in the plane are the convex Jordan curves. In 1956, Shiffman [23] studied the minimal annulus in $\mathbb{R}^3$ whose boundary consists of two closed convex curves in parallel planes P_1, P_2. He proved that the intersection of the surface with any parallel plane P, between P_1 and P_2, is a convex Jordan curve. In 1957, Gabriel [9] proved that the level sets of the Green function of a 3-dimensional bounded convex domain are strictly convex. In 1977, Lewis [14] extended Gabriel's result to p-harmonic functions in higher dimensions. Caffarelli-Spruck [5] generalized Lewis' [14] results to a class of semilinear elliptic partial differential equations. Motivated by the result of Caffarelli-Friedman [4], Korevaar [13] gave a new proof on the results of Gabriel and Lewis by applying the deformation process and the constant rank theorem of the second fundamental form of the convex level sets of p-harmonic function. A survey of this subject is given by Kawohl [12]. For more recent related extensions, please see the papers by Bian-Guan-Ma-Xu [2] and Bianchini-Longinetti-Salani [3].

Now, we turn to the curvature estimates of level sets of the solutions of elliptic partial differential equations. For 2-dimensional harmonic function and minimal surface with convex level curves, Ortel-Schneider [21], Longinetti [15], and [16] proved that the curvature of the level curves attains its minimum on the boundary (see also Talenti [24] for related results). Longinetti also studied the relation between the curvature of the convex level curves and the height of harmonic function in [16]. Jost-Ma-Ou [11] and Ma-Ye-Ye [20] proved that the Gaussian curvature and the principal curvature of the convex level sets of 3-dimensional harmonic function attain its minimum on the boundary. Then, Ma-Ou-Zhang [19] and Chang-Ma-Yang [6] got the Gaussian curvature and principal curvature estimates of the convex level sets of higher-dimensional harmonic function (in [6], they also treated a class of semilinear elliptic equations) in terms of the Gaussian curvature or principal curvature of the boundary and the norm of the gradient on the boundary. For more recent results on curvature estimates, please see the papers [7,10,25–27] and the references therein.

In this paper, utilizing the support function of the strictly convex level sets and the maximum principle, we obtain the concavity of the Gaussian curvature of the convex level sets with respect to the height of the p-harmonic function.

First, we state the following convexity theorem.

Theorem 1.1 (Gabriel [9] and Lewis [14]). *Let u satisfy*

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & in\ \ \Omega = \Omega_0 \backslash \bar{\Omega}_1, \\ u = 0 & on\ \partial\Omega_0, \\ u = 1 & on\ \partial\Omega_1, \end{cases}$$

where $1 < p < +\infty$, Ω_0, and Ω_1 are bounded smooth convex domains in $\mathbb{R}^n$, $n \geq 2$, $\bar{\Omega}_1 \subset \Omega_0$. Then, all the level sets of u are strictly convex.

Now, we state our main theorem.

Theorem 1.2 *Let u satisfy*

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & in \ \ \Omega = \Omega_0 \backslash \bar{\Omega}_1, \\ u = 0 & on \ \partial\Omega_0, \\ u = 1 & on \ \partial\Omega_1, \end{cases}$$

where $1 < p < +\infty$, Ω_0, and Ω_1 are bounded smooth convex domains in $\mathbb{R}^n$, $n \geq 2$, $\bar{\Omega}_1 \subset \Omega_0$. Let

$$\Gamma_t = \{x \in \Omega | u(x) = t\} \ \ for \ \ 0 < t < 1,$$

and K be the Gaussian curvature of the level sets. Then, the function

$$f(t) = \min_{x \in \Gamma_t} (|\nabla u|^{n+1-2p} K)^{\frac{1}{n-1}}(x)$$

is a concave function for $t \in (0, 1)$.

Under the same assumptions of Theorem 1.2, Ma-Ou-Zhang [19] proved the following statement:

> $for \ n \geq 2$ and $1 < p < +\infty$, the function $|\nabla u|^{n+1-2p} K$ attains its minimum on the boundary.

From this fact, they got the positive lower bound estimates for the Gaussian curvature of the level sets.

The following corollary is straightforward.

Corollary 1.3 *Under the conditions of Theorem 1.2, for any point $x \in \Gamma_t$, $0 < t < 1$, we have the following estimates.*

(1) For $p = 2$, we have

$$(|\nabla u|^{n-3} K)^{\frac{1}{n-1}}(x) \geq (1-t) \min_{\partial\Omega_0} (|\nabla u|^{n-3} K)^{\frac{1}{n-1}} + t \min_{\partial\Omega_1} (|\nabla u|^{n-3} K)^{\frac{1}{n-1}}.$$

(2) For $p = \frac{n+1}{2}$, we have

$$K^{\frac{1}{n-1}}(x) \geq (1-t) \min_{\partial\Omega_0} K^{\frac{1}{n-1}} + t \min_{\partial\Omega_1} K^{\frac{1}{n-1}}.$$

The estimates above contain the norm of the gradient, i.e., $|\nabla u|$, but it is well-known that $|\nabla u|$ attains its maximum and minimum on the boundary [18, 19].

Remark 1.4 Related to the case $n = 2$ in above corollary, Longinetti [16] proved that the function $f(t) = \min_{x \in \Gamma_t} \log K(x)$ is a concave function with respect to t. More precisely, for any point $x \in \Gamma_t$, $0 < t < 1$, he got the following inequality

$$\log K(x) \geq (1-t) \min_{\partial\Omega_0} \log K + t \min_{\partial\Omega_1} \log K.$$

Now, we give an example which shows that our estimates are sharp.

Remark 1.5 Let u be the standard p-Green function of the ball $B_R(0) \subset \mathbb{R}^n$, i.e.,

$$u(x) = \begin{cases} |x|^{\frac{p-n}{p-1}} - R^{\frac{p-n}{p-1}}, & \text{for} \quad 1 < p < n; \\ -\log|x| + \log R, & \text{for} \quad p = n. \end{cases}$$

Then,

$$|\nabla u|(x) = \begin{cases} \frac{n-p}{p-1}|x|^{\frac{1-n}{p-1}}, & \text{for} \quad 1 < p < n, \\ \frac{1}{|x|}, & \text{for} \quad p = n, \end{cases}$$

and the Gaussian curvature of the level set through x is

$$K(x) = |x|^{1-n}.$$

Hence, for $t = u(x)$ and $1 < p < n$,

$$\begin{aligned} \left(|\nabla u|^{n+1-2p} K\right)^{\frac{1}{n-1}}(x) &= \left(\frac{n-p}{p-1}\right)^{\frac{n+1-2p}{n-1}} |x|^{\frac{p-n}{p-1}} \\ &= \left(\frac{n-p}{p-1}\right)^{\frac{n+1-2p}{n-1}} \left[u(x) + R^{\frac{p-n}{p-1}}\right] \\ &= \left(\frac{n-p}{p-1}\right)^{\frac{n+1-2p}{n-1}} t + \left(\frac{n-p}{p-1}\right)^{\frac{n+1-2p}{n-1}} R^{\frac{p-n}{p-1}}. \end{aligned}$$

For $p = n$, we have

$$|\nabla u|^{1-n} K(x) = 1.$$

From the above calculation, we know that $(|\nabla u|^{n+1-2p} K)^{\frac{1}{n-1}}$ is an affine function on the height of the p-Green function.

To prove the theorem, let K be the Gaussian curvature of the convex level sets, and set

$$\varphi = -\alpha \log|\nabla u| - \log K.$$

For suitable choice of α and β, in Sect. 3 we will derive the following differential inequality (in the sense modulo the terms involving $\nabla_\theta \varphi$ with locally bounded coefficients)

$$L(e^{\beta\varphi}) \le 0 \qquad \text{mod } \nabla_\theta \varphi \quad \text{in } \Omega,$$

where L is an elliptic operator associated to the p-Laplace operator. The operator L is a linearized operator for the p-Laplace operator which is defined in (2.8), and $\nabla_\theta \varphi$

is defined in Sect. 2 as $\theta = (\theta_1, \cdots, \theta_{n-1})$ is a local orthogonal coordinate system on $\mathbb{S}^{n-1}$. Then, by a maximum principle argument, we can obtain the desired result.

The organization of this paper is as follows. In Sect. 2, we first give a brief definition of the support function of the convex level sets and then obtain a useful representation of the p-Laplace equation with support function. We prove Theorem 1.2 in Sect. 3. The main technique in the proof of the theorem consists of rearranging the second and third derivative terms using the equation and the first derivative condition for φ. The key idea is the Pogorelov's method in a priori estimates for fully nonlinear elliptic equations.

2 Support Function

We start by introducing some basic notations, which appeared in [8, 18].

Let Ω_0 and Ω_1 be two bounded smooth convex domains in $\mathbb{R}^n$ such that $\bar{\Omega}_1 \subset \Omega_0$ and let $\Omega = \Omega_0 \setminus \bar{\Omega}_1$. Let $u : \bar{\Omega} \to \mathbb{R}$ be a smooth function such that

$$u = 0 \quad \text{on } \partial\Omega_0, \qquad u = 1 \quad \text{on } \bar{\Omega}_1.$$

Furthermore, we assume that $|\nabla u| > 0$ in Ω and the level sets of u are strictly convex with respect to the normal direction ∇u. For $0 < t < 1$, we set

$$\bar{\Omega}_t = \{x \in \bar{\Omega}_0 | u \geq t\}.$$

Then, each point $x \in \Omega$ belongs to the boundary of $\bar{\Omega}_{u(x)}$. Under these assumptions, it is possible to define a function $H : \mathbb{R}^n \times [0, 1] \to \mathbb{R},\ (X, t) \mapsto H(X, t)$ as follows. For each $t \in [0, 1]$, $H(\cdot, t)$ is the support function of the convex body $\bar{\Omega}_t$. Denote by h the restriction of H to $\mathbb{S}^{n-1} \times [0, 1]$.

In the rest of this section, we will derive the p-Laplace equation by means of h. Before doing that, we should reformulate the first and second derivatives of u using h and its derivatives (see [8, 18, 22]). For convenience of reader, we sketch out the main steps here.

Note that h is the restriction of H to $\mathbb{S}^{n-1} \times [0, 1]$. It follows that $h(\theta, t) = H(Y(\theta), t)$, where $Y \in \mathbb{S}^{n-1}$ and $\theta = (\theta_1, \cdots, \theta_{n-1})$ is a local orthogonal coordinate system on $\mathbb{S}^{n-1}$. Since the level sets of u are strictly convex, we can define the map

$$x(X, t) = x_{\bar{\Omega}_t}(X),$$

which for every $(X, t) \in \mathbb{R}^n \setminus \{0\} \times (0, 1)$ assigns the unique point $x \in \Omega$ on the level set $\{u = t\}$ where the gradient of u is parallel to X (and orientation reversed).

If we define

$$T_i = \frac{\partial Y}{\partial \theta_i},$$

then $\{T_1, \cdots, T_{n-1}\}$ is a tangent frame field on $\mathbb{S}^{n-1}$. Furthermore, we assume that $\{T_1, \cdots, T_{n-1}, Y\}$ is an orthogonal frame positively oriented. It is easy to see that

$$\frac{\partial T_i}{\partial \theta_j} = -\delta_{ij} Y, \tag{2.1}$$

where δ_{ij} is the standard Kronecker delta symbol.

We denote

$$x(\theta, t) = x_{\bar{\Omega}_t}(Y(\theta)).$$

Since Y is orthogonal to $\partial\bar{\Omega}_t$ at $x(\theta, t)$, by differentiating the equality

$$h(\theta, t) = \langle x(\theta, t), Y(\theta)\rangle, \tag{2.2}$$

we obtain

$$h_i = \langle x, T_i\rangle. \tag{2.3}$$

Here, $\langle\cdot, \cdot\rangle$ is the usual inner product on $\mathbb{R}^n$. By (2.2) and (2.3), we have

$$x = hY + \sum_{i=1}^{n-1} h_i T_i. \tag{2.4}$$

Henceforth, we will omit the range of the summation indices if they run from 1 to $n-1$. With (2.1) in hand, by differentiating (2.4), we obtain

$$\frac{\partial x}{\partial t} = h_t Y + \sum_i h_{ti} T_i;$$
$$\frac{\partial x}{\partial \theta_j} = hT_j + \sum_i h_{ij} T_i, \quad j = 1, \cdots, n-1.$$

The inverse of the above Jacobian matrix is

$$\frac{\partial t}{\partial x_\alpha} = h_t^{-1}[Y]_\alpha, \quad \alpha = 1, \cdots, n;$$
$$\frac{\partial \theta_i}{\partial x_\alpha} = \sum_j b^{ij}[T_j - h_t^{-1} h_{tj} Y]_\alpha, \quad \alpha = 1, \cdots, n, \tag{2.5}$$

where $[\cdot]_\alpha$ denotes the α-coordinate of the vector in the bracket and (b^{ij}) denotes the inverse matrix of the inverse second fundamental form

$$b_{ij} = \left\langle \frac{\partial x}{\partial \theta_i}, \frac{\partial Y}{\partial \theta_j} \right\rangle = h\delta_{ij} + h_{ij} \tag{2.6}$$

of the level set $\partial\bar{\Omega}_t$ at $x(\theta, t)$. The eigenvalues of (b^{ij}) are the principal curvatures $k_1, \cdots, k_{n-1}$ of $\partial\bar{\Omega}_t$ at $x(\theta, t)$ (see Schneider [22]).

The first equation of (2.5) can be rewritten as

$$\nabla u = \frac{Y}{h_t},$$

where the left hand side is computed at $x(\theta, t)$ and the right hand side is computed at (θ, t). It follows that

$$|\nabla u| = -\frac{1}{h_t}.$$

By chain rule and (2.5), the second derivatives of u in terms of h and its derivatives can be computed as

$$u_{\alpha\beta} = \sum_{i,j}[-h_t^{-2}h_{ti}Y + h_t^{-1}T_i]_\alpha b^{ij}[T_j - h_t^{-1}h_{tj}Y]_\beta - h_t^{-3}h_{tt}[Y]_\alpha[Y]_\beta,$$

for $\alpha, \beta = 1, \cdots, n$.

Thus, the p-Laplace equation becomes

$$h_{tt} = \sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)b^{ij}. \tag{2.7}$$

The associated linear elliptic operator is

$$L = \sum_{i,j,p,q}\left(\frac{1}{p-1}h_t^2\delta_{pq} + h_{tp}h_{tq}\right)b^{ip}b^{jq}\frac{\partial^2}{\partial\theta_i\partial\theta_j} - 2\sum_{i,j}h_{tj}b^{ij}\frac{\partial^2}{\partial\theta_i\partial t} + \frac{\partial^2}{\partial t^2}, \tag{2.8}$$

which is a linearized operator for the p-Laplace operator.

Let $u \in C^4(\mathbb{S}^{n-1})$. The following commutation formulas for covariant derivatives of u are well-known

$$\begin{aligned} u_{ijk} - u_{ikj} &= -u_k\delta_{ij} + u_j\delta_{ik}, \\ u_{ijkl} - u_{ijlk} &= u_{ik}\delta_{jl} - u_{il}\delta_{jk} + u_{kj}\delta_{il} - u_{lj}\delta_{ik}. \end{aligned} \tag{2.9}$$

3 Concavity of the Gaussian Curvature of Level Sets

For a continuous function $f(t)$ on $[0, 1]$, we define its generalized second order derivative at any point t in $(0, 1)$ as

$$D^2 f(t) = \limsup_{h\to 0}\frac{f(t+h) + f(t-h) - 2f(t)}{h^2}.$$

The following lemma was first proved by Longinetti for the case $n = 2$ in the appendix of [16]. Here, we will need its general form.

Lemma 3.1 *Let* $\mathbb{Q} \equiv \mathbb{S}^{n-1} \times (0, 1)$ *and* $G(\theta, t)$ *be a smooth function in* $\mathbb{Q}$ *such that*

$$\mathscr{L}(G(\theta, t)) \geq 0 \quad \text{for } (\theta, t) \in \mathbb{Q},$$

where $\mathscr{L}$ *is an elliptic operator of the form*

$$\mathscr{L} = \sum_{i,j} a^{ij} \frac{\partial^2}{\partial\theta_i \partial\theta_j} + \sum_i b^i \frac{\partial^2}{\partial\theta_i \partial t} + \frac{\partial^2}{\partial t^2} + \sum_i c^i \frac{\partial}{\partial\theta_i}$$

with smooth coefficients a^{ij}, b^i, c^i. *Set*

$$\phi(t) = \max\{G(\theta, t) | \theta \in \mathbb{S}^{n-1}\}.$$

Then, ϕ *satisfies the following differential inequality*

$$D^2\phi(t) \geq 0.$$

Moreover, $\phi(t)$ *is a convex function with respect to* t.

Proof of the Theorem 1.2 Since the level sets of u are strictly convex, the inverse second fundamental form (b_{ij}) is positive definite in Ω. Set

$$\varphi = \alpha \log(-h_t) - \log K,$$

where $K = \det(b^{ij})$ is the Gaussian curvature of the level sets. For $\alpha = n + 1 - 2p$ and $\beta = -\frac{1}{n-1}$, it follows that

$$e^{\beta\varphi} = (|\nabla u|^{n+1-2p} K)^{\frac{1}{n-1}}.$$

We will derive the following differential inequality (in the sense modulo the terms involving $\nabla_\theta \varphi$ with locally bounded coefficients)

$$L(e^{\beta\varphi}) \leq 0 \qquad \text{mod } \nabla_\theta \varphi \quad \text{in } \Omega, \tag{3.1}$$

where the elliptic operator L is given in (2.8). By Lemma 3.1, we obtain the desired result.

In order to prove (3.1) at an arbitrary point $x_0 \in \Omega$, we may assume the matrix $(b_{ij}(x_0))$ is diagonal by choosing suitable orthonormal frame. From now on, all the calculation will be done at the fixed point x_0. In the following, we shall prove the theorem in three steps.

Step1. we first compute $L(\varphi)$ *in refLvarphi2.*

Since

$$\varphi = \alpha \log(-h_t) + \log\det(b_{ij}),$$

taking first derivatives of φ, we get

$$\frac{\partial\varphi}{\partial\theta_j} = \alpha h_t^{-1} h_{tj} + \sum_{k,l} b^{kl} b_{kl,j}, \tag{3.2}$$

$$\frac{\partial\varphi}{\partial t} = \alpha h_t^{-1} h_{tt} + \sum_{k,l} b^{kl} b_{kl,t}. \tag{3.3}$$

Taking derivatives of equations (3.2) and (3.3), we have

$$\frac{\partial^2\varphi}{\partial\theta_i\partial\theta_j} = -\alpha h_t^{-2} h_{ti} h_{tj} + \alpha h_t^{-1} h_{tji} - \sum_{k,l,r,s} b^{kr} b_{rs,i} b^{sl} b_{kl,j} + \sum_{k,l} b^{kl} b_{kl,ji},$$

$$\frac{\partial^2\varphi}{\partial\theta_i\partial t} = -\alpha h_t^{-2} h_{ti} h_{tt} + \alpha h_t^{-1} h_{tti} - \sum_{k,l,r,s} b^{kr} b_{rs,i} b^{sl} b_{kl,t} + \sum_{k,l} b^{kl} b_{kl,ti},$$

$$\frac{\partial^2\varphi}{\partial t^2} = -\alpha h_t^{-2} h_{tt}^2 + \alpha h_t^{-1} h_{ttt} - \sum_{k,l,r,s} b^{kr} b_{rs,t} b^{sl} b_{kl,t} + \sum_{k,l} b^{kl} b_{kl,tt},$$

and hence

$$\begin{aligned} L(\varphi) = &-\alpha h_t^{-2}\Big[\sum_{i,j}\Big(\frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj}\Big) b^{ii} b^{jj} h_{ti} h_{tj} - 2\sum_i h_{ti}^2 b^{ii} h_{tt} + h_{tt}^2\Big] \\ &+\alpha h_t^{-1}\Big[\sum_{i,j}\Big(\frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj}\Big) b^{ii} b^{jj} h_{tji} - 2\sum_i h_{ti} b^{ii} h_{tti} + h_{ttt}\Big] \\ &-\sum_{k,l} b^{kk} b^{ll}\Big[\sum_{i,j}\Big(\frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj}\Big) b^{ii} b^{jj} b_{kl,i} b_{kl,j} \\ &-2\sum_i h_{ti} b^{ii} b_{kl,i} b_{kl,t} + b_{kl,t}^2\Big] + \sum_k b^{kk} L(b_{kk}) \\ \triangleq\ & I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{3.4}$$

In the following, we will deal with the four terms above, respectively. By recalling our equation, i.e.,

$$h_{tt} = \sum_{i,j}\Big(\frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj}\Big) b^{ij}, \tag{3.5}$$

at the point x_0, we have

$$h_{tt} = \frac{1}{p-1} h_t^2 \sigma_1 + \sum_i h_{ti}^2 b^{ii}, \tag{3.6}$$

where $\sigma_1 = \sum_i b^{ii}$ is the mean curvature of the level sets.

For the term I_1, we have

$$\begin{aligned} I_1 &= -\alpha h_t^{-2} \Bigg[\sum_{i,j} \left(\frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ii} b^{jj} h_{ti} h_{tj} - 2 \sum_i h_{ti}^2 b^{ii} h_{tt} + h_{tt}^2 \Bigg] \\ &= -\alpha h_t^{-2} \Bigg[\frac{1}{p-1} h_t^2 \sum_i (h_{ti} b^{ii})^2 + \left(\sum_i h_{ti}^2 b^{ii} - h_{tt} \right)^2 \Bigg] \\ &= -\frac{\alpha}{p-1} \sum_i (h_{ti} b^{ii})^2 - \frac{\alpha}{(p-1)^2} h_t^2 \sigma_1^2. \end{aligned} \tag{3.7}$$

Now, we treat the term I_2. Differentiating (3.5) with respect to t, we have

$$h_{ttt} = \frac{2}{p-1} h_t h_{tt} \sigma_1 + 2 \sum_i h_{tti} h_{ti} b^{ii} - \sum_{i,j} \left(\frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ii} b^{jj} b_{ij,t}. \tag{3.8}$$

By inserting (3.8) into I_2, we can get

$$\begin{aligned} I_2 &= \alpha h_t^{-1} \Bigg[\sum_{i,j} \left(\frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ii} b^{jj} h_{tji} - 2 \sum_i h_{ti} b^{ii} h_{tti} + h_{ttt} \Bigg] \\ &= \alpha h_t^{-1} \Bigg[\sum_{i,j} \left(\frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ii} b^{jj} (h_{tji} - b_{ij,t}) + \frac{2}{p-1} h_t h_{tt} \sigma_1 \Bigg]. \end{aligned}$$

Recalling the definition of the inverse second fundamental form, i.e., (2.6), together with the Eq. (3.6), we obtain

$$\begin{aligned} I_2 &= \alpha h_t^{-1} \Bigg[\sum_{i,j} \left(\frac{1}{p-1} h_t^2 \delta_{ij} + h_{ti} h_{tj} \right) b^{ii} b^{jj} (-h_t \delta_{ij}) + \frac{2}{p-1} h_t h_{tt} \sigma_1 \Bigg] \\ &= -\frac{\alpha}{p-1} h_t^2 \sum_i (b^{ii})^2 - \alpha \sum_i (h_{ti} b^{ii})^2 + \frac{2\alpha}{(p-1)^2} h_t^2 \sigma_1^2 \\ &\quad + \frac{2\alpha}{p-1} \sigma_1 \sum_i h_{ti}^2 b^{ii}. \end{aligned} \tag{3.9}$$

Combining (3.7) and (3.9), we have

$$I_1 + I_2 = -\frac{p\alpha}{p-1}\sum_i (h_{ti}b^{ii})^2 + \frac{\alpha}{(p-1)^2}h_t^2\sigma_1^2 - \frac{\alpha}{p-1}h_t^2\sum_i (b^{ii})^2 + \frac{2\alpha}{p-1}\sigma_1\sum_i h_{ti}^2 b^{ii}. \tag{3.10}$$

In order to deal with the last two terms, we shall compute $L(b_{kk})$ in advance. By differentiating (3.5) twice with respect to θ_k, we have

$$h_{ttk} = \sum_{i,j}(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj})_k b^{ij} + \sum_{ij,p,q}(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj})(-b^{ip}b_{pq,k}b^{qj})$$

and

$$\begin{aligned} h_{ttkk} = &\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)_{kk} b^{ij} \\ &+2\sum_{ij,p,q}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)_k \left(-b^{ip}b_{pq,k}b^{qj}\right) \\ &+\sum_{ij,p,q,r,s}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)(2b^{ir}b_{rs,k}b^{sp}b_{pq,k}b^{qj}) \\ &+\sum_{ij,p,q}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)(-b^{ip}b_{pq,kk}b^{qj}) \\ \triangleq\ & J_1 + J_2 + J_3 + J_4. \end{aligned} \tag{3.11}$$

For the term J_1, we have

$$\begin{aligned} J_1 &= \sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)_{kk} b^{ij} \\ &= \sum_{i,j}\left(\frac{2}{p-1}h_t h_{tk}\delta_{ij} + h_{tik}h_{tj} + h_{ti}h_{tjk}\right)_k b^{ij} \\ &= \frac{2}{p-1}h_{tk}^2\sigma_1 + \frac{2}{p-1}h_t h_{tkk}\sigma_1 + 2\sum_i h_{tikk}h_{ti}b^{ii} + 2\sum_i h_{tik}^2 b^{ii}. \end{aligned}$$

Applying (2.9) for the support function h, we can get

$$\begin{aligned} b_{ij,k} &= b_{ik,j}, \\ h_{tik} &= h_{kit} = b_{ki,t} - h_t\delta_{ki}, \\ h_{tikk} &= h_{ikkt} = b_{ik,kt} - h_{kt}\delta_{ik} = b_{kk,it} - h_{kt}\delta_{ik}, \end{aligned}$$

and hence we obtain

$$J_1 = \frac{2}{p-1}h_{tk}^2\sigma_1 + \frac{2}{p-1}h_t b_{kk,t}\sigma_1 - \frac{2}{p-1}h_t^2\sigma_1 + 2\sum_i b_{kk,it}h_{ti}b^{ii} \\ -2h_{tk}^2 b^{kk} + 2\sum_l b_{kl,t}^2 b^{ll} - 4h_t b_{kk,t}b^{kk} + 2h_t^2 b^{kk}. \quad (3.12)$$

For the term J_2, we have

$$\begin{aligned} J_2 &= 2\sum_{i,j}\left(\frac{2}{p-1}h_t h_{tk}\delta_{ij} + h_{tik}h_{tj} + h_{ti}h_{tjk}\right)(-b^{ii}b_{ij,k}b^{jj}) \\ &= -\frac{4}{p-1}h_t h_{tk}\sum_i (b^{ii})^2 b_{ii,k} - 4\sum_{i,j} h_{tik}h_{tj}b^{ii}b^{jj}b_{ij,k} \\ &= -\frac{4}{p-1}h_t h_{tk}\sum_i (b^{ii})^2 b_{ii,k} - 4\sum_{i,l} h_{ti}b^{ii}b^{ll}b_{kl,i}b_{kl,t} + 4h_t\sum_i h_{ti}b^{ii}b^{kk}b_{kk,i}. \end{aligned} \quad (3.13)$$

Also, we have

$$J_3 = 2\sum_{i,j,l}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)b^{ii}b^{jj}b^{ll}b_{kl,i}b_{kl,j}. \quad (3.14)$$

Again by (2.9), we have the following commutation rule

$$b_{ij,kl} - b_{ij,lk} = b_{jk}\delta_{il} - b_{jl}\delta_{ik} + b_{ik}\delta_{jl} - b_{il}\delta_{jk},$$

and hence

$$b_{ij,kk} = b_{ki,jk} = b_{kk,ij} + b_{ij} - b_{kk}\delta_{ij} + b_{kj}\delta_{ik} - b_{ik}\delta_{jk}.$$

For the term J_4, we have

$$\begin{aligned} J_4 &= -\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)b^{ii}b^{jj}b_{ij,kk} \\ &= -\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)b^{ii}b^{jj}(b_{kk,ij} + b_{ij} - b_{kk}\delta_{ij} + b_{kj}\delta_{ik} - b_{ik}\delta_{jk}) \\ &= -\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)b^{ii}b^{jj}(b_{kk,ij} + b_{ij} - b_{kk}\delta_{ij}) \\ &= -\sum_{i,j}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)b^{ii}b^{jj}b_{kk,ij} - h_{tt} + b_{kk}\left[\frac{1}{p-1}h_t^2\sum_i (b^{ii})^2 \right. \\ &\quad \left. + \sum_i (h_{ti}b^{ii})^2\right]. \end{aligned} \quad (3.15)$$

On the other hand, one has

$$b_{kk,tt} = h_{kktt} + h_{tt} = h_{ttkk} + h_{tt}. \tag{3.16}$$

By putting (3.12)–(3.16) into (3.11), recalling the definition of the operator L, we obtain

$$\begin{aligned} L(b_{kk}) &= \sum_{i,j,p,q}\left(\frac{1}{p-1}h_t^2\delta_{pq} + h_{tp}h_{tq}\right)b^{ip}b^{jq}b_{kk,ij} - 2\sum_{i,j}h_{tj}b^{ij}b_{kk,it} + b_{kk,tt} \\ &= \frac{2}{p-1}h_{tk}^2\sigma_1 + \frac{2}{p-1}h_t\sigma_1 b_{kk,t} - \frac{2}{p-1}h_t^2\sigma_1 - 2h_{tk}^2 b^{kk} \\ &\quad + 2\sum_l b_{kl,t}^2 b^{ll} - 4h_t b^{kk}b_{kk,t} + 2h_t^2 b^{kk} - \frac{4}{p-1}h_t h_{tk}\sum_i (b^{ii})^2 b_{ii,k} \\ &\quad - 4\sum_{i,l}h_{ti}b^{ii}b^{ll}b_{kl,i}b_{kl,t} + 2\sum_{i,j,l}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)b^{ii}b^{jj}b^{ll}b_{kl,i}b_{kl,j} \\ &\quad + 4h_t\sum_i h_{ti}b^{ii}b^{kk}b_{kk,i} + b_{kk}\left[\frac{1}{p-1}h_t^2\sum_i (b^{ii})^2 + \sum_i (h_{ti}b^{ii})^2\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} I_4 &= \sum_k b^{kk}L(b_{kk}) \\ &= 2\sum_{i,j,k,l}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)b^{ii}b^{jj}b^{kk}b^{ll}b_{kl,i}b_{kl,j} \\ &\quad - 4\sum_{i,k,l}h_{ti}b^{ii}b^{kk}b^{ll}b_{kl,i}b_{kl,t} + 2\sum_{k,l}b^{kk}b^{ll}b_{kl,t}^2 \\ &\quad + \frac{2}{p-1}h_t\sigma_1\sum_k b^{kk}b_{kk,t} - 4h_t\sum_k (b^{kk})^2 b_{kk,t} \\ &\quad + \frac{4p-8}{p-1}h_t\sum_{i,k}h_{ti}b^{ii}(b^{kk})^2 b_{kk,i} \\ &\quad + \frac{2p+n-3}{p-1}h_t^2\sum_i (b^{ii})^2 - \frac{2}{p-1}h_t^2\sigma_1^2 + \frac{2}{p-1}\sigma_1\sum_i h_{ti}^2 b^{ii} \\ &\quad + (n-3)\sum_i (h_{ti}b^{ii})^2. \end{aligned} \tag{3.17}$$

By substituting (3.10) and (3.17) in (3.4), we obtain

$$\begin{aligned} L(\varphi) &= \sum_{i,j,k,l}\left(\frac{1}{p-1}h_t^2\delta_{ij} + h_{ti}h_{tj}\right)b^{ii}b^{jj}b^{kk}b^{ll}b_{kl,i}b_{kl,j} \\ &\quad -2\sum_{i,k,l}h_{ti}b^{ii}b^{kk}b^{ll}b_{kl,i}b_{kl,t} + \sum_{k,l}b^{kk}b^{ll}b_{kl,t}^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{p-1}h_t\sigma_1\sum_k b^{kk}b_{kk,t}-4h_t\sum_k(b^{kk})^2b_{kk,t}\\
&+\frac{4p-8}{p-1}h_t\sum_{i,k}h_{ti}b^{ii}(b^{kk})^2b_{kk,i}\\
&+\frac{2p+n-3-\alpha}{p-1}h_t^2\sum_i(b^{ii})^2+\frac{\alpha-2p+2}{(p-1)^2}h_t^2\sigma_1^2+\frac{2\alpha+2}{p-1}\sigma_1\sum_i h_{ti}^2b^{ii}\\
&+\left(n-3-\frac{p\alpha}{p-1}\right)\sum_i(h_{ti}b^{ii})^2. \qquad (3.18)
\end{aligned}
$$

Step 2. In this step, we shall calculate $L(e^{\beta\varphi})$ and then obtain the formula 3.24.

Note that

$$
\begin{aligned}
L(e^{\beta\varphi})&=\beta e^{\beta\varphi}\{L(\varphi)+\beta\varphi_t^2\}+\beta^2e^{\beta\varphi}\sum_{i,j}(\frac{1}{p-1}h_t^2\delta_{ij}+h_{ti}h_{tj})b^{ii}b^{jj}\frac{\partial\varphi}{\partial\theta_i}\frac{\partial\varphi}{\partial\theta_j}\\
&\quad-2\beta^2e^{\beta\varphi}\sum_i h_{ti}b^{ii}\frac{\partial\varphi}{\partial\theta_i}\frac{\partial\varphi}{\partial t}. \qquad (3.19)
\end{aligned}
$$

For $\beta=-\frac{1}{n-1}$, in order to prove

$$
L(e^{\beta\varphi})\le 0 \qquad \mathrm{mod}\ \nabla_\theta\varphi \quad \text{in } \Omega,
$$

it suffices to prove

$$
L(\varphi)+\beta\varphi_t^2\ge 0 \qquad \mathrm{mod}\ \nabla_\theta\varphi \quad \text{in } \Omega. \qquad (3.20)
$$

Now, we compute $\beta\varphi_t^2$. By (3.3) and the Eq. (3.6), we have

$$
\begin{aligned}
\beta\varphi_t^2&=\beta\alpha^2h_t^{-2}h_{tt}^2+2\beta\alpha h_t^{-1}h_{tt}\sum_k b^{kk}b_{kk,t}+\beta\left(\sum_k b^{kk}b_{kk,t}\right)^2\\
&=\frac{1}{(p-1)^2}\beta\alpha^2h_t^2\sigma_1^2+\frac{2}{p-1}\beta\alpha^2\sigma_1\sum_i h_{ti}^2b^{ii}+\beta\alpha^2h_t^{-2}\left(\sum_i h_{ti}^2b^{ii}\right)^2\\
&\quad+\frac{2}{p-1}\beta\alpha h_t\sigma_1\sum_k b^{kk}b_{kk,t}+2\beta\alpha h_t^{-1}\left(\sum_i h_{ti}^2b^{ii}\right)\left(\sum_k b^{kk}b_{kk,t}\right)\\
&\quad+\beta\left(\sum_k b^{kk}b_{kk,t}\right)^2. \qquad (3.21)
\end{aligned}
$$

Jointing (3.18) with (3.21), we regroup the terms in $L(\varphi)+\beta\varphi_t^2$ as follows

$$
L(\varphi)+\beta\varphi_t^2\triangleq P_1+P_2+P_3, \qquad (3.22)
$$

where

$$P_1 = \sum_{k\neq l}\Bigg(\sum_{i,j} h_{ti}h_{tj}b^{ii}b^{jj}b^{kk}b^{ll}b_{kl,i}b_{kl,j} - 2\sum_i h_{ti}b^{ii}b^{kk}b^{ll}b_{kl,i}b_{kl,t} + b^{kk}b^{ll}b^2_{kl,t}\Bigg) \geq 0,$$

$$P_2 = \sum_k (b^{kk}b_{kk,t})^2 + \beta\left(\sum_k b^{kk}b_{kk,t}\right)^2 + 2\sum_k\left(\frac{\beta\alpha+1}{p-1}h_t\sigma_1 + \beta\alpha h_t^{-1}\sum_i h_{ti}^2 b^{ii} - \sum_i h_{ti}b^{ii}b^{kk}b_{kk,i} - 2h_t b^{kk}\right)\cdot(b^{kk}b_{kk,t})$$

and

$$\begin{aligned}P_3 = &\frac{1}{p-1}h_t^2\sum_{i,k,l}(b^{ii})^2 b^{kk}b^{ll}b^2_{kl,i} + \sum_{i,j,k}h_{ti}h_{tj}b^{ii}b^{jj}b^{kk}b_{kk,i}b^{kk}b_{kk,j}\\ &+\frac{4p-8}{p-1}h_t\sum_{i,k}h_{ti}b^{ii}(b^{kk})^2 b_{kk,i}\\ &+\frac{2p+n-3-\alpha}{p-1}h_t^2\sum_i (b^{ii})^2 + \frac{\beta\alpha^2+\alpha-2p+2}{(p-1)^2}h_t^2\sigma_1^2\\ &+\frac{2\beta\alpha^2+2\alpha+2}{p-1}\sigma_1\sum_i h_{ti}^2 b^{ii}\\ &+\left(n-3-\frac{p\alpha}{p-1}\right)\sum_i (h_{ti}b^{ii})^2 + \beta\alpha^2 h_t^{-2}\left(\sum_i h_{ti}^2 b^{ii}\right)^2.\end{aligned}$$

In the rest of this step, we will deal with the term P_2. Let $X_k = b^{kk}b_{kk,t}(k = 1, 2, \cdots, n-1)$, then, P_2 can be rewritten as

$$P_2(X_1, X_2, \cdots, X_{n-1}) = \sum_k X_k^2 + \beta\left(\sum_k X_k\right)^2 + 2\sum_k c_k X_k,$$

where

$$c_k = \frac{\beta\alpha+1}{p-1}h_t\sigma_1 + \beta\alpha h_t^{-1}\sum_i h_{ti}^2 b^{ii} - \sum_i h_{ti}b^{ii}b^{kk}b_{kk,i} - 2h_t b^{kk}.$$

For $\alpha = n+1-2p$ and $\beta = -\frac{1}{n-1}$, by (3.2) we can obtain

$$\sum_k c_k = -\sum_i h_{ti}b^{ii}\frac{\partial\varphi}{\partial\theta_i}$$

and

$$\begin{aligned}\sum_k c_k^2 =& \sum_{i,j,k} h_{ti}h_{tj}b^{ii}b^{jj}b^{kk}b_{kk,i}b^{kk}b_{kk,j} + 4h_t\sum_{i,k} h_{ti}b^{ii}(b^{kk})^2 b_{kk,i}\\ &+4h_t^2\sum_k (b^{kk})^2 - \frac{2(1+\alpha\beta)}{p-1}h_t^2\sigma_1^2 - 4\beta\alpha\sigma_1\sum_i h_{ti}^2 b^{ii}\\ &+\beta\alpha^2 h_t^{-2}\left(\sum_i h_{ti}^2 b^{ii}\right)^2\\ &-2\left(\frac{\beta\alpha+1}{p-1}h_t\sigma_1 + \beta\alpha h_t^{-1}\sum_j h_{tj}^2 b^{jj}\right)\sum_i h_{ti}b^{ii}\frac{\partial\varphi}{\partial\theta_i}.\end{aligned}$$

By straightforward computation, we have

$$\begin{aligned}\sum_k\left(X_k + \beta\sum_i X_i + c_k\right)^2 =& P_2(X_1, X_2, \cdots, X_{n-1}) + \sum_k c_k^2\\ &-2\beta\left(\sum_k b^{kk}b_{kk,t}\right)\sum_i h_{ti}b^{ii}\frac{\partial\varphi}{\partial\theta_i},\end{aligned}$$

therefore,

$$\begin{aligned}&P_2(X_1, X_2, \cdots, X_{n-1}) \geq -\sum_k c_k^2 + 2\beta\left(\sum_k b^{kk}b_{kk,t}\right)\sum_i h_{ti}b^{ii}\frac{\partial\varphi}{\partial\theta_i}\\ &= -\sum_{i,j,k} h_{ti}h_{tj}b^{ii}b^{jj}b^{kk}b_{kk,i}b^{kk}b_{kk,j} - 4h_t\sum_{i,k} h_{ti}b^{ii}(b^{kk})^2 b_{kk,i}\\ &\quad -4h_t^2\sum_k (b^{kk})^2 + \frac{2(1+\alpha\beta)}{p-1}h_t^2\sigma_1^2 + 4\beta\alpha\sigma_1\sum_i h_{ti}^2 b^{ii} - \beta\alpha^2 h_t^{-2}\left(\sum_i h_{ti}^2 b^{ii}\right)^2\\ &\quad + 2\left(\frac{\beta\alpha+1}{p-1}h_t\sigma_1 + \beta\alpha h_t^{-1}\sum_j h_{tj}^2 b^{jj} + \beta\sum_k b^{kk}b_{kk,t}\right)\sum_i h_{ti}b^{ii}\frac{\partial\varphi}{\partial\theta_i}.\end{aligned} \tag{3.23}$$

Putting (3.23) into (3.22), for $\alpha = n+1-2p$ and $\beta = -\frac{1}{n-1}$, we obtain

$$\begin{aligned}L(\varphi) + \beta\varphi_t^2 \geq& \ P_2 + P_3\\ \geq& \frac{1}{p-1}h_t^2\sum_{i,k,l}(b^{ii})^2 b^{kk}b^{ll}b_{kl,i}^2 - \frac{4}{p-1}h_t\sum_{i,k} h_{ti}b^{ii}(b^{kk})^2 b_{kk,i}\\ &+\frac{2}{p-1}\sigma_1\sum_i h_{ti}^2 b^{ii} + (n-3-\frac{p\alpha}{p-1})\sum_i (h_{ti}b^{ii})^2\end{aligned}$$

$$+2\left(\frac{\beta\alpha+1}{p-1}h_t\sigma_1+\beta\alpha h_t^{-1}\sum_j h_{tj}^2 b^{jj}+\beta\sum_k b^{kk}b_{kk,t}\right)\sum_i h_{ti}b^{ii}\frac{\partial\varphi}{\partial\theta_i}. \tag{3.24}$$

In the next step, we will concentrate on the following term

$$R=h_t^2\sum_{i,k,l}(b^{ii})^2 b^{kk}b^{ll}b_{kl,i}^2-4h_t\sum_{i,k}h_{ti}b^{ii}(b^{kk})^2 b_{kk,i}. \tag{3.25}$$

Step 3.In this step, we first obtain a sharp lower bound on R in 3.25 *and then we get* 3.20 *and complete the proof of* 3.1.

Recalling our first derivative condition 3.2, without loss of generality, we shall isolate the term $b^{11}b_{11,j}$. Then, we have

$$b^{11}b_{11,j}=\frac{\partial\varphi}{\partial\theta_j}-\sum_{k\geq 2}b^{kk}b_{kk,j}-\alpha h_t^{-1}h_{tj},\quad \text{for } j=1,2,\cdots,n-1. \tag{3.26}$$

For the term R, we have

$$\begin{aligned}
R&=h_t^2\sum_i\sum_{k\neq l}(b^{ii})^2 b^{kk}b^{ll}b_{kl,i}^2+h_t^2\sum_{i,k}(b^{ii})^2(b^{kk}b_{kk,i})^2\\
&\quad -4h_t\sum_{i,k}h_{ti}b^{ii}(b^{kk})^2 b_{kk,i}\\
&=2h_t^2\sum_{k\geq 2}(b^{11})^2 b^{kk}b^{11}b_{k1,1}^2+2h_t^2\sum_{i,k\geq 2}(b^{ii})^2 b^{kk}b^{11}b_{k1,i}^2\\
&\quad +h_t^2\sum_i\sum_{\substack{k,l\geq 2\\ k\neq l}}(b^{ii})^2 b^{kk}b^{ll}b_{kl,i}^2\\
&\quad +h_t^2\sum_i(b^{ii})^2(b^{11}b_{11,i})^2+h_t^2\sum_i\sum_{k\geq 2}(b^{ii})^2(b^{kk}b_{kk,i})^2\\
&\quad -4h_t\sum_i h_{ti}b^{ii}(b^{11})^2 b_{11,i}-4h_t\sum_i\sum_{k\geq 2}h_{ti}b^{ii}(b^{kk})^2 b_{kk,i}\\
&\triangleq R_1+R_2+R_3,
\end{aligned} \tag{3.27}$$

where

$$\begin{aligned}
R_1&=2h_t^2\sum_{k\geq 2}(b^{11})^2 b^{kk}b^{11}b_{k1,1}^2+h_t^2\sum_i(b^{ii})^2(b^{11}b_{11,i})^2\\
&\quad -4h_t\sum_i h_{ti}b^{ii}(b^{11})^2 b_{11,i},
\end{aligned}$$

$$R_2 = 2h_t^2 \sum_{i,k\geq 2} (b^{ii})^2 b^{kk} b^{11} b_{k1,i}^2 + h_t^2 \sum_i \sum_{\substack{k,l\geq 2\\ k\neq l}} (b^{ii})^2 b^{kk} b^{ll} b_{kl,i}^2,$$

$$R_3 = h_t^2 \sum_i \sum_{k\geq 2} (b^{ii})^2 (b^{kk} b_{kk,i})^2 - 4h_t \sum_i \sum_{k\geq 2} h_{ti} b^{ii} (b^{kk})^2 b_{kk,i}.$$

By (3.26), one has

$$\begin{aligned} R_1 = {} & 2h_t^2 b^{11} \sum_{i,k,l\geq 2} b^{ii} b^{kk} b^{ll} b_{kk,i} b_{ll,i} + 4\alpha h_t b^{11} \sum_{i,k\geq 2} h_{ti} b^{ii} b^{kk} b_{kk,i} \\ & + 2\alpha^2 b^{11} \sum_{i\geq 2} h_{ti}^2 b^{ii} + h_t^2 \sum_i \sum_{k,l\geq 2} (b^{ii})^2 b^{kk} b^{ll} b_{kk,i} b_{ll,i} \\ & + 2\alpha h_t \sum_i \sum_{k\geq 2} h_{ti} (b^{ii})^2 b^{kk} b_{kk,i} \\ & + \alpha^2 \sum_i (h_{ti} b^{ii})^2 + 4h_t \sum_i \sum_{k\geq 2} h_{ti} b^{ii} b^{11} b^{kk} b_{kk,i} \\ & + 4\alpha b^{11} \sum_i h_{ti}^2 b^{ii} + R(\nabla_\theta \varphi), \end{aligned}$$

where

$$\begin{aligned} R(\nabla_\theta \varphi) = {} & 2h_t^2 b^{11} \sum_{k\geq 2} b^{kk} \left(\frac{\partial \varphi}{\partial \theta_k}\right)^2 - 4h_t^2 b^{11} \sum_{k,l\geq 2} b^{kk} b^{ll} b_{ll,k} \frac{\partial \varphi}{\partial \theta_k} \\ & - 4\alpha h_t b^{11} \sum_{k\geq 2} b^{kk} h_{tk} \frac{\partial \varphi}{\partial \theta_k} \\ & + h_t^2 \sum_i (b^{ii})^2 \left(\frac{\partial \varphi}{\partial \theta_i}\right)^2 - 2h_t^2 \sum_i \sum_{k\geq 2} (b^{ii})^2 b^{kk} b_{kk,i} \frac{\partial \varphi}{\partial \theta_i} \\ & - 2\alpha h_t \sum_i (b^{ii})^2 h_{ti} \frac{\partial \varphi}{\partial \theta_i} - 4h_t b^{11} \sum_i b^{ii} h_{ti} \frac{\partial \varphi}{\partial \theta_i}. \end{aligned} \tag{3.28}$$

We rewrite the term R_1 as follows

$$\begin{aligned} R_1 = {} & h_t^2 \sum_{k,l\geq 2} (b^{11})^2 b^{kk} b^{ll} b_{kk,1} b_{ll,1} + 2\alpha h_t \sum_{k\geq 2} h_{t1} (b^{11})^2 b^{kk} b_{kk,1} \\ & + 4h_t \sum_{k\geq 2} h_{t1} (b^{11})^2 b^{kk} b_{kk,1} \\ & + 2h_t^2 b^{11} \sum_{i,k,l\geq 2} b^{ii} b^{kk} b^{ll} b_{kk,i} b_{ll,i} + h_t^2 \sum_{i,k,l\geq 2} (b^{ii})^2 b^{kk} b^{ll} b_{kk,i} b_{ll,i} \\ & + 4\alpha h_t b^{11} \sum_{i,k\geq 2} h_{ti} b^{ii} b^{kk} b_{kk,i} + 2\alpha h_t \sum_{i,k\geq 2} h_{ti} (b^{ii})^2 b^{kk} b_{kk,i} \end{aligned}$$

$$+4h_t b^{11} \sum_{i,k\geq 2} h_{ti} b^{ii} b^{kk} b_{kk,i}$$
$$+2\alpha^2 b^{11} \sum_{i\geq 2} h_{ti}^2 b^{ii} + \alpha^2 \sum_i (h_{ti} b^{ii})^2 + 4\alpha b^{11} \sum_i h_{ti}^2 b^{ii} + R(\nabla_\theta \varphi). \tag{3.29}$$

On the other hand,

$$R_2 = 2h_t^2 b^{11} \sum_{k\geq 2} (b^{kk})^3 b_{kk,1}^2 + 2h_t^2 \sum_{\substack{i,k\geq 2\\ i\neq k}} b^{ii} (b^{kk})^3 b_{kk,i}^2$$
$$+2h_t^2 \sum_{\substack{i,k\geq 2\\ i\neq k}} (b^{ii})^2 b^{kk} b^{11} b_{k1,i}^2 + h_t^2 \sum_i \sum_{\substack{k,l\geq 2\\ k\neq l, k\neq i, l\neq i}} (b^{ii})^2 b^{kk} b^{ll} b_{kl,i}^2. \tag{3.30}$$

We rewrite R_3 as

$$R_3 = h_t^2 \sum_{k\geq 2} (b^{11})^2 (b^{kk} b_{kk,1})^2 - 4h_t \sum_{k\geq 2} h_{t1} b^{11} (b^{kk})^2 b_{kk,1}$$
$$+h_t^2 \sum_{i,k\geq 2} (b^{ii})^2 (b^{kk} b_{kk,i})^2 - 4h_t \sum_{i,k\geq 2} h_{ti} b^{ii} (b^{kk})^2 b_{kk,i}. \tag{3.31}$$

Now, we are at a stage to regroup the terms in R in a natural way by collecting (3.27)–(3.31). Let

$$R \triangleq T_1 + T_2 + T_3, \tag{3.32}$$

where T_1 are the terms involving $b_{kk,1} (k \geq 2)$, T_2 are the terms involving $b_{kk,i} (k, i \geq 2)$, and T_3 are the rest terms. More precisely,

$$T_1 = \sum_{k\geq 2} (1 + 2b_{11} b^{kk}) \cdot (h_t b^{11} b^{kk} b_{kk,1})^2 + \left(\sum_{k\geq 2} h_t b^{11} b^{kk} b_{kk,1} \right)^2$$
$$+4h_{t1} b^{11} \sum_{k\geq 2} (1 + \frac{\alpha}{2} - b_{11} b^{kk}) \cdot (h_t b^{11} b^{kk} b_{kk,1}),$$
$$T_2 = \sum_{i\geq 2} \left\{ (1 + 2b_{ii} b^{11}) \cdot \left(\sum_{k\geq 2} h_t b^{ii} b^{kk} b_{kk,i} \right)^2 + \sum_{\substack{k\geq 2\\ k\neq i}} 2b_{ii} b^{kk} \cdot (h_t b^{ii} b^{kk} b_{kk,i})^2 \right.$$
$$+ \sum_{k\geq 2} (h_t b^{ii} b^{kk} b_{kk,i})^2 + 4h_{ti} b^{ii} \sum_{k\geq 2} \Big[-b_{ii} b^{kk} + \frac{\alpha}{2}$$
$$\left. +(1+\alpha) b_{ii} b^{11} \Big] \cdot (h_t b^{ii} b^{kk} b_{kk,i}) \right\}$$

and

$$T_3 = 2\alpha^2 b^{11} \sum_{i\geq 2} h_{ti}^2 b^{ii} + \alpha^2 \sum_i (h_{ti} b^{ii})^2 + 4\alpha b^{11} \sum_i h_{ti}^2 b^{ii}$$
$$+2h_t^2 \sum_{\substack{i,k\geq 2\\ i\neq k}} (b^{ii})^2 b^{kk} b^{11} b_{k1,i}^2 + h_t^2 \sum_i \sum_{\substack{k,l\geq 2\\ k\neq l, k\neq i, l\neq i}} (b^{ii})^2 b^{kk} b^{ll} b_{kl,i}^2 + R(\nabla_\theta \varphi). \tag{3.33}$$

We shall maximize the terms T_1 and T_2 via Lemma 3.2 for different choice of parameters.

At first, let us examine the term T_1. Set $X_k = h_t b^{11} b^{kk} b_{kk,1}, \lambda = 1, \mu = h_{t1} b^{11}, b_k = 1 + 2b_{11} b^{kk}$, and $c_k = b_{11} b^{kk} - (1 + \frac{\alpha}{2})$ where $k \geq 2$. By Lemma 3.2, which will be stated in the end of this section, we have

$$-T_1 \leq 4(h_{t1} b^{11})^2 \Gamma_1,$$

where

$$\Gamma_1 = \sum_{k\geq 2} \frac{c_k^2}{b_k} - \left(1 + \sum_{k\geq 2} \frac{1}{b_k}\right)^{-1} \left(\sum_{k\geq 2} \frac{c_k}{b_k}\right)^2.$$

Next, we shall simplify Γ_1. By denoting $\beta_k = \frac{1}{b_k}$, we have

$$b_{11} b^{kk} = \frac{1}{2\beta_k} - \frac{1}{2}, \quad c_k = \frac{1}{2\beta_k} - \frac{3+\alpha}{2}.$$

Hence,

$$\Gamma_1 = \sum_{k\geq 2} \beta_k \left(\frac{1}{2\beta_k} - \frac{3+\alpha}{2}\right)^2 - \left(1 + \sum_{k\geq 2} \beta_k\right)^{-1} \left[\sum_{k\geq 2} \beta_k \left(\frac{1}{2\beta_k} - \frac{3+\alpha}{2}\right)\right]^2$$
$$= \frac{1}{4} \sum_{k\geq 2} \frac{1}{\beta_k} - \left(1 + \sum_{k\geq 2} \beta_k\right)^{-1} \frac{(n+1+\alpha)^2}{4} + \frac{(3+\alpha)^2}{4}.$$

Since

$$1 \leq 1 + \sum_{k\geq 2} \beta_k \leq n-1,$$

it follows that

$$\Gamma_1 \leq \tfrac{1}{4} \sum_{k\geq 2} \frac{1}{\beta_k} - \frac{(n+1+\alpha)^2}{4(n-1)} + \frac{(3+\alpha)^2}{4}$$
$$= \tfrac{n-2}{4(n-1)} (2+\alpha)^2 + \tfrac{1}{4}(2\sigma_1 b_{11} - 2).$$

Therefore,

$$T_1 \geq -\left[\frac{n-2}{n-1}(2+\alpha)^2 + 2\sigma_1 b_{11} - 2\right](h_{t1}b^{11})^2. \tag{3.34}$$

Now, we will deal with the term T_2. For every $i \geq 2$ fixed, set $X_k = h_t b^{ii} b^{kk} b_{kk,i}$, $\lambda = 1 + 2b_{ii}b^{11}$, $\mu = h_{ti}b^{ii}$, $b_k = 1 + 2b_{ii}b^{kk}$ $(k \neq i)$, $b_i = 1$, and $c_k = b_{ii}b^{kk} - \frac{\alpha}{2} - (1+\alpha)b_{ii}b^{11}$. By Lemma 3.2, we have

$$-T_2 \leq 4\sum_{i\geq 2}(h_{ti}b^{ii})^2\Gamma_i,$$

where

$$\Gamma_i = c_i^2 + \sum_{\substack{k\geq 2\\ k\neq i}}\frac{c_k^2}{b_k} - \left(\frac{1}{\lambda} + 1 + \sum_{\substack{k\geq 2\\ k\neq i}}\frac{1}{b_k}\right)^{-1}\left(c_i + \sum_{\substack{k\geq 2\\ k\neq i}}\frac{c_k}{b_k}\right)^2.$$

For $k \neq i$, let $\beta_k = \frac{1}{b_k}$. Then, we have

$$b_{ii}b^{kk} = \frac{1}{2\beta_k} - \frac{1}{2}, \qquad c_k = \frac{1}{2\beta_k} - \delta,$$

where

$$\delta = \frac{1+\alpha}{2} + (1+\alpha)b_{ii}b^{11}.$$

Noticed that

$$c_i = \frac{3}{2} - \delta, \qquad \frac{\delta}{\lambda} = \frac{1+\alpha}{2},$$

we obtain

$$\begin{aligned}
\Gamma_i &= c_i^2 + \sum_{\substack{k\geq 2\\ k\neq i}}\beta_k\left(\frac{1}{2\beta_k} - \delta\right)^2 - \left(\frac{1}{\lambda} + 1 + \sum_{\substack{k\geq 2\\ k\neq i}}\beta_k\right)^{-1}\left[c_i + \sum_{\substack{k\geq 2\\ k\neq i}}\beta_k\left(\frac{1}{2\beta_k} - \delta\right)\right]^2 \\
&= \frac{1}{4}\sum_{\substack{k\geq 2\\ k\neq i}}\frac{1}{\beta_k} - \left(\frac{1}{\lambda} + 1 + \sum_{\substack{k\geq 2\\ k\neq i}}\beta_k\right)^{-1}\left(\frac{n}{2} + \frac{\delta}{\lambda}\right)^2 + \frac{9}{4} + \frac{\delta^2}{\lambda} \\
&= \frac{1}{4}\sum_{\substack{k\geq 2\\ k\neq i}}\frac{1}{\beta_k} - \left(\frac{1}{\lambda} + 1 + \sum_{\substack{k\geq 2\\ k\neq i}}\beta_k\right)^{-1}\frac{(n+1+\alpha)^2}{4} + \frac{9}{4} + \frac{1+\alpha}{2}\delta.
\end{aligned}$$

Obviously,

$$1 \leq \frac{1}{\lambda} + 1 + \sum_{\substack{k\geq 2\\ k\neq i}} \beta_k \leq n-1,$$

and hence

$$\begin{aligned}\Gamma_i &\leq \frac{1}{4}\sum_{\substack{k\geq 2\\ k\neq i}}\frac{1}{\beta_k} - \frac{(n+1+\alpha)^2}{4(n-1)} + \frac{9}{4} + \frac{1+\alpha}{2}\delta \\ &= \frac{n-2}{4(n-1)}\alpha^2 - \frac{1}{n-1}\alpha + \frac{n-3}{2(n-1)} + \frac{1}{2}\sigma_1 b_{ii} + \frac{1}{2}\alpha^2 b_{ii}b^{11} + \alpha b_{ii}b^{11}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}T_2 \geq -\sum_{i\geq 2}\Big(&\frac{n-2}{n-1}\alpha^2 - \frac{4}{n-1}\alpha + \frac{2n-6}{n-1} + 2\sigma_1 b_{ii} \\ &+2\alpha^2 b_{ii}b^{11} + 4\alpha b_{ii}b^{11}\Big)(h_{ti}b^{ii})^2. \qquad (3.35)\end{aligned}$$

Combining (3.32)–(3.35), we obtain

$$R \geq \sum_i \left(\frac{1}{n-1}\alpha^2 + \frac{4}{n-1}\alpha - \frac{2n-6}{n-1} - 2\sigma_1 b_{ii}\right)(h_{ti}b^{ii})^2 + R(\nabla_\theta \varphi). \quad (3.36)$$

By (3.24)–(3.25) and (3.36),

$$\begin{aligned}L(\varphi) - \frac{1}{n-1}\varphi_t^2 \geq \frac{1}{p-1}\Bigg[&\frac{1}{n-1}\alpha^2 - (p - \frac{4}{n-1})\alpha + (n-3)(p-1) \\ &-\frac{2n-6}{n-1}\Bigg]\sum_i (h_{ti}b^{ii})^2 \\ &+2\left(\frac{\beta\alpha+1}{p-1}h_t\sigma_1 + \beta\alpha h_t^{-1}\sum_j h_{tj}^2 b^{jj} + \beta\sum_k b^{kk}b_{kk,t}\right) \\ &\sum_i h_{ti}b^{ii}\frac{\partial\varphi}{\partial\theta_i} + \frac{1}{p-1}R(\nabla_\theta\varphi). \qquad (3.37)\end{aligned}$$

For $\alpha = n+1-2p$ and $\beta = -\frac{1}{n-1}$, we have

$$\begin{aligned}L(\varphi) - \frac{1}{n-1}\varphi_t^2 &\geq \frac{2(n+1)}{n-1}\frac{(p-2)^2}{p-1}\sum_i (h_{ti}b^{ii})^2 \mod \nabla_\theta\varphi \\ &\geq 0 \mod \nabla_\theta\varphi. \qquad (3.38)\end{aligned}$$

So, we finish the proof of (3.20) and then complete the proof of Theorem 1.2. □

Now, we state the following elementary calculus lemma which appeared in [19].

Lemma 3.2 *Let* $\lambda \geq 0$, $\mu \in \mathbb{R}$, $b_k > 0$, *and* $c_k \in \mathbb{R}$ *for* $2 \leq k \leq n-1$. *Define the quadratic polynomial*

$$Q(X_2, \cdots, X_{n-1}) = -\sum_{2\leq k\leq n-1} b_k X_k^2 - \lambda\Big(\sum_{2\leq k\leq n-1} X_k\Big)^2 + 4\mu \sum_{2\leq k\leq n-1} c_k X_k.$$

Then, we have

$$Q(X_2, \cdots, X_{n-1}) \leq 4\mu^2\Gamma,$$

where

$$\Gamma = \sum_{2\leq k\leq n-1} \frac{c_k^2}{b_k} - \lambda\Big(1 + \lambda \sum_{2\leq k\leq n-1} \frac{1}{b_k}\Big)^{-1}\Big(\sum_{2\leq k\leq n-1} \frac{c_k}{b_k}\Big)^2.$$

Let us give a remark on Theorem 1.2.

Remark 3.3 In the proof of Theorem 1.2, if we choose $-\frac{1}{n-1} < \beta \leq 0$ (for $\beta = 0$ it suffices to work on $L(\varphi)$), we may use Lemma 3.2 to get a lower bound for P_2 and repeat the similar calculation as in step 2 and step 3. Then, we have

$$\begin{aligned} L(\varphi) + \beta\varphi_t^2 \geq \Big[q_1(\alpha,\beta)\sum_i (b^{ii})^2 + q_2(\alpha,\beta)\sigma_1^2\Big]h_t^2 \\ +q_3(\alpha,\beta)\sum_i (h_{ti}b^{ii})^2 \qquad \mathrm{mod}\ \nabla_\theta\varphi. \end{aligned}$$

In the last formula, we have let

$$\begin{aligned} q_1(\alpha,\beta) &= \frac{n+1-2p-\alpha}{p-1}, \\ q_2(\alpha,\beta) &= \frac{\beta}{1+(n-1)\beta}\frac{(n+1-2p-\alpha)^2}{(p-1)^2} - \frac{n+1-2p-\alpha}{(p-1)^2}, \\ q_3(\alpha,\beta) &= \frac{1}{p-1}\Big[\frac{1}{n-1}\alpha^2 - \Big(p - \frac{4}{n-1}\Big)\alpha + (n-3)(p-1) - \frac{2n-6}{n-1}\Big]. \end{aligned}$$

By a simple observation, a sufficient condition to guarantee

$$L(\varphi) + \beta\varphi_t^2 \geq 0 \qquad \mathrm{mod}\ \nabla_\theta\varphi,$$

is

$$\begin{cases} q_1(\alpha, \beta) + q_2(\alpha, \beta) \geq 0, \\ q_1(\alpha, \beta) + (n-1)q_2(\alpha, \beta) \geq 0, \\ q_3(\alpha, \beta) \geq 0. \end{cases} \tag{3.39}$$

By solving the inequalities in (3.39), we can arrive at the following conclusions:

- $p = 2, n \geq 2 : \alpha > n - 3, \beta = 0;$
- $\alpha = n + 1 - 2p, -\frac{1}{n-1} < \beta \leq 0.$

For the case $n = 2$, $p = 2$ we can choose $\alpha = \beta = 0$. Let K be the curvature of the level curves. Then, for $\varphi = -\log K$, we have

$$L(\varphi) \geq (h_{t1}b^{11})^2 \geq 0 \qquad \text{mod } \nabla_\theta \varphi.$$

We recover Longinetti's result (see Remark 1.4).

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References

1. Ahlfors, L.V.: Conformal Invariants. Topics in Geometric Function Theory. Reprint of the Original, p. 162. AMS Chelsea, Providence (1973)
2. Bian, B.-J., Guan, P., Ma, X.-N., Xu, L.: A constant rank theorem for quasiconcave solutions of fully nonlinear partial differential equations. Indiana Univ. Math. J. **60**(1), 101–119 (2011)
3. Bianchini, C., Longinetti, M., Salani, P.: Quasiconcave solutions to elliptic problems in convex rings. Indiana Univ. Math. J. **58**(4), 1565–1589 (2009)
4. Caffarelli, L.A., Friedman, A.: Convexity of solutions of semilinear elliptic equations. Duke Math. J. **52**(2), 431–456 (1985)
5. Caffarelli, L.A., Spruck, J.: Convexity properties of solutions to some classical variational problems. Comm. Partial Differ. Equ. **7**(11), 1337–1379 (1982)
6. Chang, S.-Y. A., Ma, X.-N., Yang, P.: Principal curvature estimates for the convex level sets of semilinear elliptic equations. Discrete Contin. Dyn. Syst. **28**(3), 1151–1164 (2010)
7. Chen, C.-Q., Shi, S.-J.: Curvature estimates for the level sets of spatial quasiconcave solutions to a class of parabolic equations. Sci. China Math. **54**(10), 2063–2080 (2011)
8. Chiti, G., Longinetti, M.: Differential Inequalities for Minkowski Functionals of Level Sets. General Inequalities, 6 (Oberwolfach, 1990), 109–127, Internat. Ser. Numer. Math., 103, Birkhäuser, Basel, (1992)
9. Gabriel, R.M.: A result concerning convex level surfaces of 3-dimensional harmonic functions. J. London Math. Soc. **32**, 286–294 (1957)
10. Guan, P., Xu, L.: Convexity estimates for level sets of quasiconcave solutions to fully nonlinear elliptic equations. J. Reine Angew. Math. **680**, 41–67 (2013)
11. Jost, J., Ma, X.-N., Ou, Q.-Z.: Curvature estimates in dimensions 2 and 3 for the level sets of p-harmonic functions in convex rings. Trans. Amer. Math. Soc. **364**, 4605–4627 (2012)
12. Kawohl, B.: Rearrangements and Convexity of Level Sets in PDE. Lecture Notes in Mathematics, 1150, p. 136. Springer, Berlin (1985)
13. Korevaar, N.J.: Convexity of level sets for solutions to elliptic ring problems. Comm. Partial Differ. Equ. **15**(4), 541–556 (1990)
14. Lewis, J.L.: Capacitary functions in convex rings. Arch. Rational Mech. Anal. **66**(3), 201–224 (1977)
15. Longinetti, M.: Convexity of the level lines of harmonic functions. Boll. Un. Mat. Ital. A (6) **2**(1), 71–75 (1983)

16. Longinetti, M.: On minimal surfaces bounded by two convex curves in parallel planes. J. Differ. Equ. **67**(3), 344–358 (1987)
17. Longinetti, M.: A Strict Convexity Principle for Nonlinear Elliptic Equations. Preprint, (2006)
18. Longinetti, M., Salani, P.: On the Hessian matrix and Minkowski addition of quasiconvex functions. J. Math. Pures Appl. (9) **88**(3), 276–292 (2007)
19. Ma, X.-N., Ou, Q.-Z., Zhang, W.: Gaussian curvature estimates for the convex level sets of p-harmonic functions. Comm. Pure Appl. Math. **63**(7), 935–971 (2010)
20. Ma, X.-N., Ye, J., Ye, Y.-H.: Principal curvature estimates for the level sets of harmonic functions and minimal graphs in $\mathbb{R}^3$. Commun. Pure Appl. Anal. **10**(1), 225–243 (2011)
21. Ortel, M., Schneider, W.: Curvature of level curves of harmonic functions. Canad. Math. Bull. **26**(4), 399–405 (1983)
22. Schneider, R.: Convex Bodies: The Brunn-Minkowski theory. Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, p. xiv+490 (1993)
23. Shiffman, M.: On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes. Ann. Math. **2**(63), 77–90 (1956)
24. Talenti, G.: On functions, whose lines of steepest descent bend proportionally to level lines. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **10**(4), 587–605 (1983)
25. Wang, P.-H., Zhang, W.: Gaussian curvature estimates for the convex level sets of solutions for some nonlinear elliptic partial differential equations. J. Partial Differ. Equ. **25**(3), 239–275 (2012)
26. Ye, Y.-H.: Gauss curvature estimates for the convex level sets of a minimal surface immersed in $\mathbb{R}^{n+1}$. Nonlinear Anal. **74**(16), 5663–5677 (2011)
27. Zhang, T., Zhang, W.: On convexity of level sets of p-harmonic functions. J. Differ. Equ. **255**(7), 2065–2081 (2013)