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# Curvature Estimates for the Level Sets of Solutions to the Monge-Ampère Equation det $D^2u = 1$*

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**Abstract** For the Monge-Ampère equation $\det D^2u = 1$, the authors find new auxiliary curvature functions which attain their respective maxima on the boundary. Moreover, the upper bounded estimates for the Gauss curvature and the mean curvature of the level sets for the solution to this equation are obtained.

**Keywords** Curvature estimates, Level sets, Monge-Ampère equation
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## 1 Introduction

The Monge-Ampère equation is one of the most important fully nonlinear partial differential equations. It has the general form

$$\det D^2u = f(x, u, Du).$$

Here $\det D^2u$ denotes the determinant of the Hessian matrix $D^2u$, $u$ is a function in the Euclicean space $\mathbb{R}^n$, and $f$ is a given function. It is elliptic when the Hessian matrix $D^2u$ is positive definite, namely, $u$ is strictly convex. There is an extensive literature on the research of the Monge-Ampère equation (see [5, 13] and the references therein).

In 2011, Hong, Huang, Wang [7] studied a class of degenerate elliptic Monge-Ampère equations in a smooth, bounded and strictly convex domain $\Omega$ of dimension 2. When they proved the existence of global smooth solutions to the homogeneous Dirichlet problem, they introduced the key auxiliary function $\mathcal{H}$, which is the product of curvature $\kappa$ of the level line of $u$ and the cubic of $|Du|$, and got the uniformly lower bound of $\mathcal{H}$ on $\overline{\Omega}$. These imply an estimate for the

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lower bound of the curvature of the level line in some sense, which inspires us to study the following simplest homogeneous Dirichlet problem for the elliptic Monge-Ampère equation:

$$\begin{cases} \det D^2 u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

We find appropriate functions called $P$ functions, prove that the $P$ functions attain their maxima on the boundary and get the upper bounded estimates for the Gauss curvature and the mean curvature of the level sets.

In order to state our results, we need the standard curvature formula of the level sets of a function (see [12]). Firstly, we recall the definition of elementary symmetric functions. For any $k = 1, 2, \cdots, n$, we set

$$\sigma_k(\lambda) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k} \quad \text{for any } \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n.$$

Let $W = (w_{ij})$ be a symmetric $n \times n$ matrix, and we set

$$\sigma_k(W) = \sigma_k(\lambda(W)),$$

where $\lambda(W) = (\lambda_1(W), \cdots, \lambda_n(W))$ are the eigenvalues of $W$. We also set $\sigma_0 = 1$ and $\sigma_k = 0$ for any $k > n$.

Since the level sets of the strictly convex solution to the problems (1.1) are convex with respect to the normal direction $-Du$, we have the following formula on the $m$-th curvature of the level sets of the solution $u$, $m = 1, 2, \cdots, n-1$,

$$\sum_{k,l=1}^{n} \frac{\partial \sigma_{m+1}(D^2 u)}{\partial u_{kl}} u_k u_l |Du|^{-m-2}.$$

When $m = 1$,

$$H = \sum_{k,l=1}^{n} \frac{\partial \sigma_2(D^2 u)}{\partial u_{kl}} u_k u_l |Du|^{-3}$$

is the mean curvature of the level sets; When $m = n-1$,

$$K = \sum_{k,l=1}^{n} \frac{\partial \sigma_n(D^2 u)}{\partial u_{kl}} u_k u_l |Du|^{-n-1}$$

is the Gauss curvature of the level sets.

**Theorem 1.1** *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain, $n \ge 2$, and $u$ be the strictly convex solution to* (1.1)*. Then the function*

$$\varphi = \sum_{k,l=1}^{n} \frac{\partial \sigma_n(D^2 u)}{\partial u_{kl}} u_k u_l - 2u$$

*attains its maximum on the boundary $\partial\Omega$.*

**Theorem 1.2** *Under the same assumptions as in the above theorem, we have that the function*

$$\psi = \sum_{k,l=1}^{n} \frac{\partial \sigma_2(D^2u)}{\partial u_{kl}} u_k u_l - 2(n-1)u$$

*also attains its maximum on the boundary $\partial\Omega$. Moreover, $\psi$ attains its maximum in $\Omega$ if and only if $\Omega$ is an ellipse for $n = 2$ or a ball for $n \geq 3$.*

Naturally, we have the following corollary.

**Corollary 1.1** *Let $\Omega$ be a smooth, bounded and strictly convex domain in $\mathbb{R}^n$, $n \geq 2$. If $u$ is the solution to the problem (1.1), then the functions $K|Du|^{n+1}$ and $H|Du|^3$ attain their maxima only on the boundary $\partial\Omega$. Thus, for $x \in \Omega \setminus \Omega'$, we have the following estimates*:

$$K(x) < \frac{\max\limits_{\partial\Omega} K \max\limits_{\partial\Omega'} \kappa_M^{n+1}}{\min\limits_{\partial\Omega} \kappa_m^{n+1}}$$

*and*

$$H(x) < \frac{\max\limits_{\partial\Omega} H \max\limits_{\partial\Omega'} \kappa_M^{n+1}}{\min\limits_{\partial\Omega} \kappa_m^{n+1}},$$

*where*

$$\Omega' = \{x \in \Omega \mid u(x) < c, c \in (\min_{\Omega} u, 0) \text{ is a constant}\},$$

*and $\kappa_m, \kappa_M$ are the minimal and maximal principal curvatures of the level sets at a point respectively.*

It should be mentioned that for the case $n = 2$, Ma [10] and Anedda, Porru [1] considered the problem (1.1) and arrived at the conclusion of Theorem 1.1. When $n = 2$, there is only one curvature $\kappa$ for the level sets at a point, so $\kappa = K = H$ and $\varphi = \psi$ in Theorems 1.1–1.2. And

$$\mathcal{H} = \kappa|Du|^3 = \sum_{k,l=1}^{2} \frac{\partial \sigma_2(D^2u)}{\partial u_{kl}} u_k u_l$$

is the auxiliary function introduced by Hong, Huang and Wang [7].

There are also many papers that study curvature estimates for the level sets of solutions to partial differential equations (see [2–4, 8–9, 11] etc).

This paper is organized as follows. In Section 2, we prove Theorem 1.1 by establishing a differential inequality for the given function. In Section 3, through the same process as the proof of Theorem 1.1, we prove the first result in Theorem 1.2. Through the computation of the third derivatives for the solution $u$, we prove the relation between $\psi$ attaining its maximum in the interior and the shape of the domain $\Omega$. Finally, we prove the corollary and give some remarks.

## 2 Proof of Theorem 1.1

Let $D^2u = (u_{ij}),\ (u^{ij}) = (u_{ij})^{-1}$. Because $u$ is the strictly convex solution to the equation $\sigma_n(D^2u) = \det D^2u = 1$, $(u_{ij})$ is positive definite and $\frac{\partial\sigma_n(D^2u)}{\partial u_{kl}} = u^{kl}$. Therefore,

$$\varphi = \sum_{k,l=1}^{n} \frac{\partial\sigma_n(D^2u)}{\partial u_{kl}} u_k u_l - 2u = \sum_{k,l=1}^{n} u^{kl} u_k u_l - 2u. \tag{2.1}$$

We will prove the following differential inequality:

$$\sum_{i,j=1}^{n} u^{ij}\varphi_{ij} \geq 0 \quad \text{in } \Omega. \tag{2.2}$$

From the differential inequality and by the maximum principle, $\varphi$ attains its maximum on the boundary $\partial\Omega$.

In the following, we will prove (2.2). For any $x_o \in \Omega$, we choose coordinates such that $(u_{ij}(x_o))$ is diagonal. All the following calculations are done at $x_o$.

Let ${u^{ij}}_k = \frac{\partial u^{ij}}{\partial x_k}$ and ${u^{ij}}_{kk} = \frac{\partial^2 u^{ij}}{\partial x_k^2}$. From direct computations, we have

$$\begin{aligned}
\varphi_i &= \Big(\sum_{k,l=1}^{n} u^{kl} u_k u_l\Big)_i - 2u_i \\
&= \sum_{k,l=1}^{n} ({u^{kl}}_i u_k u_l + 2u^{kl} u_{ki} u_l) - 2u_i \\
&= \sum_{k,l=1}^{n} {u^{kl}}_i u_k u_l
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
\varphi_{ii} &= \sum_{k,l=1}^{n} ({u^{kl}}_{ii} u_k u_l + 2{u^{kl}}_i u_{ki} u_l) \\
&= \sum_{k,l=1}^{n} {u^{kl}}_{ii} u_k u_l + 2\sum_{l=1}^{n} {u^{il}}_i u_{ii} u_l.
\end{aligned} \tag{2.4}$$

Thus

$$\begin{aligned}
\sum_{i,j=1}^{n} u^{ij}\varphi_{ij} &= \sum_{i=1}^{n} u^{ii}\varphi_{ii} \\
&= \sum_{i,k,l=1}^{n} u^{ii} {u^{kl}}_{ii} u_k u_l + 2\sum_{i,l=1}^{n} {u^{il}}_i u_l \\
&= \sum_{i,k,l=1}^{n} u^{ii} {u^{kl}}_{ii} u_k u_l,
\end{aligned} \tag{2.5}$$

where we have used $\sum\limits_{i=1}^{n} {u^{il}}_i = 0$ in the last equality above. Since

$${u^{kl}}_{ii} = \Big(-\sum_{p,q=1}^{n} u^{kq} u^{pl} u_{pqi}\Big)_i$$

$$
\begin{aligned}
&= \sum_{p,q,r,s=1}^{n} (u^{ks}u^{rq}u^{pl} + u^{kq}u^{ps}u^{rl})u_{pqi}u_{rsi} - \sum_{p,q=1}^{n} u^{kq}u^{pl}u_{pqii} \\
&= 2\sum_{j=1}^{n} u^{kk}u^{ll}u^{jj}u_{jki}u_{jli} - u^{kk}u^{ll}u_{klii},
\end{aligned} \tag{2.6}
$$

substituting (2.6) into (2.5), we obtain

$$
\begin{aligned}
\sum_{i,j=1}^{n} u^{ij}\varphi_{ij} &= \sum_{i=1}^{n} u^{ii}\varphi_{ii} \\
&= 2\sum_{i,j,k,l=1}^{n} u^{ii}u^{jj}u^{kk}u^{ll}u_{ijk}u_{ijl}u_k u_l - \sum_{i,k,l=1}^{n} u^{kk}u^{ll}u^{ii}u_{iikl}u_k u_l.
\end{aligned} \tag{2.7}
$$

Because of the equation $\det(u_{ij}) = 1$, differentiating it once, we can get

$$
\sum_{i,j=1}^{n} u^{ij}u_{ijk} = 0,
$$

i.e.,

$$
\sum_{i=1}^{n} u^{ii}u_{iik} = 0. \tag{2.8}
$$

Differentiating the equation once again, we have

$$
\sum_{i,j=1}^{n} u^{ij}u_{ijkl} + \sum_{i,j,p,q=1}^{n} (-u^{iq}u^{pj}u_{ijk}u_{pql}) = 0,
$$

i.e.,

$$
\sum_{i=1}^{n} u^{ii}u_{iikl} = \sum_{i,j=1}^{n} u^{ii}u^{jj}u_{ijk}u_{ijl}. \tag{2.9}
$$

Substituting (2.9) into (2.7), we obtain

$$
\begin{aligned}
\sum_{i,j=1}^{n} u^{ij}\varphi_{ij} &= \sum_{i=1}^{n} u^{ii}\varphi_{ii} \\
&= 2\sum_{i,j,k,l=1}^{n} u^{ii}u^{jj}u^{kk}u^{ll}u_{ijk}u_{ijl}u_k u_l - \sum_{i,j,k,l=1}^{n} u^{kk}u^{ll}u^{ii}u^{jj}u_{ijk}u_{ijl}u_k u_l \\
&= \sum_{i,j,k,l=1}^{n} u^{ii}u^{jj}u^{kk}u^{ll}u_{ijk}u_{ijl}u_k u_l \\
&= \sum_{i,j=1}^{n} u^{ii}u^{jj}\Big(\sum_{k=1}^{n} u_{ijk}u^{kk}u_k\Big)^2 \\
&\geq 0.
\end{aligned} \tag{2.10}
$$

We have completed the proof of Theorem 1.1.

**Remark 2.1** When $n=2$, Ma [10] and Anedda, Porru [1] gave the result of Theorem 1.1 and further pointed out that $\varphi$ assumes its minimum on $\partial\Omega$ or at the unique critical point $x_0$ of $u$, i.e., the point where $Du=0$. We can also get the conclusion from the above proof directly. In fact, from (2.3), we get

$$-\varphi_i=\sum_{k,l=1}^{2}u^{kk}u^{ll}u_{kli}u_ku_l,$$

that is

$$\begin{cases}(u^{11}u_1)^2u_{111}+2u_1u_2u_{112}+(u^{22}u_2)^2u_{122}=-\varphi_1,\\(u^{11}u_1)^2u_{112}+2u_1u_2u_{122}+(u^{22}u_2)^2u_{222}=-\varphi_2.\end{cases}$$

Here we have used the equation $\det(u_{ij})=1$. Combining with (2.8), that is

$$\begin{cases}u^{11}u_{111}+u^{22}u_{122}=0,\\u^{11}u_{112}+u^{22}u_{222}=0,\end{cases}$$

under the case of modulo $D\varphi$, we obtain the homogeneous linear algebraic system about the third derivatives $u_{111},u_{112},u_{122},u_{222}$ of $u$,

$$\begin{cases}u^{11}u_{111}+u^{22}u_{122}=0,\\u^{11}u_{112}+u^{22}u_{222}=0,\\(u^{11}u_1)^2u_{111}+2u_1u_2u_{112}+(u^{22}u_2)^2u_{122}=0,\\(u^{11}u_1)^2u_{112}+2u_1u_2u_{122}+(u^{22}u_2)^2u_{222}=0.\end{cases}\tag{2.11}$$

From direct computations, we get that the determinant of the coefficient matrix is $(u^{11}u_1^2+u^{22}u_2^2)^2$, which is greater than 0 in $\Omega\setminus\{x_0\}$. Therefore

$$u_{111}=u_{112}=u_{122}=u_{222}=0\quad\text{mod }(D\varphi).$$

Consequently, from (2.10), we have that

$$\sum_{i,j=1}^{2}u^{ij}\varphi_{ij}=0\quad\text{mod }(D\varphi)\ \text{ in }\Omega\setminus\{x_0\},$$

and by the maximum principle, $\varphi$ attains its minimum on $\partial\Omega$ or at the unique critical points $x_0$.

## 3 Proof of Theorem 1.2

Let $\frac{\partial\sigma_2(D^2u)}{\partial u_{kl}}=b^{kl}$. Then

$$\psi=\sum_{k,l=1}^{n}b^{kl}u_ku_l-2(n-1)u.\tag{3.1}$$

We will prove the following differential inequality:

$$\sum_{i,j=1}^{n}u^{ij}\psi_{ij}\geq 0\quad\text{ in }\Omega.\tag{3.2}$$

From the differential inequality, and by the maximum principle, $\psi$ attains its maximum on the boundary $\partial\Omega$.

In the following, we will prove the differential inequality (3.2). For any $x_o \in \Omega$, we choose coordinates such that $D^2u(x_o)$ is diagonal. All the following calculations are done at $x_o$.

Let ${b^{ij}}_k = \frac{\partial b^{ij}}{\partial x_k}$ and ${b^{ij}}_{kk} = \frac{\partial^2 b^{ij}}{\partial x_k^2}$. From direct computations, we have

$$\begin{aligned}
\psi_i &= \Big( \sum_{k,l=1}^n b^{kl} u_k u_l \Big)_i - 2(n-1)u_i \\
&= \sum_{k,l=1}^n ({b^{kl}}_i u_k u_l + 2b^{kl} u_{ki} u_l) - 2(n-1)u_i \\
&= \sum_{k,l=1}^n {b^{kl}}_i u_k u_l + 2b^{ii} u_{ii} u_i - 2(n-1)u_i
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
\psi_{ii} = &\sum_{k,l=1}^n ({b^{kl}}_{ii} u_k u_l + 4{b^{kl}}_i u_{ki} u_l + 2b^{kl} u_{kii} u_l + 2b^{kl} u_{ki} u_{li}) \\
&- 2(n-1)u_{ii} \\
= &\sum_{k,l=1}^n {b^{kl}}_{ii} u_k u_l + 4\sum_{l=1}^n {b^{il}}_i u_{ii} u_l + 2\sum_{k=1}^n b^{kk} u_{kii} u_k \\
&+ 2b^{ii} u_{ii}^2 - 2(n-1)u_{ii}.
\end{aligned} \tag{3.4}$$

Therefore,

$$\begin{aligned}
\sum_{i,j=1}^n u^{ij} \psi_{ij} &= \sum_{i=1}^n u^{ii} \psi_{ii} \\
&= \sum_{i,k,l=1}^n u^{ii} {b^{kl}}_{ii} u_k u_l + 4\sum_{i,l=1}^n {b^{il}}_i u_l + 2\sum_{i,k=1}^n b^{kk} u^{ii} u_{iik} u_k \\
&\quad + 2\sum_{i=1}^n b^{ii} u_{ii} - 2n(n-1) \\
&= \sum_{i,k,l=1}^n u^{ii} {b^{kl}}_{ii} u_k u_l + 4\sigma_2(D^2u) - 2n(n-1),
\end{aligned} \tag{3.5}$$

where we have used (2.8) and

$$\sum_{i=1}^n {b^{il}}_i = 0, \quad \sum_{i=1}^n b^{ii} u_{ii} = 2\sigma_2(D^2u)$$

in the last equality above. Since

$$b^{kl} = \begin{cases} \sum\limits_{\substack{j=1 \\ j\neq k}}^n u_{jj}, & k = l, \\ -u_{kl}, & k \neq l, \end{cases}$$

we have

$$b^{kl}{}_{ii} = \begin{cases} \displaystyle\sum_{\substack{j=1 \\ j\neq k}}^{n} u_{jjii}, & k = l, \\ -u_{klii}, & k \neq l. \end{cases} \tag{3.6}$$

Substituting (3.6) into (3.5), we can get

$$\begin{aligned} \sum_{i,j=1}^{n} u^{ij}\psi_{ij} &= \sum_{i=1}^{n} u^{ii}\psi_{ii} \\ &= \sum_{\substack{i,j,k=1 \\ j\neq k}}^{n} u^{ii}u_{iijj}u_k^2 - \sum_{\substack{i,k,l=1 \\ k\neq l}}^{n} u^{ii}u_{iikl}u_k u_l \\ &\quad + (4\sigma_2(D^2u) - 2n(n-1)) \\ &= \sum_{\substack{i,j,k,l=1 \\ k\neq l}}^{n} (u^{ii}u^{jj}u_{ijl}^2 u_k^2 - u^{ii}u^{jj}u_{ijk}u_{ijl}u_k u_l) \\ &\quad + (4\sigma_2(D^2u) - 2n(n-1)), \end{aligned} \tag{3.7}$$

where we have used (2.9) in the last equality above. We also have

$$\sigma_2(D^2u) \geq C_n^2(\sigma_n(D^2u))^{\frac{2}{n}} = C_n^2 = \frac{n(n-1)}{2} \tag{3.8}$$

by Newton's inequality (see [6, section 2.22]), and

$$\sum_{k,l=1}^{n} (u_{ijl}^2 u_k^2 - u_{ijk}u_{ijl}u_k u_l) \geq 0 \tag{3.9}$$

by Cauchy-Schwarz's inequality. Combining (3.7)–(3.9), We can obtain that

$$\begin{aligned} \sum_{i,j=1}^{n} u^{ij}\psi_{ij} &= \sum_{i=1}^{n} u^{ii}\psi_{ii} \\ &= \sum_{\substack{i,j,k,l=1 \\ k\neq l}}^{n} (u^{ii}u^{jj}u_{ijl}^2 u_k^2 - u^{ii}u^{jj}u_{ijk}u_{ijl}u_k u_l) \\ &\quad + (4\sigma_2(D^2u) - 2n(n-1)) \\ &= \sum_{i,j=1}^{n} \Big(u^{ii}u^{jj} \sum_{k,l=1}^{n} (u_{ijl}^2 u_k^2 - u_{ijk}u_{ijl}u_k u_l)\Big) \\ &\quad + (4\sigma_2(D^2u) - 2n(n-1)) \\ &\geq 0. \end{aligned} \tag{3.10}$$

Furthermore, $\psi = \varphi$ when $n = 2$. It is easily obtained that if $\psi$ attains its maximum in $\Omega$, then $\Omega$ is an ellipse by Remark 2.1 and vice versa. We also can get that $\Omega$ is an ellipse from

(3.10). In fact, if $\psi$ attains its maximum in $\Omega$, then $\psi$ is a constant in $\Omega$. So (3.10) is the equality, that is,

$$\begin{aligned}0 &= \sum_{i,j=1}^{2}(u^{ii}u^{jj}\sum_{k,l=1}^{n}(u_{ijl}^2u_k^2 - u_{ijk}u_{ijl}u_ku_l))\\ &= (u^{11})^2(u_{111}u_2 - u_{112}u_1)^2 + (u^{22})^2(u_{122}u_2 - u_{222}u_1)^2 + 2(u_{112}u_2 - u_{122}u_1)^2.\end{aligned}$$

Because $u^{11} > 0$, and $u^{22} > 0$,

$$\begin{cases} u_{111}u_2 - u_{112}u_1 = 0,\\ u_{122}u_2 - u_{222}u_1 = 0,\\ u_{112}u_2 - u_{122}u_1 = 0.\end{cases} \tag{3.11}$$

Since $u$ is strictly convex, it has the unique critical point $x_0$, and $(u_1, u_2) \neq (0, 0)$ in $\Omega \setminus \{x_0\}$. From the theory of linear algebraic systems, we have that the rank of the coefficient matrix of the system (3.11) about $u_1, u_2$ is less than 2, so

$$\begin{cases} u_{112}^2 - u_{111}u_{122} = 0,\\ u_{122}^2 - u_{112}u_{222} = 0,\\ u_{111}u_{222} - u_{112}u_{122} = 0.\end{cases} \tag{3.12}$$

By (2.8), we get

$$\begin{cases} u_{111}u^{11} + u_{122}u^{22} = 0,\\ u_{112}u^{11} + u_{222}u^{22} = 0.\end{cases} \tag{3.13}$$

Combining (3.12) with (3.13), we obtain

$$u_{111} = u_{112} = u_{122} = u_{222} = 0 \quad \text{in } \Omega \setminus \{x_0\},$$

so all the third derivatives of $u$ vanish in $\Omega$ by the continuousness. Consequently, $\Omega = \{u < 0\}$ must be an ellipse.

When $n \geq 3$, if $\psi$ attains its maximum in $\Omega$, then $\psi$ is a constant in $\Omega$. So (3.10) is the equality and we must have $4\sigma_2(D^2u) - 2n(n-1) = 0$, that is, the equality holds in (3.8). But the equality holds in the Newton's inequality, if and only if all the eigenvalues of $D^2u$ are equal. Therefore, the eigenvalues of $D^2u$ are equal to 1 by the equation $\det D^2u = 1$ and $D^2u$ is the unit matrix. Consequently,

$$u = \frac{1}{2}(|x - x_0|^2 - r^2),$$

where $x_0 \in \mathbb{R}^n$ is a fixed point, $r > 0$ is a constant, and $\Omega = \{u < 0\} = B_r(x_0)$ is a ball. On the other hand, if $\Omega = B_r(x_0)$ is a ball, then the solution to the problem (1.1) is $u = \frac{1}{2}(|x-x_0|^2-r^2)$ and $\psi \equiv (n-1)r^2$ is a constant.

We have completed the proof of Theorem 1.2.

## 4 Proof of Corollary 1.1

We firstly give the boundary estimate of the gradient $Du$ for the solution of (1.1).

**Lemma 4.1** *Let $\Omega$ be a smooth, bounded and strictly convex domain in $\mathbb{R}^n$, $x \in \partial\Omega$ and $\kappa_i(x)$, $i = 1, 2, \cdots, n-1$ be the principal curvatures of $\partial\Omega$ at $x$. Let*

$$\kappa_m(x) = \min\{\kappa_i(x) \mid i = 1, 2, \cdots, n-1\}, \quad \kappa_M(x) = \max\{\kappa_i(x) \mid i = 1, 2, \cdots, n-1\}.$$

*If $u$ is the smooth and strictly convex solution of (1.1), then on the boundary $\partial\Omega$, $|Du|_{\partial\Omega}$ satisfies the following estimate:*

$$\frac{1}{\max\limits_{\partial\Omega} \kappa_M} \le |Du|_{\partial\Omega} \le \frac{1}{\min\limits_{\partial\Omega} \kappa_m}. \tag{4.1}$$

*The same estimate is true for $\Omega' = \big\{x \in \Omega \mid u(x) < c, c \in \big(\min\limits_{\Omega} u, 0\big) \text{ is a constant}\big\}$, that is, on the boundary $\partial\Omega'$, $|Du|_{\partial\Omega'}$ satisfies*

$$\frac{1}{\max\limits_{\partial\Omega'} \kappa_M} \le |Du|_{\partial\Omega'} \le \frac{1}{\min\limits_{\partial\Omega'} \kappa_m}. \tag{4.2}$$

**Proof** For any boundary point $x$, let $\Omega \subseteq \Omega_0$ and $\Omega_1 \subseteq \Omega$ be two balls of radius $R = \frac{1}{\min_{\partial\Omega} \kappa_m}$ and $r = \frac{1}{\max_{\partial\Omega} \kappa_M}$ respectively and $x \in \overline{\Omega} \cap \overline{\Omega}_j$, $j = 0, 1$. Let $u_{\Omega_j}, j = 0, 1$ be the solution to the problem

$$\begin{cases} \det D^2 u = 1 & \text{in } \Omega_j, \\ u = 0 & \text{on } \partial\Omega_j. \end{cases}$$

Since $u$ vanishes on $\partial\Omega$, it follows immediately that

$$|Du_{\Omega_1}(x)| \le |Du(x)| \le |Du_{\Omega_0}(x)|.$$

An explicit calculation yields

$$|Du_{\Omega_1}(x)| = r, \quad |Du_{\Omega_0}(x)| = R,$$

and thus

$$r \le |Du(x)| \le R.$$

Therefore,

$$\frac{1}{\max\limits_{\partial\Omega} \kappa_M} \le |Du|_{\partial\Omega} \le \frac{1}{\min\limits_{\partial\Omega} \kappa_m},$$

and (4.1) holds. For the same reasons, (4.2) also holds.

Next, we start the proof of Corollary 1.1.

**Proof of Corollary 1.1** By Theorem 1.1, we have that $K|Du|^{n+1} - 2u$ takes its maximum on the boundary $\partial\Omega$. For any $x \in \Omega$, we have

$$K(x)|Du(x)|^{n+1} - 2u(x) \le \max_{\partial\Omega} K|Du|^{n+1},$$

and thus, by $u(x) < 0$,

$$K(x)|Du(x)|^{n+1} \le \max_{\partial\Omega} K|Du|^{n+1} + 2u(x) < \max_{\partial\Omega} K|Du|^{n+1}.$$

Therefore $K|Du|^{n+1}$ attains its maximum only on the boundary $\partial\Omega$. For the same reasons, by Theorem 1.2, we get that $H|Du|^3$ also attains its maximum only on the boundary $\partial\Omega$. Since $u$ is strictly convex, $|Du|$ increases along the increasing direction of the level sets. By Lemma 4.1, we have, for $x \in \Omega \setminus \Omega'$,

$$K(x) < \frac{\max\limits_{\partial\Omega}(K|Du|^{n+1})}{|Du(x)|^{n+1}} \le \frac{\max\limits_{\partial\Omega}(K|Du|^{n+1})}{\min\limits_{\partial\Omega'}|Du|^{n+1}} \le \frac{\max\limits_{\partial\Omega} K \max\limits_{\partial\Omega'} \kappa_M^{n+1}}{\min\limits_{\partial\Omega} \kappa_m^{n+1}}.$$

For the same reasons,

$$H(x) < \frac{\max\limits_{\partial\Omega} H \max\limits_{\partial\Omega'} \kappa_M^{n+1}}{\min\limits_{\partial\Omega} \kappa_m^{n+1}}.$$

**Remark 4.1** When $n = 2$, by Remark 2.1, $\varphi$ attains its minimum on $\partial\Omega$ or at the unique critical point $x_0$ of $u$. Therefore, we can furthermore give the positive lower bounded estimate for the curvature of the level lines. In fact, for any $x \in \Omega \setminus \Omega'$, we have

$$\begin{aligned}\kappa(x)|Du(x)|^3 &\ge \min\Big\{\min_{\partial\Omega}\kappa \min_{\partial\Omega}|Du|^3, -2u(x_0)\Big\} + 2u(x)\\ &\ge \min\Big\{\min_{\partial\Omega}\kappa \min_{\partial\Omega}|Du|^3 + 2c, 2c - 2u(x_0)\Big\}.\end{aligned}$$

Since

$$\begin{aligned}\min_{\partial\Omega}|Du|^3 &\ge \frac{1}{\max\limits_{\partial\Omega}\kappa^3},\\ |Du(x)|^3 \le \max_{\partial\Omega}|Du|^3 &\le \frac{1}{\min\limits_{\partial\Omega}\kappa^3},\end{aligned}$$

we obtain

$$\kappa(x) \ge \Big(\min_{\partial\Omega}\kappa^3\Big)\Big(\min\Big\{\frac{\min\limits_{\partial\Omega}\kappa}{\max\limits_{\partial\Omega}\kappa^3} + 2c, 2c - 2u(x_0)\Big\}\Big).$$

**Remark 4.2** It is more interesting to obtain the lower bounded estimate for the curvature of the level sets for Monge-Ampère equations in higher dimensions. If it is true, then it may be helpful to improve the regularity of solutions to degenerate elliptic Monge-Ampère equations in higher dimensions as in [7].

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