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The Convexity and the Gaussian Curvature Estimates for the Level Sets of Harmonic Functions on Convex Rings in Space Forms

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Abstract In this paper, we first establish a constant rank theorem for the second fundamental form of the convex level sets of harmonic functions in space forms. Applying the deformation process, we prove that the level sets of the harmonic functions on convex rings in space forms are strictly convex. Moreover, we give a lower bound for the Gaussian curvature of the convex level sets of harmonic functions in terms of the Gaussian curvature of the boundary and the norm of the gradient on the boundary.

Keywords Convexity of level sets · Curvature estimate · Harmonic function · Space form

Mathematics Subject Classification 35J05 · 53J67

1 Introduction

The geometry of the level sets of solutions to elliptic partial differential equations is a classical subject. For instance, Ahlfors [1] contains the well-known result that level curves of the Green function on a simply connected convex domain in the plane are convex Jordan curves. In 1931, Gergen [8] proved the star shape of the level sets of the Green function on a 3-dimensional star-shaped domain. In 1956, Shiffman [23]

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studied the convexity of the level sets of an immersion minimal annulus in $\mathbb{R}^3$. In 1957, Gabriel [9] proved that the level sets of the Green function on a 3-dimensional bounded convex domain are strictly convex; see also the book by Hörmander [10] for a detailed exposition of Gabriel's proof. Lewis [14] and Caffarelli-Spruck [5] generalized Gabriel's result to p-harmonic functions and a class of semilinear elliptic partial differential equations in higher dimensions. Motivated by the result of Caffarelli-Friedman [4], Korevaar [13] gave a new proof of the results of [5, 9, 14] by applying the deformation process and the constant rank theorem of the second fundamental form of the convex level sets of p-harmonic functions. A survey on this subject and the recent related extensions can be found in [2, 3, 12].

Now we turn to the curvature estimates of the level sets of solutions to elliptic partial differential equations. For 2-dimensional harmonic functions and minimal surfaces with convex level curves, Ortel–Schneider [20] and Longinetti [15, 16] proved that the curvature of the level curves attains its minimum on the boundary. Ma–Ou–Zhang [18] and Chang–Ma–Yang [6] got the lower bound estimates of Gaussian (principal) curvature of the convex level sets of higher-dimensional harmonic functions in terms of the Gaussian (principal) curvature of the boundary and the norm of the gradient on the boundary. Recently, Ma–Zhang [19] also explored the relation between the Gaussian curvature of the level sets and the height of the harmonic function, which generalized the 2-dimensional result by Longinetti [16].

In this paper we shall study the convexity of level sets of harmonic functions in space forms. In this case, Papadimitrakis [21] proved the convexity of the level curves of harmonic functions on convex rings in the hyperbolic plane via one complex variable tools; see also the related works by Rosay–Rudin [22] on the sensitivity of the curvature of the base manifold. Here we give a unified treatment on the sphere and hyperbolic space. We first establish a constant rank theorem of the second fundamental form of the convex level sets of harmonic functions in space forms. Applying the deformation process, we prove that the level sets of the harmonic functions on convex rings in space forms are strictly convex. Moreover, we give a lower bound for the Gaussian curvature of the convex level sets of harmonic functions in terms of the Gaussian curvature of the boundary and the norm of the gradient on the boundary.

Now we state our main results.

Theorem 1.1 *Let (M^n, g) be a space form of sectional curvature $K_{sec} = 1$, or -1, and Ω_0 and Ω_1 be bounded smooth convex domains in M^n, $n \geq 2$ and $\bar{\Omega}_1 \subset \Omega_0$. Let u satisfy*

$$\begin{cases} \Delta u = 0 & \text{in } \Omega = \Omega_0 \backslash \bar{\Omega}_1, \\ u = 0 & \text{on } \partial\Omega_0, \\ u = 1 & \text{on } \partial\Omega_1. \end{cases} \tag{1.1}$$

Then the level sets of u are smooth strictly convex hypersurfaces. Let K be the Gaussian curvature of the level sets; we have the following estimates.

Case 1: for $(M^n, g) = (S^n, g_{standard})$ with $K_{sec} = 1$, if $n = 2, 3$, we have

$$\min_{\Omega} K \geq \min_{\partial\Omega} K, \tag{1.2}$$

if $n \geq 4$, *we have*

$$\min_{\Omega} K \geq \min_{\partial\Omega} K \left(\frac{\min_{\partial\Omega_0} |\nabla u|}{\max_{\partial\Omega_1} |\nabla u|} \right)^{n-3}, \tag{1.3}$$

in this case, we assume Ω_0, Ω_1 *lie on the same upper-hemisphere.*

Case 2: For $(M^n, g) = (H^n, g_{standard})$ *with* $K_{sec} = -1$, *if* $n = 2$, *we have*

$$\left(\frac{\min_{\partial\Omega_0} |\nabla u|}{\max_{\partial\Omega_1} |\nabla u|} \right) \min_{\partial\Omega} K \leq K \leq \max_{\partial\Omega} K \left(\frac{\max_{\partial\Omega_1} |\nabla u|}{\min_{\partial\Omega_0} |\nabla u|} \right), \tag{1.4}$$

if $n = 3$, *we have*

$$\min_{\Omega} K \geq \min_{\partial\Omega} K, \tag{1.5}$$

if $n \geq 4$, *we have*

$$\min_{\Omega} K \geq \min_{\partial\Omega} K \left(\frac{\min_{\partial\Omega_0} |\nabla u|}{\max_{\partial\Omega_1} |\nabla u|} \right). \tag{1.6}$$

The proof of Theorem 1.1 is a generalization of the old works in [13, 14, 18] to space forms. The main points are as follows. The harmonic function u in Theorem 1.1 has no critical points in $\Omega = \Omega_0 \backslash \bar{\Omega}_1$. Here we introduce a new trick to prove it; it is new even in the Euclidean case. Then we introduce a new addition operation on the convex domains in space form; it is a substitute for the Minkowski sum in Euclidean space.

In this paper, we make the following convention. The Greek indices $1 \leq \alpha, \beta, \gamma \leq n$, the Latin indices $1 \leq i, j, k \leq n - 1$.

In Sect. 2, we first give the well-known curvature formulas for the level sets of a function. In Sect. 3, we prove that there is no critical point of the solution u in Theorem 1.1 in $\Omega = \Omega_0 \backslash \bar{\Omega}_1$. Then in Sect. 4, we introduce a new addition operation on the convex domains in space forms. In Sect. 5, we prove the constant rank theorem of the second fundamental form of the convex level sets of harmonic functions in space forms along the calculation in Xu [24], then we get the strict convexity result in Theorem 1.1 using the deformation process. We give the lower bound estimates in Sect. 5 via a calculation similar to that in [18].

2 Preliminary

In this section, we introduce the curvature formulas for the level sets of a smooth function on M^n and the relevant geometry of space forms.

Let $u : M^n \longrightarrow \mathbb{R}$ be a smooth function, for any regular value $c \in \mathbb{R}$ of u (i.e., $\nabla u(x) \neq 0$ for any $x \in M$ such that $u(x) = c$), the level set $u^{-1}(c)$ is a smooth hypersurface by the implicit function theorem. The second fundamental form of the level set $u^{-1}(c)$ is given by

$$h(V, W) = -\frac{Hess(u)(V, W)}{|\nabla u|}.$$

Indeed, $\nu = \frac{\nabla u}{|\nabla u|}$ is the normal direction for $u^{-1}(c)$; please see the details in Chow–Lu–Ni [7] (p. 40). Let $\kappa_1, \ldots, \kappa_{n-1}$ be the principle curvature of the level sets of u with respect to ν. Then the k-th curvature of the level sets, denoted by σ_k, is the k-th elementary symmetric function of $\kappa_1, \ldots, \kappa_{n-1}$. Clearly, σ_1 and σ_{n-1} are the mean curvature and the Gaussian curvature of the level sets, respectively.

Proposition 2.1 *Let $u(x) \in C^2(M^n)$ and $|\nabla u| \neq 0$ in M^n. Assume the level sets of u are convex with respect to the normal ν. Let $\{e_1, e_2, \ldots, e_n\}$ be any local orthonormal frame field on M^n. Then the k-th curvature of the level set $\Sigma_c = u^{-1}(c)$ is*

$$\sigma_k[\Sigma_c] = (-1)^k \sum_{\alpha,\beta=1}^{n} \frac{\partial \sigma_{k+1}(u_{\alpha\beta})}{\partial u_{\alpha\beta}} u_\alpha u_\beta |\nabla u|^{-(k+2)}, \tag{2.1}$$

where $(u_{\alpha\beta})$ is the Hessian of u and $1 \leq k \leq n-1$, and $\sigma_k(\nabla^2 u)$ is the k-th elementary symmetric function of the eigenvalues of the Hessian.

Remark 2.2 In [11], Jost–Ma–Ou obtained a similar formula for the k-th curvature of the level set for function $u \in C^2(\Omega)$ for $\Omega \subset \mathbb{R}^n$.

Proof of Proposition 2.1 First, we check that the right-hand side of (2.1) is independent of the choice of the frame fields $\{e_\alpha\}$ on M^n. Then we can just show (2.1) in a special frame field.

Step 1: If B is an orthogonal transformation between two tangential frame fields, i.e., $(\bar{e}_1, \ldots, \bar{e}_n) = (e_1, \ldots, e_n)B$, where $B = (b_{\alpha\beta})$ is an orthogonal matrix, we have

$$e_\beta = \sum_{\beta=1}^{n} \bar{e}_\alpha b_{\beta\alpha}, \qquad \bar{u}_\alpha = \frac{\partial u}{\partial \bar{e}_\alpha} = \sum_{\beta=1}^{n} u_\beta \frac{\partial e_\beta}{\partial \bar{e}_\alpha},$$

$$(\bar{u}_1, \ldots, \bar{u}_n) = (u_1, \ldots, u_n)B, \quad |\nabla \bar{u}|^2 = |\nabla u|^2,$$

and

$$\bar{u}_{\alpha\beta} = \sum_{\gamma,\delta=1}^{n} u_{\gamma\delta} b_{\delta\beta} b_{\gamma\alpha}, \qquad (\bar{u}_{\alpha\beta}) = B^T (u_{\alpha\beta}) B = B^{-1}(u_{\alpha\beta})B,$$

$$(u_{\alpha\beta}) = B(\bar{u}_{\alpha\beta})B^T \;\Rightarrow\; \frac{\partial u_{\gamma\delta}}{\partial \bar{u}_{\alpha\beta}} = b_{\gamma\alpha} b_{\delta\beta}.$$

As in [11], from elementary matrix theory,

$$\sum_{\alpha,\beta=1}^{n} \frac{\partial \sigma_{k+1}(\bar{u}_{\alpha\beta})}{\partial \bar{u}_{\alpha\beta}} \bar{u}_\alpha \bar{u}_\beta = \sum_{\gamma,\delta=1}^{n} \frac{\partial \sigma_{k+1}(u_{\alpha\beta})}{\partial u_{\gamma\delta}} u_\gamma u_\delta.$$

Step 2: For any point $x \in \Sigma_c$, one can choose $e_n = \nu = \frac{\nabla u}{|\nabla u|}$ as the unit inner normal of the level set. Let $\{e_k\}$, $k = 1, 2, \ldots, n-1$, be a local orthonormal frame on the level set Σ_c. We have the second fundamental form of the level sets

$$h_{ij} = g(\nabla_{e_i} e_j, \nu) = -\frac{u_{ij}}{|\nabla u|}. \tag{2.2}$$

It actually follows by

$$u_{ij} = e_i e_j u - g(\nabla_{e_i} e_j, \nu)\nu(u) = -h_{ij}|\nabla u|.$$

Since the level set of u is convex with respect to the normal ν, it follows that the σ_k curvature of its level sets is

$$\begin{aligned}\sigma_k(\kappa_1, \ldots, \kappa_{n-1}) &= \frac{(-1)^k}{|\nabla u|^k}\sigma_k(u_{ij}) = \frac{(-1)^k}{|\nabla u|^k}\frac{\partial \sigma_{k+1}(u_{\alpha\beta})}{\partial u_{nn}} \\ &= (-1)^k \frac{\partial \sigma_{k+1}(u_{\alpha\beta})}{\partial u_{nn}} u_n u_n |\nabla u|^{-(k+2)} \\ &= (-1)^k \sum_{\gamma,\delta=1}^{n} \frac{\partial \sigma_{k+1}(u_{\alpha\beta})}{\partial u_{\gamma\delta}} u_\gamma u_\delta |\nabla u|^{-(k+2)}.\end{aligned}$$

From step 1 and step 2, we get the proof of the curvature formula (2.1). □

We now give necessary formulas concerning our later computations in space forms; refer to, for example, [7].

Proposition 2.3 *Let (M^n, g) be a space form of sectional curvature $K_{sec} = \epsilon$. We have*

$$R_{\alpha\beta\gamma\delta} = \epsilon(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}), \tag{2.3}$$

$$u_{\alpha\beta\gamma} = u_{\alpha\gamma\beta} - R_{\gamma\beta\xi\alpha} u_\xi, \tag{2.4}$$

$$\sum_{\alpha=1}^{n} u_{\xi\eta\alpha\alpha} = \sum_{\alpha=1}^{n} u_{\alpha\alpha\xi\gamma} - 2\sum_{\alpha,\beta=1}^{n} R_{\xi\alpha\eta\beta} u_{\alpha\beta} + \sum_{\alpha=1}^{n} (R_{\xi\alpha} u_{\eta\alpha} + R_{\eta\alpha} u_{\xi\alpha}). \tag{2.5}$$

3 No Critical Points

Let $(M^n, g), n \geq 2$, be a space form of sectional curvature $K_{sec} = 1, 0$, or -1, and $U_2 \setminus U_1$ be an annulus bounded by two convex hypersurfaces in M^n. Suppose that u is a harmonic function defined in $U_2 \setminus U_1$ and continuous in $\overline{U_2 \setminus U_1}$ satisfying

$$u|_{\partial U_1} = 1, \quad u|_{\partial U_2} = 0.$$

We prove that in our situation u has no critical points.

In our consideration, let $x_0 \in U_1$ be a given point and $\rho(x)$ be the distance from x_0 to x. Under the identification $B_R(x_0) \setminus \{x_0\} = (0, R) \times S^{n-1}$, we express

$$g = d\rho^2 + \varphi^2(\rho) g_S,$$

where φ satisfies

$$\frac{d^2\varphi}{d\rho^2} + K_{sec}\varphi = 0, \quad \varphi(0) = 0,$$

and g_S is the standard metric on the sphere S^{n-1} of sectional curvature 1.

Let $\{E_k\}_{k=1}^{n-1}$ be a local orthonormal frame on (S^{n-1}, g_S), so that

$$e_k = \varphi^{-1} E_k, \quad e_n = \nabla\rho = \frac{\partial}{\partial\rho}$$

form a local orthonormal frame on $M^n \setminus \{x_0\}$. Note that each e_k is actually parallel along the e_n direction.

Suppose that u is a harmonic function defined on $U_2 \setminus U_1$. We define

$$F = e_n(u) = \sum_{\alpha=1}^{n} u_\alpha \rho_\alpha.$$

Lemma 3.1 *It holds that*

$$\Delta\big(\varphi^{n-1} F\big) = 2(n-2) g\big(\nabla \log\varphi, \nabla\big(\varphi^{n-1} F\big)\big).$$

Proof We first compute that

$$\begin{aligned}
\Delta F &= \Delta\left(\sum_{\alpha=1}^{n} \rho_\alpha u_\alpha\right) \\
&= \sum_{\alpha,\gamma=1}^{n} \rho_\alpha u_{\alpha\gamma\gamma} + \sum_{\alpha,\gamma=1}^{n} u_\alpha \rho_{\alpha\gamma\gamma} + 2\sum_{\alpha,\gamma=1}^{n} \rho_{\alpha\gamma} u_{\alpha\gamma} \\
&= -\sum_{\alpha,\gamma,\xi=1}^{n} R_{\gamma\alpha\xi\gamma} u_\xi \rho_\alpha - \sum_{\alpha,\gamma,\xi=1}^{n} R_{\gamma\alpha\xi\gamma} \rho_\xi u_\alpha + \sum_{\alpha,\gamma=1}^{n} u_\alpha \rho_{\gamma\gamma\alpha} + 2\sum_{\alpha,\gamma=1}^{n} \rho_{\alpha\gamma} u_{\alpha\gamma} \\
&= 2\sum_{\alpha,\xi=1}^{n} R_{\alpha\xi} u_\xi \rho_\alpha + \sum_{\alpha,\gamma=1}^{n} u_\alpha \rho_{\gamma\gamma\alpha} + 2\sum_{\alpha,\gamma=1}^{n} \rho_{\alpha\gamma} u_{\alpha\gamma}.
\end{aligned}$$

Let f' stand for the differentiation of f in e_n. Note that

$$\nabla_{e_k} e_n = (\log\varphi)' e_k, \quad \nabla_{e_k} e_l = -(\log\varphi)' \delta_{kl} e_n,$$

so we have

$$\rho_\alpha = \delta_{n\alpha}, \quad \rho_{n\alpha} = 0, \quad \rho_{kl} = (\log\varphi)' \delta_{kl}.$$

Hence we get

$$\Delta F = 2(n-1)K_{sec}F + (n-1)(\log\varphi)''F + 2(\log\varphi)'u_{kk}$$
$$= 2(n-1)K_{sec}F + (n-1)(\log\varphi)''F - 2(\log\varphi)'F'.$$

Recall that

$$\varphi'' + K_{sec}\varphi = 0,$$

so we get

$$\Delta F = (n-1)\big(K_{sec} - \varphi^{-2}\varphi'^2\big)F - 2\varphi^{-1}\varphi' F'.$$

Note that

$$\Delta = \frac{\partial^2}{\partial\rho^2} + (n-1)(\log\varphi)'\frac{\partial}{\partial\rho} + \varphi^{-2}\Delta_{gs}.$$

A straightforward computation shows that

$$\Delta\big(\varphi^{n-1}F\big) = 2(n-1)(n-2)\varphi^{n-3}\varphi'^2 F + 2(n-2)\varphi^{n-2}\varphi' F'.$$

That is,

$$\Delta\big(\varphi^{n-1}F\big) = 2(n-2)\varphi^{-1}\varphi'\big(\varphi^{n-1}F\big)'.$$

Then we complete the proof of this lemma. □

Now we use the above lemma to prove that there is no critical point of the solution u in Theorem 1.1 in $\Omega = \Omega_0 \backslash \bar{\Omega}_1$.

Proposition 3.2 *Let $U_2 \setminus U_1$ be an annulus between smooth convex hypersurfaces in a space form M^n, and u be a harmonic function in $U_2 \setminus U_1$ and continuous in $\overline{U_2 \setminus U_1}$ satisfying*

$$u|_{\partial U_1} = 1, \quad u|_{\partial U_2} = 0.$$

Then it holds that

$$|\nabla u|(x) > 0, \quad \forall x \in \overline{U_2 \setminus U_1}.$$

Proof Taking a point $x_0 \in U_1$, all the geodesics $\gamma(t)$ initiating from x_0 satisfy

$$g(\dot{\gamma}, \nu) < 0$$

on the boundary of the annulus, since the boundary is two convex hypersurfaces. On the other hand, by the Hopf Lemma, on the boundary of the annulus we have

$$\frac{\partial u}{\partial \nu} > 0.$$

Therefore, on the boundary of the annulus we have

$$\dot{\gamma}(u) = g(\dot{\gamma}, \nabla u) = g(\dot{\gamma}, \nu)\frac{\partial u}{\partial \nu} < 0.$$

Recall that

$$F = e_n(u) = \dot{\gamma}(u),$$

so we have

$$\varphi^{n-1}F < 0$$

on the boundary of the annulus. By Lemma 3.1 and the maximum principle, we see that $\varphi^{n-1}e_n(u) < 0$ in the annulus. □

In this section, we find a new test function $\varphi^{n-1}F$ which attains its maximum and minimum on the boundary. If $M^n = R^n$, then $\varphi^{n-1}F = |X|^{n-2}\sum_{\alpha=1}^n X_\alpha u_\alpha$, and it is a constant on the standard Green function on a ball. Our test function is different from the classical test function $\sum_{\alpha=1}^n X_\alpha u_\alpha$ which can be found in Caffarelli–Spruck [5] or Kawohl [12].

4 Deformations of Convex Annulus

Let (M^n, g) be a space form of sectional curvature $K_{sec} = 1, 0$, or -1, and $\Omega_2 \setminus \Omega_1$ be an annulus bounded by two convex hypersurfaces in M^n. Suppose $B_2 \setminus B_1$ is another annulus containing $\Omega_2 \setminus \Omega_1$ and $B_i, i = 1, 2$, are also convex domains. We will construct a family of annuli $V_t = V_2(t) \setminus V_1(t), 0 \le t \le 1$, bounded by two convex domains with

$$V_0 = B_2 \setminus B_1, \quad V_1 = \Omega_2 \setminus \Omega_1.$$

We use B_1 and Ω_1 to construct a family of convex domains $V_1(t)$, and $V_2(t)$ can be obtained in the same way.

Take a point $O \in B_1$ as our center. In polar coordinates centered at O, we have

$$g = d\rho^2 + \varphi^2(\rho)g_S;$$

here φ and g_S is the same as in the last section.

Let Ω be a convex domain that encloses O. Since the convexity of the domain Ω is the same as that of any geodesic which ends on the boundary $\partial\Omega$ inside the domain, we first study the geodesics which end on $\partial\Omega$. Let p, q be two different points on $\partial\Omega$. The geodesic γ connecting p and q lies on the plane spanned the geodesics Op and Oq. To be more precise, the geodesics Op and Oq give two tangent vectors in $T_O M^n$, which are assumed here to be linearly independent, hence they span a tangent plane. This tangent plane determines a surface as the image of the exponential map, which is totally geodesic.

The induced metric on this plane expressed in polar coordinates can be written as

$$g = d\rho^2 + \varphi(\rho)^2 d\theta^2, \quad \theta \in [0, 2\pi].$$

A straightforward computation shows

$$\nabla_{\frac{\partial}{\partial\rho}}\frac{\partial}{\partial\rho}=0,\quad \nabla_{\frac{\partial}{\partial\rho}}\frac{\partial}{\partial\theta}=\nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\rho}=\varphi^{-1}\varphi_\rho\frac{\partial}{\partial\theta},\quad \nabla_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta}=-\varphi\varphi_\rho\frac{\partial}{\partial\rho}.$$

Let θ be the parameterization of γ. Therefore, we have

$$\gamma(\theta)=\big(\rho(\theta),\theta\big)$$

and

$$|\dot\gamma|^2(\theta)=\rho_\theta^2+\varphi(\rho)^2.$$

By the geodesic equation

$$\nabla_{\dot\gamma/|\dot\gamma|}\dot\gamma/|\dot\gamma|=0,$$

we get

$$\nabla_{\dot\gamma}\dot\gamma=\frac{d}{d\theta}\big(|\dot\gamma|\big)\frac{\dot\gamma}{|\dot\gamma|}.$$

One can compute that

$$\nabla_{\dot\gamma}\dot\gamma=(\rho_{\theta\theta}-\varphi\varphi_\rho)\frac{\partial}{\partial\rho}+2\rho_\theta\varphi^{-1}\varphi_\rho\frac{\partial}{\partial\theta}$$

and

$$\frac{d}{d\theta}\big(|\dot\gamma|\big)\frac{\dot\gamma}{|\dot\gamma|}=|\dot\gamma|^{-2}\rho_\theta(\rho_{\theta\theta}+\varphi\varphi_\rho)\left(\rho_\theta\frac{\partial}{\partial\rho}+\frac{\partial}{\partial\theta}\right).$$

Hence we get the following characterization of geodesics in space forms

$$\rho_{\theta\theta}=2\rho_\theta^2\varphi^{-1}\varphi_\rho+\varphi\varphi_\rho.$$

Note that

$$\varphi_{\rho\rho}+K_{sec}\varphi=0,$$

we get

$$\left(\frac{\varphi_\rho}{\varphi}\right)_{\theta\theta}=-\varphi\varphi_\rho\big(K_{sec}+\varphi^{-2}\varphi_\rho^2\big).$$

One can check case by case that

$$\left(\frac{\varphi_\rho}{\varphi}\right)_{\theta\theta}+\frac{\varphi_\rho}{\varphi}=0.$$

Let

$$\sigma(\rho)=\frac{\varphi_\rho(\rho)}{\varphi(\rho)}$$

and σ^{-1} be the inverse function of σ. We now construct $V_1(t)$ as follows. Let S^{n-1} be the unit sphere of $T_O M^n$ and $(\rho,\theta)=\exp_O(\rho,\theta)$, $\theta\in S^{n-1}$. Denote the boundary of B_1 and Ω_1 by

$$\partial B_1=\left\{\left(\rho_{B_1}(\theta),\theta\right),\theta\in S^{n-1}\right\},\quad \partial\Omega_1=\left\{\left(\rho_{\Omega_1}(\theta),\theta\right),\theta\in S^{n-1}\right\}.$$

Let $V_1(t)$ be the domain enclosed by $\partial V_1(t)$, which is given by

$$\partial V_1(t)=\left\{\left(\sigma^{-1}\left[(1-t)\sigma\left(\rho_{B_1}(\theta)\right)+t\sigma\left(\rho_{\Omega_1}(\theta)\right)\right],\theta\right)\mid\theta\in S^{n-1}\right\}.$$

Clearly, $V_1(0)=B_1$, $V_1(1)=\Omega_1$ and $\partial V_1(t)\subset\Omega_1\setminus B_1$.

Proposition 4.1 *$V_1(t)$ is a smooth convex domain for each $t\in[0,1]$.*

Proof Let γ be a geodesic with ends $p_1=(\rho_\gamma(\theta_1),\theta_1)$ and $p_2=(\rho_\gamma(\theta_2),\theta_2)$ on $\partial V_1(t)$. We assume the geodesics Op_1 and Op_2 do not match into a smooth geodesic. That is, $|\theta_1-\theta_2|<\pi$. Otherwise, it is even easier to see the geodesic p_1p_2 is inside the domain $V_1(t)$. Clearly, in the situation $|\theta_1-\theta_2|<\pi$ we can express $\gamma=\{(\rho_\gamma(\theta),\theta),\theta\in[\theta_1,\theta_2]\}$.

Let γ_1 be the geodesic with ends $(\rho_{\gamma_1}(\theta_1),\theta_1)$ and $(\rho_{\gamma_1}(\theta_2),\theta_2)$ on ∂B_1; respectively, let γ_2 be the geodesic with ends $(\rho_{\gamma_2}(\theta_1),\theta_1)$ and $(\rho_{\gamma_2}(\theta_2),\theta_2)$ on $\partial\Omega_1$. Note that γ, γ_1, and γ_2 are all on the same plane spanned by Op_1 and Op_2. On the other hand,

$$\widetilde{\gamma}(\theta)=\left\{\left(\sigma^{-1}\left[(1-t)\sigma\left(\rho_{\gamma_1}(\theta)\right)+t\sigma\left(\rho_{\gamma_2}(\theta)\right)\right],\theta\right)\mid\theta\in[\theta_1,\theta_2]\right\}$$

is a geodesic with the same ends as γ. Hence we have $\gamma=\widetilde{\gamma}$.

Note that σ is a decreasing function and

$$\rho_{B_1}(\theta)>\rho_{\gamma_1}(\theta),\quad \rho_{\Omega_1}(\theta)>\rho_{\gamma_2}(\theta),\quad \theta\in(\theta_1,\theta_2).$$

Hence for any $\theta\in(\theta_1,\theta_2)$, we have

$$\begin{aligned}\rho_\gamma(\theta)&=\sigma^{-1}\left[(1-t)\sigma\left(\rho_{\gamma_1}(\theta)\right)+t\sigma\left(\rho_{\gamma_2}(\theta)\right)\right]\\&<\sigma^{-1}\left[(1-t)\sigma\left(\rho_{B_1}(\theta)\right)+t\sigma\left(\rho_{\Omega_1}(\theta)\right)\right],\end{aligned}$$

which shows the convexity of $V_1(t)$.

And from the construction, $V_1(t)$ is a smooth convex domain for each $t\in[0,1]$. □

Even in R^n, our addition operation is different from the usual Minkowski vector sum in classical convex bodies theory.

5 Constant Rank Theorem of the Second Fundamental Form

In this section, following the calculation in Xu [24], we use the curvature expression of the level sets in (2.1) to prove the following constant rank theorem for the second fundamental forms of convex level sets Σ^c.

Theorem 5.1 *Suppose $u \in C^4(\Omega)$ is a solution of the Poisson equation*

$$\Delta u = f(x) \geq 0 \quad in\ \Omega, \tag{5.1}$$

where $\Omega \subset M^n$, and (M^n, g) is a space form of sectional curvature $K_{sec} = 1$, or -1. Assume $f(x)$ satisfies the condition

$$3 f_i f_j + 4\varepsilon f^2 \delta_{ij} \leq 2 f f_{ij},$$

where $\varepsilon = \pm 1$ for $K_{sec} = \pm 1$, and we have used the local orthonormal frame on (M^n, g). If $\nabla u \neq 0$ in Ω and the set $\{x \in \Omega \mid u(x) \geq c_0\}$ is connected and locally convex for all $c \in (-\gamma_0 + c_0, \gamma_0 + c_0)$ and some $\gamma_0 > 0$. Then the second fundamental form of level surfaces $\{x \in \Omega \mid u(x) = c\}$ has the same constant rank for all $c \in (-\gamma_0 + c_0, \gamma_0 + c_0)$.

Remark 5.2 When $(M^n, g) = (S^n, g_{standard})$ with $K_{sec} = 1$ and $f > 0$, the structure condition of f is equivalent to $f^{-\frac{1}{2}}(\frac{X}{|X|})$ is a concave function in $\mathbb{R}^{n+1} \backslash \{0\}$.

Since Theorem 5.1 is of local feature, we may assume level surface $\Sigma^c = \{x \in \Omega \mid u(x) = c\}$ is connected for each $c \in (c_0 - \gamma_0, c_0 + \gamma_0)$. We will concentrate in a neighborhood of some point $x_0 \in \Omega$ such that the minimal rank l of the second fundamental form of $\Sigma^{u(x_0)}$ is attained at x_0. We may assume $l \leq n-2$, otherwise the second fundamental form of every level surface has full rank. And we assume $u \in C^4(\Omega)$ and $u_n > 0$ in the rest of this paper.

Let U be a small open neighborhood of x_0 such that for each $x \in U$, there are l "good" eigenvalues of the second fundamental form of Σ_c which are bounded from below by a positive constant, and the other $n-1-l$ "bad" eigenvalues of the second fundamental form of Σ_c are very small. Let G be the index set of these "good" eigenvalues and B be the index set of "bad" eigenvalues. For any $x \in U$ fixed, by choosing $e_1, \ldots, e_{n-1}, e_n$ such that

$$|\nabla u|(x) = u_n(x) > 0 \text{ and the matrix } (u_{ij}), i, j = 1, \ldots, n-1, \text{ is diagonal at } x, \tag{5.2}$$

we may express the second fundamental form of Σ_c in the form of (2.2). So the matrix $(h_{ij}), i, j = 1, \ldots, n-1$ is also diagonal at x, and without loss of generality we may assume $h_{11} \leq h_{22} \leq \cdots \leq h_{n-1,n-1}$. There is a positive constant $C > 0$ depending only on $\|u\|_{C^4}$ and U, such that $h_{n-1,n-1} \geq h_{n-2,n-2} \geq \cdots \geq h_{n-l,n-l} > C$ for all $x \in U$. For convenience we let $G = \{n-l, n-l+1, \ldots, n-1\}$ and $B = \{1, 2, \ldots, n-l-1\}$ be the "good" and "bad" sets of indices, respectively. If there is no confusion, we also denote

$$B = \{h_{11}, \ldots, h_{n-l-1,n-l-1}\} \quad \text{and} \quad G = \{h_{n-l,n-l}, \ldots, h_{n-1,n-1}\}. \tag{5.3}$$

Note that for any $\delta > 0$, we may choose U small enough such that $h_{jj} < \delta$ for all $j \in B$ and $x \in U$. We also let $\lambda = h_{ii}$ in the following calculation.

For each c, set

$$\varphi(h) = |\nabla u|^{l+3} \sigma_{l+1}(\lambda_1, \ldots, \lambda_{n-1}) = (-1)^{l+1} \sum_{\alpha,\beta=1}^{n} \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{\alpha\beta}} u_\alpha u_\beta. \tag{5.4}$$

Theorem 5.1 is equivalent to saying $\varphi(h) \equiv 0$ in U.

For any fixed $x \in U$, with the coordinate chosen as in (5.2) and (5.3). So by the strong maximum principle, we need only to show:

$$\Delta\varphi(x) \le 0 \,\mathrm{mod}\big\{\nabla\varphi(x), \varphi(x)\big\} \quad \text{in } U. \tag{5.5}$$

Proof of Theorem 5.1 Now all our calculations work at x with the above coordinate. In order to simplify the calculation, we introduce a new notation $a_{ij,\alpha}$,

$$u_n u_{ij\alpha} = -u_n^2 a_{ij,\alpha} + u_{ni}u_{j\alpha} + u_{nj}u_{i\alpha} + u_{n\alpha}u_{ij}. \tag{5.6}$$

Then we have the following relations

$$u_n u_{iin} = -u_n^2 a_{ii,n} + 2u_{ni}^2, \quad \text{for } i \in B, \tag{5.7}$$

$$u_n u_{iij} = -u_n^2 a_{ii,j}, \quad \text{for } i \in B, \quad 1 \le j \le n-1. \tag{5.8}$$

In the following, all the calculations will be done at x, and the terms of $\varphi(x), \nabla\varphi(x)$ will be omitted, i.e., all the equalities or inequalities should be understood in the sense of *mod*$\{\varphi(x),\ \nabla\varphi(x)\}$.

$$\begin{aligned}
\varphi_\nu &= (-1)^{l+1} \sum_{\alpha,\beta,\gamma,\delta=1}^{n} \frac{\partial^2 \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{\alpha\beta}\partial u_{\gamma\delta}} u_\alpha u_\beta u_{\gamma\delta\nu} \\
&\quad + (-1)^{l+1} \sum_{\alpha,\beta=1}^{n} \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{\alpha\beta}} (u_{\alpha\nu}u_\beta + u_\alpha u_{\beta\nu}) \\
&= (-1)^{l+1} u_n^2 \sum_{i=1}^{n-1} \sigma_l(u_{ij} \mid i) u_{ii\nu} + 2(-1)^{l+1} u_n \sum_{\alpha=1}^{n} \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{\alpha n}} u_{\alpha\nu} \\
&= -u_n^{l+2}\sigma_l(G) \sum_{i\in B} u_{ii\nu} + 2(-1)^{l+1} u_n \left[\frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{nn}} u_{n\nu} + \sum_{i=1}^{n-1} \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{in}} u_{i\nu} \right] \\
&= -u_n^{l+2}\sigma_l(G) \sum_{i\in B} u_{ii\nu} - 2(-1)^{l+1} u_n (-1)^l u_n^l \sum_{i=1}^{n-1} \sigma_l(\lambda \mid i) u_{ni} u_{i\nu} \\
&= -u_n^{l+2}\sigma_l(G) \sum_{i\in B} u_{ii\nu} + 2u_n^{l+1}\sigma_l(G) \sum_{i\in B} u_{ni}u_{i\nu}.
\end{aligned} \tag{5.9}$$

Noting (5.7)–(5.8), we deduce:

$$\sum_{i\in B} a_{ii,\alpha} = 0, \quad \forall 1 \le \alpha \le n. \tag{5.10}$$

Now we calculate that

$$\begin{aligned}
\Delta\varphi = &(-1)^{l+1}\sum_{\alpha,\beta,\gamma,\delta,\zeta,\eta,\nu=1}^{n}\frac{\partial^3\sigma_{l+2}(u_{\alpha\beta})}{\partial u_{\alpha\beta}\partial u_{\gamma\delta}\partial u_{\zeta\eta}}u_{\gamma\delta\nu}u_{\zeta\eta\nu}u_\alpha u_\beta \\
&+4(-1)^{l+1}\sum_{\alpha,\beta,\gamma,\delta,\nu=1}^{n}\frac{\partial^2\sigma_{l+2}(u_{\alpha\beta})}{\partial u_{\alpha\beta}\partial u_{\gamma\delta}}u_{\gamma\delta\nu}u_{\alpha\nu}u_\beta \\
&+(-1)^{l+1}\sum_{\alpha,\beta,\gamma,\delta,\nu=1}^{n}\frac{\partial^2\sigma_{l+2}(u_{\alpha\beta})}{\partial u_{\alpha\beta}\partial u_{\gamma\delta}}u_{\gamma\delta\nu\nu}u_\alpha u_\beta \\
&+2(-1)^{l+1}\sum_{\alpha,\beta,\nu=1}^{n}\frac{\partial\sigma_{l+2}(u_{\alpha\beta})}{\partial u_{\alpha\beta}}u_{\alpha\nu\nu}u_\beta \\
&+2(-1)^{l+1}\sum_{\alpha,\beta,\nu=1}^{n}\frac{\partial\sigma_{l+2}(u_{\alpha\beta})}{\partial u_{\alpha\beta}}u_{\alpha\nu}u_{\beta\nu} \\
:=& I+II+III+IV+V. \qquad (5.11)
\end{aligned}$$

Next we will compute the above terms step by step.

$$\begin{aligned}
I :=& (-1)^{l+1}\sum_{\alpha,\beta,\gamma,\delta,\zeta,\eta,\nu=1}^{n}\frac{\partial^3\sigma_{l+2}(u_{\alpha\beta})}{\partial u_{\alpha\beta}\partial u_{\gamma\delta}\partial u_{\zeta\eta}}u_{\gamma\delta\nu}u_{\zeta\eta\nu}u_\alpha u_\beta \\
=& u_n^{l+1}\sum_{\nu=1}^{n}\sum_{i\neq j,i,j=1}^{n-1}\sigma_{l-1}(\lambda\mid i,j)u_{ii\nu}u_{jj\nu}-u_n^{l+1}\sum_{\nu=1}^{n}\sum_{i\neq j,i,j=1}^{n-1}\sigma_{l-1}(\lambda\mid i,j)u_{ij\nu}^2 \\
=& u_n^{l+1}\sum_{\nu=1}^{n}\Bigg(\sum_{i\in G,j\in B}+\sum_{i\in B,j\in G}+\sum_{i,j\in G,i\neq j}+\sum_{i,j\in B,i\neq j}\Bigg)\sigma_{l-1}(\lambda\mid i,j)u_{ii\nu}u_{jj\nu} \\
&-u_n^{l+1}\sum_{\nu=1}^{n}\Bigg(\sum_{i\in G,j\in B}+\sum_{i\in B,j\in G}+\sum_{i,j\in G,i\neq j}+\sum_{i,j\in B,i\neq j}\Bigg)\sigma_{l-1}(\lambda\mid i,j)u_{ij\nu}^2 \\
=& 2u_n^{l-1}\sum_{\alpha=1}^{n}\sum_{i\in G,j\in B}\sigma_{l-1}(G\mid i)\big[(u_nu_{ii\alpha})(u_nu_{jj\alpha})-(u_nu_{ij\alpha})^2\big] \\
&+u_n^{l-1}\sigma_{l-1}(G)\sum_{\alpha=1}^{n}\sum_{i,j\in B,i\neq j}\big[(u_nu_{ii\alpha})(u_nu_{jj\alpha})-(u_nu_{ij\alpha})^2\big]. \qquad (5.12)
\end{aligned}$$

Using (5.6) and (5.10), it follows that

$$
\begin{aligned}
&2\sum_{\alpha=1}^{n}\sum_{i\in G, j\in B}\sigma_{l-1}(G\mid i)\big[(u_n u_{ii\alpha})(u_n u_{jj\alpha})-(u_n u_{ij\alpha})^2\big]\\
&\quad=2\sum_{\alpha=1}^{n}\sum_{i\in G, j\in B}\sigma_{l-1}(G\mid i)\big[\big(-u_n^2 a_{ii,\alpha}+2u_{ni}u_{i\alpha}+u_{n\alpha}u_{ii}\big)\\
&\qquad\times\big(-u_n^2 a_{jj,\alpha}+2u_{nj}u_{j\alpha}\big)-\big(-u_n^2 a_{ij,\alpha}+u_{ni}u_{j\alpha}+u_{nj}u_{i\alpha}\big)^2\big]\\
&\quad=-2u_n^4\sum_{\alpha=1}^{n}\sum_{i\in G, j\in B}\sigma_{l-1}(G\mid i)a_{ij,\alpha}^2-4u_n^2\sum_{i\in G, j\in B}\sigma_{l-1}(G\mid i)a_{ii,n}u_{nj}^2\\
&\qquad+4u_n^2\sum_{\alpha=1}^{n}\sum_{i\in G, j\in B}\sigma_{l-1}(G\mid i)a_{ij,\alpha}(u_{ni}u_{j\alpha}+u_{nj}u_{i\alpha})\\
&\qquad+4\sum_{i\in G, j\in B}\sigma_{l-1}(G\mid i)\big(2u_{ni}^2+u_{nn}u_{ii}\big)u_{nj}^2\\
&\qquad-2\sum_{\alpha=1}^{n}\sum_{i\in G, j\in B}\sigma_{l-1}(G\mid i)[u_{ni}u_{j\alpha}+u_{nj}u_{i\alpha}]^2
\end{aligned}
\tag{5.13}
$$

and

$$
\begin{aligned}
&\sum_{\alpha=1}^{n}\sum_{i,j\in B, i\neq j}\big[(u_n u_{ii\alpha})(u_n u_{jj\alpha})-(u_n u_{ij\alpha})^2\big]\\
&\quad=\sum_{\alpha=1}^{n}\sum_{i,j\in B, i\neq j}\big[\big(-u_n^2 a_{ii,\alpha}+2u_{ni}u_{i\alpha}\big)\big(-u_n^2 a_{jj,\alpha}+2u_{nj}u_{j\alpha}\big)\\
&\qquad-\big(-u_n^2 a_{ij,\alpha}+u_{ni}u_{j\alpha}+u_{nj}u_{i\alpha}\big)^2\big]\\
&\quad=\sum_{\alpha=1}^{n}\sum_{i,j\in B, i\neq j}\big[-u_n^4 a_{ij,\alpha}^2+u_n^4 a_{ii,\alpha}a_{jj,\alpha}-4u_n^2 a_{ii,\alpha}u_{nj}u_{j\alpha}\\
&\qquad+4u_n^2 a_{ij,\alpha}u_{ni}u_{j\alpha}+4u_{ni}u_{nj}u_{i\alpha}u_{j\alpha}-(u_{ni}u_{j\alpha}+u_{nj}u_{i\alpha})^2\big]\\
&\quad=-u_n^4\sum_{\alpha=1}^{n}\sum_{j\in B}a_{jj,\alpha}^2-u_n^4\sum_{\alpha=1}^{n}\sum_{i,j\in B, i\neq j}a_{ij,\alpha}^2
\end{aligned}
$$

$$+4u_n^2\sum_{j\in B}a_{jj,n}u_{nj}^2+4u_n^2\sum_{i,j\in B,i\neq j}a_{ij,n}u_{ni}u_{nj}$$

$$=-u_n^4\sum_{\alpha=1}^{n}\sum_{i,j\in B}a_{ij,\alpha}^2+4u_n^2\sum_{i,j\in B}a_{ij,n}u_{ni}u_{nj}. \tag{5.14}$$

It follows that

$$\begin{aligned}u_n^{1-l}I=&-2u_n^4\sum_{\alpha=1}^{n}\sum_{i\in G,j\in B}\sigma_{l-1}(G\mid i)a_{ij,\alpha}^2-u_n^4\sigma_{l-1}(G)\sum_{\alpha=1}^{n}\sum_{i,j\in B}a_{ij,\alpha}^2\\&-4u_n^2\sum_{i\in G,j\in B}\sigma_{l-1}(G\mid i)a_{ii,n}u_{nj}^2+8u_n^2\sum_{i\in G,j\in B}\sigma_{l-1}(G\mid i)a_{ij,n}u_{ni}u_{nj}\\&-4u_n^3\sigma_l(G)\sum_{i\in G,j\in B}a_{ij,i}u_{nj}+4u_n^2\sigma_{l-1}(G)\sum_{i,j\in B}a_{ij,n}u_{ni}u_{nj}\\&-4lu_nu_{nn}\sigma_l(G)\sum_{j\in B}u_{nj}^2-2u_n^2\sigma_1(G)\sigma_l(G)\sum_{j\in B}u_{nj}^2.\end{aligned}\tag{5.15}$$

To compute the second term in (5.11), we still use (5.10):

$$\begin{aligned}II:=&\,4(-1)^{l+1}\sum_{\alpha,\beta,\gamma,\delta,\nu=1}^{n}\frac{\partial^2\sigma_{l+2}(u_{\alpha\beta})}{\partial u_{\alpha\beta}\partial u_{\gamma\delta}}u_{\gamma\delta\nu}u_{\alpha\nu}u_\beta\\=&\,4(-1)^{l+1}u_n\sum_{\gamma,\delta,\alpha=1}^{n}\left[\frac{\partial^2\sigma_{l+2}(u_{\alpha\beta})}{\partial u_{nn}\partial u_{\gamma\delta}}u_{\gamma\delta\alpha}u_{n\alpha}+\sum_{i=1}^{n-1}\frac{\partial^2\sigma_{l+2}(u_{\alpha\beta})}{\partial u_{in}\partial u_{\gamma\delta}}u_{\gamma\delta\alpha}u_{i\nu}\right]\\=&\,-4u_n^{l+1}\sum_{\alpha=1}^{n}\sum_{i=1}^{n-1}\sigma_l(h_{ij}\mid i)u_{n\alpha}u_{ii\alpha}+4u_n^{l+1}\sum_{\alpha=1}^{n}\sum_{i=1}^{n-1}\sigma_l(h_{ij}\mid i)u_{i\alpha}u_{ni\alpha}\\&+4u_n^l\sum_{\alpha=1}^{n}\sum_{i\neq j,i,j=1}^{n-1}\sigma_{l-1}(h_{ij}\mid i,j)u_{i\alpha}u_{ji\alpha}u_{nj}\\&-4u_n^l\sum_{\alpha=1}^{n}\sum_{i\neq j,i,j=1}^{n-1}\sigma_{l-1}(h_{ij}\mid i,j)u_{i\alpha}u_{jj\alpha}u_{ni}\\:=&\,II_1+II_2+II_3+II_4.\end{aligned}\tag{5.16}$$

By (5.10), the first term on the right-hand side of the above equality can be treated as:

$$
\begin{aligned}
II_1 &:= -4u_n^{l+1}\sum_{\alpha=1}^{n}\sum_{i=1}^{n-1}\sigma_l(h_{ij}\mid i)u_{n\alpha}u_{ii\alpha}\\
&= -4u_n^{l+1}\sigma_l(G)\sum_{i\in B}u_{nn}u_{iin} - 4u_n^{l+1}\sigma_l(G)\sum_{j=1}^{n-1}\sum_{i\in B}u_{nj}u_{iij}\\
&= -8u_n^{l}\sigma_l(G)u_{nn}\sum_{j\in B}u_{nj}^2. \qquad (5.17)
\end{aligned}
$$

Using (5.6), (5.10), and (5.1), we get

$$
\begin{aligned}
II_2 &:= 4u_n^{l+1}\sum_{\alpha=1}^{n}\sum_{i=1}^{n-1}\sigma_l(h_{ij}\mid i)u_{i\alpha}u_{ni\alpha}\\
&= 4u_n^{l+1}\sigma_l(G)\sum_{i\in B}u_{in}u_{nin} + 4u_n^{l+1}\sigma_l(G)\sum_{j=1}^{n-1}\sum_{i\in B}u_{ij}u_{nij}\\
&= 4u_n^{l+1}\sigma_l(G)\sum_{i\in B}u_{in}u_{nni}\\
&= 4u_n^{l+1}\sigma_l(G)\sum_{j\in B}u_{nj}\Big(f_j - \sum_{i\in G}u_{iij}\Big)\\
&= 4u_n^{l+1}\sigma_l(G)\sum_{j\in B}u_{nj}f_j - 4u_n^{l}\sigma_l(G)\sum_{j\in B}\sum_{i\in G}u_{nj}\big[-u_n^2a_{ii,j}+u_{nj}u_{ii}\big]\\
&= 4u_n^{l+1}\sigma_l(G)\sum_{j\in B}u_{nj}f_j + 4u_n^{l+2}\sigma_l(G)\sum_{i\in G, j\in B}u_{nj}a_{ii,j}\\
&\quad + 4u_n^{l+1}\sigma_1(G)\sigma_l(G)\sum_{j\in B}u_{nj}^2. \qquad (5.18)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
II_3 &:= 4u_n^{l}\sum_{\alpha=1}^{n}\sum_{i\neq j, i,j=1}^{n-1}\sigma_{l-1}(h_{ij}\mid i,j)u_{i\alpha}u_{ji\alpha}u_{nj}\\
&= 4u_n^{l}\sum_{\alpha=1}^{n}\Bigg[\sum_{i\in G, j\in B}+\sum_{i\in B, j\in G}+\sum_{i,j\in G, i\neq j}+\sum_{i,j\in B, i\neq j}\Bigg]\sigma_{l-1}(h_{ij}\mid i,j)u_{i\alpha}u_{ji\alpha}u_{nj}
\end{aligned}
$$

$$
\begin{aligned}
&= 4u_n^l \sum_{\alpha=1}^{n} \sum_{i\in G, j\in B} \sigma_{l-1}(G \mid i) u_{i\alpha} u_{nj} u_{ij\alpha} \\
&\quad + 4u_n^l \sum_{\alpha=1}^{n} \sum_{i\in G, j\in B} \sigma_{l-1}(G \mid i) u_{j\alpha} u_{ni} u_{ij\alpha} \\
&\quad + 4u_n^l \sigma_{l-1}(G) \sum_{i,j\in B, i\neq j} u_{in} u_{nj} u_{ijn} \\
&= 8u_n^l \sum_{i\in G, j\in B} \sigma_{l-1}(G \mid i) u_{ni} u_{nj} u_{ijn} - 4u_n^{l+1} \sigma_l(G) \sum_{i\in G, j\in B} u_{nj} u_{iij} \\
&\quad + 4u_n^l \sigma_{l-1}(G) \sum_{i,j\in B, i\neq j} u_{in} u_{nj} u_{ijn}.
\end{aligned} \tag{5.19}
$$

Using (5.6) and (5.10) again, it follows that

$$
\begin{aligned}
II_3 :=& -8u_n^{l+1} \sum_{i\in G, j\in B} \sigma_{l-1}(G \mid i) u_{ni} u_{nj} a_{ij,n} + 16u_n^{l-1} \sum_{i\in G, j\in B} \sigma_{l-1}(G \mid i) u_{ni}^2 u_{nj}^2 \\
&+ 4u_n^{l+2} \sigma_l(G) \sum_{i\in G, j\in B} u_{nj} a_{ii,j} + 4u_n^{l+1} \sigma_1(G) \sigma_l(G) \sum_{j\in B} u_{nj}^2 \\
&- 4u_n^{l+1} \sigma_{l-1}(G) \sum_{i,j\in B, i\neq j} u_{in} u_{nj} a_{ij,n} + 8u_n^{l-1} \sigma_{l-1}(G) \sum_{i,j\in B, i\neq j} u_{ni}^2 u_{nj}^2.
\end{aligned} \tag{5.20}
$$

Now we treat the II_4 term.

$$
\begin{aligned}
II_4 :=& -4u_n^l \sum_{\alpha=1}^{n} \sum_{i\neq j, i,j=1}^{n-1} \sigma_{l-1}(h_{ij} \mid i,j) u_{i\alpha} u_{jj\alpha} u_{ni} \\
=& -4u_n^l \sum_{\alpha=1}^{n} \left[\sum_{i\in G, j\in B} + \sum_{i\in B, j\in G} + \sum_{i,j\in G, i\neq j} + \sum_{i,j\in B, i\neq j} \right] \sigma_{l-1}(h_{ij} \mid i,j) u_{i\alpha} u_{jj\alpha} u_{ni} \\
=& -4u_n^l \sum_{\alpha=1}^{n} \sum_{i\in G, j\in B} \sigma_{l-1}(G \mid i) u_{i\alpha} u_{ni} u_{jj\alpha} \\
&- 4u_n^l \sum_{\alpha=1}^{n} \sum_{i\in G, j\in B} \sigma_{l-1}(G \mid i) u_{j\alpha} u_{nj} u_{ii\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&- 4u_n^l \sigma_{l-1}(G) \sum_{\alpha=1}^{n} \sum_{i,j\in B, i\neq j} u_{in} u_{i\alpha} u_{jj\alpha} \\
&= -8u_n^{l-1}\Big(\sum_{i\in G}\sigma_{l-1}(G\mid i)u_{ni}^2\Big)\sum_{j\in B}u_{nj}^2 - 4u_n^l \sum_{i\in G, j\in B}\sigma_{l-1}(G\mid i)u_{nj}^2 u_{iin} \\
&\quad - 8u_n^{l-1}\sigma_{l-1}(G)\Big(\sum_{j\in B}u_{nj}^2\Big)^2 + 4u_n^l\sigma_{l-1}(G)\sum_{j\in B}u_{nj}^2 u_{jjn}. \qquad (5.21)
\end{aligned}
$$

Using (5.6) and (5.10) again, we have

$$
\begin{aligned}
&-4u_n^l \sum_{i\in G, j\in B}\sigma_{l-1}(G\mid i)u_{nj}^2 u_{iin} \\
&\quad = 4u_n^{l+1}\Big(\sum_{i\in G}\sigma_{l-1}(G\mid i)a_{ii,n}\Big)\sum_{j\in B}u_{nj}^2 \\
&\qquad - 8u_n^{l-1}\Big(\sum_{i\in G}\sigma_{l-1}(G\mid i)u_{ni}^2\Big)\sum_{j\in B}u_{nj}^2 + 4lu_n^l u_{nn}\sigma_l(G)\sum_{j\in B}u_{nj}^2, \qquad (5.22)
\end{aligned}
$$

and

$$
\begin{aligned}
&4u_n^l\sigma_{l-1}(G)\sum_{j\in B}u_{nj}^2 u_{jjn} \\
&\quad = -4u_n^{l+1}\sigma_{l-1}(G)\sum_{j\in B}a_{jj,n}u_{nj}^2 + 8u_n^{l-1}\sigma_{l-1}(G)\sum_{j\in B}u_{nj}^4. \qquad (5.23)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
u_n^{1-l}II &:= -8u_n^2 \sum_{i\in G, j\in B}\sigma_{l-1}(G\mid i)u_{ni}u_{nj}a_{ij,n} + 8u_n^3\sigma_l(G)\sum_{i\in G, j\in B}u_{nj}a_{ii,j} \\
&\quad + 4u_n^2\Big(\sum_{i\in G}\sigma_{l-1}(G\mid i)a_{ii,n}\Big)\sum_{j\in B}u_{nj}^2 - 4u_n^2\sigma_{l-1}(G)\sum_{j\in B}a_{jj,n}u_{nj}^2 \\
&\quad - 4u_n^2\sigma_{l-1}(G)\sum_{i,j\in B, i\neq j}u_{in}u_{nj}a_{ij,n} \\
&\quad - 8fu_n\sigma_l(G)\sum_{j\in B}u_{nj}^2 + 4lu_n u_{nn}\sigma_l(G)\sum_{j\in B}u_{nj}^2 \\
&\quad + 4u_n^2\sigma_l(G)\sum_{j\in B}u_{nj}f_j. \qquad (5.24)
\end{aligned}
$$

By (5.15) and (5.24), and using $a_{ij,i} = a_{ii,j}$ for $i \in G$, $j \in B$, we have

$$
\begin{aligned}
u_n^{1-l}(I + II) := {} & -2u_n^4 \sum_{\alpha=1}^{n} \sum_{i\in G, j\in B} \sigma_{l-1}(G \mid i) a_{ij,\alpha}^2 + 4u_n^3 \sigma_l(G) \sum_{i\in G, j\in B} a_{ij,i} u_{nj} \\
& - u_n^4 \sigma_{l-1}(G) \sum_{\alpha=1}^{n} \sum_{i,j\in B} a_{ij,\alpha}^2 \\
& - 8 f u_n \sigma_l(G) \sum_{j\in B} u_{nj}^2 + 4u_n^2 \sigma_l(G) \sum_{j\in B} u_{nj} f_j \\
& - 2u_n^2 \sigma_1(G) \sigma_l(G) \sum_{j\in B} u_{nj}^2 \\
= {} & -2u_n^4 \sum_{\alpha\neq i, \alpha=1}^{n} \sum_{i\in G, j\in B} \sigma_{l-1}(G \mid i) a_{ij,\alpha}^2 - u_n^4 \sigma_{l-1}(G) \sum_{\alpha=1}^{n} \sum_{i,j\in B} a_{ij,\alpha}^2 \\
& - 2 \sum_{i\in G, j\in B} \sigma_{l-1}(G \mid i) \big(u_n^2 a_{ii,j} + u_{ii} u_{nj}\big)^2 \\
& - u_n^4 \sigma_{l-1}(G) \sum_{\alpha=1}^{n} \sum_{i,j\in B} a_{ij,\alpha}^2 \\
& - 8 f u_n \sigma_l(G) \sum_{j\in B} u_{nj}^2 + 4u_n^2 \sigma_l(G) \sum_{j\in B} u_{nj} f_j.
\end{aligned} \tag{5.25}
$$

Now we deal with the third and fourth terms in (5.11). Using the commutator formulas (2.4)–(2.5), by (5.1) we get

$$
\sum_{\alpha=1}^{n} u_{i\alpha\alpha} = f_i, \qquad \sum_{\alpha=1}^{n} u_{ii\alpha\alpha} = f_{ii} - 2\varepsilon f. \tag{5.26}
$$

By (5.26), we have

$$
\begin{aligned}
u_n^{1-l} III &= (-1)^{l+1} u_n^{1-l} \sum_{\alpha,\beta,\gamma,\delta,\nu=1}^{n} \frac{\partial^2 \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{\alpha\beta} \partial u_{\gamma\delta}} u_{\gamma\delta\nu\nu} u_\alpha u_\beta \\
&= -u_n^3 \sum_{\alpha=1}^{n} \sum_{i=1}^{n-1} \sigma_l(h_{ij} \mid i) u_{ii\alpha\alpha} \\
&= -u_n^3 \sigma_l(G) \sum_{\alpha=1}^{n} \sum_{j\in B} u_{jj\alpha\alpha} \\
&= -u_n^3 \sigma_l(G) \sum_{j\in B} (f_{jj} - 2\varepsilon f).
\end{aligned} \tag{5.27}
$$

Similarly, by (5.26) we have

$$
\begin{aligned}
u_n^{1-l} IV &= 2(-1)^{l+1} u_n^{1-l} \sum_{\alpha,\beta,\nu=1}^{n} \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{\alpha\beta}} u_{\alpha\nu\nu} u_\beta \\
&= 2u_n^2 \sigma_l(G) \sum_{j\in B} u_{nj} \left(\sum_{\alpha=1}^{n} u_{j\alpha\alpha} \right) \\
&= 2u_n^2 \sigma_l(G) \sum_{j\in B} u_{nj} f_j. \qquad (5.28)
\end{aligned}
$$

For V, we have

$$
\begin{aligned}
V &= 2(-1)^{l+1} \sum_{\alpha,\beta,\nu=1}^{n} \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{\alpha\beta}} u_{\alpha\nu} u_{\beta\nu} \\
&= 2(-1)^{l+1} \left[\sum_{\alpha=1}^{n} \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{nn}} u_{n\alpha}^2 + \sum_{\alpha=1}^{n} \sum_{i,j=1}^{n-1} \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{ij}} u_{i\alpha} u_{j\alpha} \right. \\
&\quad \left. + 2 \sum_{\alpha=1}^{n} \sum_{j=1}^{n-1} \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{nj}} u_{n\alpha} u_{j\alpha} \right]. \qquad (5.29)
\end{aligned}
$$

Note that

$$
\sum_{\alpha=1}^{n} \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{nn}} u_{n\alpha}^2 = 0, \qquad (5.30)
$$

and

$$
4(-1)^{l+1} u_n^{1-l} \sum_{\alpha=1}^{n} \sum_{j=1}^{n-1} \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{nj}} u_{n\alpha} u_{j\alpha} = 4 u_n u_{nn} \sigma_l(G) \sum_{j\in B} u_{nj}^2. \qquad (5.31)
$$

We now compute the term

$$
\begin{aligned}
&2(-1)^{l+1} u_n^{1-l} \sum_{\alpha=1}^{n} \sum_{i,j=1}^{n-1} \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{ij}} u_{i\alpha} u_{j\alpha} \\
&= 2(-1)^{l+1} u_n^{1-l} \sum_{\alpha=1}^{n} \left[\sum_{i=j\in G} + \sum_{i=j\in B} + \sum_{i\neq j\in G} + \sum_{i\neq j\in B} + 2 \sum_{i\in G, j\in B} \right] \\
&\quad \times \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{ij}} u_{i\alpha} u_{j\alpha} \\
&:= V_1 + V_2 + V_3 + V_4 + V_5. \qquad (5.32)
\end{aligned}
$$

One can see that

$$V_1 = 2(-1)^{l+1} u_n^{1-l} \sum_{\alpha=1}^{n} \sum_{i=j\in G} \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{ij}} u_{i\alpha} u_{j\alpha}$$

$$= 2(-1)^{l+1} u_n^{1-l} \sum_{i\in G} \sigma_{l-1}(G \mid i)(-u_n)^{l-1} \sum_{j\in B} \big(-u_{nj}^2\big)\big(u_{ii}^2 + u_{in}^2\big)$$

$$= -2u_n^2 \sigma_1(G)\sigma_l(G) \sum_{j\in B} u_{nj}^2 - 2 \sum_{i\in G, j\in B} \sigma_{l-1}(G \mid i) u_{ni}^2 u_{nj}^2,$$

$$V_2 = 2(-1)^{l+1} u_n^{1-l} \sum_{\alpha=1}^{n} \sum_{i=j\in B} \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{ij}} u_{i\alpha} u_{j\alpha}$$

$$= 2(-1)^{l+1} u_n^{1-l} \sum_{j\in B} u_{nj}^2 \Bigg[\sigma_l(G)(-u_n)^l u_{nn}$$

$$+ \sum_{i\in G} \sigma_{l-1}(G \mid i)(-u_n)^{l-1} \Bigg(-u_{ni}^2 + \sum_{k\in B, k\neq j} \big(-u_{kn}^2\big) \Bigg) \Bigg]$$

$$= -2u_n u_{nn} \sigma_l(G) \sum_{j\in B} u_{nj}^2 - 2 \sum_{i\in G} \sigma_{l-1}(G \mid i) u_{ni}^2 \sum_{j\in B} u_{nj}^2$$

$$- 2 \sum_{i\in G} \sigma_{l-1}(G \mid i) \sum_{j,k\in B, j\neq k} u_{nj}^2 u_{nk}^2,$$

$$V_3 = 2(-1)^{l+1} u_n^{1-l} \sum_{\alpha=1}^{n} \sum_{i,j\in G, i\neq j} \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{ij}} u_{i\alpha} u_{j\alpha} = 0,$$

$$V_4 = 2(-1)^{l+1} u_n^{1-l} \sum_{\alpha=1}^{n} \sum_{i,j\in B, i\neq j} \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{ij}} u_{i\alpha} u_{j\alpha}$$

$$= 2 \sum_{k\in G} \sigma_{l-1}(G|k) \sum_{i,j\in B, i\neq j} u_{n_i}^2 u_{nj}^2,$$

and

$$V_5 = 4(-1)^{l+1} u_n^{1-l} \sum_{\alpha=1}^{n} \sum_{i\in G, j\in B} \frac{\partial \sigma_{l+2}(u_{\alpha\beta})}{\partial u_{ij}} u_{i\alpha} u_{j\alpha}$$

$$= 4 \sum_{i\in G} \sigma_{l-1}(G \mid i) u_{ni}^2 \sum_{j\in B} u_{nj}^2.$$

So we get

$$
\begin{aligned}
&2(-1)^{l+1}u_n^{1-l}\sum_{\alpha=1}^{n}\sum_{i,j=1}^{n-1}\frac{\partial\sigma_{l+2}(u_{\alpha\beta})}{\partial u_{ij}}u_{i\alpha}u_{j\alpha}\\
&\quad=-2u_nu_{nn}\sigma_l(G)\sum_{j\in B}u_{nj}^2-2u_n^2\sigma_1(G)\sigma_l(G)\sum_{j\in B}u_{nj}^2.
\end{aligned}
\tag{5.33}
$$

Combining (5.29)–(5.33), it follows that

$$
u_n^{1-l}V=2fu_n\sigma_l(G)\sum_{j\in B}u_{nj}^2. \tag{5.34}
$$

By (5.11), (5.25), (5.27), (5.28), and (5.34), it follows that

$$
u_n^{1-l}\Delta\varphi\le-u_n\sigma_l(G)\sum_{j\in B}\big[u_n^2(f_{jj}-2\varepsilon f)-6u_nu_{nj}f_j+6fu_{nj}^2\big]. \tag{5.35}
$$

So if $f(x)\ge0$ satisfies the following condition

$$
3f_if_j+4\varepsilon f^2\delta_{ij}\le2ff_{ij},
$$

then we get

$$
\Delta\varphi\le0. \tag{5.36}
$$

So we finish the proof of Theorem 5.1. □

Proof of the strict convexity in Theorem 1.1 In Sect. 4, we constructed a family of annuli $V_t=V_2(t)\setminus V_1(t), 0\le t\le1$, bounded by two convex domains with

$$
V_0=B_2\setminus B_1,\qquad V_1=\Omega_0\setminus\Omega_1
$$

where B_1,B_2 are convex geodesic balls with the same center and supposing that $B_2\setminus B_1$ contain $\Omega_2\setminus\Omega_1$. We assume the harmonic function u_t satisfies the homogeneous Dirichlet boundary conditions in the convex ring V_t (see (1.1)). In Sect. 3, we know $|\nabla u_t|\ne0$ in V_t, and by the standard elliptic theory we have the uniform estimates on $\|u_t\|_{C^3(V_t)}$ with the bound only depending on the geometry of Ω_0,Ω_1. Since at $t=0$, the standard solution of (1.1) on $V_0=B_2\setminus B_1$ is radial solution, and its level sets is a sphere. If $0<t_o<1$ is the first time that the level sets of u_{t_o} become convex but not strict convex. By the constant rank Theorem 5.1, we know the second fundamental form of the level sets of u_{t_o} is full rank. So we can extend to $t=1$, and know the level sets of the solution of (1.1) is strictly convex. This finish the strict convexity proof. □

6 Gaussian Curvature Estimates for the Convex Level Sets of Harmonic Functions

In this section, we prove the Gaussian curvature estimates of the strictly convex level set of harmonic functions on the space form. We follow the proof in Ma–Ou–Zhang [18].

Theorem 6.1 *Let (M^n, g) be a space form of sectional curvature $K_{sec} = 1$, or -1, and Ω be a bounded smooth domain in M^n, $n \geq 2$. Let u satisfy*

$$\Delta u = 0 \quad in\ \Omega \tag{6.1}$$

We assume $|\nabla u| \neq 0$ in Ω and the level sets of u are smooth strictly convex hypersurfaces with respect to $\frac{\nabla u}{|\nabla u|}$. Let K be the Gaussian curvature of the level sets; we have the following statement.

Case 1: for $(M^n, g) = (S^n, g_{standard})$ with $K_{sec} = 1$,

(1a) *for $n = 2, 3$, we have K attains its minimum on the boundary;*
(1b) *for $n \geq 4$, we have $|\nabla u|^{3-n} K$ attains its minimum on the boundary.*

Case 2: for $(M^n, g) = (H^n, g_{standard})$ with $K_{sec} = -1$,

(2a) *for $n = 2$, we have $|\nabla u| K$ attains its minimum on the boundary;*
(2b) *for $n = 3$, we have K attains its minimum on the boundary;*
(2c) *for $n \geq 4$, we have $|\nabla u|^{-1} K$ attains its minimum on the boundary.*

Proof of Theorem 6.1 Since the level sets of u are strictly convex with respect to the normal $\frac{\nabla u}{|\nabla u|}$. We set $\psi(x) = |\nabla u|^{2\alpha} K(x)$, and let

$$\varphi = \log \psi(x) = \log K(x) + \alpha \log |\nabla u|^2,$$

where

$$K(x) = (-1)^{n-1} \sum_{\alpha,\beta=1}^{n} \frac{\partial \sigma_n(u_{\alpha\beta})}{\partial u_{\alpha\beta}} u_\alpha u_\beta |\nabla u|^{-(n+1)}$$

is the Gaussian curvature of the level sets. For a suitable choice of θ, we will derive the following elliptic inequality

$$\Delta \varphi \leq 0, \quad \mod \nabla \varphi \text{ in } \Omega, \tag{6.2}$$

where we modular the terms of $\nabla \varphi$ with locally bounded coefficients. Then by the standard strong minimum principle, we get the result immediately.

In order to prove (6.2) at an arbitrary point $x_o \in \Omega$, as in Caffarelli–Friedman [4], we choose the normal coordinate at x_o, by rotating the coordinate system suitably by T_{x_o}, we may assume that $u_i(x_o) = 0$, $1 \leq i \leq n-1$ and $u_n(x_o) = |\nabla u| > 0$. And we can further assume that the matrix $(u_{ij}(x_o))$ $(1 \leq i, j \leq n-1)$ is diagonal and $u_{ii}(x_o) < 0$. We also choose T_{x_o} to vary smoothly in x_o. If we can establish (6.2) at x_o under the above assumptions, then going back to the original coordinate we find that (6.2) remains valid with new locally bounded coefficients on $\nabla\varphi$ in (6.2), depending smoothly on the independent variables. Thus it suffices to establish (6.2) under the above assumptions.

In the following calculation we shall let

$$g = (-1)^{n-1} \sum_{\alpha,\beta=1}^{n} \frac{\partial \sigma_n(u_{\alpha\beta})}{\partial u_{\alpha\beta}} u_\alpha u_\beta, \quad \text{and} \quad \theta = \alpha - \frac{n+1}{2} \tag{6.3}$$

then

$$\varphi = \log \psi(x) = \log g + \theta \log |\nabla u|^2. \tag{6.4}$$

From now on, all the calculations will be done at the fixed point x_o. In the following, we shall prove the theorem in three steps.

Step 1: We first compute the formula (6.12).

In the following, all the calculations will be done at x_o,

$$\varphi_\nu = \theta \frac{|\nabla u|^2{}_\nu}{|\nabla u|^2} + \frac{g_\nu}{g}, \tag{6.5}$$

and using (6.5)

$$\begin{aligned}
\Delta\varphi &= \sum_{\alpha=1}^{n} \frac{g_{\alpha\alpha}}{g} - \sum_{\alpha=1}^{n} \frac{g_\alpha^2}{g^2} + \theta \frac{\Delta|\nabla u|^2}{|\nabla u|^2} - \theta \sum_{\alpha=1}^{n} \frac{(|\nabla u|^2{}_\alpha)^2}{|\nabla u|^4} \\
&= \sum_{\alpha=1}^{n} \frac{g_{\alpha\alpha}}{g} + \theta \frac{\Delta|\nabla u|^2}{|\nabla u|^2} - \left(\theta + \theta^2\right) \sum_{\alpha=1}^{n} \frac{(|\nabla u|^2{}_\alpha)^2}{|\nabla u|^4} \\
&\quad - \sum_{\alpha=1}^{n} \varphi_\alpha^2 + 2\theta \sum_{\alpha=1}^{n} \frac{|\nabla u|^2{}_\alpha}{|\nabla u|^2} \varphi_\alpha.
\end{aligned} \tag{6.6}$$

Let $\lambda_i = -\frac{u_{ii}}{u_n}$, $\sigma_l := \sigma_l(\lambda)$, and using (6.1), we have

$$\begin{aligned}
&\sum_{\alpha=1}^{n} u_{n\alpha}^2 = u_{nn}^2 + \sum_{j=1}^{n-1} u_{nj}^2 = u_n^2 \sigma_1^2 + \sum_{j=1}^{n-1} u_{nj}^2 \\
&\sum_{\alpha,\beta=1}^{n} u_{\alpha\beta}^2 = u_{nn}^2 + 2\sum_{j=1}^{n-1} u_{nj}^2 + \sum_{j=1}^{n-1} u_{jj}^2 = u_n^2 \sigma_1^2 + u_n^2 \sum_{j=1}^{n-1} \lambda_j^2 + 2\sum_{j=1}^{n-1} u_{nj}^2.
\end{aligned} \tag{6.7}$$

Since $|\nabla u|_\alpha^2 = 2u_n u_{n\alpha}$, it follows that

$$\begin{aligned}
-\left(\theta + \theta^2\right) \sum_{\alpha=1}^{n} \frac{(|\nabla u|^2{}_\alpha)^2}{|\nabla u|^4} &= -4\left(\theta + \theta^2\right) \sum_{\alpha=1}^{n} \frac{u_{n\alpha}^2}{u_n^2} \\
&= -4\left(\theta + \theta^2\right)\sigma_1^2 - 4\left(\theta + \theta^2\right) \sum_{j=1}^{n-1} \frac{u_{nj}^2}{u_n^2}.
\end{aligned} \tag{6.8}$$

Using the commutator formulas, (6.1), (6.7), and $R_{\alpha\beta} = (n-1)\varepsilon\delta_{\alpha\beta}$, we have

$$\sum_{\alpha=1}^{n} u_{n\alpha\alpha} = \sum_{\alpha=1}^{n} u_{\alpha\alpha n} + \sum_{\alpha=1}^{n} R_{n\alpha} u_\alpha$$
$$= (n-1)\varepsilon u_n, \tag{6.9}$$

and

$$\Delta|\nabla u|^2 = 2\sum_{\alpha,\beta=1}^{n} u_{\alpha\beta}^2 + 2\sum_{\alpha,\beta=1}^{n} u_\alpha u_{\alpha\beta\beta}$$
$$= 2u_n \sum_{\alpha=1}^{n} u_{n\alpha\alpha} + 2\sum_{\alpha,\beta=1}^{n} u_{\alpha\beta}^2$$
$$= 2(n-1)\varepsilon u_n^2 + 2u_n^2\sigma_1^2 + 2u_n^2 \sum_{j=1}^{n-1} \lambda_j^2 + 4\sum_{j=1}^{n-1} u_{nj}^2. \tag{6.10}$$

It follows that

$$\theta\frac{\Delta|\nabla u|^2}{|\nabla u|^2} = 2\theta(n-1)\varepsilon + 2\theta\sigma_1^2 + 2\theta\sum_{j=1}^{n-1}\lambda_j^2 + 4\theta u_n^{-2}\sum_{j=1}^{n-1} u_{nj}^2. \tag{6.11}$$

And we have

$$\sum_{\alpha=1}^{n} \frac{|\nabla u|^2{}_\alpha}{|\nabla u|^2}\varphi_\alpha = 2\sigma_1\varphi_n + 2u_n^{-1}\sum_{j=1}^{n-1} u_{nj}\varphi_j$$

By (6.6)–(6.8) and (6.11), we obtain

$$\Delta\varphi = \sum_{\alpha=1}^{n}\frac{g_{\alpha\alpha}}{g} + 2\theta(n-1)\varepsilon - \left(2\theta + 4\theta^2\right)\sigma_1^2 - 4\theta^2 u_n^{-2}\sum_{j=1}^{n-1} u_{nj}^2 + 2\theta\sum_{j=1}^{n-1}\lambda_j^2$$
$$- \sum_{\alpha=1}^{n}\varphi_\alpha^2 + 4\theta\sigma_1\varphi_n + 4\theta u_n^{-1}\sum_{j=1}^{n-1} u_{nj}\varphi_j. \tag{6.12}$$

Step 2: In this step, using a calculation similar to the one in Sect. 5, we compute $\sum_{\alpha=1}^{n}\frac{g_{\alpha\alpha}}{g}$. We get the formula in (6.37).

First, at x_o, $g = u_n^{n+1}K$. Similar to (5.9), we have

$$g_\nu = (-1)^{n-1}\sum_{\alpha,\beta,\gamma,\delta=1}^{n}\frac{\partial^2\sigma_n(u_{\alpha\beta})}{\partial u_{\alpha\beta}\partial u_{\gamma\delta}} u_\alpha u_\beta u_{\gamma\delta\nu}$$
$$+ (-1)^{n-1}\sum_{\alpha,\beta=1}^{n-1}\frac{\partial\sigma_n(u_{\alpha\beta})}{\partial u_{\alpha\beta}}(u_{\alpha\nu}u_\beta + u_\alpha u_{\beta\nu})$$

$$= -u_n^n \sum_{i=1}^{n-1} \sigma_{n-2}(\lambda \mid i) u_{ii\nu} + 2u_n^n K u_{n\nu}$$
$$+ 2u_n^{n-1} \sum_{i=1}^{n-1} \sigma_{n-2}(\lambda \mid i) u_{ni} u_{i\nu}, \tag{6.13}$$

then

$$\frac{g_\alpha}{g} = -u_n^{-1} \sum_{i=1}^{n-1} \frac{u_{ii\alpha}}{\lambda_i} + 2\frac{u_{n\alpha}}{u_n} + 2u_n^{-2} \sum_{i=1}^{n-1} \frac{u_{ni}u_{i\alpha}}{\lambda_i}. \tag{6.14}$$

From (6.5) and (6.14), we get

$$\sum_{i=1}^{n-1} \frac{u_{ii\alpha}}{\lambda_i} = 2(1+\theta)u_{n\alpha} + 2u_n^{-1} \sum_{i=1}^{n-1} \frac{u_{ni}u_{i\alpha}}{\lambda_i} - u_n\varphi_\alpha, \tag{6.15}$$

so we have the following formulas

$$\begin{aligned} \sum_{i=1}^{n-1} \frac{u_{iin}}{u_n\lambda_i} &= 2(1+\theta)\sigma_1 + 2u_n^{-2} \sum_{j=1}^{n-1} \frac{u_{nj}^2}{\lambda_j} - \varphi_n, \\ \sum_{i=1}^{n-1} \frac{u_{iij}}{u_n\lambda_i} &= 2\theta u_n^{-1} u_{nj} - \varphi_j \quad \text{for } 1 \le j \le n-1. \end{aligned} \tag{6.16}$$

Now we calculate that

$$\begin{aligned} \Delta g &= (-1)^{n-1} \sum_{\alpha,\beta,\gamma,\delta,\zeta,\eta,\nu=1}^{n} \frac{\partial^3 \sigma_n(u_{\alpha\beta})}{\partial u_{\alpha\beta}\partial u_{\gamma\delta}\partial u_{\zeta\eta}} u_{\gamma\delta\nu} u_{\zeta\eta\nu} u_\alpha u_\beta \\ &\quad + 4(-1)^{n-1} \sum_{\alpha,\beta,\gamma,\delta,\nu=1}^{n} \frac{\partial^2 \sigma_n(u_{\alpha\beta})}{\partial u_{\alpha\beta}\partial u_{\gamma\delta}} u_{\gamma\delta\nu} u_{\alpha\nu} u_\beta \\ &\quad + (-1)^{n-1} \sum_{\alpha,\beta,\gamma,\delta,\nu=1}^{n} \frac{\partial^2 \sigma_n(u_{\alpha\beta})}{\partial u_{\alpha\beta}\partial u_{\gamma\delta}} u_{\gamma\delta\nu\nu} u_\alpha u_\beta \\ &\quad + 2(-1)^{n-1} \sum_{\alpha,\beta,\gamma,\delta,\nu=1}^{n} \frac{\partial \sigma_n(u_{\alpha\beta})}{\partial u_{\alpha\beta}} u_{\alpha\nu\nu} u_\beta \\ &\quad + 2(-1)^{n-1} \sum_{\alpha,\beta,\nu=1}^{n} \frac{\partial \sigma_n(u_{\alpha\beta})}{\partial u_{\alpha\beta}} u_{\alpha\nu} u_{\beta\nu} \\ &:= I + II + III + IV + V, \end{aligned} \tag{6.17}$$

and we write

$$\frac{\Delta g}{g} = \frac{III + IV + V}{g} + \frac{I + II}{g}. \tag{6.18}$$

As in Sect. 5, we will compute the above terms step by step. First we deal with the terms $III + IV$ in (6.17). Using the commutator formulas (2.4)–(2.5), we have

$$\sum_{\alpha=1}^{n} u_{i\alpha\alpha} = 0, \tag{6.19}$$

and for $1 \le i \le n-1$,

$$\begin{aligned}\sum_{\alpha=1}^{n} u_{ii\alpha\alpha} &= -2R_{inin}u_{nn} - 2\sum_{j,k=1}^{n-1} R_{ijik}u_{jk} + 2\sum_{\alpha=1}^{n} R_{i\alpha}u_{\alpha i}\\ &= -2\varepsilon u_{nn} + 2(n-1)\varepsilon u_{ii} - 2\varepsilon \sum_{j\neq i, j=1}^{n-1} u_{jj}\\ &= 2n\varepsilon u_{ii}. \end{aligned}\tag{6.20}$$

By (6.20), we have

$$\begin{aligned} III &= (-1)^{n-1} \sum_{\alpha,\beta,\gamma,\delta,\nu=1}^{n} \frac{\partial^2 \sigma_n(u_{\alpha\beta})}{\partial u_{\alpha\beta}\partial u_{\gamma\delta}} u_{\gamma\delta\nu\nu}u_\alpha u_\beta\\ &= -u_n^n \sum_{i=1}^{n-1} \sigma_{n-2}(\lambda \mid i) \sum_{\alpha=1}^{n} u_{ii\alpha\alpha}\\ &= -u_n^n \sum_{i=1}^{n-1} \sigma_{n-2}(\lambda \mid i)(2n\varepsilon u_{ii})\\ &= 2n(n-1)\varepsilon u_n^{n+1} K. \end{aligned}\tag{6.21}$$

Similarly, by (6.9), (6.19), and (5.28), we have

$$\begin{aligned} IV &= 2(-1)^{n-1} \sum_{\alpha,\beta,\nu=1}^{n} \frac{\partial \sigma_n(u_{\alpha\beta})}{\partial u_{\alpha\beta}} u_{\alpha\nu\nu}u_\beta\\ &= 2(-1)^{n-1}u_n \left[\sum_{\alpha=1}^{n} \frac{\partial \sigma_n(u_{\alpha\beta})}{\partial u_{nn}} u_{n\alpha\alpha} + \sum_{i=1}^{n-1}\sum_{\alpha=1}^{n} \frac{\partial \sigma_n(u_{\alpha\beta})}{\partial u_{in}} u_{i\alpha\alpha}\right]\\ &= 2(n-1)\varepsilon u_n^{n+1} K. \end{aligned}\tag{6.22}$$

For V, as in (5.29) and using (6.1), we have

$$
\begin{aligned}
V &= 2(-1)^{n-1}\frac{\partial \sigma_n(u_{\alpha\beta})}{\partial u_{\alpha\beta}}u_{\alpha\nu}u_{\beta\nu}\\
&= 2(-1)^{n-1}\left[\sum_{\alpha=1}^{n}\frac{\partial \sigma_n(u_{\alpha\beta})}{\partial u_{nn}}u_{n\alpha}^2 + 2\sum_{\alpha=1}^{n}\sum_{j=1}^{n-1}\frac{\partial \sigma_n(u_{\alpha\beta})}{\partial u_{nj}}u_{n\alpha}u_{j\alpha}\right.\\
&\quad\left.+\sum_{\alpha=1}^{n}\sum_{i,j=1}^{n-1}\frac{\partial \sigma_n(u_{\alpha\beta})}{\partial u_{ij}}u_{i\alpha}u_{j\alpha}\right]\\
&= 2(-1)^{n-1}\left[\sigma_{n-1}(u_{ij})\sum_{\alpha=1}^{n}u_{n\alpha}^2 - 2\sum_{j=1}^{n-1}\sum_{\alpha=1}^{n}\sigma_{n-2}(u_{ii}\mid j)u_{nj}u_{n\alpha}u_{j\alpha}\right.\\
&\quad\left.+\sum_{\alpha=1}^{n}\sum_{i=1}^{n-1}\frac{\partial \sigma_n(u_{\alpha\beta})}{\partial u_{ii}}u_{i\alpha}^2 + \sum_{i\neq j,i,j=1}^{n-1}\sum_{\alpha=1}^{n}\frac{\partial \sigma_n(u_{\alpha\beta})}{\partial u_{ij}}u_{i\alpha}u_{j\alpha}\right].
\end{aligned}
\tag{6.23}
$$

Note that

$$
\begin{aligned}
&\sum_{\alpha=1}^{n}\sum_{i=1}^{n-1}\frac{\partial \sigma_n(u_{\alpha\beta})}{\partial u_{ii}}u_{i\alpha}^2\\
&= u_{nn}\sum_{\alpha=1}^{n}\sum_{i=1}^{n-1}\sigma_{n-2}(u_{ii}\mid i)u_{i\alpha}^2 - \sum_{\alpha=1}^{n}\sum_{i\neq j,i,j=1}^{n-1}\sigma_{n-3}(u_{ii}\mid i,j)u_{nj}^2u_{i\alpha}^2\\
&= u_{nn}\sum_{i=1}^{n-1}\sigma_{n-2}(u_{ii}\mid i)u_{in}^2 + u_{nn}\sigma_{n-1}(u_{ij})\sum_{i=1}^{n-1}u_{ii}\\
&\quad - \sum_{i\neq j,i,j=1}^{n-1}\sigma_{n-3}(u_{ii}\mid i,j)u_{nj}^2u_{in}^2 - \sum_{j=1}^{n-1}\sigma_{n-2}(u_{ii}\mid j)u_{nj}^2(-u_{nn}-u_{jj}),
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{i\neq j,i,j=1}^{n-1}\sum_{\alpha=1}^{n}\frac{\partial \sigma_n(u_{\alpha\beta})}{\partial u_{ij}}u_{i\alpha}u_{j\alpha} &= \sum_{\alpha=1}^{n}\sum_{i\neq j,i,j=1}^{n-1}\sigma_{n-3}(u_{ii}\mid i,j)u_{ni}u_{nj}u_{i\alpha}u_{j\alpha}\\
&= \sum_{i\neq j,i,j=1}^{n-1}\sigma_{n-3}(u_{ii}\mid i,j)u_{nj}^2u_{in}^2.
\end{aligned}
$$

By the equation, one can see that

$$
\begin{aligned}
V = {} & 2(-1)^{n-1}\Bigg[\sigma_{n-1}(u_{ij})\left(u_{nn}^2+\sum_{j=1}^{n-1}u_{nj}^2\right)-2u_{nn}\sum_{j=1}^{n-1}\sigma_{n-2}(u_{ii}\mid j)u_{nj}^2 \\
& -2\sigma_{n-1}(u_{ij})\sum_{j=1}^{n-1}u_{nj}^2+u_{nn}\sum_{i=1}^{n-1}\sigma_{n-2}(u_{ii}\mid i)u_{in}^2 \\
& +u_{nn}\sigma_{n-1}(u_{ij})\sum_{i=1}^{n-1}u_{ii}-\sum_{j=1}^{n-1}\sigma_{n-2}(u_{ii}\mid j)u_{nj}^2(-u_{nn}-u_{jj})\Bigg] \\
= {} & 0.
\end{aligned}
\tag{6.24}
$$

Combining (6.21)–(6.22) and (6.24), it follows that

$$
\frac{III+IV+V}{g}=2(n+1)(n-1)\varepsilon. \tag{6.25}
$$

Now we treat the term I in (6.17).

$$
\begin{aligned}
I := {} & (-1)^{n-1}\sum_{\alpha,\beta,\gamma,\delta,\zeta,\eta,\nu=1}^{n}\frac{\partial^3\sigma_n(u_{\alpha\beta})}{\partial u_{\alpha\beta}\partial u_{\gamma\delta}\partial u_{\zeta\eta}}u_{\gamma\delta\nu}u_{\zeta\eta\nu}u_\alpha u_\beta \\
= {} & u_n^{n-1}\sum_{\alpha=1}^{n}\sum_{i\neq j,i,j=1}^{n-1}\sigma_{n-3}(\lambda\mid i,j)u_{ii\alpha}u_{jj\alpha} \\
& -u_n^{n-1}\sum_{\alpha=1}^{n}\sum_{i\neq j,i,j=1}^{n-1}\sigma_{n-3}(\lambda\mid i,j)u_{ij\alpha}^2,
\end{aligned}
\tag{6.26}
$$

so we have

$$
\begin{aligned}
\frac{I}{g} = {} & u_n^{-2}\sum_{\alpha=1}^{n}\sum_{i\neq j,i,j=1}^{n-1}\frac{u_{ii\alpha}u_{jj\alpha}}{\lambda_i\lambda_j}-u_n^{-2}\sum_{\alpha=1}^{n}\sum_{i\neq j,i,j=1}^{n-1}\frac{u_{ij\alpha}^2}{\lambda_i\lambda_j} \\
= {} & u_n^{-2}\sum_{\alpha=1}^{n}\left(\sum_{i=1}^{n-1}\frac{u_{ii\alpha}}{\lambda_i}\right)^2-u_n^{-2}\sum_{\alpha=1}^{n}\sum_{i,j=1}^{n-1}\frac{u_{ij\alpha}^2}{\lambda_i\lambda_j}.
\end{aligned}
\tag{6.27}
$$

Using (6.16), we get

$$
\begin{aligned}
u_n^{-2}\left(\sum_{i=1}^{n-1}\frac{u_{iin}}{\lambda_i}\right)^2 = {} & \left[2(1+\theta)\sigma_1+2u_n^{-2}\sum_{i=1}^{n-1}\frac{u_{ni}^2}{\lambda_i}-\varphi_n\right]^2 \\
= {} & 4(1+\theta)^2\sigma_1^2+4\left(\sum_{i=1}^{n-1}\frac{u_{ni}^2}{u_n^2\lambda_i}\right)^2+8(1+\theta)\sigma_1\sum_{i=1}^{n-1}\frac{u_{ni}^2}{u_n^2\lambda_i} \\
& -4\varphi_n\left[(1+\theta)\sigma_1+u_n^{-2}\sum_{i=1}^{n-1}\frac{u_{ni}^2}{\lambda_i}\right]+\varphi_n^2,
\end{aligned}
\tag{6.28}
$$

and

$$u_n^{-2}\sum_{j=1}^{n-1}\left(\sum_{i=1}^{n-1}\frac{u_{iij}}{\lambda_i}\right)^2=\sum_{j=1}^{n-1}\left(2\theta u_n^{-1}u_{nj}-\varphi_j\right)^2$$
$$=4\theta^2\sum_{j=1}^{n-1}\frac{u_{nj}^2}{u_n^2}-4\theta u_n^{-1}\sum_{j=1}^{n-1}u_{nj}\varphi_j+\sum_{j=1}^{n-1}\varphi_j^2, \quad (6.29)$$

so we have

$$u_n^{-2}\sum_{\alpha=1}^{n}\left(\sum_{i=1}^{n-1}\frac{u_{ii\alpha}}{\lambda_i}\right)^2=u_n^{-2}\left(\sum_{i=1}^{n-1}\frac{u_{iin}}{\lambda_i}\right)^2+u_n^{-2}\sum_{j=1}^{n-1}\left(\sum_{i=1}^{n-1}\frac{u_{iij}}{\lambda_i}\right)^2$$
$$=4(1+\theta)^2\sigma_1^2+4\left(\sum_{i=1}^{n-1}\frac{u_{ni}^2}{u_n^2\lambda_i}\right)^2+8(1+\theta)\sigma_1\sum_{i=1}^{n-1}\frac{u_{ni}^2}{u_n^2\lambda_i}$$
$$+4\theta^2\sum_{j=1}^{n-1}\frac{u_{nj}^2}{u_n^2}-4\varphi_n\left[(1+\theta)\sigma_1+u_n^{-2}\sum_{i=1}^{n-1}\frac{u_{ni}^2}{\lambda_i}\right]$$
$$-4\theta u_n^{-1}\sum_{j=1}^{n-1}u_{nj}\varphi_j+\sum_{\alpha=1}^{n}\varphi_\alpha^2. \quad (6.30)$$

To compute the second term in (6.17), we still use (5.16)

$$II:=4(-1)^{n-1}\sum_{\alpha,\beta,\gamma,\delta,\nu=1}^{n}\frac{\partial^2\sigma_n(u_{\alpha\beta})}{\partial u_{\alpha\beta}\partial u_{\gamma\delta}}u_{\gamma\delta\nu}u_{\alpha\nu}u_\beta$$
$$=4(-1)^{n-1}u_n\sum_{\gamma,\delta,\alpha=1}^{n}\left[\frac{\partial^2\sigma_n(u_{\alpha\beta})}{\partial u_{nn}\partial u_{\gamma\delta}}u_{\gamma\delta\alpha}u_{n\alpha}+\sum_{i=1}^{n-1}\frac{\partial^2\sigma_n(u_{\alpha\beta})}{\partial u_{in}\partial u_{\gamma\delta}}u_{\gamma\delta\alpha}u_{i\alpha}\right]$$
$$=-4u_n^{n-1}\sum_{\alpha=1}^{n}\sum_{i=1}^{n-1}\sigma_{n-2}(\lambda\mid i)u_{n\alpha}u_{ii\alpha}+4u_n^{n-1}\sum_{\alpha=1}^{n}\sum_{i=1}^{n-1}\sigma_{n-2}(\lambda\mid i)u_{i\alpha}u_{ni\alpha}$$
$$+4u_n^{n-2}\sum_{\alpha=1}^{n}\sum_{i\neq j,i,j=1}^{n-1}\sigma_{n-3}(\lambda\mid i,j)u_{i\alpha}u_{nj}u_{ij\alpha}$$
$$-4u_n^{n-2}\sum_{\alpha=1}^{n}\sum_{i\neq j,i,j=1}^{n-1}\sigma_{n-3}(\lambda\mid i,j)u_{i\alpha}u_{ni}u_{jj\alpha}. \quad (6.31)$$

Using the equation and the commutator formula we have

$$u_{nin} = u_{nni} - \sum_{\alpha=1}^{n} R_{ni\alpha n} u_\alpha = -\sum_{j=1}^{n-1} u_{jji}, \quad u_{iji} = u_{iij},$$
$$u_{nii} = u_{ini} = u_{iin} - \sum_{\alpha=1}^{n} R_{in\alpha i} u_\alpha = u_{iin} + \varepsilon u_n. \tag{6.32}$$

Via a calculation similar to (5.24), it follows that

$$\begin{aligned}
\frac{II}{g} :=& -4\frac{\sigma_1}{u_n}\sum_{i=1}^{n-1}\frac{u_{iin}}{\lambda_i} - \frac{4}{u_n}\sum_{i=1}^{n-1}u_{iin} + \frac{4}{u_n^3}\sum_{i\neq j,i,j=1}^{n-1}\frac{u_{ni}u_{nj}}{\lambda_i\lambda_j}u_{ijn} \\
&- \frac{4}{u_n^3}\sum_{i\neq j,i,j=1}^{n-1}\frac{u_{ni}^2}{\lambda_i\lambda_j}u_{jjn} \\
&- \frac{4}{u_n^2}\left[\sum_{i,j=1}^{n-1}\frac{u_{iij}}{\lambda_i}u_{nj} + \sum_{i,j=1}^{n-1}\frac{u_{jji}}{\lambda_i}u_{ni} + \sum_{i\neq j,i,j=1}^{n-1}\frac{u_{iij}}{\lambda_j}u_{nj} - \sum_{i\neq j,i,j=1}^{n-1}\frac{u_{jji}}{\lambda_j}u_{ni}\right] \\
&- 4(n-1)\varepsilon \\
:=& II_1 + II_2 - 4(n-1)\varepsilon.
\end{aligned} \tag{6.33}$$

By (6.16),

$$\begin{aligned}
II_1 =& -4\left(\sigma_1 + \sum_{i=1}^{n-1}\frac{u_{ni}^2}{u_n^2\lambda_i}\right)\sum_{j=1}^{n-1}\frac{u_{jjn}}{u_n\lambda_j} - \frac{4}{u_n}\sum_{i=1}^{n-1}u_{iin} + \frac{4}{u_n^3}\sum_{i,j=1}^{n-1}\frac{u_{ni}u_{nj}}{\lambda_i\lambda_j}u_{ijn} \\
=& -8(1+\theta)\sigma_1^2 - 8(2+\theta)\sigma_1\sum_{i=1}^{n-1}\frac{u_{ni}^2}{u_n^2\lambda_i} - 8\left(\sum_{i=1}^{n-1}\frac{u_{ni}^2}{u_n^2\lambda_i}\right)^2 \\
&- \frac{4}{u_n}\sum_{i=1}^{n-1}u_{iin} + \frac{4}{u_n^3}\sum_{i,j=1}^{n-1}\frac{u_{ni}u_{nj}}{\lambda_i\lambda_j}u_{ijn} \\
&+ 4\left(\sigma_1 + \sum_{i=1}^{n-1}\frac{u_{ni}^2}{u_n^2\lambda_i}\right)\varphi_n.
\end{aligned} \tag{6.34}$$

And

$$II_2 = -8\sum_{i,j=1}^{n-1}\frac{u_{ni}u_{jji}}{u_n^2\lambda_i}. \tag{6.35}$$

Then by (6.33)–(6.35), we get

$$\frac{II}{g} = -8(1+\theta)\sigma_1^2 - 8(2+\theta)\sigma_1 \sum_{i=1}^{n-1} \frac{u_{ni}^2}{u_n^2\lambda_i} - 8\left(\sum_{i=1}^{n-1} \frac{u_{ni}^2}{u_n^2\lambda_i}\right)^2$$

$$- \frac{4}{u_n}\sum_{i=1}^{n-1} u_{iin} + \frac{4}{u_n^3}\sum_{i,j=1}^{n-1} \frac{u_{ni}u_{nj}}{\lambda_i\lambda_j}u_{ijn} - 8\sum_{i,j=1}^{n-1}\frac{u_{ni}u_{jji}}{u_n^2\lambda_i}$$

$$- 4(n-1)\varepsilon + 4\left(\sigma_1 + \sum_{i=1}^{n-1}\frac{u_{ni}^2}{u_n^2\lambda_i}\right)\varphi_n. \tag{6.36}$$

Combining (6.18), (6.25), (6.27), (6.30), and (6.36), it follows that

$$\frac{\Delta g}{g} = -u_n^{-2}\sum_{i,j=1}^{n-1}\frac{u_{ijn}^2}{\lambda_i\lambda_j} - \frac{4}{u_n}\sum_{i=1}^{n-1}u_{iin} + \frac{4}{u_n^3}\sum_{i,j=1}^{n-1}\frac{u_{ni}u_{nj}}{\lambda_i\lambda_j}u_{ijn}$$

$$- u_n^{-2}\sum_{i,j,k=1}^{n-1}\frac{u_{ijk}^2}{\lambda_i\lambda_j} - 8\sum_{i,j=1}^{n-1}\frac{u_{ni}u_{jji}}{u_n^2\lambda_i}$$

$$+ 4\big(\theta^2-1\big)\sigma_1^2 - 4\left(\sum_{i=1}^{n-1}\frac{u_{ni}^2}{u_n^2\lambda_i}\right)^2 + 4u^{-2}\sum_{i=1}^{n-1}\left(\theta^2 - \frac{2\sigma_1}{\lambda_i}\right)u_{ni}^2$$

$$+ 2(n-1)^2\varepsilon - 4\theta\varphi_n\sigma_1 - 4\theta u_n^{-1}\sum_{j=1}^{n-1}u_{nj}\varphi_j + \sum_{\alpha=1}^{n}\varphi_\alpha^2. \tag{6.37}$$

Step 3: In this step, using a calculation similar to the one in Ma–Ou–Zhang [18], we complete the proof of this theorem. Combining (6.12) and (6.37), it follows that

$$u_n^2\Delta\varphi = -\sum_{i,j=1}^{n-1}\frac{u_{ijn}^2}{\lambda_i\lambda_j} - 4u_n\sum_{i=1}^{n-1}u_{iin} + \frac{4}{u_n}\sum_{i,j=1}^{n-1}\frac{u_{ni}u_{nj}}{\lambda_i\lambda_j}u_{ijn}$$

$$- \sum_{i,j,k=1}^{n-1}\frac{u_{ijk}^2}{\lambda_i\lambda_j} - 8\sum_{i,j=1}^{n-1}\frac{u_{ni}u_{jji}}{\lambda_i}$$

$$- 4u_n^{-2}\left(\sum_{i=1}^{n-1}\frac{u_{ni}^2}{\lambda_i}\right)^2 - 8\sigma_1\sum_{j=1}^{n-1}\frac{u_{nj}^2}{\lambda_j}$$

$$- (4+2\theta)u_n^2\sigma_1^2 + 2\theta u_n^2\sum_{j=1}^{n-1}\lambda_j^2 + 2(n-1)(\theta+n-1)\varepsilon u_n^2. \tag{6.38}$$

As in Sect. 5, in order to simplify the calculation, we introduce a new notation $a_{ij,\alpha}$,

$$u_n u_{ij\alpha} = -u_n^2 a_{ij,\alpha} + u_{ni}u_{j\alpha} + u_{nj}u_{i\alpha} + u_{n\alpha}u_{ij}. \tag{6.39}$$

Then we know $a_{ij,k}$ is a commutator for $1 \le i, j, k \le n-1$. Using (6.39) and (6.16), we have

$$\begin{aligned}
&\sum_{i=1}^{n-1} \frac{a_{ii,n}}{\lambda_i} = -(n+2\theta+1)\sigma_1 + \varphi_n, \\
&\sum_{i=1}^{n-1} \frac{a_{ii,j}}{\lambda_i} = -(n+2\theta+1)u_n^{-1}u_{nj} + \varphi_j \quad \text{for } 1 \le j \le n-1.
\end{aligned} \tag{6.40}$$

By (6.39) and (6.40), it follows that

$$\begin{aligned}
&-\sum_{i,j=1}^{n-1} \frac{u_{ijn}^2}{\lambda_i\lambda_j} - 4u_n \sum_{i=1}^{n-1} u_{iin} + \frac{4}{u_n} \sum_{i,j=1}^{n-1} \frac{u_{ni}u_{nj}}{\lambda_i\lambda_j} u_{ijn} \\
&= -u_n^2 \sum_{i,j=1}^{n-1} \frac{a_{ij,n}^2}{\lambda_i\lambda_j} + 4u_n^2 \sum_{i=1}^{n-1} a_{ii,n} \\
&\quad + (n+4\theta+7)u_n^2\sigma_1^2 - 8\sum_{j=1}^{n-1} u_{nj}^2 + 4u_n^{-2}\left(\sum_{j=1}^{n-1} \frac{u_{nj}^2}{\lambda_j}\right)^2 - 2u_n^2\sigma_1\varphi_n,
\end{aligned} \tag{6.41}$$

and

$$\begin{aligned}
&-\sum_{i,j,k=1}^{n-1} \frac{u_{ijk}^2}{\lambda_i\lambda_j} - 8\sum_{i,j=1}^{n-1} \frac{u_{ni}u_{jji}}{\lambda_i} \\
&= -u_n^2 \sum_{i,j,k=1}^{n-1} \frac{a_{ij,k}^2}{\lambda_i\lambda_j} + 4u_n \sum_{i,j=1}^{n-1} \frac{u_{ni}a_{jj,i}}{\lambda_i} \\
&\quad + 6\sigma_1 \sum_{i=1}^{n-1} \frac{u_{nj}^2}{\lambda_j} + (n+4\theta+13)\sum_{j=1}^{n-1} u_{nj}^2 - 2u_n \sum_{j=1}^{n-1} u_{nj}\varphi_j.
\end{aligned} \tag{6.42}$$

Combining (6.38) and (6.41)–(6.42), we have

$$\begin{aligned}
u_n^2\Delta\varphi &= -u_n^2 \sum_{i,j=1}^{n-1} \frac{a_{ij,n}^2}{\lambda_i\lambda_j} + 4u_n^2 \sum_{i=1}^{n-1} a_{ii,n} - u_n^2 \sum_{i,j,k=1}^{n-1} \frac{a_{ij,k}^2}{\lambda_i\lambda_j} + 4u_n \sum_{i,j=1}^{n-1} \frac{u_{ni}a_{jj,i}}{\lambda_i} \\
&\quad + (n+4\theta+5)\sum_{j=1}^{n-1} u_{nj}^2 - 2\sigma_1 \sum_{i=1}^{n-1} \frac{u_{nj}^2}{\lambda_j} + (n+2\theta+3)u_n^2\sigma_1^2 + 2\theta u_n^2 \sum_{j=1}^{n-1} \lambda_j^2 \\
&\quad + 2(n-1)(\theta+n-1)u_n^2\varepsilon - 2u_n^2\sigma_1\varphi_n - 2u_n \sum_{j=1}^{n-1} u_{nj}\varphi_j.
\end{aligned} \tag{6.43}$$

Recall that $\theta = \alpha - \frac{n+1}{2}$, then (6.40) becomes

$$\sum_{i=1}^{n-1} \frac{a_{ii,\alpha}}{\lambda_i} = -2\alpha u_n^{-1} u_{n\alpha} + \varphi_\alpha, \tag{6.44}$$

and if we let

$$\begin{aligned} Q := & -u_n^2 \sum_{i,j=1}^{n-1} \frac{a_{ij,n}^2}{\lambda_i \lambda_j} + 4u_n^2 \sum_{i=1}^{n-1} a_{ii,n} - u_n^2 \sum_{i,j,k=1}^{n-1} \frac{a_{ij,k}^2}{\lambda_i \lambda_j} + 4u_n \sum_{i,j=1}^{n-1} \frac{u_{ni} a_{jj,i}}{\lambda_i} \\ & + (4\alpha + 3 - n) \sum_{j=1}^{n-1} u_{nj}^2 - 2\sigma_1 \sum_{i=1}^{n-1} \frac{u_{nj}^2}{\lambda_j} + (2\alpha + 2) u_n^2 \sigma_1^2 \\ & + (2\alpha - n - 1) u_n^2 \sum_{j=1}^{n-1} \lambda_j^2 - 2u_n^2 \sigma_1 \varphi_n - 2u_n \sum_{j=1}^{n-1} u_{nj} \varphi_j, \end{aligned} \tag{6.45}$$

then we have

$$u_n^2 \Delta\varphi = Q + 2(n-1)\left(\alpha - \frac{3-n}{2}\right) u_n^2 \varepsilon. \tag{6.46}$$

Notice that (6.44) and (6.45) are the same as (3.6) and (3.47) in Ma–Ou–Zhang [18]. By the analysis in [18] we get the following sufficient condition on α to guarantee

$$Q \le 0$$

(where we have suppressed the terms containing the gradient of φ with locally bounded coefficients), which turns out to be

$$\begin{aligned} & n = 2, \quad -\infty < \alpha < +\infty, \\ & n \ge 3, \quad \alpha \ge \frac{n-3}{2} \quad \text{or} \quad \alpha \le -\frac{1}{2}. \end{aligned} \tag{6.47}$$

Combining (6.46) and (6.47), we get the following sufficient condition on α to guarantee

$$\Delta\varphi \le 0 \operatorname{mod} \nabla\varphi.$$

Case 1: for $(M^n, g) = (S^n, g_o)$ with $K_{sec} = 1$, i.e., $\varepsilon = 1$, $n = 2, 3$, we choose $\alpha = 0$, $n \ge 4$, we choose $\alpha = -\frac{n-3}{2}$.

Case 2: for $(M^n, g) = (H^n, g_o)$ with $K_{sec} = -1$, i.e., $\varepsilon = -1$, $n = 2$, we choose $\alpha = \frac{1}{2}$, $n = 3$, we choose $\alpha = 0$, $n \ge 4$, we choose $\alpha = -\frac{1}{2}$.

By the maximum principle, we complete the proof of Theorem 6.1. □

Now we give a discussion for the H^2 case. We shall get a new proof of Papadimitrakis's [21] result, and using the following conclusion we can obtain an upper bound for the curvature of level curves in terms of the boundary data.

Theorem 6.2 *Let (H^2, g) be the hyperbolic space form of sectional curvature -1. Let u satisfy (1.1) in a convex ring $\Omega = \Omega_0 \backslash \bar{\Omega}_1$ in H^2 and k be the curvature of the level curves with respect to $\frac{\nabla u}{|\nabla u|}$. For $\varphi = |\nabla u|^{-1} k$, the following relation is satisfied*:

$$\Delta\varphi = 2\varphi. \tag{6.48}$$

From (6.48), we have two conclusions.

(1) *If $k > 0$ on $\partial\Omega$, then $k > 0$ in Ω.*
(2) *φ attains its maximum on $\partial\Omega$.*

Proof From Sect. 2, we know the curvature of the level curve of u with respect to the normal $\frac{\nabla u}{|\nabla u|}$ is

$$k = -\sum_{\alpha,\beta=1}^{2} \frac{\partial\sigma_2(u_{\alpha\beta})}{\partial u_{\alpha\beta}} u_\alpha u_\beta |\nabla u|^{-3}.$$

We set $\varphi(x) = |\nabla u|^{2\alpha} k(x)$,

$$g = -\sum_{\alpha,\beta=1}^{2} \frac{\partial\sigma_2(u_{\alpha\beta})}{\partial u_{\alpha\beta}} u_\alpha u_\beta$$

and $\theta = \alpha - \frac{3}{2}$. Then we can rewrite $\varphi(x) = |\nabla u|^{2\alpha} k(x)$ as

$$\varphi(x) = |\nabla u|^{2\theta} g.$$

In order to prove (6.48) at an arbitrary point $x_o \in \Omega$, as in the proof of Theorem 6.1, we choose the normal coordinate at x_o, by rotating the coordinate system suitably by T_{x_o}, we may assume that $u_1(x_o) = 0$ and $u_2(x_o) = |\nabla u| > 0$. We also choose T_{x_o} to vary smoothly with x_o. We only need to establish (6.48) at x_o under the above assumptions.

In the following, all the calculations will be done at x_o,

$$\varphi_\alpha = \left(|\nabla u|^{2\theta}\right)_\alpha g + |\nabla u|^{2\theta} g_\alpha, \tag{6.49}$$

and

$$\Delta\varphi = g\Delta|\nabla u|^{2\theta} + 2\sum_{\alpha=1}^{2} g_\alpha \left(|\nabla u|^{2\theta}\right)_\alpha + |\nabla u|^{2\theta}\Delta g. \tag{6.50}$$

Similar to the calculation in Theorem 6.1, at x_o, $g = -u_2^2 u_{11}$, and we have

$$\Delta|\nabla u|^{2\theta} = -2\theta u_2^{2\theta} + 4\theta^2 u_2^{2\theta-2}\left(u_{11}^2 + u_{12}^2\right). \tag{6.51}$$

Using (6.49), we get

$$2\sum_{\alpha=1}^{2} g_\alpha\big(|\nabla u|^{2\theta}\big)_\alpha = 8\theta^2 u_2^{2\theta} u_{11}\big(u_{11}^2+u_{12}^2\big) + 4\theta u_2^{-1}\sum_{\alpha=1}^{2} u_{2\alpha}\varphi_\alpha. \tag{6.52}$$

It follows that

$$\begin{aligned} & g\Delta|\nabla u|^{2\theta} + 2\sum_{\alpha=1}^{2} g_\alpha\big(|\nabla u|^{2\theta}\big)_\alpha \\ &\quad = 4\theta^2 u_2^{2\theta} u_{11}\big(u_{11}^2+u_{12}^2\big), +2\theta u_2^{2\theta+2} u_{11} + 4\theta u_2^{-1}\sum_{\alpha=1}^{2} u_{2\alpha}\varphi_\alpha. \end{aligned} \tag{6.53}$$

Note that

$$g = -u_{11}u_2u_2 - u_{22}u_1u_1 + 2u_{12}u_1u_2,$$

so we get

$$g_\alpha = -u_2^2 u_{11\alpha} - 2u_2u_{11}u_{2\alpha} + 2u_2u_{12}u_{1\alpha}. \tag{6.54}$$

By (6.49), one can see that

$$g_\alpha = u_2^{-2\theta}\varphi_\alpha + 2\theta u_2 u_{11} u_{2\alpha}. \tag{6.55}$$

So we have

$$\begin{aligned} u_2u_{111} &= -2\theta u_{11}u_{12} - u_2^{-2\theta-1}\varphi_1, \\ u_2u_{112} &= 2u_{12}^2 + 2(1+\theta)u_{11}^2 - u_2^{-2\theta-1}\varphi_2. \end{aligned} \tag{6.56}$$

We now compute that

$$\begin{aligned} \Delta g = & -\sum_{\alpha,\beta,\gamma,\delta,\nu=1}^{2} \frac{\partial^2\sigma_2(u_{\alpha\beta})}{\partial u_{\alpha\beta}\partial u_{\gamma\delta}} u_{\gamma\delta\nu\nu}u_\alpha u_\beta - 2\sum_{\alpha,\beta,\nu=1}^{n} \frac{\partial\sigma_2(u_{\alpha\beta})}{\partial u_{\alpha\beta}} u_{\alpha\nu\nu}u_\beta \\ & -2\sum_{\alpha,\beta,\nu=1}^{n} \frac{\partial\sigma_2(u_{\alpha\beta})}{\partial u_{\alpha\beta}} u_{\alpha\nu}u_{\beta\nu} - 4\sum_{\alpha,\beta,\gamma,\delta,\nu=1}^{2} \frac{\partial^2\sigma_2(u_{\alpha\beta})}{\partial u_{\alpha\beta}\partial u_{\gamma\delta}} u_{\gamma\delta\nu}u_{\alpha\nu}u_\beta \\ := & \, I + II + III + IV. \end{aligned} \tag{6.57}$$

One can check that

$$I = 4u_2^2u_{11}, \quad II = 2u_2^2u_{11}, \quad III = 0,$$

$$IV = -4u_2^2u_{11} - 8u_{12}(u_2u_{111}) + 8u_{11}(u_2u_{112}).$$

It follows that

$$\Delta g = 2u_2^2 u_{11} - 8u_{12}(u_2 u_{111}) + 8u_{11}(u_2 u_{112}). \tag{6.58}$$

Using (6.56), we obtain

$$\begin{aligned} \Delta g = {} & 2u_2^2 u_{11} + 16(1+\theta)u_{11}\big(u_{11}^2 + u_{12}^2\big) \\ & + 8u_2^{-2\theta-1} u_{12}\varphi_1 - 8u_2^{-2\theta-1} u_{11}\varphi_2. \end{aligned} \tag{6.59}$$

Combining (6.50), (6.53), and (6.59), it follows that

$$\begin{aligned} u_2^{-2\theta}\Delta\varphi = {} & 2(1+\theta)u_2^2 u_{11} + 4(2+\theta)^2 u_{11}\big(u_{11}^2 + u_{12}^2\big) \\ & + (8+4\theta)u_2^{-2\theta-1} u_{12}\varphi_1 - (8+4\theta)u_2^{-2\theta-1} u_{11}\varphi_2. \end{aligned} \tag{6.60}$$

Let $\theta = -2$, i.e., $\alpha = -\frac{1}{2}$. In this case $\varphi = |\nabla u|^{-1} k = -u_2^{-2} u_{11}$ satisfies

$$\Delta\varphi = 2\varphi. \tag{6.61}$$

□

We use the observation on the monotonicity of the norm of the gradient along the gradient direction, which also appeared in [15, 17, 18]. It follows that $|\nabla u|$ attains its minimum on $\partial\Omega_0$, and attains its maximum on $\partial\Omega_1$.

Now we combine Theorem 6.1, Theorem 6.2, and the strict convexity of the level sets to obtain the estimates in the second part of Theorem 1.1.

Proof of estimates in Theorem 1.1 If u is the smooth solution of (1.1), then from the first part we know the level sets of u are strictly convex with respect to the normal $\frac{\nabla u}{|\nabla u|}$. Since $|\nabla u|$ attains its minimum on $\partial\Omega_0$, and attains its maximum on $\partial\Omega_1$, from Theorems 6.1 and 6.2 we can get the estimates in Theorem 1.1. □

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