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Gradient estimates of mean curvature equations with Neumann boundary value problems ☆

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ABSTRACT

In this paper, we use the maximum principle to get the gradient estimate for the solutions of the prescribed mean curvature equation with Neumann boundary value problem, which gives a positive answer for the question raised by Lieberman in 2013. As a consequence, we obtain the corresponding existence theorem for a class of mean curvature equations.

1. Introduction

Gradient estimate for the prescribed mean curvature equation has been extensively studied. The interior gradient estimate, for the minimal surface equation was obtained in the case of two variables by Finn [3]. Bombieri, De Giorgi and M. Miranda [1] obtained the estimate for high dimensional case. For the general mean curvature equation, such

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estimate had also been obtained by Ladyzhenskaya and Ural'tseva [11], Trudinger [22] and Simon [19]. All their methods were used by test function argument and a resulting Sobolev inequality. In 1983, Korevaar [8] introduced the normal variation technique and got the maximum principle proof for the interior gradient estimate on the minimal surface equation. Trudinger [23] also studied the curvature equations and got the interior gradient estimates for a class curvature equation. In 1998, Wang [25] gave a new proof for the mean curvature equation via standard Bernstein technique. The Dirichlet problem for the prescribed mean curvature equation has been studied by Jenkins–Serrin [6] and Serrin [18]. A more detailed history could be found in Gilbarg and Trudinger [5].

For the mean curvature equation with prescribed contact angle boundary value problem, Ural'tseva [24] first got the boundary gradient estimates and the corresponding existence theorem. At the same time, Simon–Spruck [20] and Gerhardt [4] also obtained existence theorem on the positive gravity case. For the σ_k-Yamabe problem on compact manifolds with boundary, Li–Zhu [12] got some existence theorem. For more general quasilinear divergence structure equation with conormal derivative boundary value problem, Lieberman [13] gave the gradient estimate. They obtained these estimates also via test function technique.

Spruck [21] used the maximum principle to obtain boundary gradient estimate in two dimensions for the positive gravity capillary problems. Korevaar [9] generalized his normal variation technique and got the gradient estimates for the positive gravity case in high dimensional case. In [14,15], Lieberman developed the maximum principle approach on the boundary gradient estimates to the quasilinear elliptic equation with oblique derivative boundary value problem, and in [16] he got the maximum principle proof for the gradient estimates on the general quasilinear elliptic equation with capillary boundary value problems.

In a recent book ([17], page 360), Lieberman posed the following question, how to get the gradient estimates for the mean curvature equation with Neumann boundary value problem. In this paper we use the technique developed by Spruck [21], Lieberman [16], Wang [25] and Jin–Li–Li [7] to get a positive answer. As a consequence, we obtain an existence theorem for a class of mean curvature equations with Neumann boundary value problem.

We first consider the boundary gradient estimates for the mean curvature equation with Neumann boundary value problem. Now let's state our main gradient estimates.

Theorem 1.1. *Suppose $u \in C^2(\overline{\Omega}) \bigcap C^3(\Omega)$ is a bounded solution for the following boundary value problem*

$$\mathtt{div}(\frac{Du}{\sqrt{1+|Du|^2}}) = f(x,u) \quad in \quad \Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial \gamma} = \psi(x,u) \quad on \quad \partial\Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $n \geq 2$, $\partial\Omega \in C^3$, γ is the inward unit normal to $\partial\Omega$.

We assume $f(x,z) \in C^1(\overline{\Omega} \times [-M_0, M_0])$ and $\psi(x,z) \in C^3(\overline{\Omega} \times [-M_0, M_0])$, and there exist positive constants M_0, L_1, L_2 such that

$$|u| \leq M_0 \quad in \quad \overline{\Omega}, \tag{1.3}$$

$$f_z(x,z) \geq 0 \quad in \quad \overline{\Omega} \times [-M_0, M_0], \tag{1.4}$$

$$|f(x,z)| + |f_x(x,z)| \leq L_1 \quad in \quad \overline{\Omega} \times [-M_0, M_0], \tag{1.5}$$

$$|\psi(x,z)|_{C^3(\overline{\Omega}\times[-M_0,M_0])} \leq L_2. \tag{1.6}$$

Then there exists a small positive constant μ_0 such that we have the following estimate

$$\sup_{\overline{\Omega}_{\mu_0}} |Du| \leq \max\{M_1, M_2\},$$

where M_1 is a positive constant depending only on n, μ_0, M_0, L_1, which is from the interior gradient estimates; M_2 is a positive constant depending only on $n, \Omega, \mu_0, M_0, L_1, L_2$, and $d(x) = \mathtt{dist}(x, \partial\Omega), \Omega_{\mu_0} = \{x \in \Omega : d(x) < \mu_0\}$.

As we stated before, there is a standard interior gradient estimates for the mean curvature equation.

Remark 1.2. (See [5].) If $u \in C^3(\Omega)$ is a bounded solution for the equation (1.1) with (1.3), and if $f \in C^1(\overline{\Omega} \times [-M_0, M_0])$ satisfies the conditions (1.4)–(1.5), then for any subdomain $\Omega' \subset\subset \Omega$, we have

$$\sup_{\Omega'} |Du| \leq M_1,$$

where M_1 is a positive constant depending only on $n, M_0, \mathtt{dist}(\Omega', \partial\Omega), L_1$.

From the standard bounded estimates for the prescribed mean curvature equation in Concus–Finn [2] (see also Spruck [21]), we can get the following existence theorem for the Neumann boundary value problem of mean curvature equation.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $n \geq 2$, $\partial\Omega \in C^3$, γ is the inward unit normal to $\partial\Omega$. If $\psi \in C^3(\overline{\Omega})$, is a given function, then the following boundary value problem*

$$\mathtt{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = u \quad in \quad \Omega, \tag{1.7}$$

$$\frac{\partial u}{\partial \gamma} = \psi(x) \quad on \quad \partial\Omega, \tag{1.8}$$

exists a unique solution $u \in C^2(\overline{\Omega})$.

Remark 1.4. In [17], Lieberman also considered the following oblique boundary value problem for prescribed mean curvature equation

$$\mathtt{div}(\frac{Du}{\sqrt{1+|Du|^2}}) = f(x,u) \quad \text{in} \quad \Omega, \tag{1.9}$$

$$v^{q-1}\frac{\partial u}{\partial \gamma} + \psi(x,u) = 0 \quad \text{on} \quad \partial\Omega, \tag{1.10}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $n \geq 2$, γ is the inward unit normal to $\partial\Omega$ and $q \geq 0$, $v = \sqrt{1+|Du|^2}$.

Lieberman ([17], page 360) got the gradient estimates for $q > 1$ or $q = 0$. When $q = 1$, it corresponds to the Neumann boundary value problem which we treat it in Theorem 1.1. Recently in [26], the second author used the maximum principle to obtain a new proof of gradient estimates for the problem (1.9), (1.10) with $q > 1$ or $q = 0$.

The rest of the paper is organized as follows. In section 2, we first give the definitions and some notations. We prove the main Theorem 1.1 in section 3 under the help of one lemma. In section 4, we give the proof of Theorem 1.3.

2. Preliminaries

We denote by Ω a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $\partial\Omega \in C^3$, set

$$d(x) = \mathtt{dist}(x, \partial\Omega),$$

and

$$\Omega_\mu = \{x \in \Omega : d(x) < \mu\}.$$

Then it is well known that there exists a positive constant $\mu_1 > 0$ such that $d(x) \in C^3(\overline{\Omega}_{\mu_1})$. As in Simon–Spruck [20] or Lieberman ([17] page 331), we can take $\gamma = Dd$ in Ω_{μ_1} and note that γ is a $C^2(\overline{\Omega}_{\mu_1})$ vector field. As mentioned in [16] and the book [17], we also have the following formulas

$$\begin{gathered} |D\gamma| + |D^2\gamma| \leq C(n,\Omega) \quad \text{in} \quad \Omega_{\mu_1}, \\ \sum_{1\leq i\leq n} \gamma^i D_j \gamma^i = 0, \sum_{1\leq i\leq n} \gamma^i D_i \gamma^j = 0, |\gamma| = 1 \quad \text{in} \quad \Omega_{\mu_1}. \end{gathered} \tag{2.1}$$

As in [17], we define

$$c^{ij} = \delta_{ij} - \gamma^i\gamma^j \quad \text{in} \quad \Omega_{\mu_1}, \tag{2.2}$$

and for a vector $\zeta \in R^n$, we write ζ' for the vector with i-th component $\sum_{1\le j\le n} c^{ij}\zeta_j$. So

$$|D'u|^2 = \sum_{1\le i,j\le n} c^{ij} u_i u_j. \tag{2.3}$$

Let

$$a^{ij}(Du) = v^2\delta_{ij} - u_i u_j, \quad v = (1+|Du|^2)^{\frac{1}{2}}. \tag{2.4}$$

Then the equations (1.1), (1.2) are equivalent to the following boundary value problem

$$\sum_{i,j=1}^{n} a^{ij} u_{ij} = f(x,u)v^3 \quad \text{in } \Omega, \tag{2.5}$$

$$u_\gamma = \psi(x,u) \quad \text{on } \partial\Omega. \tag{2.6}$$

3. Proof of Theorem 1.1

Now we begin to prove Theorem 1.1, as mentioned in introduction, using the technique developed by Spruck [21], Lieberman [16] and Wang [25]. We shall choose an auxiliary function which contains $|Du|^2$ and other lower order terms. Then we use the maximum principle for this auxiliary function in $\overline{\Omega}_{\mu_0}$, $0 < \mu_0 < \mu_1$. At last, we get our estimates.

Proof of Theorem 1.1.

Setting $w = u - \psi(x,u)d$, we choose the following auxiliary function

$$\Phi(x) = \log|Dw|^2 e^{1+M_0+u} e^{\alpha_0 d}, \quad x \in \overline{\Omega}_{\mu_0},$$

where $\alpha_0 = |\psi|_{C^0(\overline{\Omega}\times[-M_0,M_0])} + C_0 + 1$, C_0 is a positive constant depending only on n, Ω.

In order to simplify the computation, we let

$$\varphi(x) = \log\Phi(x) = \log\log|Dw|^2 + h(u) + g(d). \tag{3.1}$$

In our case, we take

$$h(u) = 1 + M_0 + u, \quad g(d) = \alpha_0 d. \tag{3.2}$$

We assume that $\varphi(x)$ attains its maximum at $x_0 \in \overline{\Omega}_{\mu_0}$, where $0 < \mu_0 < \mu_1$ is a sufficiently small number which we shall decide it later.

Now we divide three cases to complete the proof of Theorem 1.1.

Case I: If $\varphi(x)$ attains its maximum at $x_0 \in \partial\Omega$, then we shall use the Hopf Lemma to get the bound of $|Du|(x_0)$.

Case II: If $\varphi(x)$ attains its maximum at $x_0 \in \partial\Omega_{\mu_0} \bigcap \Omega$, then we shall get the estimates via the standard interior gradient bound [5].

Case III: If $\varphi(x)$ attains its maximum at $x_0 \in \Omega_{\mu_0}$, in this case for the sufficiently small constant $\mu_0 > 0$, then we can use the maximum principle to get the bound of $|Du|(x_0)$.

Now all computations work at the point x_0.

Case 1. If $\varphi(x)$ attains its maximum at $x_0 \in \partial\Omega$, we shall get the bound of $|Du|(x_0)$. We differentiate φ along the normal direction.

$$\frac{\partial\varphi}{\partial\gamma} = \frac{\sum_{1\le i\le n}(|Dw|^2)_i\gamma^i}{|Dw|^2\log|Dw|^2} + h'u_\gamma + g'. \tag{3.3}$$

Since

$$w_i = u_i - \psi_u u_i d - \psi_{x_i} d - \psi\gamma^i, \tag{3.4}$$

$$|Dw|^2 = |D'w|^2 + w_\gamma^2, \tag{3.5}$$

we have

$$w_\gamma = u_\gamma - \psi_u u_\gamma d - \sum_{1\le i\le n}\psi_{x_i}\gamma^i d - \psi = 0 \quad \text{on} \quad \partial\Omega, \tag{3.6}$$

$$(|Dw|^2)_i = (|D'w|^2)_i \quad \text{on} \quad \partial\Omega. \tag{3.7}$$

Applying (2.1), (2.3) and (3.7), it follows that

$$\begin{aligned}\sum_{1\le i\le n}(|Dw|^2)_i\gamma^i &= \sum_{1\le i\le n}(|D'w|^2)_i\gamma^i\\ &= 2\sum_{1\le i,k,l\le n}c^{kl}w_{ki}w_l\gamma^i\\ &= 2\sum_{1\le i,k,l\le n}c^{kl}u_{ki}u_l\gamma^i - 2\sum_{1\le k,l\le n}c^{kl}u_l D_k\psi,\end{aligned} \tag{3.8}$$

where

$$D_k\psi = \psi_{x_k} + \psi_u u_k.$$

Differentiating (2.6) with respect to tangential direction, we have

$$\sum_{1\le k\le n}c^{kl}(u_\gamma)_k = \sum_{1\le k\le n}c^{kl}D_k\psi. \tag{3.9}$$

It follows that

$$\sum_{1\le i,k\le n}c^{kl}u_{ik}\gamma^i = -\sum_{1\le i,k\le n}c^{kl}u_i(\gamma^i)_k + \sum_{1\le k\le n}c^{kl}D_k\psi. \tag{3.10}$$

Inserting (3.10) into (3.8) and combining (2.6), (3.3), we have

$$|Dw|^2 \log |Dw|^2 \frac{\partial \varphi}{\partial \gamma}(x_0) = (g'(0) + h'\psi)|Dw|^2 \log |Dw|^2 - 2 \sum_{1 \le i,k,l \le n} c^{kl} u_i u_l (\gamma^i)_k. \tag{3.11}$$

From (3.4), we obtain

$$|Dw|^2 = |Du|^2 - \psi^2 \quad \text{on} \quad \partial\Omega. \tag{3.12}$$

Assume $|Du|(x_0) \ge \sqrt{100 + 2|\psi|^2_{C^0(\overline{\Omega}\times[-M_0,M_0])}}$, otherwise we get the estimates. At x_0, we have

$$\frac{1}{2}|Du|^2 \le |Dw|^2 \le |Du|^2, \tag{3.13}$$

$$|Dw|^2 \ge 50. \tag{3.14}$$

Inserting (3.13) and (3.14) into (3.11), we have

$$\begin{aligned} |Dw|^2 \log |Dw|^2 \frac{\partial \varphi}{\partial \gamma}(x_0) &\ge (\alpha_0 - |\psi|_{C^0(\overline{\Omega}\times[-M_0,M_0])} - C_0)|Dw|^2 \log |Dw|^2 \\ &= |Dw|^2 \log |Dw|^2 \\ &> 0. \end{aligned} \tag{3.15}$$

On the other hand, from Hopf Lemma, we have

$$\frac{\partial \varphi}{\partial \gamma}(x_0) \le 0,$$

it is a contradiction to (3.15).

Then we have

$$|Du|(x_0) \le \sqrt{100 + 2|\psi|^2_{C^0(\overline{\Omega}\times[-M_0,M_0])}}. \tag{3.16}$$

Case 2. $x_0 \in \partial\Omega_{\mu_0} \bigcap \Omega$. This is due to interior gradient estimates. From Remark 1.2, we have

$$\sup_{\partial\Omega_{\mu_0} \bigcap \Omega} |Du| \le \tilde{M}_1, \tag{3.17}$$

where $\tilde{M}_1$ is a positive constant depending only on n, M_0, μ_0, L_1.

Case 3. $x_0 \in \Omega_{\mu_0}$.

In this case, x_0 is a critical point of φ. We choose the normal coordinate at x_0, by rotating the coordinate system suitably, we may assume that $u_i(x_0) = 0$, $2 \le i \le n$ and

$u_1(x_0) = |Du| > 0$. And we can further assume that the matrix $(u_{ij}(x_0))$ $(2 \le i, j \le n)$ is diagonal. Let

$$\mu_2 \le \frac{1}{100L_2}$$

be such that

$$|\psi_u|\mu_2 \le \frac{1}{100}, \quad \text{then} \quad \frac{99}{100} \le 1 - \psi_u\mu_2 \le \frac{101}{100}. \tag{3.18}$$

We can choose

$$\mu_0 = \frac{1}{2}\min\{\mu_1, \mu_2, 1\}.$$

In order to simplify the calculations, we let

$$w = u - G, \quad G = \psi(x, u)d.$$

Then we have

$$w_k = (1 - G_u)u_k - G_{x_k}. \tag{3.19}$$

Since at x_0,

$$|Dw|^2 = w_1^2 + \sum_{2\le i\le n} w_i^2, \tag{3.20}$$

$$w_i = -G_{x_i} = -\psi_{x_i}d - \psi\gamma^i, \quad i = 2, \dots, n, \tag{3.21}$$

$$w_1 = (1 - G_u)u_1 - G_{x_1} = (1 - G_u)u_1 - \psi_{x_1}d - \psi\gamma^1. \tag{3.22}$$

So from the above relation, at x_0, we can assume

$$u_1 = |Du|(x_0) \ge \sqrt{3000(1 + |\psi|^2_{C^1(\overline{\Omega}\times[-M_0,M_0])})}, \tag{3.23}$$

then

$$\frac{9}{10}u_1^2 \le |Dw|^2 \le \frac{11}{10}u_1^2, \quad \frac{9}{10}u_1^2 \le w_1^2 \le \frac{11}{10}u_1^2, \tag{3.24}$$

and by the choice of μ_0 and (3.18), we have

$$\frac{99}{100} \le 1 - G_u \le \frac{101}{100}. \tag{3.25}$$

From the above choice, we shall prove Theorem 1.1 with three steps. As we mentioned before, all the calculations will be done at the fixed point x_0.

Step 1: We first get the formula (3.53).

Taking the first derivatives of φ,

$$\varphi_i = \frac{(|Dw|^2)_i}{|Dw|^2 \log |Dw|^2} + h' u_i + g' \gamma^i. \tag{3.26}$$

From $\varphi_i(x_0) = 0$, we have

$$(|Dw|^2)_i = -|Dw|^2 \log |Dw|^2 (h' u_i + g' \gamma^i). \tag{3.27}$$

Take the derivatives again for φ_i,

$$\begin{aligned}\varphi_{ij} = {} & \frac{(|Dw|^2)_{ij}}{|Dw|^2 \log |Dw|^2} - (1 + \log |Dw|^2) \frac{(|Dw|^2)_i (|Dw|^2)_j}{(|Dw|^2 \log |Dw|^2)^2} \\ & + h' u_{ij} + h'' u_i u_j + g'' \gamma^i \gamma^j + g' (\gamma^i)_j.\end{aligned} \tag{3.28}$$

Using (3.27), it follows that

$$\begin{aligned}\varphi_{ij} = {} & \frac{(|Dw|^2)_{ij}}{|Dw|^2 \log |Dw|^2} + h' u_{ij} + \left[h'' - (1 + \log |Dw|^2) h'^{\,2}\right] u_i u_j \\ & + \left[g'' - (1 + \log |Dw|^2) g'^{\,2}\right] \gamma^i \gamma^j - (1 + \log |Dw|^2) h' g' (\gamma^i u_j + \gamma^j u_i) + g' (\gamma^i)_j.\end{aligned} \tag{3.29}$$

Then we get

$$0 \geq \sum_{1 \leq i,j \leq n} a^{ij} \varphi_{ij} =: I_1 + I_2, \tag{3.30}$$

where

$$I_1 = \frac{1}{|Dw|^2 \log |Dw|^2} \sum_{1 \leq i,j \leq n} a^{ij} (|Dw|^2)_{ij}, \tag{3.31}$$

and

$$\begin{aligned}I_2 = \sum_{1 \leq i,j \leq n} a^{ij} \Big\{ & h' u_{ij} + \left[h'' - (1 + \log |Dw|^2) h'^{\,2}\right] u_i u_j + \left[g'' - (1 + \log |Dw|^2) g'^{\,2}\right] \gamma^i \gamma^j \\ & - 2(1 + \log |Dw|^2) h' g' \gamma^i u_j + g' (\gamma^i)_j \Big\}.\end{aligned} \tag{3.32}$$

From the choice of the coordinate, we have

$$a^{11} = 1,\ a^{ii} = v^2 = 1 + u_1^2\ (2 \leq i \leq n), a^{ij} = 0\ (i \neq j, 1 \leq i,j \leq n). \tag{3.33}$$

Now we first treat I_2.

From the choice of the coordinate and the equations (2.5), (3.33), we have

$$\begin{aligned} I_2 &= h' f v^3 - h'^2 u_1^2 \log|Dw|^2 + (h'' - h'^2) u_1^2 + \big[g'' - (1 + \log|Dw|^2) g'^2\big] \sum_{1 \le i \le n} a^{ii} (\gamma^i)^2 \\ &\quad - 2(1 + \log|Dw|^2) h' g' \gamma^1 u_1 + g' \sum_{1 \le i \le n} a^{ii} (\gamma^i)_i \\ &= h' f v^3 - (h'^2 + c^{11} g'^2) u_1^2 \log|Dw|^2 + \big[h'' - h'^2 + c^{11}(g'' - g'^2) + g' \sum_{2 \le i \le n} (\gamma^i)_i\big] u_1^2 \\ &\quad - 2h' g' \gamma^1 u_1 \log|Dw|^2 - 2h' g' \gamma^1 u_1 + g'' - g'^2 + g' \sum_{1 \le i \le n} (\gamma^i)_i. \end{aligned} \tag{3.34}$$

Hence from (3.2), we have

$$\begin{aligned} I_2 &= f v^3 - (1 + c^{11} \alpha_0^2) u_1^2 \log|Dw|^2 + \big[\alpha_0 \sum_{2 \le i \le n} (\gamma^i)_i - 1 - c^{11} \alpha_0^2\big] u_1^2 \\ &\quad - 2\alpha_0 \gamma^1 u_1 \log|Dw|^2 - 2\alpha_0 \gamma^1 u_1 - \alpha_0^2 + \alpha_0 \sum_{1 \le i \le n} (\gamma^i)_i \\ &\ge f v^3 - (1 + c^{11} \alpha_0^2) u_1^2 \log|Dw|^2 - C_1 u_1^2, \end{aligned} \tag{3.35}$$

here C_1 is a positive constant depending only on $n, \Omega, M_0, \mu_0, L_2$.

Next, we calculate I_1 and get the formula (3.52).

Taking the first derivatives of $|Dw|^2$, we have

$$(|Dw|^2)_i = 2 \sum_{1 \le k \le n} w_k w_{ki}. \tag{3.36}$$

Taking the derivatives of $|Dw|^2$ once more, we have

$$(|Dw|^2)_{ij} = 2 \sum_{1 \le k \le n} w_k w_{kij} + 2 \sum_{1 \le k \le n} w_{ki} w_{kj}. \tag{3.37}$$

By (3.31) and (3.37), we can rewrite I_1 as

$$\begin{aligned} I_1 &= \frac{1}{|Dw|^2 \log|Dw|^2} \big[2 \sum_{1 \le i,j,k \le n} w_k a^{ij} w_{ijk} + 2 \sum_{1 \le i,j,k \le n} a^{ij} w_{ki} w_{kj}\big] \\ &=: \frac{1}{|Dw|^2 \log|Dw|^2} \big[I_{11} + I_{12}\big]. \end{aligned} \tag{3.38}$$

In the following, we shall deal with I_{11} and I_{12} respectively.

For the term I_{11}: as we have let

$$w = u - G, \quad G = \psi(x, u) d, \tag{3.39}$$

then we have

$$\begin{aligned} w_k &= (1-G_u)u_k - G_{x_k}, \\ w_{ki} &= (1-G_u)u_{ki} - G_{uu}u_k u_i - G_{ux_i}u_k - G_{ux_k}u_i - G_{x_k x_i}, \end{aligned} \tag{3.40}$$

$$\begin{aligned} w_{kij} &= (1-G_u)u_{kij} - G_{uu}(u_{ki}u_j + u_{kj}u_i + u_{ij}u_k) \\ &\quad - G_{ux_i}u_{kj} - G_{ux_j}u_{ki} - G_{ux_k}u_{ij} \\ &\quad - G_{uuu}u_k u_i u_j - G_{uux_i}u_j u_k - G_{uux_j}u_i u_k - G_{uux_k}u_i u_j \\ &\quad - G_{ux_i x_j}u_k - G_{ux_k x_j}u_i - G_{ux_i x_k}u_j - G_{x_i x_j x_k}. \end{aligned} \tag{3.41}$$

So from the choice of the coordinate and the equations (2.5), (3.33), we have

$$\begin{aligned} \sum_{1\le i,j,k\le n} w_k a^{ij} w_{ijk} &= \sum_{1\le i,j,k\le n} \big[(1-G_u)w_k a^{ij}u_{ijk} - G_{uu}w_k u_k a^{ij}u_{ij} - 2G_{uu}w_k a^{ij}u_{ki}u_j \\ &\quad - w_k G_{ux_k}a^{ij}u_{ij} - 2w_k a^{ij}G_{ux_i}u_{kj} - G_{uuu}w_k u_k a^{ij}u_i u_j \\ &\quad - 2w_k u_k a^{ij}G_{uux_i}u_j - w_k G_{uux_k}a^{ij}u_i u_j - w_k u_k a^{ij}G_{ux_i x_j} \\ &\quad - 2w_k a^{ij}G_{ux_i x_k}u_j - w_k a^{ij}G_{x_i x_j x_k}\big] \\ &= (1-G_u)\sum_{1\le i,j,k\le n} w_k a^{ij}u_{ijk} - 2(G_{uu}u_1 + G_{ux_1})\sum_{1\le k\le n} w_k u_{k1} \\ &\quad - 2v^2 \sum_{2\le i\le n} G_{ux_i}\sum_{1\le k\le n} w_k u_{ki} - G_{uu}fv^3 u_1 w_1 - G_{uuu}u_1^3 w_1 \\ &\quad - v^2 u_1 w_1 \sum_{2\le i\le n} G_{ux_i x_i} - fv^3 \sum_{1\le k\le n} w_k G_{ux_k} - 2G_{uux_1}u_1^2 w_1 \\ &\quad - u_1^2 \sum_{1\le k\le n} w_k G_{uux_k} - v^2 \sum_{1\le k\le n} w_k \sum_{2\le i\le n} G_{x_i x_i x_k} - G_{ux_1 x_1}u_1 w_1 \\ &\quad - 2u_1 \sum_{1\le k\le n} G_{ux_1 x_k}w_k - \sum_{1\le k\le n} G_{x_1 x_1 x_k}w_k. \end{aligned} \tag{3.42}$$

By the equation (2.5), we have

$$u_{11} = fv^3 - v^2 \sum_{2\le i\le n} u_{ii}, \tag{3.43}$$

and

$$\Delta u = fv + \frac{u_1^2}{v^2}u_{11}. \tag{3.44}$$

Differentiating (2.5), we have

$$\sum_{1\le i,j\le n} a^{ij}u_{ijk} = -\sum_{1\le i,j,l\le n} a^{ij}_{p_l}u_{lk}u_{ij} + v^3 D_k f + 3fv^2 v_k. \tag{3.45}$$

From (2.4), we have

$$a^{ij}_{p_l} = 2u_l\delta_{ij} - \delta_{il}u_j - \delta_{jl}u_i. \tag{3.46}$$

By the definition of v, we have

$$vv_k = u_1u_{1k}. \tag{3.47}$$

Since

$$D_k f = f_u u_k + f_{x_k}, \tag{3.48}$$

from (3.46), (3.43) and (3.47), we have

$$\begin{aligned}\sum_{1\le i,j\le n} a^{ij}u_{ijk} &= -2u_1u_{1k}\Delta u + 2u_1\sum_{1\le i\le n} u_{1i}u_{ik} + v^3 D_k f + 3fvu_1u_{1k},\\ &= \frac{2u_1}{v^2}u_{11}u_{1k} + 2u_1\sum_{2\le i\le n} u_{1i}u_{ik} + v^3 D_k f + fvu_1u_{1k}\\ &= \frac{2u_1}{v^2}u_{11}u_{1k} + 2u_1\sum_{2\le i\le n} u_{1i}u_{ik} + f_u v^3 u_k + f_{x_k}v^3 + fvu_1u_{1k}.\end{aligned} \tag{3.49}$$

Inserting (3.49) into (3.42), we rewrite I_{11} as

$$\begin{aligned}I_{11} &= 4(1-G_u)\frac{u_1}{v^2}u_{11}\sum_{1\le k\le n} w_k u_{k1} + 4(1-G_u)u_1\sum_{2\le i\le n} u_{1i}\sum_{1\le k\le n} w_k u_{ki}\\ &\quad + [2(1-G_u)fu_1v - 4G_{uu}u_1 - 4G_{ux_1}]\sum_{1\le k\le n} w_k u_{k1} - 4v^2\sum_{2\le i\le n} G_{ux_i}\sum_{1\le k\le n} w_k u_{ki}\\ &\quad + 2(1-G_u)f_u v^3 u_1 w_1 - 2G_{uu} f v^3 u_1 w_1 + 2(1-G_u)v^3\sum_{1\le k\le n} f_{x_k} w_k\\ &\quad - 2fv^3\sum_{1\le k\le n} G_{ux_k} w_k - 2G_{uuu}u_1^3 w_1 - 2v^2 u_1 w_1\sum_{2\le i\le n} G_{ux_ix_i} - 4G_{uux_1}u_1^2 w_1\\ &\quad - 2u_1^2\sum_{1\le k\le n} w_k G_{uux_k} - 2v^2\sum_{1\le k\le n} w_k\sum_{2\le i\le n} G_{x_ix_ix_k} - 2G_{ux_1x_1}u_1w_1\\ &\quad - 4u_1\sum_{1\le k\le n} G_{ux_1x_k} w_k - 2\sum_{1\le k\le n} G_{x_1x_1x_k} w_k.\end{aligned} \tag{3.50}$$

For the term I_{12}: applying (3.33) and (3.40), we have

$$
\begin{aligned}
I_{12} &= 2\sum_{1\le i,k\le n} a^{ii}w_{ki}^2 \\
&= 2\sum_{1\le k\le n} w_{k1}^2 + 2v^2\sum_{1\le k\le n}\sum_{2\le i\le n} w_{ki}^2 \\
&= 2w_{11}^2 + 2(v^2+1)\sum_{2\le i\le n} w_{1i}^2 + 2v^2\sum_{2\le i,k\le n} w_{ki}^2 \\
&= 2[(1-G_u)u_{11} - (G_{uu}u_1^2 + 2G_{ux_1}u_1 + G_{x_1x_1})]^2 \\
&\quad + 2(v^2+1)\sum_{2\le i\le n}[(1-G_u)u_{1i} - (G_{ux_i}u_1 + G_{x_1x_i})]^2 \\
&\quad + 2v^2\sum_{2\le i,k\le n}[(1-G_u)u_{ki} - G_{x_kx_i}]^2 \\
&= 2(1-G_u)^2u_{11}^2 + 2(1-G_u)^2(v^2+1)\sum_{2\le i\le n} u_{1i}^2 + 2(1-G_u)^2v^2\sum_{2\le i\le n} u_{ii}^2 \\
&\quad - 4(1-G_u)(G_{uu}u_1^2 + 2G_{ux_1}u_1 + G_{x_1x_1})u_{11} \\
&\quad - 4(1-G_u)(v^2+1)\sum_{2\le i\le n}(G_{ux_i}u_1 + G_{x_1x_i})u_{1i} \\
&\quad - 4(1-G_u)v^2\sum_{2\le i\le n} G_{x_ix_i}u_{ii} + 2(G_{uu}u_1^2 + 2G_{ux_1}u_1 + G_{x_1x_1})^2 \\
&\quad + 2(v^2+1)\sum_{2\le i\le n}(G_{ux_i}u_1 + G_{x_1x_i})^2 + 2v^2\sum_{2\le i,k\le n} G_{x_kx_i}^2. \qquad (3.51)
\end{aligned}
$$

Combining (3.50), (3.51), it follows that

$$
\begin{aligned}
I_1 = \frac{1}{|Dw|^2\log|Dw|^2}\Bigg[& 2(1-G_u)^2u_{11}^2 + 2(1-G_u)^2(v^2+1)\sum_{2\le i\le n} u_{1i}^2 \\
& + 2(1-G_u)^2v^2\sum_{2\le i\le n} u_{ii}^2 + 4(1-G_u)\frac{u_1}{v^2}u_{11}\sum_{1\le k\le n} w_ku_{k1} \\
& + 4(1-G_u)u_1\sum_{2\le i\le n} u_{1i}\sum_{1\le k\le n} w_ku_{ki} \\
& + [2(1-G_u)fu_1v - 4G_{uu}u_1 - 4G_{ux_1}]\sum_{1\le k\le n} w_ku_{k1} \\
& - 4v^2\sum_{2\le i\le n} G_{ux_i}\sum_{1\le k\le n} w_ku_{ki} \\
& - 4(1-G_u)(G_{uu}u_1^2 + 2G_{ux_1}u_1 + G_{x_1x_1})u_{11} \\
& - 4(1-G_u)(v^2+1)\sum_{2\le i\le n}(G_{ux_i}u_1 + G_{x_1x_i})u_{1i}
\end{aligned}
$$

$$
\begin{aligned}
&- 4(1-G_u)v^2 \sum_{2\le i\le n} G_{x_ix_i}u_{ii} + 2(1-G_u)f_u v^3 u_1 w_1 - 2G_{uu} f v^3 u_1 w_1 \\
&+ 2(G_{uu}u_1^2 + 2G_{ux_1}u_1 + G_{x_1x_1})^2 + 2(v^2+1) \sum_{2\le i\le n} (G_{ux_i}u_1 + G_{x_1x_i})^2 \\
&+ 2(1-G_u)v^3 \sum_{1\le k\le n} f_{x_k} w_k - 2fv^3 \sum_{1\le k\le n} G_{ux_k} w_k - 2G_{uuu}u_1^3 w_1 \\
&- 2v^2 u_1 w_1 \sum_{2\le i\le n} G_{ux_ix_i} - 4G_{uux_1}u_1^2 w_1 - 2u_1^2 \sum_{1\le k\le n} w_k G_{uux_k} \\
&- 2v^2 \sum_{1\le k\le n} w_k \sum_{2\le i\le n} G_{x_ix_ix_k} - 2G_{ux_1x_1}u_1 w_1 + 2v^2 \sum_{2\le i,k\le n} G_{x_kx_i}^2 \\
&- 4u_1 \sum_{1\le k\le n} G_{ux_1x_k} w_k - 2 \sum_{1\le k\le n} G_{x_1x_1x_k} w_k \Bigg]. \qquad (3.52)
\end{aligned}
$$

Inserting (3.52) and (3.34) into (3.30), we can obtain the following formula

$$
0 \ge \sum_{1\le i,j\le n} a^{ij}\varphi_{ij} =: Q_1 + Q_2 + Q_3, \qquad (3.53)
$$

where Q_1 contains all the quadratic terms of u_{ij}; Q_2 is the term which contains all linear terms of u_{ij}; and the remaining terms are denoted by Q_3. Then we have

$$
\begin{aligned}
Q_1 = &\frac{1}{|Dw|^2 \log |Dw|^2} \Bigg[2(1-G_u)^2 u_{11}^2 + 2(1-G_u)^2(v^2+1) \sum_{2\le i\le n} u_{1i}^2 \\
&+ 4(1-G_u)\frac{u_1}{v^2} u_{11} \sum_{1\le k\le n} w_k u_{k1} + 4(1-G_u)u_1 \sum_{2\le i\le n} u_{1i} \sum_{1\le k\le n} w_k u_{ki} \\
&+ 2(1-G_u)^2 v^2 \sum_{2\le i\le n} u_{ii}^2 \Bigg]. \qquad (3.54)
\end{aligned}
$$

The linear terms of u_{ij} are

$$
\begin{aligned}
Q_2 = &\frac{1}{|Dw|^2 \log |Dw|^2} \Bigg[[2(1-G_u) f u_1 v - 4G_{uu}u_1 - 4G_{ux_1}] \sum_{1\le k\le n} w_k u_{k1} \\
&- 4v^2 \sum_{2\le i\le n} G_{ux_i} \sum_{1\le k\le n} w_k u_{ki} - 4(1-G_u)(G_{uu}u_1^2 + 2G_{ux_1}u_1 + G_{x_1x_1})u_{11} \\
&- 4(1-G_u)(v^2+1) \sum_{2\le i\le n} (G_{ux_i}u_1 + G_{x_1x_i})u_{1i} - 4(1-G_u)v^2 \sum_{2\le i\le n} G_{x_ix_i}u_{ii} \Bigg]; \\
& \qquad (3.55)
\end{aligned}
$$

and the remaining terms are

$$\begin{aligned}Q_3 = I_2 + \frac{1}{|Dw|^2 \log|Dw|^2}\Big[&2(1-G_u)f_u v^3 u_1 w_1 - 2G_{uu} f v^3 u_1 w_1 \\ &+ 2(G_{uu}u_1^2 + 2G_{ux_1}u_1 + G_{x_1x_1})^2 + 2(v^2+1)\sum_{2\le i\le n}(G_{ux_i}u_1 + G_{x_1x_i})^2 \\ &+ 2(1-G_u)v^3 \sum_{1\le k\le n} f_{x_k} w_k - 2fv^3 \sum_{1\le k\le n} G_{ux_k} w_k - 2G_{uuu}u_1^3 w_1 \\ &- 2v^2 u_1 w_1 \sum_{2\le i\le n} G_{ux_ix_i} - 4G_{uux_1}u_1^2 w_1 - 2u_1^2 \sum_{1\le k\le n} w_k G_{uux_k} \\ &- 2v^2 \sum_{1\le k\le n} w_k \sum_{2\le i\le n} G_{x_ix_ix_k} - 2G_{ux_1x_1}u_1 w_1 + 2v^2 \sum_{2\le i,k\le n} G^2_{x_kx_i} \\ &- 4u_1 \sum_{1\le k\le n} G_{ux_1x_k} w_k - 2\sum_{1\le k\le n} G_{x_1x_1x_k} w_k\Big]. \end{aligned} \tag{3.56}$$

From the estimate on I_2 in (3.35), we have

$$Q_3 \ge fv^3 - \frac{2fG_{uu}}{|Dw|^2\log|Dw|^2} v^3 u_1 w_1 - (1 + c^{11}\alpha_0^2)u_1^2 \log|Dw|^2 - C_2 u_1^2, \tag{3.57}$$

in the computation of Q_3, we use the relation $f_u \ge 0$, where C_2 is a positive constant which depends only on $n, \Omega, M_0, \mu_0, L_1, L_2$.

Step 2: In this step we shall treat the terms Q_1, Q_2, using the first order derivative condition

$$\varphi_i(x_0) = 0,$$

and let

$$A = |Dw|^2 \log|Dw|^2. \tag{3.58}$$

By (3.27) and (3.36), we have

$$\begin{aligned}\sum_{1\le k\le n} w_k w_{ki} &= -\frac{h'}{2}|Dw|^2 \log|Dw|^2 u_i - \frac{g'\gamma^i}{2}|Dw|^2\log|Dw|^2 \\ &= -\frac{h'}{2}Au_i - \frac{g'\gamma^i}{2}A, \qquad i = 1,2,\dots,n. \end{aligned} \tag{3.59}$$

Putting (3.40) into (3.59), by the choice of coordinate, we get

$$(1-G_u)\sum_{1\le k\le n} w_k u_{ki} = -\frac{h'}{2}Au_i + G_{uu}\sum_{1\le k\le n} w_k u_k u_i - \frac{g'\gamma^i}{2}A + \sum_{1\le k\le n} w_k u_k G_{ux_i}$$

$$
\begin{aligned}
&+ \sum_{1\le k\le n} w_k G_{ux_k} u_i + \sum_{1\le k\le n} w_k G_{x_k x_i} \\
&= -\frac{h'}{2} A u_i + G_{uu} w_1 u_1 u_i - \frac{g'\gamma^i}{2} A + w_1 u_1 G_{ux_i} \\
&+ \sum_{1\le k\le n} w_k G_{ux_k} u_i + \sum_{1\le k\le n} w_k G_{x_k x_i}, \quad i = 1, 2, \ldots, n.
\end{aligned} \tag{3.60}
$$

By (3.60), we have

$$
\begin{aligned}
(1 - G_u) \sum_{1\le k\le n} w_k u_{k1} = -\frac{h'}{2} A u_1 + G_{uu} u_1^2 w_1 - \frac{g'\gamma^1}{2} A + G_{ux_1} w_1 u_1 \\
+ \sum_{1\le k\le n} w_k G_{ux_k} u_1 + \sum_{1\le k\le n} w_k G_{x_k x_1},
\end{aligned} \tag{3.61}
$$

and

$$
(1 - G_u) \sum_{1\le k\le n} w_k u_{ki} = -\frac{g'\gamma^i}{2} A + G_{ux_i} w_1 u_1 + \sum_{1\le k\le n} w_k G_{x_k x_i}, \quad i = 2, \ldots, n. \tag{3.62}
$$

Through (3.62) and the choice of the coordinate at x_0, we have

$$
\begin{aligned}
& u_{1i} = -\frac{1}{w_1} w_i u_{ii} - \frac{g'\gamma^i}{2(1 - G_u)} \frac{A}{w_1} + \frac{G_{ux_i}}{1 - G_u} u_1 + \frac{1}{(1 - G_u) w_1} \sum_{1\le k\le n} w_k G_{x_k x_i}, \\
& i = 2, 3, \ldots, n.
\end{aligned} \tag{3.63}
$$

Using (3.61) and (3.63), it follows that

$$
\begin{aligned}
u_{11} &= \sum_{2\le i\le n} \frac{w_i^2}{w_1^2} u_{ii} - \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} + \frac{G_{uu}}{1 - G_u} u_1^2 - \frac{g'\gamma^1}{2(1 - G_u)} \frac{A}{w_1} \\
&+ \frac{G_{ux_1}}{1 - G_u} u_1 + \frac{u_1}{(1 - G_u) w_1} \sum_{1\le k\le n} w_k G_{ux_k} + \frac{g'}{2(1 - G_u)} \sum_{2\le i\le n} w_i \gamma^i \frac{A}{w_1^2} \\
&- \frac{u_1}{(1 - G_u) w_1} \sum_{2\le i\le n} G_{ux_i} w_i + \frac{1}{(1 - G_u) w_1} \sum_{1\le k\le n} w_k G_{x_k x_1} \\
&- \frac{1}{(1 - G_u) w_1^2} \sum_{1\le k, i\le n} w_i w_k G_{x_k x_i} \\
&=: \sum_{2\le i\le n} \frac{w_i^2}{w_1^2} u_{ii} - \frac{h'}{2(1 - G_u)} \frac{A u_1}{w_1} + \frac{D}{1 - G_u},
\end{aligned} \tag{3.64}
$$

where we have let

$$D = G_{uu}u_1^2 - \frac{g'\gamma^1}{2}\frac{A}{w_1} + G_{ux_1}u_1 + \frac{u_1}{w_1}\sum_{1\le k\le n} w_k G_{ux_k} + \frac{g'}{2}\sum_{2\le i\le n} w_i\gamma^i\frac{A}{w_1^2}$$
$$- \frac{u_1}{w_1}\sum_{2\le i\le n} G_{ux_i}w_i + \frac{1}{w_1}\sum_{1\le k\le n} w_k G_{x_kx_1} - \frac{1}{w_1^2}\sum_{1\le k,i\le n} w_iw_kG_{x_kx_i}. \tag{3.65}$$

It follows that

$$|D| \le C_3u_1^2, \tag{3.66}$$

where C_3 is a positive constant which depends only on $n, \Omega, M_0, \mu_0, L_1, L_2$.

By (3.43) and (3.64), we have

$$\sum_{2\le i\le n}(v^2 + \frac{w_i^2}{w_1^2})u_{ii} = fv^3 + \frac{h'}{2(1-G_u)}\frac{Au_1}{w_1} - \frac{D}{1-G_u}. \tag{3.67}$$

Now we use the formulas (3.61)–(3.64) to treat each term in Q_1, Q_2. At first, we treat the first four terms of Q_1 in (3.54), and get (3.68)–(3.71).

By (3.64), we have

$$\begin{aligned}2(1-G_u)^2u_{11}^2 &= 2(1-G_u)^2\Big[\sum_{2\le i\le n}\frac{w_i^2}{w_1^2}u_{ii} - \frac{h'}{2(1-G_u)}\frac{Au_1}{w_1} + \frac{D}{1-G_u}\Big]^2\\
&= 2(1-G_u)^2(\sum_{2\le i\le n}\frac{w_i^2}{w_1^2}u_{ii})^2 - 2(1-G_u)(h'\frac{Au_1}{w_1^3} - \frac{2D}{w_1^2})\sum_{2\le i\le n} w_i^2u_{ii}\\
&\quad + \frac{h'^2}{2}\frac{A^2u_1^2}{w_1^2} - 2h'\frac{Au_1}{w_1}D + 2D^2\\
&= 2(1-G_u)^2(\sum_{2\le i\le n}\frac{w_i^2}{w_1^2}u_{ii})^2 - 2(1-G_u)(h'\frac{Au_1}{w_1^3} - \frac{2D}{w_1^2})\sum_{2\le i\le n} w_i^2u_{ii}\\
&\quad + \frac{h'^2}{2}\frac{A^2u_1^2}{w_1^2} + AO(u_1^2).\end{aligned} \tag{3.68}$$

By (3.63), we have

$$\begin{aligned}&2(1-G_u)^2(v^2+1)\sum_{2\le i\le n}u_{1i}^2\\
&= 2(v^2+1)\sum_{2\le i\le n}\Big[(1-G_u)\frac{w_i}{w_1}u_{ii} + \frac{g'\gamma^i}{2}\frac{A}{w_1} - G_{ux_i}u_1 - \frac{1}{w_1}\sum_{1\le k\le n}w_kG_{x_kx_i}\Big]^2\\
&= 2(1-G_u)^2(v^2+1)\sum_{2\le i\le n}\frac{w_i^2}{w_1^2}u_{ii}^2 + 2(1-G_u)g'A\frac{v^2+1}{w_1^2}\sum_{2\le i\le n}w_i\gamma^iu_{ii}\end{aligned}$$

$$
\begin{aligned}
&- 4(1-G_u)\frac{v^2+1}{w_1^2}\sum_{2\le i\le n}(G_{ux_i}u_1w_1+\sum_{1\le k\le n}w_kG_{x_kx_i})w_iu_{ii}\\
&+\frac{c^{11}}{2}g'^{\,2}\frac{v^2+1}{w_1^2}A^2-2g'A\frac{v^2+1}{w_1^2}\sum_{2\le i\le n}(G_{ux_i}u_1w_1+\sum_{1\le k\le n}w_kG_{x_kx_i})\gamma^i\\
&+\frac{2(v^2+1)}{w_1^2}\sum_{2\le i\le n}(G_{ux_i}u_1w_1+\sum_{1\le k\le n}w_kG_{x_kx_i})^2\\
=\ &2(1-G_u)^2(v^2+1)\sum_{2\le i\le n}\frac{w_i^2}{w_1^2}u_{ii}^2+2(1-G_u)g'A\frac{v^2+1}{w_1^2}\sum_{2\le i\le n}w_i\gamma^iu_{ii}\\
&-4(1-G_u)\frac{v^2+1}{w_1^2}\sum_{2\le i\le n}(G_{ux_i}u_1w_1+\sum_{1\le k\le n}w_kG_{x_kx_i})w_iu_{ii}\\
&+\frac{c^{11}}{2}g'^{\,2}\frac{v^2+1}{w_1^2}A^2+AO(u_1^2). \qquad (3.69)
\end{aligned}
$$

By (3.64) and (3.61), we have

$$
\begin{aligned}
&4(1-G_u)\frac{u_1}{v^2}u_{11}\sum_{1\le k\le n}w_ku_{k1}\\
=\ &\frac{4u_1}{v^2}\Big[\sum_{2\le i\le n}\frac{w_i^2}{w_1^2}u_{ii}-\frac{h'}{2(1-G_u)}\frac{Au_1}{w_1}+\frac{D}{1-G_u}\Big]\Big[-\frac{h'}{2}Au_1-\frac{g'\gamma^1}{2}A+G_{uu}u_1^2w_1\\
&+G_{ux_1}w_1u_1+\sum_{1\le k\le n}w_kG_{ux_k}u_1+\sum_{1\le k\le n}w_kG_{x_kx_1}\Big]\\
=\ &\Big[-2h'\frac{Au_1^2}{v^2w_1^2}+4G_{uu}\frac{u_1^3}{v^2w_1}-2g'\gamma^1\frac{Au_1}{v^2w_1^2}+4G_{ux_1}\frac{u_1^2}{v^2w_1}+4\sum_{1\le k\le n}w_kG_{ux_k}\frac{u_1^2}{v^2w_1^2}\\
&+4\sum_{1\le k\le n}w_kG_{x_kx_1}\frac{u_1}{v^2w_1^2}\Big]\sum_{2\le i\le n}w_i^2u_{ii}+\frac{h'^{\,2}}{1-G_u}\frac{A^2u_1^3}{v^2w_1}-\frac{2h'}{1-G_u}\frac{Au_1^2}{v^2}D\\
&-\frac{4u_1}{v^2}\Big[\frac{h'}{2(1-G_u)}\frac{Au_1}{w_1}-\frac{D}{1-G_u}\Big]\Big[\frac{g'\gamma^1}{2}A-G_{uu}u_1^2w_1-G_{ux_1}w_1u_1\\
&-\sum_{1\le k\le n}w_kG_{ux_k}u_1-\sum_{1\le k\le n}w_kG_{x_kx_1}\Big]\\
=\ &\Big[-2h'\frac{Au_1^2}{v^2w_1^2}+4G_{uu}\frac{u_1^3}{v^2w_1}-2g'\gamma^1\frac{Au_1}{v^2w_1^2}+4G_{ux_1}\frac{u_1^2}{v^2w_1}+4\sum_{1\le k\le n}w_kG_{ux_k}\frac{u_1^2}{v^2w_1^2}\\
&+4\sum_{1\le k\le n}w_kG_{x_kx_1}\frac{u_1}{v^2w_1^2}\Big]\sum_{2\le i\le n}w_i^2u_{ii}+\frac{h'^{\,2}}{1-G_u}\frac{A^2u_1^3}{v^2w_1}+AO(u_1^2). \qquad (3.70)
\end{aligned}
$$

By (3.63) and (3.62), we have

$$
\begin{aligned}
&4(1-G_u)u_1 \sum_{2\le i\le n} u_{1i} \sum_{1\le k\le n} w_k u_{ki} \\
&= 4u_1 \sum_{2\le i\le n} \Big[-\frac{g'\gamma^i}{2}A + G_{ux_i}w_1u_1 + \sum_{1\le k\le n} w_k G_{x_kx_i}\Big]\Big[-\frac{1}{w_1}w_iu_{ii} - \frac{g'\gamma^i}{2(1-G_u)}\frac{A}{w_1} \\
&\quad + \frac{G_{ux_i}}{1-G_u}u_1 + \frac{1}{(1-G_u)w_1}\sum_{1\le k\le n} w_kG_{x_kx_i}\Big] \\
&= 2g'\frac{Au_1}{w_1}\sum_{2\le i\le n} w_i\gamma^i u_{ii} - 4u_1^2\sum_{2\le i\le n} w_iG_{ux_i}u_{ii} - \frac{4u_1}{w_1}\sum_{1\le k\le n} w_k \sum_{2\le i\le n} G_{x_kx_i}w_iu_{ii} \\
&\quad + \frac{c^{11}g'^2}{1-G_u}\frac{A^2u_1}{w_1} - \sum_{2\le i\le n}\Big[2g'\gamma^iAu_1 - 4G_{ux_i}w_1u_1^2 - 4u_1\sum_{1\le k\le n} w_kG_{x_kx_i}\Big]\Big[\frac{G_{ux_i}}{1-G_u}u_1 \\
&\quad + \frac{1}{(1-G_u)w_1}\sum_{1\le k\le n} w_kG_{x_kx_i}\Big] - \frac{2g'}{1-G_u}\frac{Au_1}{w_1}\sum_{2\le i\le n}\Big[G_{ux_i}u_1w_1 + \sum_{1\le k\le n} w_kG_{x_kx_i}\Big]\gamma^i \\
&= 2g'\frac{Au_1}{w_1}\sum_{2\le i\le n} w_i\gamma^i u_{ii} - 4u_1^2\sum_{2\le i\le n} w_iG_{ux_i}u_{ii} - \frac{4u_1}{w_1}\sum_{1\le k\le n} w_k \sum_{2\le i\le n} G_{x_kx_i}w_iu_{ii} \\
&\quad + \frac{c^{11}g'^2}{1-G_u}\frac{A^2u_1}{w_1} + AO(u_1^2). \qquad (3.71)
\end{aligned}
$$

Now we treat the first four terms of Q_2 in (3.55), and get (3.72)–(3.75). From (3.61), we get

$$
\begin{aligned}
&\big[2(1-G_u)fu_1v - 4G_{uu}u_1 - 4G_{ux_1}\big]\sum_{1\le k\le n} w_ku_{k1} \\
&= \frac{1}{1-G_u}\big[2(1-G_u)fu_1v - 4G_{uu}u_1 - 4G_{ux_1}\big]\Big[-\frac{h'}{2}Au_1 + G_{uu}u_1^2w_1 \\
&\quad - \frac{g'\gamma^1}{2}A + G_{ux_1}w_1u_1 + \sum_{1\le k\le n} w_kG_{ux_k}u_1 + \sum_{1\le k\le n} w_kG_{x_kx_1}\Big] \\
&= -h'fAvu_1^2 + 2G_{uu}fvu_1^3w_1 - fg'\gamma^1Avu_1 \\
&\quad + 2fvu_1\Big[G_{ux_1}w_1u_1 + \sum_{1\le k\le n} w_kG_{ux_k}u_1 + \sum_{1\le k\le n} w_kG_{x_kx_1}\Big] \\
&\quad - \frac{4}{1-G_u}(G_{uu}u_1 + G_{ux_1})\Big[-\frac{h'}{2}Au_1 + G_{uu}u_1^2w_1 - \frac{g'\gamma^1}{2}A \\
&\quad + G_{ux_1}w_1u_1 + \sum_{1\le k\le n} w_kG_{ux_k}u_1 + \sum_{1\le k\le n} w_kG_{x_kx_1}\Big] \\
&= -h'fAvu_1^2 + 2G_{uu}fvu_1^3w_1 + AO(u_1^2). \qquad (3.72)
\end{aligned}
$$

From (3.62), we have

$$
\begin{aligned}
&-4v^2 \sum_{2\le i\le n} G_{ux_i} \sum_{1\le k\le n} w_k u_{ki}\\
&= -\frac{4v^2}{1-G_u} \sum_{2\le i\le n} G_{ux_i}\big[-\frac{g'\gamma^i}{2}A + G_{ux_i}w_1u_1 + \sum_{1\le k\le n} w_k G_{x_kx_i}\big]\\
&= \frac{2g'}{1-G_u}\sum_{2\le i\le n} G_{ux_i}\gamma^i A v^2 - \frac{4}{1-G_u}\sum_{2\le i\le n} G_{ux_i}^2 v^2 u_1 w_1\\
&\quad - \frac{4}{1-G_u}\sum_{2\le i\le n} G_{ux_i}\sum_{1\le k\le n} w_k G_{x_kx_i} v^2\\
&= AO(u_1^2).
\end{aligned}
\tag{3.73}
$$

From (3.64), we have

$$
\begin{aligned}
&-4(1-G_u)(G_{uu}u_1^2 + 2G_{ux_1}u_1 + G_{x_1x_1})u_{11}\\
&= -4(G_{uu}u_1^2 + 2G_{ux_1}u_1 + G_{x_1x_1})\big[(1-G_u)\sum_{2\le i\le n}\frac{w_i^2}{w_1^2}u_{ii} - \frac{h'}{2}\frac{Au_1}{w_1} + D\big]\\
&= -4(1-G_u)(G_{uu}u_1^2 + 2G_{ux_1}u_1 + G_{x_1x_1})\sum_{2\le i\le n}\frac{w_i^2}{w_1^2}u_{ii}\\
&\quad + 4(G_{uu}u_1^2 + 2G_{ux_1}u_1 + G_{x_1x_1})(\frac{h'}{2}\frac{Au_1}{w_1} - D)\\
&= -4(1-G_u)(G_{uu}u_1^2 + 2G_{ux_1}u_1 + G_{x_1x_1})\sum_{2\le i\le n}\frac{w_i^2}{w_1^2}u_{ii} + AO(u_1^2).
\end{aligned}
\tag{3.74}
$$

From (3.63), we have

$$
\begin{aligned}
&-4(1-G_u)(v^2+1)\sum_{2\le i\le n}(G_{ux_i}u_1 + G_{x_1x_i})u_{1i}\\
&= 4(v^2+1)\sum_{2\le i\le n}(G_{ux_i}u_1 + G_{x_1x_i})\big[\frac{1-G_u}{w_1}w_iu_{ii} + \frac{g'\gamma^i}{2}\frac{A}{w_1}\\
&\quad - G_{ux_i}u_1 - \frac{1}{w_1}\sum_{1\le k\le n} w_k G_{x_kx_i}\big]\\
&= 4(1-G_u)\frac{v^2+1}{w_1}\sum_{2\le i\le n}(G_{ux_i}u_1 + G_{x_1x_i})w_iu_{ii}\\
&\quad + 4(v^2+1)\sum_{2\le i\le n}(G_{ux_i}u_1 + G_{x_1x_i})\big[\frac{g'\gamma^i}{2}\frac{A}{w_1} - G_{ux_i}u_1 - \frac{1}{w_1}\sum_{1\le k\le n} w_k G_{x_kx_i}\big]\\
&= 4(1-G_u)\frac{v^2+1}{w_1}\sum_{2\le i\le n}(G_{ux_i}u_1 + G_{x_1x_i})w_iu_{ii} + AO(u_1^2).
\end{aligned}
\tag{3.75}
$$

We treat the term Q_1 using the relations (3.68)–(3.71) and use the formulas (3.72)–(3.75) to treat the term Q_2. By the formula on Q_3 in (3.56), we can get the following new formula of (3.53),

$$0 \geq \sum_{1\leq i,j\leq n} a^{ij}\varphi_{ij} =: J_1 + J_2, \tag{3.76}$$

where J_1 only contains the terms with $u_{ii}\,(i \geq 2)$, the other terms belong to J_2. We can write

$$J_1 =: \frac{1}{A}\big[J_{11} + J_{12}\big], \tag{3.77}$$

here J_{11} contains the quadratic terms of $u_{ii}\,(i \geq 2)$, and J_{12} is the term including linear terms of $u_{ii}\,(i \geq 2)$. It follows that

$$\begin{aligned}
J_{11} &= 2(1-G_u)^2\{(\sum_{2\leq i\leq n}\frac{w_i^2}{w_1^2}u_{ii})^2 + (v^2+1)\sum_{2\leq i\leq n}\frac{w_i^2}{w_1^2}u_{ii}^2 + v^2\sum_{2\leq i\leq n}u_{ii}^2\}\\
&= 2(1-G_u)^2\{\sum_{2\leq i\leq n}\frac{w_i^4}{w_1^4}u_{ii}^2 + 2\sum_{2\leq i<j\leq n}\frac{w_i^2w_j^2}{w_1^4}u_{ii}u_{jj} + \sum_{2\leq i\leq n}[v^2+(v^2+1)\frac{w_i^2}{w_1^2}]u_{ii}^2\}\\
&= 2(1-G_u)^2\{\sum_{2\leq i\leq n}(v^2+\frac{w_i^2}{w_1^2})(1+\frac{w_i^2}{w_1^2})u_{ii}^2 + 2\sum_{2\leq i<j\leq n}\frac{w_i^2w_j^2}{w_1^4}u_{ii}u_{jj}\}\\
&= 2(1-G_u)^2\{\sum_{2\leq i\leq n}d_ie_iu_{ii}^2 + 2\sum_{2\leq i<j\leq n}\frac{w_i^2w_j^2}{w_1^4}u_{ii}u_{jj}\},
\end{aligned} \tag{3.78}$$

where

$$d_i = v^2 + \frac{w_i^2}{w_1^2}, \quad i = 2,3,\ldots,n, \tag{3.79}$$

$$e_i = 1 + \frac{w_i^2}{w_1^2}, \quad i = 2,3,\ldots,n. \tag{3.80}$$

And

$$\begin{aligned}
J_{12} \;=\; & -2(1-G_u)(h'\frac{Au_1}{w_1^3} - \frac{2D}{w_1^2})\sum_{2\leq i\leq n}w_i^2u_{ii} + 2(1-G_u)g'A\frac{v^2+1}{w_1^2}\sum_{2\leq i\leq n}w_i\gamma^iu_{ii}\\
& - 4(1-G_u)\frac{v^2+1}{w_1^2}\sum_{2\leq i\leq n}(\sum_{1\leq k\leq n}w_kG_{x_kx_i} - G_{x_1x_i}w_1)w_iu_{ii}\\
& - \big[2h'\frac{Au_1^2}{v^2w_1^2} - 4G_{uu}\frac{u_1^3}{v^2w_1} + 2g'\gamma^1\frac{Au_1}{v^2w_1^2} - 4G_{ux_1}\frac{u_1^2}{v^2w_1} - 4\sum_{1\leq k\leq n}w_kG_{ux_k}\frac{u_1^2}{v^2w_1^2}
\end{aligned}$$

$$
\begin{aligned}
&- 4 \sum_{1\le k\le n} w_k G_{x_k x_1} \frac{u_1}{v^2 w_1^2}\Big] \sum_{2\le i\le n} w_i^2 u_{ii} + 2g' \frac{A u_1}{w_1} \sum_{2\le i\le n} w_i \gamma^i u_{ii} \\
&- 4u_1^2 \sum_{2\le i\le n} w_i G_{u x_i} u_{ii} - \frac{4u_1}{w_1} \sum_{1\le k\le n} w_k \sum_{2\le i\le n} G_{x_k x_i} w_i u_{ii} \\
&- 4(1-G_u)v^2 \sum_{2\le i\le n} G_{x_i x_i} u_{ii} \\
&- 4(1-G_u)(G_{uu}u_1^2 + 2G_{u x_1}u_1 + G_{x_1 x_1}) \sum_{2\le i\le n} \frac{w_i^2}{w_1^2} u_{ii} \\
=:& \sum_{2\le i\le n} K_i u_{ii}, \qquad (3.81)
\end{aligned}
$$

where

$$
\begin{aligned}
K_i =& -2(1-G_u)\Big(h' \frac{A u_1}{w_1^3} - \frac{2D}{w_1^2}\Big) w_i^2 + 2(1-G_u) g' A \frac{v^2+1}{w_1^2} w_i \gamma^i \\
&- 4(1-G_u)\frac{v^2+1}{w_1^2}\Big(G_{u x_i} u_1 w_1 + \sum_{1\le k\le n} w_k G_{x_k x_i}\Big) w_i \\
&- \Big[2h' \frac{A u_1^2}{v^2 w_1^2} - 4G_{uu}\frac{u_1^3}{v^2 w_1} + 2g'\gamma^1 \frac{A u_1}{v^2 w_1^2} - 4G_{u x_1}\frac{u_1^2}{v^2 w_1} - 4\sum_{1\le k\le n} w_k G_{u x_k} \frac{u_1^2}{v^2 w_1^2} \\
&- 4 \sum_{1\le k\le n} w_k G_{x_k x_1} \frac{u_1}{v^2 w_1^2}\Big] w_i^2 + 2g' \frac{A u_1}{w_1} w_i \gamma^i - 4u_1^2 w_i G_{u x_i} - \frac{4u_1}{w_1} \sum_{1\le k\le n} w_k G_{x_k x_i} w_i \\
&- 4(1-G_u)v^2 G_{x_i x_i} - 4(1-G_u)(G_{uu}u_1^2 + 2G_{u x_1}u_1 + 4G_{x_1 x_1})\frac{w_i^2}{w_1^2} \\
&+ 4(1-G_u)\frac{v^2+1}{w_1}(G_{u x_i}u_1 + G_{x_1 x_i}) w_i. \qquad (3.82)
\end{aligned}
$$

It follows that

$$
|K_i| \le C_4 A, \quad i = 2, \dots, n. \qquad (3.83)
$$

We write other terms as J_2, then

$$
\begin{aligned}
J_2 =& Q_3 - h' f v u_1^2 + \frac{2fG_{uu}}{A} v u_1^3 w_1 + \frac{h'^2}{2} \frac{A u_1^2}{w_1^2} \\
&+ \frac{h'^2}{1-G_u} \frac{A u_1^3}{v^2 w_1} + \frac{c^{11}}{2} g'^2 \frac{u_1^2}{w_1^2} A + \frac{c^{11} g'^2}{1-G_u} \frac{A u_1}{w_1} + O(u_1^2). \qquad (3.84)
\end{aligned}
$$

So by the choice of μ_0 and the formula on Q_3 in (3.57), (3.58) and (3.2), we get the following estimate on J_2,

$$
\begin{aligned}
J_2 \geq & -(1+c^{11}\alpha_0^2)u_1^2\log|Dw|^2 + \frac{h'^2}{2}\frac{Au_1^2}{w_1^2} + \frac{h'^2}{1-G_u}\frac{Au_1^3}{v^2 w_1} \\
& + \frac{c^{11}}{2}g'^2\frac{u_1^2}{w_1^2}A + \frac{c^{11}g'^2}{1-G_u}\frac{Au_1}{w_1} - C_5 u_1^2 \\
\geq & \frac{1}{4}(1+c^{11}\alpha_0^2)u_1^2\log|Dw|^2 - C_6 u_1^2,
\end{aligned} \tag{3.85}
$$

where C_4, C_5, C_6 are positive constants which only depend on $n, \Omega, \mu_0, M_0, L_1, L_2$.

Step 3: In this step, we concentrate on J_1. We first treat the terms J_{11} and J_{12} and obtain the formula (3.98), then we complete the proof of Theorem 1.1 through Lemma 3.1.

By (3.67), we have

$$
u_{22} = -\frac{1}{d_2}\sum_{3\leq i\leq n} d_i u_{ii} + \frac{1}{d_2}[fv^3 + \frac{h'}{2(1-G_u)}\frac{Au_1}{w_1} - \frac{D}{1-G_u}]. \tag{3.86}
$$

We first treat the term J_{11}: using (3.86) to simplify (3.78), we get

$$
\begin{aligned}
J_{11} = & 2(1-G_u)^2\Big\{d_2 e_2 u_{22}^2 + \sum_{3\leq i\leq n} d_i e_i u_{ii}^2 + 2\frac{w_2^2}{w_1^2}u_{22}\sum_{3\leq j\leq n}\frac{w_j^2}{w_1^2}u_{jj} \\
& + 2\sum_{3\leq i<j\leq n}\frac{w_i^2 w_j^2}{w_1^4}u_{ii}u_{jj}\Big\} \\
= & 2(1-G_u)^2\Big\{\frac{e_2}{d_2}[-\sum_{3\leq i\leq n} d_i u_{ii} + fv^3 + \frac{h'}{2(1-G_u)}\frac{Au_1}{w_1} - \frac{D}{1-G_u}]^2 + \sum_{3\leq i\leq n} d_i e_i u_{ii}^2 \\
& + \frac{2w_2^2}{d_2 w_1^2}[-\sum_{3\leq i\leq n} d_i u_{ii} + fv^3 + \frac{h'}{2(1-G_u)}\frac{Au_1}{w_1} - \frac{D}{1-G_u}]\sum_{3\leq j\leq n}\frac{w_j^2}{w_1^2}u_{jj} \\
& + 2\sum_{3\leq i<j\leq n}\frac{w_i^2 w_j^2}{w_1^4}u_{ii}u_{jj}\Big\} \\
= & \frac{2(1-G_u)^2}{d_2}\Big\{\sum_{3\leq i\leq n}[e_2 d_i^2 + e_i d_i d_2 - 2\frac{w_2^2 w_i^2}{w_1^4}d_i]u_{ii}^2 \\
& + 2\sum_{3\leq i<j\leq n}[e_2 d_i d_j - \frac{w_2^2 w_i^2}{w_1^4}d_j - \frac{w_2^2 w_j^2}{w_1^4}d_i + \frac{w_i^2 w_j^2}{w_1^4}]u_{ii}u_{jj} \\
& - 2e_2[fv^3 + \frac{h'}{2(1-G_u)}\frac{Au_1}{w_1} - \frac{D}{1-G_u}]\sum_{3\leq i\leq n} d_i u_{ii} \\
& + 2\frac{w_2^2}{w_1^4}[fv^3 + \frac{h'}{2(1-G_u)}\frac{Au_1}{w_1} - \frac{D}{1-G_u}]\sum_{3\leq i\leq n} w_i^2 u_{ii} \\
& + e_2[fv^3 + \frac{h'}{2(1-G_u)}\frac{Au_1}{w_1} - \frac{D}{1-G_u}]^2\Big\}.
\end{aligned} \tag{3.87}
$$

We can rewrite it as the following

$$
\begin{aligned}
J_{11} =: \frac{2(1-G_u)^2}{d_2}\Big\{ &\sum_{3\le i\le n} b_{ii}u_{ii}^2 + 2\sum_{3\le i<j\le n} b_{ij}u_{ii}u_{jj} \\
&- 2e_2\big[fv^3 + \frac{h'}{2(1-G_u)}\frac{Au_1}{w_1} - \frac{D}{1-G_u}\big]\sum_{3\le i\le n} d_iu_{ii} \\
&+ 2\frac{w_2^2}{w_1^4}\big[fv^3 + \frac{h'}{2(1-G_u)}\frac{Au_1}{w_1} - \frac{D}{1-G_u}\big]\sum_{3\le i\le n} w_i^2u_{ii} \\
&+ e_2\big[fv^3 + \frac{h'}{2(1-G_u)}\frac{Au_1}{w_1} - \frac{D}{1-G_u}\big]^2\Big\},
\end{aligned} \tag{3.88}
$$

where

$$
\begin{aligned}
b_{ii} &= e_2d_i^2 + e_id_id_2 - 2\frac{w_2^2w_i^2}{w_1^4}d_i \\
&=: 2u_1^4 + A_{1i}u_1^2 + A_{2i}, \quad i\ge 3 \\
b_{ij} &= e_2d_id_j - \frac{w_2^2w_i^2}{w_1^4}d_j - \frac{w_2^2w_j^2}{w_1^4}d_i + \frac{w_i^2w_j^2}{w_1^4} \\
&=: u_1^4 + G_{ij}u_1^2 + \hat{G}_{ij}, \qquad i\ne j,\ i,j\ge 3,
\end{aligned} \tag{3.89}
$$

and

$$
\begin{aligned}
A_{1i} &= 4 + (w_2^2+w_i^2)\frac{u_1^2}{w_1^2}, \\
A_{2i} &= 2 + (3w_2^2+5w_i^2)\frac{u_1^2}{w_1^2} + w_i^2(w_2^2+w_i^2)\frac{u_1^2}{w_1^4} + \frac{2w_2^2+4w_i^2}{w_1^2} + \frac{2w_i^2(w_2^2+w_i^2)}{w_1^4}, \\
G_{ij} &= 2 + w_2^2\frac{u_1^2}{w_1^2}, \\
\hat{G}_{ij} &= 1 + (2w_2^2+w_i^2+w_j^2)\frac{u_1^2}{w_1^2} + \frac{w_2^2+w_i^2+w_j^2}{w_1^2} + \frac{2w_i^2w_j^2}{w_1^4} - \frac{w_2^2w_i^2w_j^2}{w_1^6}.
\end{aligned} \tag{3.90}
$$

Now we simplify the terms in J_{12}: by (3.86), we can rewrite (3.81) as

$$
\begin{aligned}
J_{12} &= K_2u_{22} + \sum_{3\le i\le n} K_iu_{ii} \\
&= \frac{K_2}{d_2}\big[-\sum_{3\le i\le n} d_iu_{ii} + fv^3 + \frac{h'}{2(1-G_u)}\frac{Au_1}{w_1} - \frac{D}{1-G_u}\big] + \sum_{3\le i\le n} K_iu_{ii} \\
&= \sum_{3\le i\le n}\big[K_i - \frac{K_2d_i}{d_2}\big]u_{ii} + \frac{K_2}{d_2}\big[fv^3 + \frac{h'}{2(1-G_u)}\frac{Au_1}{w_1} - \frac{D}{1-G_u}\big].
\end{aligned} \tag{3.91}
$$

Using (3.87) and (3.91) to treat (3.77), we have

$$J_1 = \frac{2(1-G_u)^2}{Ad_2}\Big[\sum_{3\le i\le n} b_{ii}u_{ii}^2 + 2\sum_{3\le i<j\le n} b_{ij}u_{ii}u_{jj} + \sum_{3\le i\le n}\tilde{K}_i u_{ii}\Big] + R, \tag{3.92}$$

where

$$\begin{aligned}\tilde{K}_i &= -2e_2\Big[fv^3 + \frac{h'}{2(1-G_u)}\frac{Au_1}{w_1} - \frac{D}{1-G_u}\Big]d_i \\ &\quad + \frac{2w_2^2}{w_1^4}\Big[fv^3 + \frac{h'}{2(1-G_u)}\frac{Au_1}{w_1} - \frac{D}{1-G_u}\Big]w_i^2 + K_i d_2 - K_2 d_i.\end{aligned} \tag{3.93}$$

We also have let

$$\begin{aligned}R &= \frac{2e_2(1-G_u)^2}{Ad_2}\Big[fv^3 + \frac{h'}{2(1-G_u)}\frac{Au_1}{w_1} - \frac{D}{1-G_u}\Big]^2 \\ &\quad + \frac{K_2}{Ad_2}\Big[fv^3 + \frac{h'}{2(1-G_u)}\frac{Au_1}{w_1} - \frac{D}{1-G_u}\Big].\end{aligned}$$

For $\tilde{K}_i$ and R, using the formulas on D, K_i in (3.66), (3.83); the formula of A in (3.58); d_i, e_i in (3.79)–(3.80), and $h(u), g(d)$ in (3.2), we have the following estimates

$$|\tilde{K}_i| \le C_7 u_1^5, \quad i = 3,\dots,n; \tag{3.94}$$

$$|R| \le C_8\frac{u_1^2}{\log u_1}. \tag{3.95}$$

Now we use Lemma 3.1, if there is a sufficiently large positive constant C_9 such that

$$|Du|(x_0) \ge C_9, \tag{3.96}$$

then we have

$$\begin{aligned}J_1 &\ge -\frac{C_{10}u_1^6}{Ad_2} \\ &\ge -C_{11}u_1^2,\end{aligned} \tag{3.97}$$

where we use the formulas d_2 in (3.79) and A in (3.58).

Using the estimates on J_1 in (3.97) and J_2 in (3.85), from (3.76) we obtain

$$\begin{aligned}0 &\ge \sum_{1\le i,j\le n} a^{ij}\varphi_{ij} \\ &\ge \frac{1}{4}(1 + c^{11}\alpha_0^2)u_1^2\log|Dw|^2 - C_{12}u_1^2 \\ &\ge \frac{1}{4}u_1^2\log|Dw|^2 - C_{12}u_1^2.\end{aligned} \tag{3.98}$$

There exists a positive constant C_{13} such that

$$|Du|(x_0) \le C_{13}. \tag{3.99}$$

So from Case 1, Case 2, and (3.99), we have

$$|Du|(x_0) \le C_{14}, \qquad x_0 \in \Omega_{\mu_0} \bigcup \partial\Omega.$$

Here the above $C_7, \ldots, C_{14}$ are positive constants which depend only on $n, \Omega, \mu_0, M_0, L_1, L_2$.

Since $\varphi(x) \le \varphi(x_0)$, for $x \in \Omega_{\mu_0}$, there exists M_2 such that

$$|Du|(x) \le M_2, \quad \text{in} \quad \Omega_{\mu_0} \bigcup \partial\Omega, \tag{3.100}$$

where M_2 depends only on $n, \Omega, \mu_0, M_0, L_1, L_2$.

So at last we get the following estimate

$$\sup_{\overline{\Omega}_{\mu_0}} |Du| \le \max\{M_1, M_2\},$$

where the positive constant M_1 depends only on n, μ_0, M_0, L_1; and M_2 depends only on $n, \Omega, \mu_0, M_0, L_1, L_2$.

So we complete the proof of Theorem 1.1. □

Now we prove the main Lemma 3.1 which was used to estimate J_1 defined in (3.97).

Lemma 3.1. *We define (b_{ij}) as in (3.89); d_i, e_i defined as in (3.79)–(3.80); A_{1i}, A_{2i}, $G_{ij}, \hat{G}_{ij}$ defined as in (3.90). We study the following quadratic form*

$$Q(x_3, x_4, \ldots, x_n) = \sum_{3 \le i \le n} b_{ii} x_i^2 + 2 \sum_{3 \le i < j \le n} b_{ij} x_i x_j + \sum_{3 \le i \le n} \tilde{K}_i x_i, \tag{3.101}$$

where $\tilde{K}_i$ defined in (3.93) and we have the estimate (3.94) for $\tilde{K}_i$. Then there exists a sufficiently large positive constant C_{15} which depends only on $n, \Omega, \mu_0, M_0, L_1, L_2$, such that if

$$|Du|(x_0) = u_1(x_0) \ge C_{15}, \tag{3.102}$$

then the following hold.

(I): The matrix (b_{ij}) is positive definite since the matrix $(b^1_{ij}) = (1 + \delta_{ij})$ is positive definite.

(II): We have

$$Q(x_3, x_4, \ldots, x_n) \ge -C_{16} u_1^6, \tag{3.103}$$

where positive constant C_{16} also depends only on $n, \Omega, \mu_0, M_0, L_1, L_2$.

Proof. Let

$$B = (b_{ij}) = B_1 + B_2, B_1 = u_1^4(b_{ij}^1), B_2 = (O(u_1^2)\delta_{ij}).$$

We first prove (I):

$$\begin{aligned}\sigma_k(B) &= \sigma_k(B_1 + B_2)\\ &= \sigma_k(B_1) + \sigma_k(B_1, B_1, \ldots, B_1, B_2)\\ &\quad + \cdots + \sigma_k(B_1, B_2, \ldots, B_2, B_2) + \sigma_k(B_2)\\ &= u_1^{4k}\sigma_k(b_{ij}^1) + O(u_1^{4k-2}),\end{aligned} \tag{3.104}$$

so if u_1 is sufficiently large, then $\sigma_k(B) > 0 \Longleftrightarrow \sigma_k(b_{ij}^1) > 0$.

Now we prove (II): Since $B_1 = u_1^4(b_{ij}^1)_{3\le i,j\le n}$ is positive definite, from the argument in (I), we get

$$B^{-1} = (B_1 + B_2)^{-1} = B_1^{-1}(I + B_1^{-1}B_2)^{-1} = u_1^{-4}(b_{ij}^1)^{-1}\big(1 + o(1)\big). \tag{3.105}$$

Then we have

$$(b_{ij}^1)^{-1} = \begin{pmatrix} 2 & 1 & \cdots & 1\\ 1 & 2 & \cdots & 1\\ \vdots & \vdots & \vdots & \vdots\\ 1 & 1 & \cdots & 2\end{pmatrix}^{-1} = \frac{1}{n-1}\begin{pmatrix} n-2 & -1 & \cdots & -1\\ -1 & n-2 & \cdots & -1\\ \vdots & \vdots & \vdots & \vdots\\ -1 & -1 & \cdots & n-2\end{pmatrix}. \tag{3.106}$$

Now we solve the following linear algebra equation

$$\frac{\partial Q}{\partial x_k} = 0, \quad k = 3, 4, \ldots, n. \tag{3.107}$$

We assume $(\bar{x}_3, \bar{x}_4, \ldots, \bar{x}_n)$ is the extreme point of the quadratic form $Q(x_3, x_4, \ldots, x_n)$. From the definition of $b_{ij}, \tilde{K}_i$ in (3.89), (3.93) and the estimate for $\tilde{K}_i$ in (3.94), using the formulas (3.105) and (3.106), it follows that

$$\begin{pmatrix}\bar{x}_3\\ \bar{x}_4\\ \vdots\\ \bar{x}_n\end{pmatrix} = O(u_1^5)B^{-1}\begin{pmatrix}1\\1\\ \vdots\\ 1\end{pmatrix} = O(u_1)\begin{pmatrix}1\\1\\ \vdots\\ 1\end{pmatrix}. \tag{3.108}$$

It follows that we have the following minimum of the quadratic Q,

$$Q(\bar{x}_3, \bar{x}_4, \ldots, \bar{x}_n) \ge -C_{17}u_1^6. \tag{3.109}$$

In this computation, the bounds in the coefficient on $O(u_1^5), O(u_1)$ depend only on $n, \Omega, M_0, \mu_0, L_1, L_2$. Thus we complete this proof. □

4. The proof of Theorem 1.3

In this section we first prove Theorem 1.3.

In the proof of the existence theorem for the Neumann boundary value problem, we need the apriori estimates. For the C^0 estimates we use the methods introduced by Concus–Finn [2] and Spruck [21]. As in Simon–Spruck [20], we use the continuity method to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. We consider the following family of problems on the mean curvature equations with Neumann boundary value:

$$\mathtt{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = u \quad \text{in} \quad \Omega, \tag{4.1}$$

$$\frac{\partial u}{\partial \gamma} = \tau\psi(x) \quad \text{on} \quad \partial\Omega, \tag{4.2}$$

where $\tau \in [0, 1]$.

For $\tau = 0$, then $u = 0$ is the unique solution. And we need to find the solution for $\tau = 1$. By the standard existence theorem [24,10], as in Simon–Spruck [20], if we can get the a priori estimates for the $C^2(\overline{\Omega})$ solution of the equations (4.1) and (4.2)

$$\sup_{\Omega} |u| \leq \hat{K}_1, \tag{4.3}$$

$$\sup_{\Omega} |Du| \leq \hat{K}_2, \tag{4.4}$$

where $\hat{K}_1, \hat{K}_2$ are independent of τ. Then we can get the existence theorem. From the interior gradient estimates and our boundary gradient estimates, we only need get the C^0 estimates for the solution u in (4.1) and (4.2).

In the paper by Spruck [21], he used the comparison theorem developed by Concus–Finn [2] to get the C^0 estimates for the mean curvature equation with prescribed contact angle boundary value problem. In our case, his proof is still true, so we complete the proof of Theorem 1.3. □

We give a remark to compare our result with the one in the book by Lieberman [17].

Remark 4.1. For the mean curvature equation with the following boundary condition

$$b(x, z, p) = v^{q-1}u_\gamma + \psi(x, z) = 0 \quad \text{on } \partial\Omega. \tag{4.5}$$

In Lieberman book ([17] page 360), he can get the gradient estimates when $q > 1$ or $q = 0$, see also Lieberman [17] page 356, (9.64g), (9.64h).

So for $q = 0$, this is the prescribed contact angle boundary value problem.

For $q = 1$, it is corresponding to Neumann boundary value problem, we have gotten the gradient estimates in Theorem 1.1. If we use the notation from the book [17], then

$$b(x, z, p) = u_\gamma + \psi(x, z) \quad \text{and} \quad b_{p_i} = \gamma^i.$$

So we have

$$b_p \cdot \gamma = 1, \quad \bar{\delta} b(x, z, p) = u_\gamma = -\psi,$$

where we define the operator $\bar{\delta}$ as $\bar{\delta} f(x, z, p) = p \cdot f_p(x, z, p)$.

In order to get the gradient estimates, in Lieberman [17] book, he need the following condition which appears in page 356, the formula (9.64h) i.e.

$$\bar{\delta} b \leq o(b_p \cdot \gamma).$$

In the Neumann boundary value, it doesn't satisfy this condition. □

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