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Constant mean curvature surfaces and mean curvature flow with non-zero Neumann boundary conditions on strictly convex domains

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ABSTRACT

In this paper, we study nonparametric surfaces over strictly convex bounded domains in $\mathbb{R}^n$, which are evolving by the mean curvature flow with Neumann boundary value. We prove that solutions converge to the ones moving only by translation. And we will prove the existence and uniqueness of the constant mean curvature equation with Neumann boundary value on strictly convex bounded domains.

1. Introduction

In this paper, we firstly consider the asymptotic behavior of mean curvature flow

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$$\begin{cases} u_t = (\delta_{ij} - \dfrac{u_i u_j}{1+|Du|^2})u_{ij} & \text{in } \Omega \times (0,\infty), \\ u_\nu = \varphi(x) & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \bar{\Omega}, \end{cases} \tag{1.1}$$

where Ω is a strictly convex bounded domain in $\mathbb{R}^n$ with smooth boundary for $n \geq 2$ and ν is an inward unit normal vector to $\partial\Omega$, $Du = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n})$ denotes the spatial gradient of u; $u_0(x)$ and $\varphi(x)$ are smooth functions satisfying

$$u_{0,\nu} = \varphi(x) \text{ on } \partial\Omega.$$

Then we consider the existence of constant mean curvature surface with prescribed Neumann boundary value on a strictly convex bounded domain Ω in $\mathbb{R}^n$,

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = \lambda & \text{in } \Omega\,, \\ \qquad\qquad u_\nu = \varphi\left(x\right) & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\varphi(x)$ and ν are the same as stated above.

In [2,9,10], Brakke and Huisken studied the parametric surfaces moving by their mean curvature. Their work suggested that it is geometrically more natural to consider the surfaces whose speed in direction of their unit normal is equal to the mean curvature, since in this case the mean curvature flow is the negative gradient flow of the area functional of the hypersurface, and some authors investigated the other nonparametric evolutionary problem (see for example [6]) with Dirichlet boundary value.

In [11], Huisken studied the equation (1.1) with $\varphi(x) = 0$ and proved that the solutions asymptotically converge to constant functions. In his paper, he used integral methods to prove a time-independent gradient bound by Sobolev inequality and iteration method. When $n = 2$, Altschuler and Wu [1] considered the mean curvature flow with the prescribed contact angle boundary value

$$\begin{cases} u_t = (\delta_{ij} - \dfrac{u_i u_j}{1+|Du|^2})u_{ij} & \text{in } \Omega \times (0,\infty), \\ u_\nu = \varphi(x)\sqrt{1+|Du|^2} & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \Omega. \end{cases} \tag{1.3}$$

They proved that if Ω is strictly convex bounded planar domain and $|D\varphi| < \min_{\partial\Omega} k$, where k is the curvature of $\partial\Omega$, the solution of mean curvature flow of (1.3) converges to a surface which moves at a constant speed up to a translation. Up to now it is still open to generalize the Altschuler and Wu's [1] result to high-dimension case.

For $n \geq 2$, Guan [8] studied the more generalized mean curvature type evolution equation with the prescribed contact angle problem as below

$$\begin{cases} u_t = (\delta_{ij} - \dfrac{u_i u_j}{1+|Du|^2})u_{ij} + \phi(x,u,Du) & \text{in } \Omega \times (0,\infty), \\ <\gamma,\nu>=\varphi(x) & \text{on } \partial\Omega \times \ (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{1.4}$$

where γ denotes the downward unit normal to graph $u(\cdot,t)$. He considered the asymptotic behavior of the solutions as $t \to \infty$ for two special cases: (i) $\phi(x,u,Du) = -ku\sqrt{1+|Du|^2}$ for $k > 0$, (ii) $\phi(x,u,Du) = n/u$ for $u > 0$. Guan proved that in each case the solution asymptotically approaches the solution to the corresponding stationary equation. As a consequence of the second case, he obtained some existence results for minimal surfaces in hyperbolic space $\mathbb{H}^{n+1}$ with a prescribed contact angle condition. In the work of [8], $\phi(x,u,Du)$ must satisfy the crucial monotonicity requirement with respect to u.

The existence and the asymptotic behavior of the solution to equation (1.3) or (1.4) are related to the mean curvature type equation with prescribed contact angle condition, which have been well studied in the last 40 years (for example see the book by Finn [4]). Recently, Ma and Xu studied the mean curvature equation with the Neumann boundary value in [16], and Xu obtained the corresponding existence theorem for the mean curvature flow of graphs with the Neumann boundary value in [19]. Their work inspires us to consider what the asymptotic behavior of the solution to equation (1.1) is.

In this paper, we show that up to a translation the solutions to equation (1.1) converge to solutions which move at a constant speed of translation by adopting [1]'s method. And we will also discuss the existence and uniqueness of the constant mean curvature equation with Neumann boundary value on strictly convex bounded domain in Theorem 1.3.

At first we have the following convergence theorem.

Theorem 1.1. *Let Ω be a strictly convex bounded domain in $\mathbb{R}^n$ with smooth boundary, $n \geq 2$. For $\varphi(x) \in C^\infty(\overline{\Omega})$, the unique smooth solution $u(x,t)$ to equation (1.1) converges to $\lambda t + w$, it means that*

$$\lim_{t\to\infty} \|u(x,t) - (\lambda t + w(x))\|_{C^0(\overline{\Omega})} = 0,$$

where (λ, w) is a suitable solution to (1.5).

For completeness, we state the following existence theorem for (1.5).

Theorem 1.2. *Let Ω be a strictly convex bounded domain in $\mathbb{R}^n$ with C^3 boundary, $n \geq 2$. For $\varphi(x) \in C^3(\overline{\Omega})$, there exists a unique $\lambda \in \mathbb{R}$ and $w \in C^{2,\alpha}(\overline{\Omega})$ solving*

$$\begin{cases} (\delta_{ij} - \frac{u_i u_j}{1+|Du|^2})u_{ij} = \lambda & \text{in } \Omega, \\ u_\nu = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where ν is an inward unit normal vector to $\partial\Omega$ and $0 < \alpha < 1$.

Moreover, the solution w is unique up to a constant. Here λ is called the additive eigenvalue.

Remark 1.1. By

$$\operatorname{div}(\frac{Dw}{\sqrt{1+|Dw|^2}}) = \frac{\lambda}{\sqrt{1+|Dw|^2}},$$

we integrate two sides in Ω and obtain

$$\lambda = -\frac{\int_{\partial\Omega} \frac{\varphi(x)}{\sqrt{1+|Dw|^2}} \mathrm{d}\sigma}{\int_\Omega (1+|Dw|^2)^{-1/2} \mathrm{d}x}.$$

Remark 1.2. When $\varphi(x) = 0$, Theorem 1.1 implies that $u(x,t)$ converges to a constant as $t \to \infty$. This is Huisken's result in [11].

The study of the equation (1.5) is connected to the capillary surface without gravity, which is the constant mean curvature equation with prescribed contact angle boundary value condition

$$\begin{cases} \operatorname{div}(\frac{Du}{\sqrt{1+|Du|^2}}) = \lambda & \text{in } \Omega, \\ u_\nu = \cos\theta_0 \sqrt{1+|Du|^2} & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where ν is an inward unit normal vector to $\partial\Omega$. The existence of solution to the equation (1.6) has been studied by many people, one can refer to the related papers [3], [7] and the book [4]. Finn–Giusti [5] gave an example of nonexistence for the equation (1.6) when the domain is non-convex. If Ω is a convex bounded domain in $\mathbb{R}^2$ with smooth boundary, θ_0 is a constant and it satisfies the compatibility condition $\lambda|\Omega| = -\cos\theta_o|\partial\Omega|$, Giusti [7] got an existence theorem when the curvature of the boundary of Ω, denoted by k, satisfies an additional condition $0 < k < \frac{|\partial\Omega|}{|\Omega|}$.

But for the Neumann boundary value condition, we can get the existence and uniqueness of the constant mean curvature equation with Neumann boundary value on strictly convex bounded domain in $\mathbb{R}^n$.

Theorem 1.3. *Let Ω be a strictly convex bounded domain in $\mathbb{R}^n$ with smooth boundary, $n \geq 2$. For any $\varphi \in C^\infty(\overline{\Omega})$, there exists a unique $\lambda \in \mathbb{R}$ and a function $u \in C^\infty(\overline{\Omega})$ solving*

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = \lambda & \text{in } \Omega\,, \\ \qquad\qquad u_\nu = \varphi\,(x) & \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

where ν is an inward unit normal vector to $\partial\Omega$.

Moreover, the solution u is unique up to a constant.

Remark 1.3. By integrating two sides of equation (1.7), we obtain

$$\lambda = -\frac{\int_{\partial\Omega} \frac{\varphi(x)}{\sqrt{1+|Du|^2}} d\sigma}{|\Omega|}.$$

Remark 1.4. In [16], the first author and Xu proved that

$$\begin{cases} \operatorname{div}(\dfrac{Du}{\sqrt{1+|Du|^2}}) = \epsilon u & \text{in } \Omega, \\ \qquad\qquad u_\nu = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

has a unique solution for $\epsilon > 0$. To get the existence, they need to get C^0 estimate firstly and then obtain C^1 estimate. But for the solution to (1.7), we obviously have no C^0 estimate.

Our method to prove Theorem 1.2 and Theorem 1.3 is to give the uniform C^1 estimate (independent of ε) for the solution to quasilinear equation

$$\begin{cases} H(x, Du, D^2u) = \varepsilon u & \text{in } \Omega, \\ \qquad u_\nu = \varphi(x) & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

In order to obtain the uniform C^1 estimate, we need the key condition that Ω is a strictly convex bounded domain in $\mathbb{R}^n$ with smooth boundary. By the maximum principle, we can give the C^0 uniform estimate for εu. From the Schauder theory, we get uniform high order estimates. Let $\varepsilon \to 0$, Theorem 1.2 and Theorem 1.3 are proved. This similar problem is called additive eigenvalue problem which appears in ergodic optimal control or the homogenization of Hamilton–Jacobi equations, and it has been studied by so many mathematicians, such as Lions [15], Ishii [12] etc. They applied the additive eigenvalue problem to study the large time behavior of the Cauchy problem of Hamilton–Jacobi equations. More introduction can be found in [12] and the references therein.

In this paper, in order to simplify the proof of the theorems, we write $O(z)$ as an expression that there exists a constant $C > 0$ such that $|O(z)| \le Cz$.

We also note the following facts when Ω is a strictly convex smooth domain. There exists a smooth defining function h for Ω such that $h < 0$ in Ω and $h = 0$ on $\partial\Omega$, $\{h_{ij}\} \ge k_0\{\delta_{ij}\}$ for a constant $k_0 > 0$ and $\sup_\Omega |Dh| \le 1$, $h_\nu = -1$ and $|Dh| = 1$ on $\partial\Omega$. Because of the strict convexity of the domain, we may assume that the curvature matrix of $\partial\Omega$ satisfies $\{\kappa_{ij}\} \geq k_1\{\delta_{ij}\}_{1\leq i,j\leq n-1}$, where $k_1 > 0$ is the minimum principal curvature of the boundary.

For the arrangement we proceed as below. In Section 2, we firstly give the uniform estimates for $|u_t|$ and $|Du|$ for equation (1.1) in the strictly convex bounded domain and then prove Theorem 1.1 and Theorem 1.2. In Section 3, we give the uniform C^1 estimate for the solution to equation (1.8) and prove Theorem 1.3.

2. The asymptotic behavior of mean curvature flow

We study the asymptotic behavior of the following nonparametric mean curvature flow with Neumann boundary value

$$\begin{cases} u_t = (\delta_{ij} - \dfrac{u_i u_j}{1+|Du|^2})u_{ij} & \text{in } Q_T, \\ u_\nu = \varphi(x) & \text{on } \partial\Omega \times [0,T), \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $\partial\Omega \in C^\infty$, $Q_T = \Omega \times [0,T)$. $\varphi(x), u_0(x) \in C^\infty(\overline{\Omega})$ and $u_{0,\nu} = \varphi(x)$ on $\partial\Omega$.

The existence for short time and uniqueness of the solution to (2.1) follow from the classic theory in [13] and the implicit function theorem. We assume that smooth solutions exist on the time interval $[0,T)$. In the following, we will establish a time independent estimate of $|u_t|^2$, and a time independent a priori bound on the gradient of the solution when Ω is a strictly convex bounded domain in $\mathbb{R}^n$. This will turn the quasilinear evolution equation into a uniformly parabolic equation. The higher order regularity follows from the standard theory and then the infinite time existence of the smooth solution follows.

Firstly we use the maximum principle to establish an a priori bound on $|u_t|^2$,

Lemma 2.1. *If $u(x,t)$ is a smooth solution to (2.1), then $\sup_{Q_T} |u_t|^2 = \sup_{\Omega_0} |u_t|^2$, so there exists a constant $C = C(u_0) > 0$ such that*

$$\sup_{Q_T} |u_t| \leq C.$$

The proof of this lemma is almost the same as in [1], we omit it.

Now we get a time independent a priori bound on the gradient of the solution to (2.1). This is the crucial step in establishing the infinite time existence of solutions. In this step we will make strong use of the strict convexity of the domain.

Lemma 2.2. *Let Ω be a smooth strictly convex bounded domain in $\mathbb{R}^n$ and $n \geq 2$. Suppose that $u(x,t) \in C^{3,2}(Q_T)$ is a solution to (2.1). Then there exists a constant $C_0 = C_0(n, \Omega, u_0, \varphi(x)) > 0$ such that*

$$\sup_{Q_T} |Du| \leq C_0.$$

Along the similar approach in [1], [15] and [17], we get the time independent a priori bound on the gradient of the solution to equation (2.1).

Proof. To reach the conclusion of the lemma, we only need to prove that for $0 < T' < T$ we can bound $|Du|$ on $\overline{Q_{T'}}$ independent of T' and then take a limit argument.

Let

$$\Phi(x) = \log |Dw|^2 + f(h),$$

where

$$w = u + \varphi(x)h, \; f = \alpha h,$$

and α is a positive constant which will be determined later. For convenience we denote by $G = -\varphi(x)h$.

We firstly show the maximum of $\Phi(x)$ on $\overline{\Omega} \times [0, T']$ can not be achieved at the boundary $\partial\Omega \times [0, T']$.

Let n denote the unit inner normal vector and $1 \le i \le n-1$ denote the tangential derivative. D denotes the derivative in $\mathbb{R}^n$, $\bigtriangledown$ denotes the derivative on the boundary. We also denote $\bigtriangledown_i(u_n) := u_{ni}$ for $1 \le i \le n-1$. By the boundary condition, $w_n = 0$ on $\partial\Omega$, which means that $Dw|_{\partial\Omega}$ is a tangent vector along $\partial\Omega$. If $\Phi(x)$ attains its maximum at $(x_0, t_0) \in \partial\Omega \times [0, T']$, then at (x_0, t_0), we have

$$\begin{aligned} 0 \ge \Phi_n =& \frac{|Dw|^2_n}{|Dw|^2} - \alpha \\ =& \sum_{k=1}^{n-1} \frac{2w_k D_{kn} w}{|Dw|^2} - \alpha \\ =& \sum_{k=1}^{n-1} \frac{2w_k w_{nk} + 2\sum_{i=1}^{n-1} w_k w_i \kappa_{ik}}{|Dw|^2} - \alpha \\ =& 2\sum_{i,k=1}^{n-1} \frac{w_k w_i \kappa_{ik}}{|Dw|^2} - \alpha \\ \ge& 2k_1 - \alpha. \end{aligned} \tag{2.2}$$

By taking $0 < \alpha < 2k_1$, the maximum of Φ can only be achieved in $\Omega \times [0, T']$.

Now, only the following two cases are left to be discussed.

Case 1: Φ attains its maximum $(x_0, 0) \in \Omega \times \{0\}$, then there exists a constant $C = C(u_0) > 0$ such that

$$\max_{\overline{Q_{T'}}} |v| \le C. \tag{2.3}$$

Case 2: $\Phi(x)$ attains its maximum at $(x_0, t_0) \in \Omega \times (0, T']$. We have

$$\Phi_t(x_0, t_0) = \frac{|Dw|_t^2}{|Dw|^2}, \tag{2.4}$$

$$\Phi_i(x_0, t_0) = \frac{|Dw|_i^2}{|Dw|^2} + \alpha h_i = 0 \tag{2.5}$$

and

$$\begin{aligned} \Phi_{ij}(x_0, t_0) &= \frac{|Dw|_{ij}^2}{|Dw|^2} - \frac{|Dw|_i^2 |Dw|_j^2}{|Dw|^4} + \alpha h_{ij} \\ &= \frac{|Dw|_{ij}^2}{|Dw|^2} + \alpha h_{ij} - \alpha^2 h_i h_j. \end{aligned} \tag{2.6}$$

Let $a_{ij} = \delta_{ij} - \frac{u_i u_j}{1+|Du|^2}$, then it follows at (x_0, t_0),

$$\begin{aligned} 0 &\geq \sum_{1 \leq i,j \leq n} a_{ij} \Phi_{ij} - \Phi_t \\ &= \sum_{1 \leq i,j \leq n} a_{ij} \frac{|Dw|_{ij}^2}{|Dw|^2} - \frac{|Dw|_t^2}{|Dw|^2} + \alpha \sum_{1 \leq i,j \leq n} a_{ij} h_{ij} - \alpha^2 \sum_{1 \leq i,j \leq n} a_{ij} h_i h_j \\ &\triangleq I_1 + I_2, \end{aligned} \tag{2.7}$$

where

$$I_1 = \sum_{1 \leq i,j \leq n} a_{ij} \frac{|Dw|_{ij}^2}{|Dw|^2} - \frac{|Dw|_t^2}{|Dw|^2}$$

and

$$I_2 = \sum_{1 \leq i,j \leq n} (\alpha a_{ij} h_{ij} - \alpha^2 a_{ij} h_i h_j).$$

At (x_0, t_0), we choose coordinates such that $|Du| = u_1$ and $(u_{ij})_{2 \leq i,j \leq n}$ is diagonal. Then

$$a_{11} = \frac{1}{v^2},\ a_{ij} = 0 \text{ for } i \neq j \text{ and } a_{ii} = 1 \text{ for } i \geq 2,$$

where we denote by $v = \sqrt{1 + |Du|^2}$.

We always assume u_1 is big enough such that u_1, w_1, $|Dw|$ and v are equivalent with each other at (x_0, t_0). Otherwise, the Theorem is proved. It is also noticeable that $|w_i| \leq C$ for $i = 2, \cdots, n$. In our proof, C is denoted to be a positive constant which may be changed in different places but has nothing to do with T'.

$$I_2 \geq \alpha\big[(n-1)k_0 + \frac{k_0}{v^2}\big] - \alpha^2 \big(\frac{h_1^2}{v^2} + \sum_{i=2}^{n} h_i^2\big). \tag{2.8}$$

We denote by $J = \sum_{i,j=1}^{n} a_{ij}|Dw|^2_{ij} - |Dw|^2_t$ and compute this term carefully. By differentiating the equation (2.1),

$$\begin{aligned}
J =& \sum_{i,j,k=1}^{n} a_{ij}(2w_k w_{kij} + 2w_{ki}w_{kj}) - 2\sum_{k=1}^{n} w_k w_{tk} \\
=& 2\sum_{k=1}^{n} w_k[\sum_{i,j=1}^{n} a_{ij}w_{ijk} - w_{tk}] + 2\sum_{i,k=1}^{n} a_{ii}w_{ki}^2 \\
=& 2\sum_{k=1}^{n} w_k[\sum_{i,j=1}^{n} a_{ij}(u_{kij} - G_{kij}) - u_{tk}] + 2\sum_{i,k=1}^{n} a_{ii}w_{ki}^2 \\
=& -2\sum_{i,j,k=1}^{n} a_{ij}w_k G_{kij} - 2\sum_{i,j,k=1}^{n} w_k(a_{ij})_k u_{ij} + 2\sum_{i,k=1}^{n} a_{ii}w_{ki}^2 \\
\triangleq& J_1 + J_2 + J_3.
\end{aligned}$$

It is obvious that

$$J_1 \geq -Cv. \tag{2.9}$$

Next we deal with J_2.

$$\begin{aligned}
J_2 =& \sum_{i,j,k=1}^{n} -2w_k(\delta_{ij} - \frac{u_i u_j}{v^2})_k u_{ij} \\
=& 4\sum_{i,j,k=1}^{n} w_k \frac{u_{ik}u_j u_{ij}}{v^2} - 4\sum_{i,j,k,l=1}^{n} w_k \frac{u_i u_j u_{ij} u_l u_{lk}}{v^4} \\
=& 4\sum_{i,k=1}^{n} w_k \frac{u_{ik}u_1 u_{i1}}{v^2} - 4\sum_{k=1}^{n} w_k \frac{u_{11}u_1^3 u_{1k}}{v^4} \\
=& \frac{4w_1 u_1 u_{11}^2}{v^4} + 4\sum_{i=2}^{n} \frac{w_1 u_1 u_{1i}^2}{v^2} + 4\sum_{i=2}^{n} \frac{w_i u_1 u_{1i} u_{ii}}{v^2} + 4\sum_{i=2}^{n} \frac{w_i u_{1i} u_{11} u_1}{v^4} \\
=& J_{21} + J_{22} + J_{23} + J_{24}.
\end{aligned}$$

By using (2.5), for $2 \leq i \leq n$, we have

$$u_{1i} - G_{1i} = \frac{-\alpha h_i |Dw|^2 - 2w_i u_{ii} + 2\sum_{k=2}^{n} w_k G_{ik}}{2w_1} \tag{2.10}$$

and

$$\sum_{i=2}^{n} \frac{2w_i(u_{1i} - G_{1i})}{|Dw|^2} = -\alpha h_1 - \frac{2w_1(u_{11} - G_{11})}{|Dw|^2}. \tag{2.11}$$

By (2.10), for $2 \leq i \leq n$,

$$u_{1i} = O(1) - \frac{1}{2}\alpha h_i v - \frac{w_i u_{ii}}{w_1}. \tag{2.12}$$

By (2.10),

$$\begin{aligned}\sum_{i=2}^{n} \frac{2w_i(u_{1i} - G_{i1})}{|Dw|^2} =& \sum_{i=2}^{n} \frac{2w_i}{|Dw|^2}\Big(\frac{-\alpha h_i |Dw|^2 - 2w_i u_{ii} + 2\sum_{k=2}^{n} w_k G_{ik}}{2w_1}\Big) \\ =& O\big(\frac{|\alpha h_i|}{v}\big) - \sum_{i=2}^{n} \frac{2w_i^2 u_{ii}}{|Dw|^2 w_1} + \sum_{i,k=2}^{n} \frac{2w_i w_k G_{ki}}{w_1 |Dw|^2}.\end{aligned} \tag{2.13}$$

By (2.11) and (2.13), we have

$$(2v + O(1))u_{11} = -\alpha h_1 v^2 + O(v) + \sum_{i=2}^{n} O\big(\frac{1}{v}\big)u_{ii}. \tag{2.14}$$

So,

$$u_{11} = -\frac{1}{2}\alpha h_1 v + \sum_{i=2}^{n} O\big(\frac{1}{v^2}\big)u_{ii} + O(1). \tag{2.15}$$

We deal with J_{21}, J_{22}, J_{23}, J_{24} respectively.
It's obvious that

$$J_{21} + J_{22} \geq 0. \tag{2.16}$$

For the term J_{23},

$$\begin{aligned}J_{23} =& 4\sum_{i=2}^{n} \frac{w_i u_1 u_{ii}}{v^2}\Big(O(1) - \frac{1}{2}\alpha h_i v - \frac{w_i u_{ii}}{w_1}\Big) \\ =& \sum_{i=2}^{n}\Big[O\big(\frac{1}{v^2}\big)u_{ii}^2 + O(|\alpha h_i|)u_{ii}\Big].\end{aligned} \tag{2.17}$$

And

$$\begin{aligned}J_{24} =& 4\sum_{i=2}^{n} \frac{w_i u_1}{v^4}\Big(O(1) - \frac{1}{2}\alpha h_i v - \frac{w_i u_{ii}}{w_1}\Big)\Big(-\frac{1}{2}\alpha h_1 v + \sum_{i=2}^{n} O\big(\frac{1}{v^2}\big)u_{ii} + O(1)\Big) \\ =& O\big(\frac{\alpha^2 |h_i h_1|}{v}\big) + O\big(\frac{1}{v^2}\big) + \sum_{i=2}^{n}\Big[O\big(\frac{|\alpha h_1|}{v^2}\big) + O\big(\frac{|\alpha h_i|}{v^4}\big)\Big]u_{ii} + \sum_{i=2}^{n} O\big(\frac{1}{v^6}\big)u_{ii}^2.\end{aligned} \tag{2.18}$$

By (2.16)–(2.18), it follows that

$$\begin{aligned} J_2 \geq & \sum_{i=2}^{n}[O(\frac{1}{v^6})+O(\frac{1}{v^2})]u_{ii}^2+\sum_{i=2}^{n}\big[O(\frac{|\alpha h_1|}{v^2})+O(\frac{|\alpha h_i|}{v^4})+O(|\alpha h_i|)\big]u_{ii} \\ & +O(\frac{\alpha^2|h_ih_1|}{v})+O(\frac{1}{v^2}). \end{aligned} \tag{2.19}$$

It is obvious that

$$\begin{aligned} J_3 = & \frac{2}{v^2}\sum_{k=1}^{n}(u_{1k}-G_{1k})^2+2\sum_{i=2}^{n}\sum_{k=1}^{n}(u_{ki}-G_{ik})^2 \\ \geq & (1+\frac{1}{v^2})\sum_{k=2}^{n}u_{1k}^2+\sum_{i=2}^{n}u_{ii}^2-C. \end{aligned} \tag{2.20}$$

From (2.9), (2.19) and (2.20), we write all the terms containing u_{ii} in J as below and by the fact that $ax^2+bx \geq -\frac{b^2}{4a}$ for $a>0$ we have

$$\begin{aligned} & \sum_{i=2}^{n}[1+O(\frac{1}{v^6})+O(\frac{1}{v^2})]u_{ii}^2+\sum_{i=2}^{n}\big[O(\frac{|\alpha h_1|}{v^2})+O(\frac{|\alpha h_i|}{v^4})+O(|\alpha h_i|)\big]u_{ii} \\ & \qquad \geq -\sum_{i=2}^{n}O(|\alpha h_i|^2). \end{aligned} \tag{2.21}$$

Combining (2.9) and (2.19)–(2.21), we obtain

$$J \geq -\sum_{i=2}^{n}O(|\alpha h_i|^2)-Cv-C. \tag{2.22}$$

So we have

$$I_1=\frac{J}{|Dw|^2} \geq -\frac{\sum_{i=2}^{n}O(|\alpha h_i|^2)+Cv+C}{|Dw|^2}. \tag{2.23}$$

Finally, by (2.7), (2.8) and (2.23), at the maximum point (x_0,t_0), we obtain in this case

$$\begin{aligned} 0 \geq & \sum_{1\leq i,j\leq n}a_{ij}\Phi_{ij}-\Phi_t \\ \geq & \frac{-\sum_{i=2}^{n}O(|\alpha h_i|^2)-Cv-C}{|Dw|^2}+\alpha\big[(n-1)k_0+\frac{k_0}{v^2}\big]-\alpha^2(\frac{h_1^2}{v^2}+\sum_{i=2}^{n}h_i^2) \\ \geq & -\frac{C}{v}+\alpha\big[(n-1)k_0+\frac{k_0}{v^2}\big]-\alpha^2(\frac{h_1^2}{v^2}+\sum_{i=2}^{n}h_i^2). \end{aligned} \tag{2.24}$$

Taking $0 < \alpha < \min\{(n-1)k_0, 2k_1\}$, we can obtain

$$v(x_0, t_0) \leq C, \tag{2.25}$$

where C is independent of T'.

Combining the two cases above, we get the uniform estimate for $|Du|$ which is independent of T' and then Lemma 2.2 is proved. □

We would like to point out now that the method above can also be used to obtain the uniform (w.r.t. $\varepsilon > 0$) gradient estimates for the elliptic version of the problem.

Lemma 2.3. *Let Ω be a strictly convex bounded domain in $\mathbb{R}^n$ and $\partial\Omega \in C^3$, $n \geq 2$. Suppose that $\varepsilon > 0$ and φ is a function defined on $\overline{\Omega}$. Assume that there exists a positive constant $L > 0$ such that*

$$|\varphi|_{C^3(\overline{\Omega})} \leq L. \tag{2.26}$$

Let $u \in C^2(\overline{\Omega}) \bigcap C^3(\Omega)$ be a solution to the following equation

$$\begin{cases} (\delta_{ij} - \dfrac{u_i u_j}{1+|\nabla u|^2})u_{ij} = \varepsilon u & \text{in } \Omega, \\ \qquad\qquad\qquad u_\nu = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{2.27}$$

then there exists a constant $C_0 = C_0(n, \Omega, L) > 0$ such that

$$\sup_{\overline{\Omega}} |Du| \leq C_0,$$

here C_0 is independent of ε.

For completeness, we sketch the proof.

Proof. Let $\Phi(x) = \log|Dw|^2 + \alpha h$, where $w = u + \varphi(x)h$, and α will be determined later. We denote by $G = -\varphi(x)h$.

If we choose $0 < \alpha < 2k_1$, almost the same procedure as in (2.2) then shows that the maximum of Φ can only be achieved in the interior of Ω.

Then we assume $\Phi(x)$ attains its maximum at $x_0 \in \Omega$, then we have at this point that

$$\Phi_i(x_0) = \frac{|Dw|_i^2}{|Dw|^2} + \alpha h_i = 0 \tag{2.28}$$

and

$$\begin{aligned} 0 \geq \Phi_{ij}(x_0) &= \frac{|Dw|^2_{ij}}{|Dw|^2} - \frac{|Dw|^2_i|Dw|^2_j}{|Dw|^4} + \alpha h_{ij} \\ &= \frac{|Dw|^2_{ij}}{|Dw|^2} + \alpha h_{ij} - \alpha^2 h_i h_j. \end{aligned} \tag{2.29}$$

It follows that

$$0 \geq \sum_{1\leq i,j\leq n} a_{ij}\Phi_{ij} = \sum_{1\leq i,j\leq n} a_{ij}\frac{|Dw|^2_{ij}}{|Dw|^2} + \alpha \sum_{1\leq i,j\leq n} a_{ij}h_{ij} - \alpha^2 \sum_{1\leq i,j\leq n} a_{ij}h_i h_j \triangleq I_1 + I_2, \tag{2.30}$$

where

$$I_1 = \sum_{1\leq i,j\leq n} a_{ij}\frac{|Dw|^2_{ij}}{|Dw|^2}$$

and

$$I_2 = \sum_{1\leq i,j\leq n} \alpha a_{ij}h_{ij} - \alpha^2 a_{ij}h_i h_j.$$

At x_0, we choose a coordinate such that $|Du| = u_1$ and $(u_{ij})_{2\leq i,j\leq n}$ is diagonal. For the term I_2, we have

$$I_2 \geq \alpha\big[(n-1)k_0 + \frac{k_0}{v^2}\big] - \alpha^2(\frac{h_1^2}{v^2} + \sum_{i=2}^n h_i^2).$$

Denote by $J = \sum_{i,j=1}^n a_{ij}|Dw|^2_{ij}$. By differentiating the equation (2.27), we derive

$$\begin{aligned} J =& \sum_{i,j,k=1}^n a_{ij}(2w_k w_{kij} + 2w_{ki}w_{kj}) \\ =&2\sum_{i,j,k=1}^n a_{ij}w_k(u_{kij} - G_{kij}) + 2\sum_{i,k=1}^n a_{ii}w_{ki}^2 \\ =&-2\sum_{i,j,k=1}^n a_{ij}w_k G_{kij} - 2\sum_{i,j,k=1}^n w_k(a_{ij})_k u_{ij} + 2\sum_{i,k=1}^n a_{ii}w_{ki}^2 + 2\sum_{k=1}^n w_k(\varepsilon u_k) \\ \triangleq& J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Without loss of generality, we assume that u_1 is big enough, then

$$J_4 = 2\varepsilon(u_1 - G_1)u_1 \geq 0. \tag{2.31}$$

The other terms J_1, J_2, J_3 can be treated by the same computation as (2.9)–(2.25). Taking $0 < \alpha < \min\{(n-1)k_0, 2k_1\}$, we can obtain

$$v(x_0) \le C, \tag{2.32}$$

where C is independent of ε. So the proof is completed. $\square$

Using the above Lemma 2.3, we will firstly give the proof of Theorem 1.2 for the mean curvature type equation.

Proof of Theorem 1.2. For each fixed $\varepsilon > 0$, we firstly prove the existence of the solution to equation (2.27). Basing on the C^1 estimate in Lemma 2.3, the only task remained is to derive a priori C^0 estimate for the solution to (2.27) which is denoted to be $u_\varepsilon(x)$.

Using the technique in [1], let g be a smooth function on $\overline{\Omega}$ satisfying $D_\nu g < -\sup_{\overline{\Omega}} |\varphi(x)|$. Let ζ be a point where $g - u_\varepsilon$ achieves its minimum. If $\zeta \in \partial\Omega$, then $D_T g(\zeta) = D_T u_\varepsilon(\zeta)$ and $D_\nu g(\zeta) \ge D_\nu u_\varepsilon(\zeta) = \varphi(\zeta)$, where T denotes the tangent vector to $\partial\Omega$. It's contradicted to g's definition. So $\zeta \in \Omega$, then $Dg(\zeta) = Du_\varepsilon(\zeta)$ and $D^2 g(\zeta) \ge D^2 u_\varepsilon(\zeta)$. Therefore, there exists a constant $c = c(g)$ such that

$$c \ge a_{ij}(Dg)g_{ij}(\zeta) \ge a_{ij}(Du_\varepsilon)(u_\varepsilon)_{ij}(\zeta) = \varepsilon u_\varepsilon(\zeta).$$

Therefore, combining with $g(x) - u_\varepsilon(x) \ge g(\zeta) - u_\varepsilon(\zeta)$ for $x \in \Omega$, it implies that

$$\varepsilon u_\varepsilon(x) \le \varepsilon g(x) - \varepsilon g(\zeta) + c.$$

By the similar method, we can derive the lower bound for $\varepsilon u_\varepsilon(x)$. So $\sup_{\overline{\Omega}} |\varepsilon u_\varepsilon| \le C$ and it then follows the existence result according to the standard theory of elliptic partial differential equations.

Secondly, we consider the limit behavior as $\varepsilon \to 0^+$.

Let $w_\varepsilon = u_\varepsilon - \frac{1}{|\Omega|}\int_\Omega u_\varepsilon \mathrm{d}x$. We know w_ε satisfies

$$\begin{cases} (\delta_{ij} - \dfrac{(w_\varepsilon)_i (w_\varepsilon)_j}{1 + |Dw_\varepsilon|^2})(w_\varepsilon)_{ij} = \varepsilon w_\varepsilon + \varepsilon \dfrac{1}{|\Omega|}\displaystyle\int_\Omega u_\varepsilon \mathrm{d}x & \text{in } \Omega, \\ (w_\varepsilon)_\nu = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{2.33}$$

where ν is an inward unit normal vector to $\partial\Omega$.

As

$$\sup_{\overline{\Omega}} |Dw_\varepsilon| = \sup_{\overline{\Omega}} |Du_\varepsilon| \le C,$$

and the fact that w_ε has at least one zero point, we have $|w_\varepsilon| \le C$. Also $\frac{1}{|\Omega|}\int_\Omega (\varepsilon u_\varepsilon)\mathrm{d}x \le C$. By Schauder theory, we know $|w_\varepsilon|_{C^{2,\alpha}(\overline{\Omega})} \le C$ for some $\alpha \in (0,1)$. Taking $\varepsilon \to 0$, we have $w_\varepsilon \to w$ and $\varepsilon w_\varepsilon + \varepsilon \frac{1}{|\Omega|}\int_\Omega u_\varepsilon \mathrm{d}x \to \lambda$, where (λ, w) solves the equation (1.5).

Next we come to prove the uniqueness. Assume there exist two pairs $(\lambda_1,\ u_1)$ and $(\lambda_2,\ u_2)$ solving (1.5). Without loss of generality, we may assume that $\lambda_1 \le \lambda_2$.

Let $w = u_1 - u_2$, it is obvious that w satisfies

$$\begin{cases} \widetilde{a}_{ij}w_{ij} + b_i w_i = \lambda_1 - \lambda_2 \le 0 & \text{in } \Omega, \\ w_\nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.34}$$

where $\widetilde{a}_{ij} = a_{ij}(Du_1)$ and $b_i = (u_2)_{kl}\int_0^1 a_{kl,p_i}(\eta Du_1 + (1-\eta)Du_2)\mathrm{d}\eta$.

By Hopf's lemma, w must be a constant. Consequently, we have $\lambda_1 = \lambda_2$. □

Now we study the asymptotic behavior of the solution to equation (1.1) on the strictly convex bounded domain in $\mathbb{R}^n$. Remark that we have already obtained uniform estimates on $\frac{\partial u}{\partial t}, |Du|$ as long as a smooth solution exists in Lemma 2.1 and Lemma 2.2. Applying the standard theory of quasilinear parabolic differential equations, we get the longtime existence of the solution to (1.1).

Let

$$\widetilde{w}(x,t) = w + \lambda t, \tag{2.35}$$

where (λ, w) is the solution to equation (1.5). It's easy to check that $\widetilde{w}$ solves the parabolic problem

$$\begin{cases} u_t = (\delta_{ij} - \dfrac{u_i u_j}{1+|Du|^2})u_{ij} & \text{in } \Omega\times(0,\infty), \\ u_\nu = \varphi(x) & \text{on } \partial\Omega\times(0,\infty), \\ u(x,0) = w(x) & \text{on } \Omega. \end{cases} \tag{2.36}$$

As in [1] we have

Corollary 2.4. *For a solution $u = u(x,t)$ to equation (1.1), there exists a positive constant C, independent of t, such that*

$$|u(x,t) - \lambda t| \le C.$$

Proof. Let $z(x,t) = u(x,t) - \widetilde{w}(x,t)$ and it satisfies the following equation

$$\begin{cases} z_t = \widetilde{a}_{ij}z_{ij} + b_i z_i & \text{in } \Omega\times(0,\infty), \\ z_\nu = 0 & \text{on } \partial\Omega\times(0,\infty), \\ z(x,0) = u_0(x) - w(x) & \text{on } \Omega, \end{cases} \tag{2.37}$$

where $\widetilde{a}_{ij} = a_{ij}(Du)$ and $b_i = (\widetilde{w})_{kl}\int_0^1 a_{kl,p_i}(\eta Du + (1-\eta)D\widetilde{w})\mathrm{d}\eta$.

By the maximum principle, z achieves its maximum and minimum on $\Omega \times \{0\}$. Therefore,

$$\sup_{\Omega\times(0,\infty)} |u - \lambda t| \le \sup_{\Omega} |w| + \sup_{\Omega} |u_0 - w|. \quad \square$$

Using the technique in [1], the uniform estimates in Lemma 2.1, Lemma 2.2 and Schauder estimates, we get the following result.

Lemma 2.5. *Let u_1 and u_2 be any two solutions to equation (1.1) with initial data $u_{0,1}$ and $u_{0,2}$ respectively. Let $u = u_1 - u_2$, then u converges to a constant function as $t \to \infty$. In particular, the limit of any solution to equation (1.1) is $\widetilde{w}$ up to a constant.*

Proof. As the proof of Corollary 2.4, u satisfies

$$\begin{cases} z_t = \widetilde{a}_{ij} z_{ij} + b_i z_i & \text{in } \Omega \times (0, \infty), \\ z_\nu = 0 & \text{on } \partial\Omega \times (0, \infty), \\ z(x, 0) = u_{0,1}(x) - u_{0,2}(x) & \text{on } \Omega, \end{cases} \tag{2.38}$$

where $\widetilde{a}_{ij} = a_{ij}(Du_1)$ and $b_i = (u_2)_{kl} \int_0^1 a_{kl,p_i}(\eta Du_1 + (1 - \eta)Du_2)d\eta$.

Let $osc(u)(t) = \max_\Omega u(x, t) - \min_\Omega u(x, t)$. By the strong maximum principle and the Hopf lemma, $osc(u)(t)$ is a strictly decreasing function unless u is a constant.

We claim that

$$\lim_{t\to\infty} osc(u)(t) = 0.$$

Otherwise, if $\lim\limits_{t\to\infty} osc(u)(t) = \delta$ for some $\delta > 0$, we will reach a contradiction. In fact, given a sequence $t_n \to +\infty$, we define

$$u_{1,n}(\cdot, t) = u_1(\cdot, t + t_n) - \lambda t_n$$

and

$$u_{2,n}(\cdot, t) = u_2(\cdot, t + t_n) - \lambda t_n.$$

By Corollary 2.4, for $i = 1, 2$, we know $|u_{i,n} - \lambda t| \le C$, remark that the uniform (independent of n) estimates on $\frac{\partial u_{i,n}}{\partial t}$, $|Du_{i,n}|$ have already been obtained in Lemma 2.1 and Lemma 2.2. According to Schauder theory ([14]), $u_{1,n}(\cdot, t)$ and $u_{2,n}(\cdot, t)$ are locally (in time) C^k uniformly bounded with respect to n for any k.

So, there exists a subsequence (still denoted by t_n) such that $u_{1,n}(\cdot, t)$ and $u_{2,n}(\cdot, t)$ converge locally uniformly in any C^k to $u_1^*(\cdot, t)$ and $u_2^*(\cdot, t)$ respectively. That is

$$u_1^*(\cdot, t) = \lim_{n\to\infty} u_{1,n}(\cdot, t), \quad u_2^*(\cdot, t) = \lim_{n\to\infty} u_{2,n}(\cdot, t).$$

Let $u^* = u_1^* - u_2^*$, then we deduce that

$$\begin{aligned} osc(u^*)(t) &= osc(u_1^* - u_2^*) \\ &= \lim_{n\to\infty} osc(u_1(x, t+t_n) - \lambda t_n - u_2(x, t+t_n) + \lambda t_n) \\ &= \lim_{n\to\infty} osc(u_1(x, t+t_n) - u_2(x, t+t_n)) \\ &= \lim_{n\to\infty} osc(u)(t+t_n) = \delta, \end{aligned} \tag{2.39}$$

where the second equality holds because of the uniform convergence of $u_{1,n}(\cdot, t)$ and $u_{2,n}(\cdot, t)$.

But u^* satisfies the uniform parabolic equation

$$\begin{cases} z_t = \widetilde{a}_{ij} z_{ij} + b_i z_i & \text{in } \Omega \times (-\infty, \infty), \\ z_\nu = 0 & \text{on } \partial\Omega \times (-\infty, \infty), \end{cases} \tag{2.40}$$

where $\widetilde{a}_{ij} = a_{ij}(Du_1^*)$ and $b_i = (u_2^*)_{kl} \int_0^1 a_{kl,p_i}(\eta Du_1^* + (1-\eta)Du_2^*) \mathrm{d}\eta$.

By the strong maximum principle and Hopf's lemma, we know u^* is a constant. This makes contradiction to $osc(u^*)(t) \equiv \delta$ and the claim now is proved.

According to the claim, we have $\lim\limits_{t\to\infty} \max_\Omega u = \lim\limits_{t\to\infty} \min_\Omega u = c_0$ for some constant c_0. It then follows that $\lim\limits_{t\to\infty} |u - c_0| = 0$ and we finish the proof of the Lemma. □

Now we use the Lemma 2.1, Lemma 2.2 and Lemma 2.5 to complete the proof of Theorem 1.1.

Proof of Theorem 1.1. From the Lemma 2.1 and Lemma 2.2 and the Schauder estimate, we have uniform estimates in any C^k-norm for the derivatives of u, and locally (in time) uniform bounds for the C^0-norm. So we get longtime existence with uniform bounds on all higher derivatives of u. From Corollary 2.4 and Lemma 2.5, the limit of any solution to equation (1.1) is $\widetilde{w} = w + \lambda t$ up to a constant, where (λ, w) is the solution to equation (1.5) by Theorem 1.2. □

3. The additive eigenvalue problem for mean curvature equation

In this section, we will firstly give the uniform C^1 estimate for equation (3.2) and then prove Theorem 1.3 by following the method of [18]. Note that the existence of the solution to equation (3.2) has been derived by [16] for fixed $\varepsilon > 0$.

Lemma 3.1. *Let Ω be a strictly convex bounded domain in $\mathbb{R}^n$ and $\partial\Omega \in C^3$, $n \geq 2$. ν is an inward unit normal vector to $\partial\Omega$. Suppose that $\varepsilon > 0$ and $\varphi \in C^3(\overline{\Omega})$. Assume that there exists a positive constant L such that*

$$|\varphi|_{C^3(\overline{\Omega})} \leq L. \tag{3.1}$$

Let u be the solution to the following mean curvature type equation with Neumann boundary value

$$\begin{cases} \operatorname{div}(\dfrac{Du}{\sqrt{1+|Du|^2}}) = \varepsilon u & \text{in } \Omega, \\ u_\nu = \varphi(x) & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

then there exists a constant $C = C(n, \Omega, L)$ such that

$$\sup_{\overline{\Omega}} |Du| \leq C.$$

Proof. Denoting by $a_{ij} = (1 + |Du|^2)\delta_{ij} - u_i u_j$, $f = \varepsilon u$ and $v = \sqrt{1 + |Du|^2}$, the equation (3.2) now can be expressed to be

$$\sum_{i,j=1}^{n} a_{ij} u_{ij} = f v^3.$$

Let

$$\Phi = \log |Dw|^2 + \alpha h,$$

where $w = u + \varphi(x)h$ and α is a positive constant (will be chosen small) determined later. For convenience, we denote $G = -\varphi(x)h$ in this section. During the proof of this theorem, A is denoted to be $|Dw|^2$ for simplicity.

We assume $x_0 \in \overline{\Omega}$ is the maximum point of Φ.

We firstly declare that x_0 can not be located on $\partial\Omega$ once $0 < \alpha < 2k_1$. This is almost the same as the procedure in (2.2) and we skip it.

The remained case we set out to consider is $x_0 \in \Omega$. By rotating the coordinate around x_0, we have $u_1 = |Du|$, $u_i = 0\,(i = 2, \cdots, n)$ and $\{u_{ij}\}_{2\leq i,j\leq n}$ is diagonal. Thus at x_0, we have

$$a_{11} = 1, \ a_{ii} = 1 + u_1^2 \text{ for } i = 2, \cdots, n.$$

Remark that in the following all the calculations will be done at this fixed point. Without loss of generality, we may assume that $|Du|$ is large enough, otherwise we have already reached the conclusion of this lemma.

At x_0, we have

$$0 = \Phi_i = \frac{(|Dw|^2)_i}{A} + \alpha h_i \tag{3.3}$$

and

$$0 \geq \sum_{i,j=1}^{n} a_{ij}\Phi_{ij} = \frac{\sum_{i,j=1}^{n} a_{ij}(|Dw|^2)_{ij}}{A} - \alpha^2 \sum_{i,j=1}^{n} a_{ij}h_i h_j + \alpha \sum_{i,j=1}^{n} a_{ij}h_{ij} \\ \triangleq I + II + III. \tag{3.4}$$

From (3.3), we deduce that for $i = 1, 2, \cdots, n$,

$$\sum_{l=1}^{n} w_l u_{li} = \sum_{l=1}^{n} w_l w_{li} + \sum_{l=1}^{n} w_l G_{li} = -\frac{\alpha A}{2} h_i + O(v). \tag{3.5}$$

Also it is remarkable that as v is large enough, $u_1,\ v,\ w_1$ and $|Dw|$ are equivalent with each other.

It follows that for $i > 1$,

$$\begin{aligned} w_1 u_{1i} + w_i u_{ii} &= O(v) - \frac{\alpha A}{2} h_i, \\ u_{1i} &= O(1) - \frac{\alpha A}{2w_1} h_i - \frac{w_i}{w_1} u_{ii}, \end{aligned} \tag{3.6}$$

and for $i = 1$,

$$w_1 u_{11} + \sum_{l=2}^{n} w_l u_{l1} = O(v) - \frac{\alpha A}{2} h_1. \tag{3.7}$$

Combining (3.6) with (3.7), we then have

$$\begin{aligned} u_{11} =& O(1) - \frac{\alpha A}{2w_1} h_1 - \sum_{l=2}^{n} \frac{w_l}{w_1} \left(O(1) - \frac{\alpha A}{2w_1} h_l - \frac{w_l}{w_1} u_{ll} \right) \\ =& O(1) - \frac{\alpha A}{2w_1} h_1 + \sum_{l=2}^{n} \left(\frac{w_l}{w_1} \right)^2 u_{ll}. \end{aligned} \tag{3.8}$$

From the equation $u_{11} + (1 + {u_1}^2) \sum\limits_{l=2}^{n} u_{ll} = f v^3$ and (3.8), we derive

$$\Delta u = f v + \frac{{u_1}^2}{v^2} u_{11} = f v + O(1) - \frac{{u_1}^2 \alpha A h_1}{2v^2 w_1} + \sum_{l=2}^{n} \left(\frac{u_1 w_l}{v w_1} \right)^2 u_{ll}, \tag{3.9}$$

and

$$f v = \frac{u_{11}}{v^2} + \sum_{l=2}^{n} u_{ll} = O(v^{-2}) - \frac{\alpha A h_1}{2v^2 w_1} + \sum_{l=2}^{n} \left[1 + \left(\frac{w_l}{v w_1} \right)^2 \right] u_{ll}. \tag{3.10}$$

In the following, we come to settle (3.4).

It's easy to get

$$II = -\alpha^2 \left({h_1}^2 + (1 + {u_1}^2) \sum_{i=2}^{n} {h_i}^2 \right), \tag{3.11}$$

and

$$III = \sum_{1 \le i,j \le n} \alpha a_{ij} h_{ij} \ge \alpha k_0 \left(n + (n-1) u_1^2 \right). \tag{3.12}$$

We settle the term I in the rest.

Direct calculation shows that

$$\begin{aligned} \sum_{i,j=1}^{n} a_{ij} (|Dw|^2)_{ij} =& 2 \sum_{i,j,l=1}^{n} a_{ij} u_{ijl} w_l - 2 \sum_{i,j,l=1}^{n} a_{ij} G_{ijl} w_l + 2 \sum_{i,j,l=1}^{n} a_{ij} u_{il} u_{jl} \\ & - 4 \sum_{i,j,l=1}^{n} a_{ij} u_{il} G_{jl} + 2 \sum_{i,j,l=1}^{n} a_{ij} G_{il} G_{jl} \\ =& I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{3.13}$$

In the following, we compute these terms one by one.

For the term I_1, by differentiating the equation and the condition $f' = \varepsilon > 0$, we have

$$\begin{aligned} I_1 =& 2 \sum_{l=1}^{n} w_l \left(\left(f v^3 \right)_l - \sum_{i,j=1}^{n} a_{ij,l} u_{ij} \right) \\ =& 2 \sum_{l=1}^{n} \left(f' w_l u_l v^3 + 3 f v \sum_{k=1}^{n} u_k u_{kl} w_l \right) - 4 \Delta u \sum_{k,l=1}^{n} u_k u_{kl} w_l + 4 \sum_{i,j,l=1}^{n} u_i u_{ij} u_{jl} w_l \\ \ge & \left(6 f v - 4 \Delta u \right) u_1 \sum_{l=1}^{n} u_{1l} w_l + 4 u_1 \sum_{j,l=1}^{n} u_{1j} u_{jl} w_l \\ =& I_{11} + I_{12}. \end{aligned} \tag{3.14}$$

From [1], as in the proof of the Theorem 1.2 in section 2, we conclude that $|f| = |\varepsilon u| \le C(n, \Omega)$, thus for the term I_{11}, jointing with (3.5), (3.9) and (3.10), we derive

$$\begin{aligned} I_{11} =& u_1 \left[6 f v - 4 \left(f v + O(1) - \frac{{u_1}^2 \alpha A h_1}{2 v^2 w_1} + \sum_{l=2}^{n} \left(\frac{u_1 w_l}{v w_1} \right)^2 u_{ll} \right) \right] \left(- \frac{\alpha A}{2} h_1 + O(v) \right) \\ =& u_1 \left(2 f v + O(1) + \frac{2 \alpha A {u_1}^2}{v^2 w_1} h_1 - 4 \sum_{l=2}^{n} \left(\frac{u_1 w_l}{v w_1} \right)^2 u_{ll} \right) \left(- \frac{\alpha A}{2} h_1 + O(v) \right) \end{aligned}$$

$$\begin{aligned}
&= - f v u_1 \alpha A h_1 + O(v^3) - \frac{\alpha^2 A^2 {u_1}^3}{v^2 w_1} {h_1}^2 + \sum_{l=2}^n O_l(v) u_{ll} \\
&= \left(O(v^{-2}) + \frac{\alpha A}{2 v^2 w_1} h_1 - \sum_{l=2}^n \left[1 + \left(\frac{w_l}{v w_1} \right)^2 \right] u_{ll} \right) u_1 \alpha A h_1 \\
&\quad + O(v^3) - \frac{\alpha^2 A^2 {u_1}^3}{v^2 w_1} {h_1}^2 + \sum_{l=2}^n O_l(v) u_{ll} \\
&= O(v^3) - \frac{{u_1}^3 \alpha^2 A^2 {h_1}^2}{v^2 w_1} + \sum_{l=2}^n \left(O_l(v) - u_1 \alpha A h_1 \right) u_{ll},
\end{aligned} \tag{3.15}$$

and for the term I_{12}, also by (3.5), (3.6) and (3.8), we obtain

$$\begin{aligned}
I_{12} =& 4 u_1 \sum_{j=1}^n u_{1j} \left(- \frac{\alpha A}{2} h_j + O(v) \right) \\
=& 4 u_1 \left(O(1) - \frac{\alpha A}{2 w_1} h_1 + \sum_{l=2}^n \left(\frac{w_l}{w_1} \right)^2 u_{ll} \right) \left(- \frac{\alpha A}{2} h_1 + O(v) \right) \\
& + 4 u_1 \sum_{l=2}^n \left(O(1) - \frac{\alpha A}{2 w_1} h_l - \frac{w_l}{w_1} u_{ll} \right) \left(- \frac{\alpha A}{2} h_l + O(v) \right) \\
=& O(v^3) + \frac{u_1 \alpha^2 A^2 {h_1}^2}{w_1} + \sum_{l=2}^n \frac{u_1 \alpha^2 A^2 {h_l}^2}{w_1} + \sum_{l=2}^n O_l(v^2) u_{ll}.
\end{aligned} \tag{3.16}$$

Thus by (3.15) and (3.16), we have

$$I_1 \geq O(v^3) + \frac{u_1 \alpha^2 A^2 {h_1}^2}{v^2 w_1} + \sum_{j=2}^n \frac{u_1 \alpha^2 A^2 {h_j}^2}{w_1} + \sum_{l=2}^n \left(O(v^2) - u_1 \alpha A h_1 \right) u_{ll}. \tag{3.17}$$

It is easy to observe that

$$I_2 = O(v^3), \quad I_5 = O(v^2). \tag{3.18}$$

For the term I_4 we get

$$\begin{aligned}
I_4 =& - 4 u_{11} G_{11} - 4(1+v^2) \sum_{l=2}^n u_{1l} G_{1l} - 4 v^2 \sum_{l=2}^n u_{ll} G_{ll} \\
\geq& - \left(2 {u_{11}}^2 + 2 {G_{11}}^2 + \frac{1+v^2}{2} \sum_{l=2}^n {u_{1l}}^2 + 8(1+v^2) \sum_{l=2}^n {G_{1l}}^2 \right) \\
& - \left(\frac{v^2}{2} \sum_{l=2}^n {u_{ll}}^2 + 8 v^2 \sum_{l=2}^n {G_{ll}}^2 \right).
\end{aligned} \tag{3.19}$$

Therefore we have by (3.6)

$$\begin{aligned}\sum_{i=2}^{5} I_i \geq & \frac{3}{2}(1+v^2)\sum_{l=2}^{n} {u_{1l}}^2 + \frac{3v^2}{2}\sum_{l=2}^{n} {u_{ll}}^2 + O(v^3)\\ =&\frac{3}{2}(1+v^2)\sum_{l=2}^{n}\left(O(1) - \frac{\alpha A}{2w_1}h_l - \frac{w_l}{w_1}u_{ll}\right)^2 + \frac{3v^2}{2}\sum_{l=2}^{n} {u_{ll}}^2 + O(v^3)\\ =&O(v^3) + \frac{3(1+v^2)}{8}\sum_{l=2}^{n}\frac{\alpha^2 A^2}{{w_1}^2}{h_l}^2 + \sum_{l=2}^{n}\left(\frac{3v^2}{2} + O(1)\right){u_{ll}}^2 + \sum_{l=2}^{n} O(v^2)u_{ll}.\end{aligned} \tag{3.20}$$

Combining (3.17) and (3.20), we then have

$$\begin{aligned}\sum_{i=1}^{5} I_i \geq & O(v^3) + \frac{u_1\alpha^2 A^2 {h_1}^2}{v^2 w_1} + \sum_{j=2}^{n}\frac{u_1\alpha^2 A^2 {h_j}^2}{w_1} + \frac{3(1+v^2)}{8}\sum_{l=2}^{n}\frac{\alpha^2 A^2}{{w_1}^2}{h_l}^2\\ &+\sum_{l=2}^{n}\left(\frac{3v^2}{2} + O(1)\right){u_{ll}}^2 + \sum_{l=2}^{n}\left(O(v^2) - u_1\alpha A h_1\right)u_{ll}\\ \geq & O(v^3) + \frac{u_1\alpha^2 A^2 {h_1}^2}{v^2 w_1} + \sum_{j=2}^{n}\frac{u_1\alpha^2 A^2 {h_j}^2}{w_1} + \frac{3(1+v^2)}{8}\sum_{l=2}^{n}\frac{\alpha^2 A^2}{{w_1}^2}{h_l}^2\\ &-\sum_{l=2}^{n}\frac{\left[O(v^2) - u_1\alpha A h_1\right]^2}{6v^2 + O(1)},\end{aligned} \tag{3.21}$$

where in the last formula, for each term $2 \leq l \leq n$, we once again use the fact that $at^2 + bt \geq -\frac{b^2}{4a}$ for $a > 0$.

So we have

$$I = \frac{\sum_{i,j=1}^{n} a_{ij}(|Dw|^2)_{ij}}{A} = \frac{\sum\limits_{i=1}^{5} I_i}{A}. \tag{3.22}$$

Since v has been assumed to be large enough, we have

$$\frac{u_1\alpha^2 A {h_1}^2}{v^2 w_1} + \sum_{j=2}^{n}\frac{u_1\alpha^2 A {h_j}^2}{w_1} + \frac{3(1+v^2)}{8}\sum_{l=2}^{n}\frac{\alpha^2 A}{{w_1}^2}{h_l}^2 \geq \alpha^2 v^2 \sum_{l=2}^{n}{h_l}^2, \tag{3.23}$$

and

$$-\sum_{l=2}^{n}\frac{\left[O(v^2) - u_1\alpha A h_1\right]^2}{6v^2 A + AO(1)} \geq -\frac{n-1}{5}\alpha^2 v^2 h_1^2 + O(1). \tag{3.24}$$

By (3.21) and (3.22)–(3.24), it follows that

$$I \geq O(v) + \alpha^2 v^2 \sum_{l=2}^{n} {h_l}^2 - \frac{n-1}{5}\alpha^2 v^2 h_1^2. \tag{3.25}$$

Then by (3.4), (3.11), (3.12) and (3.25), we obtain

$$\begin{aligned} 0 \geq \sum_{i,j=1}^{n} a_{ij}\Phi_{ij} \geq & O(v) + \alpha^2 v^2 \sum_{i=2}^{n} {h_i}^2 - \frac{n-1}{5}\alpha^2 v^2 h_1^2 + \alpha k_0 \left[n + (n-1){u_1}^2\right] \\ & - \alpha^2 \left({h_1}^2 + (1+{u_1}^2)\sum_{i=2}^{n} {h_i}^2\right) \\ \geq & O(v) + \alpha(n-1)k_0 v^2 - \frac{n-1}{5}\alpha^2 v^2 h_1^2. \end{aligned} \tag{3.26}$$

Taking $0 < \alpha < \min\{2k_0, 2k_1\}$, we know $|Du|$ must be bounded at this point. And by an easy argument we then reach

$$\max_{x\in\overline{\Omega}} |Du| \leq C$$

for a universal constant C depending upon the quantities described in the lemma. □

Remark 3.1. Comparing to the proof of Theorem 1.2, the main difference here is that when we deal with the third order derivatives, there exists a bad term which appears in I_{11}.

Proof of Theorem 1.3. Firstly, we show that for any given $\varepsilon > 0$ and $v \in \mathbb{R}$, there exists a unique solution to

$$(*_{\varepsilon,v}) \begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = v + \varepsilon u & \text{in } \Omega\,, \\ u_\nu = \varphi(x) & \text{on } \partial\Omega. \end{cases} \tag{3.27}$$

Actually, for fixed ε, if $v = 0$, as in the proof of the Theorem 1.2 via the technique by [1], we conclude the C^0 estimate of the solution to $(*_{\varepsilon,0})$. Joining with Lemma 3.1, it follows the existence of the solution. The uniqueness is due to the Hopf's lemma. Let

$$u_{\varepsilon,v}(x) = u_{\varepsilon,0}(x) - \frac{v}{\varepsilon},$$

then $u_{\varepsilon,v}(x)$ solves the equation $(*_{\varepsilon,v})$. Uniqueness is apparent.

Obviously, $u_{\varepsilon,v}(x)$ is strictly decreasing with respect to v. In the following we deduce that for any $\varepsilon > 0$, there exists a unique constant v_ε which is uniformly bounded such that $|u_{\varepsilon,v_\varepsilon}|_{C^1(\overline{\Omega})}$ is uniformly bounded.

Let $u_0(x) \in C^\infty(\overline{\Omega})$ be a fixed function with $\frac{\partial u_0}{\partial \nu} = \varphi(x)$.

If we choose

$$M = 1 + \max_{\overline{\Omega}} |u_0| + \max_{\overline{\Omega}} \left| \operatorname{div}\left(\frac{Du_0}{\sqrt{1+|Du_0|^2}} \right) \right| \quad \text{and} \quad u_\varepsilon^+ = u_0 + \frac{M}{\varepsilon},$$

we then derive that

$$\begin{aligned} 0 <& M - \operatorname{div}\left(\frac{Du_0}{\sqrt{1+|Du_0|^2}} \right) + \varepsilon u_0 \\ =& \left[\operatorname{div}\left(\frac{Du_{\varepsilon,0}}{\sqrt{1+|Du_{\varepsilon,0}|^2}} \right) - \varepsilon u_{\varepsilon,0} \right] - \left[\operatorname{div}\left(\frac{Du_\varepsilon^+}{\sqrt{1+|Du_\varepsilon^+|^2}} \right) - \varepsilon u_\varepsilon^+ \right] \\ =& \sum_{\alpha,\beta=1}^{n} D_\alpha \left[m_{\alpha\beta}(x) D_\beta (u_{\varepsilon,0} - u_\varepsilon^+) \right] - \varepsilon (u_{\varepsilon,0} - u_\varepsilon^+) \end{aligned} \tag{3.28}$$

where

$$m_{\alpha\beta}(x) = \int_0^1 \frac{\partial A^\alpha}{\partial p_\beta} \left(sDu_{\varepsilon,0} + (1-s)Du_\varepsilon^+ \right) \mathrm{d}s$$

and

$$A^\alpha(\vec{p}) = \frac{p_\alpha}{\sqrt{1+|\vec{p}|^2}}.$$

Therefore, by the boundary condition

$$\frac{\partial (u_{\varepsilon,0} - u_\varepsilon^+)}{\partial \nu} = 0$$

and strong maximal principle, we can get that u_ε^+ is a supersolution of $(*_{\varepsilon,0})$. Similarly, $u_\varepsilon^- = u_0 - \frac{M}{\varepsilon}$ is a subsolution of $(*_{\varepsilon,0})$. It then yields that $u_{\varepsilon,M} < u_0 < u_{\varepsilon,-M}$.

By strictly decreasing property of $u_{\varepsilon,v}$, for any $\varepsilon \in (0,1)$, there exists a unique $v_\varepsilon \in (-M, M)$ such that $u_{\varepsilon,v_\varepsilon}(0) = u_0(0)$. By Lemma 3.1, we conclude $|Du_{\varepsilon,v_\varepsilon}|$ can be uniformly bounded independent of ε, thus it also yields the uniform C^0 bound for the above fact that $u_{\varepsilon,v_\varepsilon}(0) = u_0(0)$. In a word, we have obtained the uniform bound of $|u_{\varepsilon,v_\varepsilon}|_{C^1(\overline{\Omega})}$ and Schauder theory then ensures the uniform higher order derivative estimates.

Now let $\varepsilon \to 0$, extracting subsequence if necessary, we may deduce that there exists a constant λ and a smooth function $u^\infty(x)$ such that

$$v_\varepsilon + \varepsilon u_{\varepsilon,v_\varepsilon} \to \lambda, \quad u_{\varepsilon,v_\varepsilon} \to u^\infty,$$

and it is obvious that $(\lambda,\, u^\infty)$ satisfies (1.7).

In the following, we come to prove the uniqueness. If another (μ, u^μ) also solves (1.7), then almost the same discussion as in (3.28) yields that

$$\begin{cases} \sum_{\alpha,\beta=1}^{n} D_\alpha \left[g_{\alpha\beta}(x) D_\beta (u^\infty - u^\mu) \right] = \lambda - \mu & \text{in } \Omega\,, \\ \dfrac{\partial (u^\infty - u^\mu)}{\partial \nu} = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $\{g_{\alpha\beta}(x)\}$ is positive definite and

$$g_{\alpha\beta}(x) = \int_0^1 \frac{\partial A^\alpha}{\partial p_\beta} \left(sDu_\infty + (1-s) Du_\mu \right) \mathrm{d}s.$$

Integrating by parts gives that $\lambda = \mu$ and then Hopf's lemma shows that $u^\infty - u^\mu$ must be a constant. Thus we complete the proof of the theorem. $\square$

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