



The Neumann Problem for Hessian Equations

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Abstract: In this paper, we prove the existence of a classical solution to a Neumann boundary value problem for Hessian equations in uniformly convex domain. The method depends upon the establishment of a priori derivative estimates up to second order. So we give an affirmative answer to a conjecture of N. Trudinger in 1987.

1. Introduction

Hessian equation is an important nonlinear elliptic partial differential equation. It appears naturally in classical geometry, conformal geometry, calibrated geometry and Kahler geometry. For the Dirichlet problem on the k -Hessian equation, Caffarelli–Nirenberg–Spruck [1] have studied the following boundary value problem

$$\begin{cases} \sigma_k(D^2u) = f(x) & \text{in } \Omega, \\ u = \varphi(x) & \text{on } \partial\Omega, \end{cases}$$

and they obtained the existence of the classical admissible solution when the smooth domain $\Omega \subset \mathbb{R}^n$ is uniformly $k - 1$ convex, see also the related works by Ivochkina [11], Trudinger [25] and Guan [10].

In this paper, we study the existence of the classical admissible solution of the k -Hessian equation with the following Neumann boundary value problem:

$$\begin{cases} \sigma_k(D^2u) = f(x) & \text{in } \Omega, \\ u_\nu = \varphi(x, u) & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where ν is outer unit normal vector of $\partial\Omega$.

When $k = 1$, this is the well-known Laplace equation with Neumann boundary value condition. For a priori estimates and the existence theorem we refer to the book [9]. For $k = n$, the a priori estimates and existence result were obtained by Lions, Trudinger, and Urbas [19]. For $2 \leq k \leq n - 1$, the problem has been studied by Trudinger [24], and

he solved the problem for the k -Hessian equation when the domain is a ball. At the end of his paper, Trudinger (in [24] page 305) stated that "It is plausible to conjecture that Theorems 1 and 2 extend at least from balls to sufficiently smooth uniformly convex domains but even in the Monge–Ampere case, this is an open question." In this paper we extend Trudinger's Theorem 2 to uniformly convex domains for the k -Hessian equations.

Now we state our main theorem which gives a positive answer to the conjecture of Trudinger in [24].

Theorem 1. *Let Ω be a C^4 bounded uniformly convex domain in \mathbb{R}^n . For any positive function $f \in C^2(\overline{\Omega})$ and any function $\varphi \in C^3(\overline{\Omega})$, there exists a unique k admissible solution $u \in C^{3,\alpha}(\overline{\Omega})$ to the boundary value problem,*

$$\begin{cases} \sigma_k(D^2u) = f(x) & \text{in } \Omega, \\ u_\nu = -u + \varphi(x) & \text{on } \partial\Omega. \end{cases} \quad (2)$$

For the related works on k -Hessian equations, we mention that Chou–Wang [6] got the Pogorelov type interior estimates and the existence of variational solutions. Trudinger–Wang [26] developed a Hessian measure theory for Hessian operator. For more details please see the survey paper by Wang [31].

The Neumann or oblique derivative problems on linear and quasilinear elliptic equations were widely studied for a long time. One can see the recent book written by Lieberman [15]. The capillary problems is a very natural boundary value problem for mean curvature equation, Ural'tseva [27] first got the boundary gradient estimates and the corresponding existence theorem. At the same time, Simon–Spruck [23] and Gerhard [8] also obtained existence theorem on the positive gravity case, the book by Finn [7] gives a description of this theory as was known in 1986. In the study of the reflected shocks in transonic flow, one part of the free boundary is the position of a transonic shock dividing two regions of smooth flow and it reduce to the Neumann boundary value problem for the quasilinear elliptic equation (see Canic–Keyfitz–Lieberman [2]). Lott [20] consider a weighted Gibbons–Hawking–York mass on a Riemannian manifold with boundary and the weight function satisfies Neumann boundary conditions.

In [22], Ma–Xu got the boundary gradient estimates and the corresponding existence theorem for the Neumann boundary value problem on mean curvature equations. Naturally, the Neumann boundary value problem for Hessian type equations also appears in the fully nonlinear Yamabe problem for manifolds with boundary, which is to find a conformal metric such that the k -th elementary symmetric function of eigenvalues of Schouten tensor is constant with the constant mean curvature on the boundary of manifold, see Jin–Li–Li [13] and Chen [4] for reference. In Chen–Chang [3] and Chen [5], they consider natural conformal invariants arising from the Gauss–Bonnet formulas on manifolds with boundary, and study conformal deformation problems associated to them. The boundary condition that they found there in general involves second derivatives, which is highly nonlinear, so our paper would be the first step to solve their non-Dirichlet problems. Related results on the Neumann or oblique derivative problem for some class fully nonlinear elliptic equations can be found in Urbas [28, 29].

We give a brief description of our procedures and ideas to this problem. By the standard theory of Lieberman–Trudinger [16] (see also [15, 18]), it is well known that the solvability of the Hessian equations with Neumann boundary value can be reduced to the a priori global second order derivative estimates. We have done C^1 estimate (jointed with Xu) in [21] a year ago, there we constructed a suitable auxiliary function and used a particular coordinate to let the estimate computable.

For C^2 estimate, we first reduce the global estimate to the boundary double normal derivative estimates in Lemma 13. This estimate also plays an important role in our boundary double normal estimate. This is the only place where we need the domain is uniformly convex. If the boundary condition $u_\nu = \varphi(x, u)$ satisfies that $\varphi_u(x, u) \leq -C_0$, where C_0 is a large enough positive constant, it is well known from [19] or [30] that we can delete the uniformly convex condition on domain in Lemma 13.

In order to get the estimates for the boundary double normal derivative, the main difficulty lies to construct the barrier functions of u_ν . The Neumann boundary condition will bring us a trouble term as " $\sum_{ijk} F^{ij} u_{ik} D_j v^k$ ". Motivated by Lions–Trudinger–Urbas

[19], Trudinger [24], Ivochkina–Trudinger–Wang [12] and Urbas [28], in Lemma 15 and Lemma 16 we introduce a new barrier function when the domain is uniformly $k-1$ convex, then we can extract a good term and control this trouble term. At last we get the estimates for boundary double normal derivative in Theorem 17. For C^0 estimate, we deal with a particular form of f and φ as in [24] where the boundary condition has a negative constant in front of u . Because it is easy to handle in this case while we do not want to emphasize C^0 estimate in this paper (see [19] for more general cases).

The rest of the paper is organized as follows. In Sect. 2, we first give the definitions and some notations. We get the C^0 and C^1 in Sect. 3, which was obtained by Trudinger [24] and Ma–Qiu–Xu [21]. In Sect. 4, we obtain the C^2 estimates, which is the main estimates in this paper. In the last section, we prove the main Theorem 1.

2. Preliminaries

In this section, we give the definitions of the admissible solution to the k -Hessian equation and introduce some elementary properties for k -th elementary symmetric functions.

Definition 2. For any $k = 1, 2, \dots, n$, and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ we set

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$

We also denote $\sigma_0(\lambda) = 1$.

Let $\lambda(D^2u) = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be the eigenvalues of D^2u and $\sigma_k(D^2u) = \sigma_k(\lambda(D^2u))$. And we denote that

$$\Gamma_k = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0, \quad \forall \quad j = 1, \dots, k\}.$$

We say that a function u is k -admissible if $\lambda(D^2u) \in \Gamma_k$.

In later we shall always assume that u is admissible. And the Hessian equation (1) is elliptic if u is an admissible solution. A C^2 bounded domain $\Omega \subset \mathbb{R}^n$ is uniformly $k-1$ convex if its boundary $\partial\Omega$ satisfies a geometric condition, that is $\kappa \in \Gamma_{k-1}$ and

$$\sigma_{k-1}(\kappa) \geq c_0 > 0, \quad \text{on } \partial\Omega,$$

for some positive constant c_0 , where $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n-1})$ denote the principal curvature of $\partial\Omega$ with respect to its inner normal.

We denote by $\sigma_k(\lambda|i)$ the symmetric function with $\lambda_i = 0$ and $\sigma_k(\lambda|ij)$ the symmetric function with $\lambda_i = \lambda_j = 0$.

Let $F^{ij} := \frac{\partial \sigma_k(D^2u)}{\partial u_{ij}}$, $\mathcal{F} := \sum_{1 \leq i \leq n} F^{ii}$. Sometimes we write the equation (1) in the form

$$\tilde{F}(D^2u) := \sigma_k^{\frac{1}{k}}(D^2u) = f^{\frac{1}{k}} =: \tilde{f}. \quad (3)$$

It is convenience to use the notation

$$\tilde{F}^{ij} := \frac{\partial \tilde{F}}{\partial u_{ij}}, \quad \tilde{F}^{ij,pq} := \frac{\partial^2 \tilde{F}}{\partial u_{ij} \partial u_{pq}}.$$

The σ_k operators have the following simple properties from [14].

Proposition 3. *If u is a solution to the k -Hessian equation, we have*

$$\begin{aligned} \sigma_k(\lambda) &= \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|i), \quad \forall 1 \leq i \leq n, \\ F^{ij} u_{ij} &= k \sigma_k, \end{aligned} \quad (4)$$

and

$$\mathcal{F} = (n - k + 1) \sigma_{k-1}. \quad (5)$$

We also have

Proposition 4. *If $\lambda \in \Gamma_k$, then*

$$\sigma_h(\lambda|i) > 0, \quad \forall h < k \quad \text{and} \quad 1 \leq i \leq n, \quad (6)$$

and $\sigma_k^{\frac{1}{k}}$ is a concave function in Γ_k .

Proof. See [14]. \square

The following proposition is so called MacLaurin inequality.

Proposition 5. *For $\lambda \in \Gamma_k$ and $k \geq l \geq 1$, we have*

$$\left[\frac{\sigma_k(\lambda)}{C_n^k} \right]^{\frac{1}{k}} \leq \left[\frac{\sigma_l(\lambda)}{C_n^l} \right]^{\frac{1}{l}}.$$

Moreover,

$$\sum_{i=1}^n \frac{\partial \sigma_k^{\frac{1}{k}}(\lambda)}{\partial \lambda_i} \geq [C_n^k]^{\frac{1}{k}}. \quad (7)$$

Proof. See [14]. \square

Proposition 6. *Suppose that $\lambda \in \Gamma_k$ and*

$$\lambda_1 \geq \cdots \geq \lambda_k \geq \cdots \geq \lambda_n,$$

then we have

$$\lambda_1 \sigma_{k-1}(\lambda|1) \geq \frac{k}{n} \sigma_k(\lambda). \quad (8)$$

Moreover, for $\forall i > k$, we have

$$\sigma_{k-1}(\lambda|i) \geq \sigma_{k-1}(\lambda|k) \geq c(n, k) \sigma_{k-1}(\lambda) > 0, \quad (9)$$

and

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0. \quad (10)$$

Proof. See [17] for these inequalities. \square

3. C^0 and C^1 Estimates

In this section we get a priori bound estimate and gradient estimate for the k -admissible solution to the equation (2). The following C^0 estimate was obtained by Trudinger [24].

Theorem 7 [24]. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 domain, and ν is the outer unit normal vector of $\partial\Omega$. Suppose $u \in C^2(\bar{\Omega}) \cap C^3(\Omega)$ is a k -admissible solution of the following Neumann boundary problems of Hessian equation*

$$\begin{cases} \sigma_k(D^2u) = f(x) & \text{in } \Omega, \\ u_\nu = -u + \varphi(x) & \text{on } \partial\Omega. \end{cases}$$

Then

$$\sup_{\bar{\Omega}} |u| \leq M_0,$$

where M_0 depends on $k, n, \text{diam}\Omega, \varphi, \sup f$.

Proof. Taking $o \in \Omega$ and let us consider $u - A|x|^2$. Choosing A large depending on k, n and $\sup f$ so that we have

$$F[D^2u] = f \leq F[D^2(A|x|^2)].$$

Comparison principle tells us that $u - A|x|^2$ attains its minimum point at x_0 on the boundary.

$$0 \geq (u - A|x|^2)_\nu(x_0) = -u + \varphi - 2Ax \cdot \nu.$$

Similarly, we consider u which attains its maximum on the boundary. Then we get

$$\inf_{\partial\Omega} \varphi - 4A\text{diam}\Omega \leq u \leq \sup_{\partial\Omega} \varphi.$$

□

From now on, we shall denote $M_0 := \sup_{\bar{\Omega}} |u|$. The gradient estimate was done in [21]. Since that paper was written in Chinese, for completeness we contain its proof in this section. We set

$$d(x) = \text{dist}(x, \partial\Omega),$$

and

$$\Omega_\mu := \{x \in \Omega : d(x) < \mu\}.$$

Then it is well known that there exists a positive constant $1 \geq \tilde{\mu} > 0$ such that $d(x) \in C^4(\bar{\Omega}_{\tilde{\mu}})$. As in Lieberman [15] (in page 331), we can extend ν by $\nu = -Dd$ in $\Omega_{\tilde{\mu}}$ and note that ν is a $C^2(\bar{\Omega}_{\tilde{\mu}})$ vector field. And we also have the following formulas

$$\begin{aligned} |D\nu| + |D^2\nu| &\leq C_0(n, \Omega) \text{in } \Omega_{\tilde{\mu}}, \\ \sum_{1 \leq i \leq n} \nu^i D_j \nu^i &= 0, \quad \sum_{1 \leq i \leq n} \nu^i D_i \nu^j = 0, \quad |\nu| = 1 \text{ in } \Omega_{\tilde{\mu}}. \end{aligned}$$

We define

$$c^{ij} = \delta_{ij} - v^i v^j \quad \text{in } \Omega_{\tilde{\mu}}.$$

And for a vector $\zeta \in \mathbb{R}^n$, we write ζ' for the vector with i -th component $\sum_{1 \leq j \leq n} c^{ij} \zeta^j$. Then we have

$$|D'u|^2 = \sum_{1 \leq i, j \leq n} c^{ij} u_i u_j.$$

We first state an useful lemma from [6].

Lemma 8 (Chou–Wang) [6]. *If u is k -admissible and $u_{11} < -\frac{h'|Du|^2}{128}$, here h' is any positive function. Then*

$$\frac{1}{n-k+1} \mathcal{F} \leq F^{11}, \quad (11)$$

and

$$\mathcal{F} \geq C_{n-1}^{k-1} \left[\frac{h'}{128 C_{n-1}^k} \right]^{k-1} |Du|^{2k-2}. \quad (12)$$

To state the gradient estimate of Neumann problems, we need first recall an interior estimate in [6].

Lemma 9 (Chou–Wang) [6]. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose $u \in C^3(\Omega)$ is a k -admissible solution of Hessian equation*

$$\sigma_k(D^2u) = f(x, u) \quad \text{in } \Omega$$

satisfying $|u| \leq M_0$. If $f \in C^2(\bar{\Omega} \times [-M_0, M_0])$ satisfies the conditions that there exist a positive constant L_1 such that

$$\begin{aligned} f(x, z) &\geq 0 \text{ in } \bar{\Omega} \times [-M_0, M_0], \\ |f(x, z)| + |f_x(x, z)| + |f_z(x, z)| &\leq L_1 \text{ in } \bar{\Omega} \times [-M_0, M_0]. \end{aligned}$$

Then for $\forall \Omega' \subset \subset \Omega$, it has

$$\sup_{\Omega'} |Du| \leq \tilde{M}_1,$$

where \tilde{M}_1 is a positive constant which depends on $n, k, M_0, \text{dist}(\Omega', \partial\Omega), L_1$.

Now we state the global gradient estimate which was done in [21].

Theorem 10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^3 domain, and v is the outer unit normal vector of $\partial\Omega$. Suppose $u \in C^2(\bar{\Omega}) \cap C^3(\Omega)$ is a k -admissible solution to the following Neumann boundary problems of Hessian equations*

$$\begin{cases} \sigma_k(D^2u) = f(x, u) & \text{in } \Omega, \\ u_\nu = \varphi(x, u) & \text{on } \partial\Omega, \end{cases}$$

satisfying $|u| \leq M_0$, where f, φ are given functions defined on $\bar{\Omega} \times [-M_0, M_0]$. If f, φ satisfy the conditions: \exists two positive constants L_1, L_2 such that

$$\begin{aligned} f(x, z) &> 0 \text{ in } \bar{\Omega} \times [-M_0, M_0], \\ |f(x, z)| + |f_x(x, z)| + |f_z(x, z)| &\leq L_1 \text{ in } \bar{\Omega} \times [-M_0, M_0], \\ |\varphi(x, z)|_{C^3(\bar{\Omega} \times [-M_0, M_0])} &\leq L_2. \end{aligned}$$

Then there exists a small positive constant μ_0 which depends only on $n, k, \Omega, M_0, L_1, L_2$ such that

$$\sup_{\bar{\Omega}_{\mu_0}} |Du| \leq \max\{\tilde{M}_1, \tilde{M}_2\},$$

where \tilde{M}_1 is a positive constant depending only on n, k, μ_0, M_0, L_1 , which is from the interior gradient estimates; \tilde{M}_2 is a positive constant depending only on $n, k, \Omega, \mu_0, M_0, L_1, L_2$.

3.1. Proof of Theorem 10.

Proof. We consider the auxiliary function

$$G(x) := \log |Dw|^2 + h(u) + g(d),$$

where

$$w(x) := u(x) + \varphi(x, u)d(x); \quad (13)$$

$$h(u) := -\log(1 + 4M_0 - u); \quad (14)$$

and

$$g(d) := \alpha_0 d,$$

in which α_0 large to be chosen later.

By (14) we have

$$\begin{aligned} -\log(1 + 5M_0) &\leq h \leq -\log(1 + 3M_0), \\ \frac{1}{1 + 5M_0} &\leq h' \leq \frac{1}{1 + 3M_0}, \\ \frac{1}{(1 + 5M_0)^2} &\leq h'' \leq \frac{1}{(1 + 3M_0)^2}. \end{aligned}$$

By (13) we have

$$w_i = u_i + (\varphi_i + \varphi_z u_i)d + \varphi d_i. \quad (15)$$

If we assume that $|Du| > 8nL_2$ and $\mu_0 \leq \frac{1}{2L_2}$, it follows from (15) that

$$\frac{1}{4}|Du| \leq |Dw| \leq 2|Du|.$$

These inequalities will be used below.

We assume that $G(x)$ attains its maximum at $x_0 \in \bar{\Omega}_{\mu_0}$, where $0 < \mu_0 < \tilde{\mu} \leq 1$ is a sufficiently small number which we shall decide it later.

Now we divide into three cases to complete the proof of Theorem 10.

Case I: If $G(x)$ attains its maximum at $x_0 \in \partial\Omega$, then we shall use the Hopf Lemma to get the bound of $G(x_0)$.

Case II: If $G(x)$ attains its maximum at $x_0 \in \Omega_{\mu_0}$, in this case for a sufficiently small constant $\mu_0 > 0$, then we can use the maximum principle to get the bound of $G(x_0)$.

Case III: If $G(x)$ attains its maximum at $x_0 \in \partial\Omega_{\mu_0} \cap \Omega$, then we shall get the estimates of $|Du|(x_0)$ via the standard interior gradient bound as in [6]. Which in turn give the bound for G at point x_0 .

Since $G(x) \leq G(x_0)$, we get the bound of G , which in turn give the bound of $|\nabla u|$ in $\bar{\Omega}_{\mu_0}$.

Now all computations work at the point x_0 . We use Einstein's summation convention. All repeated indices come from 1 to n .

Case I: Boundary estimates

The maximum of G is attained on the boundary. At the maximum point we have

$$0 \leq G_v = \frac{|Dw|_p^2 v^p}{|Dw|^2} - g' + h' u_v.$$

We have decomposition $|Dw|^2 = |D'w|^2 + w_v^2$. Because

$$w_v = u_v + D_v \varphi d - \varphi = 0$$

on the boundary, so we have

$$\begin{aligned} |Dw|_p^2 v^p &= C_p^{ij} w_i w_j v^p + 2C^{ij} w_{ip} w_j v^p + 2w_v D_p w v^p, \\ &= C_p^{ij} w_i w_j v^p + 2C^{ij} (u_{ip} + D_{ip} \varphi d + D_i \varphi d_p + D_p \varphi d_i + \varphi d_{ip}) w_j v^p, \\ &= C_p^{ij} w_i w_j v^p + 2C^{ij} u_{iv} w_j - 2C^{ij} D_i \varphi w_j + 2C^{ij} D_p \varphi v^p d_i w_j \\ &\quad + 2C^{ij} \varphi d_{ip} w_j v^p. \end{aligned} \tag{16}$$

On the other hand, take tangential derivative to the Neumann boundary condition:

$$C^{pq} D_q (u_i v^i) = C^{pq} D_q \varphi,$$

then we have

$$C^{pq} u_{qv} + C^{pq} u_i D_q v^i = C^{pq} D_q \varphi. \tag{17}$$

Then contracting (17) with w_p and inserting it into (16), we can cancel the term with the second derivative of u ,

$$|Dw|_p^2 v^p \leq C(n, \Omega, L_2) |Dw|^2 + C(n, \Omega, L_2) |Dw|.$$

So we choose $\alpha_0 = 2C + \frac{L_2}{1+3M_0} + 1$, such that

$$0 \leq G_v \leq -\alpha_0 + C + \frac{C}{|Dw|} + h' |\varphi|_{C^0}$$

$$\leq -C + \frac{C}{|Dw|}.$$

Thus we have estimate $|Dw|(x_0) \leq 1$, and $G(x_0) \leq -\log(1 + 3M_0) + 2C + \frac{L_2}{1+3M_0} + 1$.

Case II: Near boundary estimates

If G attains its maximum in Ω_{μ_0} . We take the first derivative and second derivative to the auxiliary function:

$$0 = G_i = \frac{2 \sum_{p=1}^n w_p w_{pi}}{|Dw|^2} + g' D_i d + h' u_i, \quad (18)$$

and

$$G_{ij} = \frac{\sum_{p=1}^n 2w_{pj} w_{pi} + 2w_p w_{pji}}{|Dw|^2} - \frac{4 \sum_{p,q=1}^n w_p w_{pi} w_q w_{qj}}{|Dw|^4} + g'' D_i d D_j d + g' D_{ij} d + h'' u_i u_j + h' u_{ij}.$$

Because $F^{ij}(D^2 u) > 0$ if we assume u is a k -admissible solution. At the maximum point of G , we get

$$0 \geq F^{ij} G_{ij} = \frac{2 \sum_{p=1}^n F^{ij} w_{pi} w_{pj}}{|Dw|^2} + \frac{2 \sum_{p=1}^n F^{ij} w_p w_{pji}}{|Dw|^2} - \frac{4 \sum_{p,q=1}^n F^{ij} w_p w_{pi} w_q w_{qj}}{|Dw|^4} + g'' F^{ij} D_i d D_j d + g' F^{ij} D_{ij} d + h'' F^{ij} u_i u_j + h' F^{ij} u_{ij}.$$

Recalling $w = u + \varphi d$, its second derivatives are

$$w_{ij} = u_{ij} + (\varphi_{ij} + \varphi_{iz} u_j + \varphi_{zj} u_i + \varphi_{zz} u_i u_j + \varphi_z u_{ij}) d + (\varphi_i + \varphi_z u_i) d_j + \varphi_j d_i + \varphi_z u_j d_i + \varphi d_{ij}. \quad (19)$$

w_{ij} has a relation with u_{ij} that

$$w_{ij} \leq (1 + \varphi_z d) u_{ij} + C(L_2, n) \mu_0 |Du|^2 + C(L_2, n) |Du| + C(L_2, n),$$

and

$$w_{ij} \geq (1 + \varphi_z d) u_{ij} - C(L_2, n) \mu_0 |Du|^2 - C(L_2, n) |Du| - C(L_2, n).$$

Let us differentiate w_{ij} again,

$$\begin{aligned} w_{ijp} = & u_{ijp} + (\varphi_{ijp} + \varphi_{ijz} u_p + \varphi_{izp} u_j + \varphi_{izz} u_p u_j + \varphi_{iz} u_{jp} + \varphi_{zjp} u_i \\ & + \varphi_{zzj} u_p u_i + \varphi_{zj} u_{ip} + \varphi_{zjp} u_i u_j + \varphi_{zzz} u_p u_i u_j + \varphi_{zz} u_{ip} u_j \\ & + \varphi_{zz} u_i u_{jp} + \varphi_{zp} u_{ij} + \varphi_{zz} u_p u_{ij} + \varphi_z u_{ijp}) d \\ & + (\varphi_{ij} + \varphi_{iz} u_j + \varphi_{zj} u_i + \varphi_{zz} u_i u_j + \varphi_z u_{ij}) d_p \\ & + (\varphi_{ip} + \varphi_{iz} u_p + \varphi_{zp} u_i + \varphi_{zz} u_p u_i + \varphi_z u_{ip}) d_j \\ & + (\varphi_i + \varphi_z u_i) d_{jp} + \varphi_{jp} d_i + \varphi_{jz} u_p d_i + \varphi_j d_{ip} \\ & + \varphi_{zp} u_j d_i + \varphi_{zz} u_p u_j d_i + \varphi_z u_{jp} d_i \\ & + \varphi_z u_j d_{ip} + \varphi_p d_{ij} + \varphi_z u_p d_{ij} + \varphi d_{ijp}. \end{aligned} \quad (20)$$

Now we choose a coordinate at x_0 such that $|\nabla w| = w_1$ and $(u_{ij})_{2 \leq i, j \leq n}$ is diagonal.

So from (15) and (18), we have for $i = 1$,

$$\begin{aligned} u_1 &= \frac{w_1 - \varphi_1 d - \varphi d_1}{1 + \varphi_z d}, \\ w_{11} &= -\frac{1}{2}(g' d_1 + h' u_1) w_1, \end{aligned} \quad (21)$$

and for $2 \leq i \leq n$,

$$\begin{aligned} u_i &= \frac{-\varphi_i d - \varphi d_i}{1 + \varphi_z d}, \\ w_{1i} &= -\frac{1}{2}(g' d_i + h' u_i) w_1, \end{aligned}$$

here we assume $\mu_0 \leq \mu_1 := \frac{1}{2L_2}$ such that $\frac{3}{2} \geq 1 + \varphi_z d \geq \frac{1}{2}$.

Suppose that $|Du|(x_0) > M_1 := 64nL_2$, we have for $i \geq 2$,

$$|u_i| \leq \frac{1}{16n} |Du|,$$

and

$$u_1 \geq \frac{1}{2} |Du|.$$

Moreover,

$$|Du|(x_0) \geq M_2 := 32n(1 + 5M_0)\alpha_0 + 128C + (1 + 5M_0) + 1$$

implies

$$|g' d_i| \leq \frac{h' u_1}{16n}.$$

So from (19) and (21) we get the key fact that

$$u_{11} \leq -\frac{1}{128} h' |Du|^2 < 0,$$

here we assume that $\mu_0 \leq \mu_2 := \frac{1}{64C(1+5M_0)}$.

For $i \geq 2$, we have

$$|w_{1i}| \leq \frac{h' |Dw|^2}{32n}, \quad (22)$$

and

$$|u_{1i}| \leq (C\mu_0 + \frac{1}{1+3M_0}) |Du|^2 + 2C |Du|. \quad (23)$$

Then we continue to compute $F^{ij}G_{ij}$. By using (15), (19) and (20) it follows that

$$\begin{aligned} F^{ij}G_{ij} &\geq -C(n, k, L_2, \Omega)\mu_0\mathcal{F}|Du|^2 + \frac{2F^{ij}u_{ij1}(1 + \varphi_z d)}{w_1} \\ &\quad - \frac{4F^{ij}(\varphi_{iz}u_{j1}d + \varphi_{zz}u_{i1}u_{jd} + \varphi_z u_{i1}d_j)}{w_1} \\ &\quad - \frac{2F^{ij}u_{ij}[(\varphi_{zp} + \varphi_{zz}u_p)d + \varphi_z d_p]}{w_1} - 2\frac{F^{ij}w_{1i}w_{1j}}{w_1^2} \\ &\quad - C(n, k, L_2, \alpha_0, \Omega)\mathcal{F}|Du| + h''F^{11}u_1^2 + h'F^{ij}u_{ij}. \end{aligned}$$

The equation (1) is k -homogenous. And differentiating it gives

$$F^{ij}u_{ij} = kf, \quad (24)$$

$$F^{ij}u_{ij1} = f_1 + f_u u_1. \quad (25)$$

We obtain from (22), (23), (24), and (25) that

$$\begin{aligned} F^{ij}G_{ij} &\geq -C(n, k, L_2, M_0, \Omega)\mu_0\mathcal{F}|Du|^2 + h''F^{11}u_1^2 - \frac{(h')^2\mathcal{F}|Dw|^2}{32} \\ &\quad - C(n, k, L_2, L_1, M_0, \alpha_0, \Omega)\mathcal{F}|Du| - C(L_1, n, L_2). \end{aligned}$$

(11) tells us if $\mu_0 \leq \mu_3 := \frac{1}{32C(1+5M_0)^2(n-k+1)}$ is small, we get

$$\frac{h''F^{11}u_1^2}{8} \geq C\mu_0\mathcal{F}|Du|^2. \quad (26)$$

By the definition of h , we have $h'' = (h')^2$. Thus from (11)

$$\frac{h''F^{11}u_1^2}{8} \geq \frac{(h')^2\mathcal{F}|Du|^2}{32}. \quad (27)$$

If we assume further $|Du|^2(x_0) \geq M_3 := 32(n-k+1)(1+5M_0)^2C$, we get

$$\frac{h''F^{11}u_1^2}{8} \geq C\mathcal{F}|Du|. \quad (28)$$

From the above estimates (11), (26), (27), and (28), we obtain

$$0 \geq F^{ij}G_{ij} \geq \frac{h''\mathcal{F}|Du|^2}{32(n-k+1)} - C.$$

Finally, the inequality (12) in the Lemma 8 implies that

$$0 \geq \frac{h''\mathcal{F}|Du|^2}{32(n-k+1)} - C > 0, \quad (29)$$

provided that

$$|Du|(x_0) \geq M_4 := \frac{32(n-k+1)(1+5M_0)^2C[(1+5M_0)128C_{n-1}^k]^{k-1}}{C_{n-1}^{k-1}} + 1.$$

So Inequality (29) is a contradiction.

We conclude that if $\mu_0 = \min\{\bar{\mu}, \mu_1, \mu_2, \mu_3\}$, we have the estimate

$$|Du|(x_0) \leq \max\{M_1, M_2, M_3, M_4\}.$$

Thus we get the estimate of $G(x_0)$.

Because G attains its maximum at x_0 and h, g is bounded from below, the gradient estimate of u follows the above three cases. \square

4. C^2 a Priori Estimates

Now we come to the a priori estimates of second derivative necessary for our existence theorem. For these bounds we restrict our attention to the following problem

$$\begin{cases} \sigma_k(D^2u) = f(x, u) & \text{in } \Omega \subset \mathbb{R}^n, \\ u_v = \varphi(x, u) & \text{on } \partial\Omega. \end{cases} \quad (30)$$

Theorem 11. *Let Ω be a bounded C^4 uniformly convex domain in \mathbb{R}^n , v is the unit outer normal vector of $\partial\Omega$. If $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$ a k -admissible solution to the Neumann problem (30), where $f \in C^2(\bar{\Omega} \times \mathbb{R})$ is positive and $\varphi \in C^3(\bar{\Omega} \times \mathbb{R})$ is non-increasing in z . Then we have*

$$\sup_{\bar{\Omega}} |D^2u| \leq C,$$

where C depends only on $n, k, \|u\|_{C^1(\bar{\Omega})}, \|f\|_{C^2(\bar{\Omega} \times [-M_0, M_0])}, \min f, \|\varphi\|_{C^3(\bar{\Omega} \times [-M_0, M_0])}$ and convexity of Ω .

It is well known that it is easy to get the estimates for second tangential-normal derivative of the solution on the boundary. We here follow the same line as in Lions–Trudinger–Urbas [19] with some minor changes.

Lemma 12. *Denoting the tangential direction τ at any point $y \in \partial\Omega$, under the condition of Theorem 11, we have*

$$|D_{\tau v}u(y)| \leq C, \quad (31)$$

where the constant C only depends on $\|u\|_{C^1}, \|\varphi\|_{C^1}$ and $\|\partial\Omega\|_{C^2}$.

Proof. Taking tangential derivative to the boundary condition

$$u_v = \varphi,$$

as in (17) we have

$$C^{ij}u_{jv} + C^{ij}u_l D_j v^l = C^{ij}D_j \varphi. \quad (32)$$

Taking inner product with τ^i , it follows that

$$\tau^i u_{li} v^l + u_l D_i v^l \tau^i = D_i \varphi \tau^i.$$

So

$$|u_{\tau v}| \leq |D_i \varphi \tau^i - u_l D_i v^l \tau^i| \leq C.$$

\square

Now we again use the technique of Lions–Trudinger–Urbas [19], we can reduce the second derivative estimates of the solution to the boundary double normal derivative bounds.

Lemma 13. *Let $M = \sup_{\partial\Omega} |u_{\nu\nu}|$ and $u(x)$ be an admissible solution in Theorem 11, then we have*

$$\sup_{\overline{\Omega}, \xi \in \mathbb{S}^{n-1}} u_{\xi\xi} \leq C_0(1 + M), \quad (33)$$

where C_0 depends only on $\|u\|_{C^1}, \|\varphi\|_{C^3}, \|\partial\Omega\|_{C^4}, \|f\|_{C^2}, \min f$, and uniformly convexity of $\partial\Omega$.

Proof. We consider the function

$$v(x, \xi) := u_{\xi\xi} - v'(x, \xi) + K_1|x|^2 + K_2|Du|^2,$$

where $v'(x, \xi) := 2(\xi \cdot \nu)\xi' \cdot (D\varphi - u_l Dv^l) = a^l u_l + b$, $\xi' = \xi - (\xi \cdot \nu)\nu$, $a^l = 2(\xi \cdot \nu)(\xi'^l \varphi_z - \xi'^l D_i v^l)$, and $b = 2(\xi \cdot \nu)\xi'^l \varphi_{x_l}$. We compute

$$v_i = u_{\xi\xi i} - D_i a^l u_l - a^l u_{li} - D_i b + 2K_1 x_i + 2 \sum_l K_2 u_l u_{li},$$

and

$$\begin{aligned} v_{ij} = & u_{\xi\xi ij} - D_{ij} a^l u_l - D_i a^l u_{lj} - D_j a^l u_{li} - a^l u_{lij} - D_{ij} b \\ & + 2K_1 \delta_{ij} + 2K_2 \sum_l u_{li} u_{lj} + 2K_2 \sum_l u_l u_{lij}. \end{aligned} \quad (34)$$

Taking first derivative of equation (3), we have

$$\tilde{F}^{ij} u_{ijl} = \tilde{f}_{x_l} + \tilde{f}_z u_l. \quad (35)$$

And we have from the concavity of $\sigma_k^{\frac{1}{k}}$

$$\tilde{F}^{ij} u_{ij\xi\xi} \geq \tilde{F}^{ij} u_{ij\xi\xi} + \tilde{F}^{ij, pq} u_{ij\xi} u_{pq\xi} = \tilde{f}_{x_\xi x_\xi} + 2\tilde{f}_{x_\xi z} u_\xi + \tilde{f}_z u_{\xi\xi}. \quad (36)$$

Then we contract (34) with the \tilde{F}^{ij} , using (36) and (35),

$$\begin{aligned} \tilde{F}^{ij} v_{ij} = & \tilde{F}^{ij} u_{\xi\xi ij} - \tilde{F}^{ij} D_{ij} a^l u_l - 2\tilde{F}^{ij} u_{lj} D_i a^l - \tilde{F}^{ij} u_{lij} a^l \\ & - \tilde{F}^{ij} D_{ij} b + 2K_1 \sum_i \tilde{F}^{ii} + 2K_2 \sum_l \tilde{F}^{ij} u_{lj} u_{li} + 2K_2 \sum_{ijl} \tilde{F}^{ij} u_{lij} u_l \\ \geq & -C_1(\|u\|_{C^1}, \|\varphi\|_{C^3}, \|\partial\Omega\|_{C^4}, \|f\|_{C^2}, \min f, K_2) \left(\sum_i \tilde{F}^{ii} + 1 \right) \\ & + \tilde{f}_z u_{\xi\xi} + 2K_1 \sum_i \tilde{F}^{ii} + 2K_2 \sum_l \tilde{F}^{ij} u_{li} u_{lj} - 2\tilde{F}^{ij} u_{lj} D_i a^l. \end{aligned}$$

At the interior maximum point, we assume (u_{ij}) is diagonal and $u_{11} \geq u_{22} \geq \dots \geq u_{nn}$. So we have by (8)

$$\begin{aligned}
2K_2 \sum_i \tilde{F}^{ii} u_{ii}^2 &\geq 2K_2 \sigma_k^{\frac{1}{k}-1} F^{11} u_{11}^2 \\
&\geq 2K_2 \frac{\sigma_k^{\frac{1}{k}}}{n} u_{11} \\
&\geq 2K_2 \frac{\sigma_k^{\frac{1}{k}}}{n} u_{\xi\xi}.
\end{aligned}$$

We can assume $u_{\xi\xi} \geq 0$, otherwise we have the estimate (33). If we choose $K_2 \geq \frac{n|\tilde{f}_z|}{2\min \tilde{f}} + 2$, we continue

$$\begin{aligned}
\sum_{ij} \tilde{F}^{ij} v_{ij} &\geq 2 \sum_i \tilde{F}^{ii} u_{ii}^2 - 2C_2(\|u\|_{C^1}, \|\varphi\|_{C^3}, \|\partial\Omega\|_{C^3}) \sum_i \tilde{F}^{ii} |u_{ii}| \\
&\quad + 2K_1 \sum_i \tilde{F}^{ii} - C_1(\sum_i \tilde{F}^{ii} + 1) \\
&\geq 2 \sum_i \tilde{F}^{ii} (|u_{ii}| - \frac{C_2}{2})^2 + (2K_1 - \frac{C_2}{2} - C_1) \sum_i \tilde{F}^{ii} - C_1.
\end{aligned}$$

Now if we choose K_1 large such that $K_1 \geq \frac{C_2^2}{2} + C_1$ and $K_1(C_n^k)^{\frac{1}{k}} > C_1$, by (7) we have

$$\sum_{ij} \tilde{F}^{ij} v_{ij} > 0.$$

So $v(x, \xi)$ attains its maximum on $\partial\Omega$.

Case a: ξ is tangential

We shall take tangential derivative twice to the boundary condition. First we rewrite (32) as following

$$u_{li} v^l = C^{ij} D_j \varphi - C^{ij} u_l D_j v^l + v^i v^j v^l u_{lj}. \quad (37)$$

So let's take tangential derivative (37) and we get

$$C^{pq} D_q (u_{li} v^l) = C^{pq} D_q (C^{ij} D_j \varphi - C^{ij} u_l D_j v^l + v^i v^j v^l u_{lj}).$$

It follows that

$$u_{lip} v^l = C^{pq} D_q (C^{ij} D_j \varphi - C^{ij} u_l D_j v^l + v^i v^j v^l u_{lj}) + v^p v^q v^l u_{liq} - C^{pq} u_{li} D_q v^l.$$

In the above formula we take sum with $\xi^i \xi^p$, then we obtain

$$\begin{aligned}
u_{\xi\xi v} &= -2\xi^p \xi^i u_{li} D_p v^l - u_l \xi^p D_{ip} v^l \xi^i + u_{vv} \sum_i \xi^p D_p v^i \xi^i \\
&\quad - \sum_i \xi^p \xi^i v^j D_p v^i D_j \varphi + \varphi_z u_{\xi\xi} + \xi^p \xi^i \varphi_{ip} \\
&\quad + \varphi_{zz} u_{\xi\xi}^2 + 2u_{\xi\xi} \xi^i \varphi_{zi}.
\end{aligned}$$

So we have

$$\begin{aligned} u_{\xi\xi v} &\leq -2\xi^p \xi^i u_{li} D_p v^l + \varphi_z u_{\xi\xi} \\ &\quad + C(\|u\|_{C^1}, \|\partial\Omega\|_{C^3}, \|\varphi\|_{C^2}) + C(\|\partial\Omega\|_{C^2})|u_{vv}| \\ &\leq -2\xi^p \xi^i u_{li} D_p v^l + C + C|u_{vv}|. \end{aligned}$$

Here in the second inequality we assume that φ is non-increasing in z .

If we assume $\xi = e_1$, it is easy to get the bound for $u_{1i}(x_0)$ for $i \neq 1$ from the maximum of $v(x, \xi)$ in the ξ direction. In fact, we can assume $\xi(t) = \frac{(1, t, 0, \dots, 0)}{\sqrt{1+t^2}}$. Because $v(x, \xi)$ attains its maximum at $\xi(0)$. Then we have

$$\begin{aligned} 0 &= \frac{\partial v(x_0, \xi(t))}{\partial t} \Big|_{t=0} \\ &= 2u_{ij}(x_0) \frac{d\xi^i(t)}{dt} \Big|_{t=0} \xi^j(0) - \frac{\partial v'(x_0, \xi(t))}{\partial t} \Big|_{t=0} \\ &= 2u_{11} \frac{-t}{(1+t^2)^{\frac{3}{2}}} \Big|_{t=0} + 2u_{12} \left(\frac{1}{\sqrt{1+t^2}} + \frac{-t^2}{(1+t^2)^{\frac{3}{2}}} \right) \Big|_{t=0} - \frac{\partial v'}{\partial t} \Big|_{t=0}. \end{aligned}$$

So we have

$$|u_{12}| \leq C(\|\varphi\|_{C^1}, \|u\|_{C^1}, \|\partial\Omega\|_{C^2}).$$

Similarly, we have for all $i \neq 1$,

$$|u_{1i}| \leq C(\|\varphi\|_{C^1}, \|u\|_{C^1}, \|\partial\Omega\|_{C^2}).$$

Due to $D_1 v_1 \geq \kappa > 0$, we have

$$u_{\xi\xi v} \leq -2\kappa u_{\xi\xi} + C(1 + |u_{vv}|). \quad (38)$$

On the other hand, we have from the Hopf lemma, (31) and $\sum_i a^i v^i = 0$,

$$\begin{aligned} 0 &\leq v_v \\ &= u_{\xi\xi v} - D_v a^l u_l - a^l u_{lv} - b_v + 2K_1(x \cdot v) + 2K_2 \sum_l u_l u_{lv} \\ &\leq u_{\xi\xi v} + C(\|u\|_{C^1}, \|\partial\Omega\|_{C^2}, \|\varphi\|_{C^2}, K_1, K_2) + 2K_2 \varphi u_{vv}. \end{aligned} \quad (39)$$

Combining (38) and (39), we therefore deduce

$$u_{\xi\xi}(x_0) \leq C(1 + |u_{vv}|(x_0)).$$

Case b: ξ is non-tangential

We write $\xi = \alpha\tau + \beta v$, where $\alpha = \xi \cdot \tau$, $|\tau| = 1$, $\tau \cdot v = 0$, $\beta = \xi \cdot v \neq 0$ and $\alpha^2 + \beta^2 = 1$.

$$\begin{aligned} u_{\xi\xi} &= \alpha^2 u_{\tau\tau} + \beta^2 u_{vv} + 2\alpha\beta u_{\tau v} \\ &= \alpha^2 u_{\tau\tau} + \beta^2 u_{vv} + 2\alpha\beta (D_i \varphi \tau^i - u_l D_i v^l \tau^i). \end{aligned}$$

By definition of $v(x, \xi)$, we have

$$\begin{aligned} v(x_0, \xi) &= \alpha^2 v(x_0, \tau) + \beta^2 v(x_0, \nu) \\ &\leq \alpha^2 v(x_0, \xi) + \beta^2 v(x_0, \nu). \end{aligned}$$

Hence

$$v(x_0, \xi) \leq v(x_0, \nu).$$

Then we get the estimate,

$$u_{\xi\xi}(x_0) \leq C_0(\|u\|_{C^1}, \|\varphi\|_{C^3}, \|\partial\Omega\|_{C^4}, \|f\|_{C^2}, \min f, \kappa)(1 + |u_{\nu\nu}(x_0)|).$$

So that this case is also reduced to the purely normal case. \square

4.1. Second Normal Derivative Bounds On The Boundary. In this section, we consider the double normal derivative estimate which is the most difficulty part in the Neumann problem for Hessian equations. As we know for the Dirichlet problem on k -Hessian equation, Caffarelli–Nirenberg–Spruck [1] (see also [25]) obtained the existence of the classical solution under the assumption on the domain $\Omega \subset \mathbb{R}^n$ is uniformly $k - 1$ convex. In this section, still under the uniformly $k - 1$ convexity of the boundary, we get the double normal derivative estimate on the boundary for the k -admissible solution u of (2) in Theorem 1.

We give some definitions first.

We introduce a well known defining function (see for example in Wang [31], section 2) for the Hessian equation

$$h(x) = -d(x) + K_3 d^2(x),$$

for some K_3 to be determined later.

We know from the classic book [9] section 14.6 that h is C^4 in

$$\Omega_\mu := \{x \in \bar{\Omega} : 0 < d(x) < \mu\}$$

for some constant μ small depending on Ω . And h also satisfied the following properties in Ω_μ :

$$-\mu + K_3 \mu^2 \leq h \leq 0.$$

It is easily to see that

$$\frac{Dh}{|Dh|} = \nu,$$

for unit outer normal on the boundary.

Lemma 14. *If Ω is a C^4 uniformly $k - 1$ -convex domain, and let $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$ be a k -admissible solution of Neumann problem (2). Where $f \in C^2(\bar{\Omega})$ is positive and $\varphi \in C^3(\mathbb{R})$ is non-increasing in z . There exist $\delta > 0$ small and K_3 large depending on the curvature of $\partial\Omega$, n , k , $\min f$. If we choose $\mu \leq \frac{1}{4K_3}$, we have*

$$F^{ij} h_{ij} \geq \delta(\mathcal{F} + 1).$$

Moreover, $h = 0$ on $\partial\Omega$ and $h \leq -\frac{\mu}{2} < 0$ on $\partial\Omega_\mu/\partial\Omega$. We also have

$$2 \geq |Dh| \geq \frac{1}{2}.$$

Proof. For $x_0 \in \Omega_\mu$, $y_0 \in \partial\Omega$ be such that $|x_0 - y_0| = d(x_0)$. Then, in terms of a principal coordinate system, see [9] section 14.6, we have

$$[-D^2d(x_0)] = \text{diag}\left[\frac{\kappa_1(y_0)}{1 - \kappa_1(y_0)d(x_0)}, \dots, \frac{\kappa_{n-1}(y_0)}{1 - \kappa_{n-1}(y_0)d(x_0)}, 0\right],$$

and

$$-Dd(x_0) = v(y_0) = (0, 0, \dots, 1).$$

Since Ω is strictly $k - 1$ -convex, i. e. $\sigma_{k-1}(\kappa) > 2b_0 > 0$, we have

$$\sigma_{k-1}(\kappa - 8\delta) > b_0$$

for small δ depending on κ, k, n . In principal coordinate system, we can easily check that $h - \delta|x|^2$ is k -admissible, provided that K_3 large and $\mu \leq \frac{1}{4K_3}$. We deduce from the concavity of \tilde{F} ,

$$\begin{aligned} \tilde{F}^{ij}(h - \delta|x|^2)_{ij} &\geq \tilde{F}[D^2(u + h - \delta|x|^2)] - \tilde{F}[D^2u] \\ &\geq \tilde{F}[D^2(h - \delta|x|^2)] \\ &\geq b_0^{\frac{1}{k}} K_3^{\frac{1}{k}} - C(\kappa, k, n, \delta) \\ &\geq \frac{1}{k} \sigma_k^{\frac{1}{k}-1} \delta, \end{aligned}$$

provided K_3 sufficiently large, and $f > 0$. So we obtain

$$F^{ij}(h - \delta|x|^2 + \delta|x|^2)_{ij} \geq \delta(\mathcal{F} + 1).$$

On $\partial\Omega$, $h = 0$ is obvious.

On $\partial\Omega_\mu/\partial\Omega$, we have

$$\begin{aligned} h &= -\mu + K_3\mu^2 \\ &\leq -\frac{\mu}{2}. \end{aligned}$$

And it is also easy to see

$$2 \geq |Dh| \geq \frac{1}{2},$$

if we choose $\mu \leq \frac{1}{4K_3}$. \square

In order to do this estimate we construct barrier functions of u_v on the boundary. Motivated by [12, 19, 24] and [28], we introduce the following functions. In $\overline{\Omega}_\mu$, we denote

$$\begin{aligned} g(x) &:= 1 - \beta h, \\ G(x) &:= (A + \sigma M)h(x), \\ \psi(x) &:= |Dh|(x)\varphi(x, u) \end{aligned}$$

where σ, β, μ, A are positive constants to be chosen later.

Now we consider the sub barrier function,

$$P(x) := g(x)(Du \cdot Dh(x) - \psi(x)) - G(x).$$

And we want to prove the following lemma.

Lemma 15. Fix $\sigma = \frac{1}{2}$, under the condition of Lemma 14, for any $x \in \overline{\Omega}_\mu$, if we choose β large, μ small, A large in a proper sequence, we have

$$P(x) \geq 0.$$

Proof. We use maximum principle to prove this lemma. First we assume the function attains its minimum point x_0 in the interior of Ω_μ . We differentiate this function twice,

$$P_i = g_i \left(\sum_l u_l h_l - \psi \right) + g \left(\sum_l u_{li} h_l + \sum_l u_l h_{li} - \psi_i \right) - G_i,$$

and

$$\begin{aligned} P_{ij} &= g_{ij} \left(\sum_l u_l h_l - \psi \right) + g_i \left(\sum_l u_{lj} h_l + \sum_l u_l h_{lj} - \psi_j \right) \\ &\quad + g \left(\sum_l u_{lij} h_l + \sum_l u_{li} h_{lj} + \sum_l u_{lj} h_{li} + \sum_l u_l h_{lij} - \psi_{ij} \right) \\ &\quad + g_j \left(\sum_l u_{li} h_l + \sum_l u_l h_{li} - \psi_i \right) - G_{ij}. \end{aligned} \quad (40)$$

At the minimum point x_0 , as before we can assume that $(u_{ij}(x_0))$ is diagonal. Contracting (40) with F^{ij} , we get

$$\begin{aligned} F^{ij} P_{ij} &= F^{ij} g_{ij} \left(\sum_l u_l h_l - \psi \right) + 2g_i F^{ij} \left(\sum_l u_{lj} h_l + \sum_l u_l h_{lj} - \psi_j \right) \\ &\quad + g F^{ij} \left(\sum_l u_{lij} h_l + 2 \sum_l u_{li} h_{lj} + \sum_l u_l h_{lij} - \psi_{ij} \right) \\ &\quad - F^{ij} G_{ij} \\ &\leq \beta C_3 (\|u\|_{C^1}, \|\partial \Omega\|_{C^3}, \|\varphi\|_{C^2}, \|f\|_{C^1}) (\mathcal{F} + 1) \\ &\quad - (A + \sigma M) k_0 (\mathcal{F} + 1) - 2\beta F^{ii} u_{ii} h_i^2 + 2F^{ii} u_{ii} h_{ii} g. \end{aligned}$$

Where in the second inequality we use

$$|\beta h| \leq \beta \frac{\mu}{2} \leq \frac{1}{2}, \quad (41)$$

which in turn implies that

$$1 \leq g \leq \frac{3}{2}. \quad (42)$$

We choose $\mu \leq \frac{1}{\beta}$ in (41).

Then we divided the index $1 \leq i \leq n$ into two categories.

(i) If

$$|\beta h_i^2| \leq \frac{k_0}{2},$$

we say $i \in \mathbf{B}$.

We choose $\beta \geq 2nk_0$, in order to let

$$|h_i^2| \leq \frac{1}{4n}. \quad (43)$$

(ii) If

$$|\beta h_i^2| \geq \frac{k_0}{2},$$

we denote $i \in \mathbf{G}$.

For any $i \in \mathbf{G}$, we use $P_i(x_0) = 0$ to get

$$u_{ii} = \frac{A + \sigma M}{g} + \frac{\beta(\sum_l u_l h_l - \psi)}{g} - \frac{\sum_l u_l h_{li}}{h_i} + \frac{\psi_i}{h_i}.$$

Because $|h_i|^2 > \frac{k_0}{2\beta}$ and (42), we have that

$$\left| \frac{\beta(\sum_l u_l h_l - \psi)}{g} - \frac{\sum_l u_l h_{li}}{h_i} + \frac{\psi_i}{h_i} \right| \leq \beta C_4(k_0, \|u\|_{C^1}, \|\partial\Omega\|_{C^2}, \|\varphi\|_{C^1}).$$

By chosen A large such that $\frac{A}{3} \geq \beta C_4$, we infer

$$\frac{4A}{3} + \sigma M \geq u_{ii} \geq \frac{A}{3} + \frac{2\sigma M}{3}, \quad \text{for } i \in \mathbf{G}. \quad (44)$$

Due to $2 \geq |Dh| \geq \frac{1}{2}$ and (43), there is a $i_0 \in \mathbf{G}$, say $i_0 = 1$ such that

$$h_1^2 \geq \frac{1}{4n}.$$

Then we continue to compute the equation of P ,

$$\begin{aligned} F^{ij} P_{ij} &\leq [\beta C_3 - (A + \sigma M)k_0](\mathcal{F} + 1) - 2\beta \sum_{i \in \mathbf{G}} F^{ii} u_{ii} h_i^2 - 2\beta \sum_{i \in \mathbf{B}} F^{ii} u_{ii} h_i^2 \\ &\quad + k_1 \sum_{u_{ii} \geq 0} F^{ii} u_{ii} + 4k_0 \sum_{u_{ii} < 0} F^{ii} u_{ii}. \end{aligned} \quad (45)$$

Since

$$-2\beta \sum_{i \in \mathbf{G}} F^{ii} u_{ii} h_i^2 \leq -2\beta F^{11} u_{11} h_1^2 \leq -\frac{\beta}{2n} F^{11} u_{11},$$

and

$$-2\beta \sum_{i \in \mathbf{B}} F^{ii} u_{ii} h_i^2 \leq -2\beta \sum_{i \in \mathbf{B}, u_{ii} < 0} F^{ii} u_{ii} h_i^2 \leq -k_0 \sum_{u_{ii} < 0} F^{ii} u_{ii},$$

it follows that

$$-2\beta \sum_{i \in \mathbf{G}} F^{ii} u_{ii} h_i^2 - 2\beta \sum_{i \in \mathbf{B}} F^{ii} u_{ii} h_i^2 + 4k_0 \sum_{u_{ii} < 0} F^{ii} u_{ii} \leq -\frac{\beta}{2n} F^{11} u_{11}. \quad (46)$$

From (45) and (46), we have

$$F^{ij} P_{ij} \leq [\beta C_3 - (A + \sigma M)k_0](\mathcal{F} + 1) - \frac{\beta}{2n} F^{11} u_{11} + k_1 \sum_{u_{ii} \geq 0} F^{ii} u_{ii}. \quad (47)$$

Now we analysis the above terms case by case. Without loss of generality, we assume that $u_{22} \geq \dots \geq u_{nn}$.

Case 1: $u_{ii} \geq 0$, for all i .

This is the easiest case. Using equation, we get

$$kf = \sum_{u_{ii} \geq 0} F^{ii} u_{ii}.$$

If we choose $A > \frac{(C_3\beta + k_1 k \max f)}{k_0}$, then from (47) we have

$$F^{ij} P_{ij} < 0. \quad (48)$$

In the following cases we can assume $u_{nn} < 0$.

Case 2: $\frac{k_0}{2k_1} u_{11} \geq |u_{nn}|$.

Due to the equation, we have

$$kf = \sum_{u_{ii} \geq 0} F^{ii} u_{ii} + \sum_{u_{ii} < 0} F^{ii} u_{ii}.$$

The terms in line (47) become

$$\begin{aligned} -\frac{\beta}{2n} F^{11} u_{11} + k_1 \sum_{u_{ii} \geq 0} F^{ii} u_{ii} &\leq k_1 (kf - \sum_{u_{ii} < 0} F^{ii} u_{ii}) \\ &\leq k_1 kf - k_1 \mathcal{F} u_{nn} \\ &\leq k_1 kf + \frac{k_0}{2} \mathcal{F} u_{11} \\ &\leq k_1 kf + k_0 \mathcal{F} \left(\frac{2A}{3} + \frac{\sigma M}{2} \right). \end{aligned} \quad (49)$$

Using (49) and choosing $A > \frac{3(C_3\beta + k_1 k \max f)}{k_0}$ in (47), then we obtain the result (48).

In the following cases we assume

$$u_{nn} < 0, \quad |u_{nn}| \geq \frac{k_0}{2k_1} u_{11}.$$

We denote $\lambda := (u_{11}, \dots, u_{nn})$ and choose $A \geq 2\sigma$.

Case 3: $\sigma_{k-1}(\lambda|1) \geq \delta_1(-u_{nn})\sigma_{k-2}(\lambda|1n)$, for a small positive constant δ_1 chosen in the later case.

If $u_{11} \geq u_{22}$, we know from (8) that,

$$u_{11}\sigma_{k-2}(\lambda|1n) \geq \frac{k-1}{n-1}\sigma_{k-1}(\lambda|n). \quad (50)$$

Otherwise $u_{11} \leq u_{22}$, we have from (44), (33) and (8) that

$$\begin{aligned}
 u_{11}\sigma_{k-2}(\lambda|1n) &\geq \left(\frac{A}{3} + \frac{2\sigma M}{3}\right)\sigma_{k-2}(\lambda|2n) \\
 &\geq \frac{2\sigma}{3C_0}u_{22}\sigma_{k-2}(\lambda|2n) \\
 &\geq \frac{k-1}{n-1}\frac{2\sigma}{3C_0}\sigma_{k-1}(\lambda|n).
 \end{aligned} \tag{51}$$

We infer from the hypothesis

$$\begin{aligned}
 F^{11} &= \sigma_{k-1}(\lambda|1) \\
 &\geq \delta_1(-u_{nn})\sigma_{k-2}(\lambda|1n) \\
 &\geq \delta_1\frac{k_0}{2k_1}u_{11}\sigma_{k-2}(\lambda|1n).
 \end{aligned} \tag{52}$$

Note we only use the hypothesis of Case 3 in the first inequality above.

Using (4) and the assumption $u_{nn} < 0$, we have from (5) that

$$\frac{1}{n-k+1}\mathcal{F} \leq F^{nn}. \tag{53}$$

Assuming $C_0 \geq 1$ such that $\sigma = \frac{1}{2} \leq \frac{3C_0}{2}$, then we substitute (50) and (51) into (52) and use (53),

$$\begin{aligned}
 F^{11} &\geq \delta_1\frac{k_0}{2k_1}u_{11}\sigma_{k-2}(\lambda|1n) \\
 &\geq \delta_1\frac{k-1}{n-1}\frac{k_0}{2k_1}\frac{2\sigma}{3C_0}\sigma_{k-1}(\lambda|n) \\
 &\geq \frac{k-1}{(n-1)(n-k+1)}\frac{k_0\delta_1\sigma}{3k_1C_0}\mathcal{F}.
 \end{aligned} \tag{54}$$

Using (33), and we choose $\beta \geq \frac{9n(n-k+1)(n-1)k_1^2C_0^2}{(k-1)k_0\delta_1\sigma^2}$ such that for the last two terms in (47). Then we have

$$\begin{aligned}
 &-\frac{\beta}{2n}F^{11}u_{11} + k_1 \sum_{u_{ii} \geq 0} F^{ii}u_{ii} \\
 &\leq \left[-\frac{(k-1)\beta k_0\delta_1\sigma}{6n(n-1)(n-k+1)k_1C_0}\left(\frac{A}{3} + \frac{2\sigma M}{3}\right) + k_1C_0(M+1)\right]\mathcal{F} \\
 &\leq \left[-\left(\frac{(k-1)\beta k_0\delta_1\sigma^2}{9n(n-k+1)(n-1)k_1C_0} - k_1C_0\right)M + \left(-\frac{A\beta k_0(k-1)\delta_1\sigma}{18n(n-k+1)(n-1)k_1C_0} + k_1C_0\right)\right]\mathcal{F} \\
 &\leq 0.
 \end{aligned} \tag{55}$$

So choosing $A > \frac{(C_3\beta+k_1k \max f)}{k_0} + 2\sigma$ in (47), and using (55), we obtain the inequality (48).

Case 4: $0 \leq \sigma_{k-1}(\lambda|1) \leq \delta_1(-u_{nn})\sigma_{k-2}(\lambda|1n)$.

By the hypothesis and for $i \geq 2$,

$$\sigma_{k-1}(\lambda|1) - u_{ii}\sigma_{k-2}(\lambda|1i) = \sigma_{k-1}(\lambda|1i).$$

We compute as follows,

$$\begin{aligned} k\sigma_k(\lambda|1) &= \sum_{i=2}^n u_{ii}\sigma_{k-1}(\lambda|1i) \\ &\leq \sum_{u_{ii} \geq 0, i \neq 1} u_{ii}[\delta_1(-u_{nn})\sigma_{k-2}(\lambda|1n) - u_{ii}\sigma_{k-2}(\lambda|1i)] \\ &\quad + \sum_{u_{ii} < 0, i \neq 1} u_{ii}(-u_{ii}\sigma_{k-2}(\lambda|1i)) \\ &\leq -u_{nn} \sum_{u_{ii} \geq 0, i \neq 1} \delta_1 u_{ii}\sigma_{k-2}(\lambda|1n) - u_{nn}^2 \sigma_{k-2}(\lambda|1n) \\ &\leq -n\delta_1 u_{nn} u_{22} \sigma_{k-2}(\lambda|1n) - u_{nn}^2 \sigma_{k-2}(\lambda|1n). \end{aligned}$$

Using (33) and (44), we continue

$$\begin{aligned} k\sigma_k(\lambda|1) &\leq -n\delta_1 C_0(M+1)u_{nn}\sigma_{k-2}(\lambda|1n) - u_{nn}^2 \sigma_{k-2}(\lambda|1n) \\ &\leq -n\delta_1 C_0 \frac{3}{2\sigma} u_{11} u_{nn} \sigma_{k-2}(\lambda|1n) - u_{nn}^2 \sigma_{k-2}(\lambda|1n) \\ &\leq \frac{3k_1 n \delta_1 C_0}{k_0 \sigma} u_{nn}^2 \sigma_{k-2}(\lambda|1n) - u_{nn}^2 \sigma_{k-2}(\lambda|1n). \end{aligned}$$

Now we let $\delta_1 = \frac{k_0 \sigma}{6k_1 n C_0}$. As in (52) and (54), we obtain

$$\begin{aligned} k\sigma_k(\lambda|1) &\leq -\frac{u_{nn}^2}{2} \sigma_{k-2}(\lambda|1n) \\ &\leq u_{nn} \frac{k-1}{(n-k+1)(n-1)} \frac{\sigma k_0}{6k_1 C_0} \mathcal{F} \\ &\leq -\frac{k-1}{(n-k+1)(n-1)} \frac{\sigma k_0^2}{12k_1^2 C_0} u_{11} \mathcal{F}. \end{aligned}$$

Inserting (44) into the above inequality, we have

$$\begin{aligned} -\frac{\beta}{2n} F^{11} u_{11} &= -\frac{\beta}{2n} (f - \sigma_k(\lambda|1)) \\ &\leq -\frac{\beta}{2n} f - \frac{\beta \sigma k_0^2}{24kn(n-k+1)k_1^2 C_0} \frac{k-1}{n-1} \left(\frac{A}{3} + \frac{2\sigma M}{3} \right) \mathcal{F}. \end{aligned}$$

If we choose $\beta \geq \frac{36kn(n-k+1)(n-1)k_1^3 C_0^2}{(k-1)\sigma^2 k_0^2}$ such that for the last two terms in (47) we get

$$-\frac{\beta}{2n} F^{11} u_{11} + k_1 \sum_{u_{ii} \geq 0} F^{ii} u_{ii} \leq -\frac{\beta}{2n} f < 0. \quad (56)$$

Finally, choosing $A \geq \frac{3(C_3\beta+k_1k \max f)}{k_0}$ in (47) and using (56), we obtain the inequality (48) which contradicts to $0 \leq F^{ij} P_{ij}$ at the minimum point x_0 .

Then the function P attains its minimum on the boundary of Ω_μ .

Now we treat the boundary value of P . On $\partial\Omega$, it is easy to see

$$P = 0.$$

On the $\partial\Omega_\mu/\partial\Omega$, we have

$$P \geq -C_5(k, \max f, \|u\|_{C^1}, \|\varphi\|_{C^0}) + (A + \sigma M) \frac{\mu}{2} \geq 0,$$

provided $A \geq \frac{2C_5}{\mu}$.

We conclude that we first choose $\delta_1 = \frac{k_0\sigma}{6k_1nC_0}$, then $\beta = \frac{36kn(n-k+1)(n-1)k_1^3C_0^2}{(k-1)\sigma^2k_0^2} + \frac{9n(n-k+1)(n-1)k_1^2C_0^2}{(k-1)k_0\delta_1\sigma^2} + 2nk_0$, then $\mu = \min\{\mu_0, \frac{1}{\beta}\}$, finally $A = \frac{3(C_3\beta+k_1k \max f)}{k_0} + 3\beta C_4 + 2\sigma + 1 + \frac{2C_5}{\mu}$. Using the maximal principle for the function $P(x)$, we get

$$P(x) \geq 0, \quad \text{in } \Omega.$$

□

Similarly, we can also find a super barrier function of u_v .

Lemma 16. Let $\bar{P} := g(x)(Du \cdot Dh(x) - \psi(x)) + G(x)$. Fix $\sigma = \frac{1}{2}$, under the condition of Lemma 14, for any $x \in \bar{\Omega}_\mu$, if we choose β large, μ small, A large in proper sequence, we have

$$\bar{P}(x) \leq 0.$$

Proof. We assume the function attains its maximum point x_0 in the interior of Ω_μ . We differentiate this function twice,

$$\bar{P}_i = g_i(\sum_l u_l h_l - \psi) + g(\sum_l u_{li} h_l + \sum_l u_l h_{li} - \psi_i) + G_i,$$

and

$$\begin{aligned} \bar{P}_{ij} &= g_{ij}(\sum_l u_l h_l - \psi) + g_i(\sum_l u_{lj} h_l + \sum_l u_l h_{lj} - \psi_j) \\ &\quad + g(\sum_l u_{lij} h_l + \sum_l u_{li} h_{lj} + \sum_l u_{lj} h_{li} + \sum_l u_l h_{lij} - \psi_{ij}) \\ &\quad + g_j(\sum_l u_{li} h_l + \sum_l u_l h_{li} - \psi_i) + G_{ij}. \end{aligned} \tag{57}$$

At the maximum point x_0 , as before we can assume $(u_{ij}(x_0))$ is diagonal. Contracting (57) with F^{ij} , we get

$$\begin{aligned}
F^{ij}\bar{P}_{ij} &= F^{ij}g_{ij}\left(\sum_l u_l h_l - \psi\right) + 2g_i F^{ij}\left(\sum_l u_{lj} h_l + \sum_l u_l h_{lj} - \psi_j\right) \\
&\quad + g F^{ij}\left(\sum_l u_{lij} h_l + 2\sum_l u_{li} h_{lj} + \sum_l u_l h_{lij} - \psi_{ij}\right) + F^{ij}G_{ij} \\
&\geq -\beta C_6(\|u\|_{C^1}, \|\partial\Omega\|_{C^3}, \|\varphi\|_{C^2}, \|f\|_{C^1})(\mathcal{F} + 1) \\
&\quad + (A + \sigma M)k_0(\mathcal{F} + 1) - 2\beta F^{ii}u_{ii}h_i^2 + 2F^{ii}u_{ii}h_{ii}g.
\end{aligned}$$

As before we divided the index $1 \leq i \leq n$ into two categories.

(i) If

$$|\beta h_i^2| \leq \frac{k_0}{2},$$

we say $i \in \mathbf{B}$.

We choose $\beta \geq 2nk_0$, in order to let

$$|h_i^2| \leq \frac{1}{4n}. \quad (58)$$

(ii) If

$$|\beta h_i^2| \geq \frac{k_0}{2},$$

we denote $i \in \mathbf{G}$.

For any $i \in \mathbf{G}$, we use $\bar{P}_i(x_0) = 0$ to get

$$u_{ii} = -\frac{A + \sigma M}{g} + \frac{\beta(\sum_l u_l h_l - \psi)}{g} - \frac{\sum_l u_l h_{li}}{h_i} + \frac{\psi_i}{h_i}. \quad (59)$$

Because $|h_i|^2 > \frac{k_0}{2\beta}$ and (59), we have that

$$\left| \frac{\beta(\sum_l u_l h_l - \psi)}{g} - \frac{\sum_l u_l h_{li}}{h_i} + \frac{\psi_i}{h_i} \right| \leq \beta C_4(k_0, \|u\|_{C^1}, \|\partial\Omega\|_{C^2}, \|\varphi\|_{C^1}).$$

By chosen A large such that $\frac{A}{3} \geq \beta C_4$, it infers

$$-\frac{4A}{3} - \sigma M \leq u_{ii} \leq -\frac{A}{3} - \frac{2\sigma M}{3}, \quad \text{for } i \in \mathbf{G}. \quad (60)$$

Due to $2 \geq |Dh| \geq \frac{1}{2}$ and (58), there is a $i_0 \in \mathbf{G}$, say $i_0 = 1$, such that

$$h_1^2 \geq \frac{1}{4n}.$$

Then we continue to compute the equation of \overline{P} ,

$$\begin{aligned} F^{ij}\overline{P}_{ij} &\geq [-\beta C_6 + (A + \sigma M)k_0](\mathcal{F} + 1) \\ &\quad - 2\beta \sum_{i \in \mathbf{G}} F^{ii} u_{ii} h_i^2 - 2\beta \sum_{i \in \mathbf{B}} F^{ii} u_{ii} h_i^2 \\ &\quad + 4k_0 \sum_{u_{ii} \geq 0} F^{ii} u_{ii} + k_1 \sum_{u_{ii} < 0} F^{ii} u_{ii}. \end{aligned}$$

We treat some terms in the last formula.

First, we have

$$-2\beta \sum_{i \in \mathbf{G}} F^{ii} u_{ii} h_i^2 \geq -2\beta F^{11} u_{11} h_1^2 \geq -\frac{\beta}{2n} F^{11} u_{11},$$

then

$$\begin{aligned} -2\beta \sum_{i \in \mathbf{B}} F^{ii} u_{ii} h_i^2 &\geq -2\beta \sum_{i \in \mathbf{B}, u_{ii} \geq 0} F^{ii} u_{ii} h_i^2 \\ &\geq -k_0 \sum_{i \in \mathbf{B}, u_{ii} \geq 0} F^{ii} u_{ii} \\ &= -k_0 \sum_{u_{ii} \geq 0} F^{ii} u_{ii}. \end{aligned}$$

It follows that

$$\begin{aligned} &-2\beta \sum_{i \in \mathbf{G}} F^{ii} u_{ii} h_i^2 - 2\beta \sum_{i \in \mathbf{B}} F^{ii} u_{ii} h_i^2 \\ &\quad + 4k_0 \sum_{u_{ii} \geq 0} F^{ii} u_{ii} + k_1 \sum_{u_{ii} < 0} F^{ii} u_{ii} \\ &\geq -\frac{\beta}{2n} F^{11} u_{11} + k_1 \sum_{u_{ii} < 0} F^{ii} u_{ii}. \end{aligned}$$

Then we have

$$\begin{aligned} F^{ij}\overline{P}_{ij} &\geq [-\beta C_6 + (A + \sigma M)k_0](\mathcal{F} + 1) \\ &\quad - \frac{\beta}{2n} F^{11} u_{11} + k_1 \sum_{u_{ii} < 0} F^{ii} u_{ii}. \end{aligned} \tag{61}$$

This is easy when $u_{11} < 0$, because we have by (10) and (9) that

$$F^{11} \geq c(k, n)\mathcal{F}.$$

From (33) and (60) we obtain

$$-\frac{\beta}{2n} F^{11} u_{11} + k_1 \sum_{u_{ii} < 0} F^{ii} u_{ii} \geq \frac{\beta c}{2n} \mathcal{F} \left(\frac{A}{3} + \frac{2\sigma M}{3} \right) - k_1 \mathcal{F} C_0 (1 + M). \tag{62}$$

If we choose $\beta \geq \frac{3nk_1C_0}{c\sigma}$ and $A \geq 2\sigma + \frac{\beta C_6}{k_0}$, then by (61) and (62) we get

$$F^{ij}\bar{P}_{ij} > 0.$$

Then the function \bar{P} attains its maximum on the boundary of Ω_μ .

On $\partial\Omega$, it is easy to see

$$\bar{P} = 0.$$

On the $\partial\Omega_\mu/\partial\Omega$, we have

$$\bar{P} \leq C_7(k, \max f, \|u\|_{C^1}, \|\varphi\|_{C^0}) - (A + \sigma M)\frac{\mu}{2} \leq 0,$$

provided $A \geq \frac{2C_7}{\mu}$.

We conclude that we first choose $\beta \geq \frac{3nk_1C_0}{c\sigma}$, then $\mu = \min\{\mu_0, \frac{1}{\beta}\}$, finally $A \geq 2\sigma + \frac{\beta C_6}{k_0} + 3\beta C_4 + 1 + \frac{2C_7}{\mu}$. Using the maximal principle for the function $\bar{P}(x)$, we get

$$\bar{P}(x) \leq 0, \quad \text{in } \Omega.$$

□

Using the barrier functions, we have the main normal-normal second derivative estimate in this section.

Lemma 17. *Let Ω be a bounded C^4 uniformly convex domain in \mathbb{R}^n , ν is the outer unit normal vector of $\partial\Omega$. If $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$ a k -admissible solution of Neumann problem (30). Where $f \in C^2(\bar{\Omega} \times \mathbb{R})$ is positive and $\varphi \in C^3(\bar{\Omega} \times \mathbb{R})$ is non-increasing in z . Then we have*

$$\sup_{\partial\Omega} |u_{\nu\nu}| \leq C,$$

where constant C depends on $n, k, \|u\|_{C^1}, \min f, \|\varphi\|_{C^3}, \|f\|_{C^2}$, convexity of $\partial\Omega$ and $\|\partial\Omega\|_{C^4}$.

Proof. Assume z_0 is the maximum point of $u_{\nu\nu}$, we have

$$\begin{aligned} 0 &\geq P_\nu(z_0) \\ &\geq g\left(\sum_l u_{lv}h_l + u_l h_{lv} - \psi_\nu\right) - (A + \sigma M)h_\nu \\ &\geq u_{\nu\nu} - C(\|u\|_{C^1}, \|\partial\Omega\|_{C^2}, \|\psi\|_{C^2}) - (A + \sigma M). \end{aligned}$$

In the second inequality we assume $u_{\nu\nu}(z_0) \geq 0$. Then we get

$$\sup_{\partial\Omega} u_{\nu\nu} \leq C + \sigma M.$$

Similarly, by $0 \leq \bar{P}_\nu(z_0)$ here z_0 is the minimum point of $u_{\nu\nu}$, we get

$$\inf_{\partial\Omega} u_{\nu\nu} \geq -C - \sigma M.$$

So chosen $\sigma = \frac{1}{2}$ as in the previous lemmas, we get the estimate

$$\sup_{\partial\Omega} |u_{\nu\nu}| \leq C.$$

□

Proof of Theorem 11. Combining Lemmas 12, 13 and 17, we complete the proof of Theorem 11. \square

Remark 1. If the boundary condition $u_v = \varphi(x, u)$ satisfies that $\varphi_u(x, u) \leq -C_0$, where C_0 is large enough positive constant, it is well known from [19] or [30] that we can delete the uniformly convex condition on domain in Lemma 13. So in this case we can obtain the C^2 estimates in Theorem 11 if Ω is uniformly $k - 1$ -convex domain.

5. Existence of the Boundary Problem

In this section we complete the proof of Theorem 1. As in [19], by combining Theorems 7, 10 and 11 with the global second derivative Holder estimates (see [18] or [16]), we get a global estimate

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C$$

for k -admissible solutions, where C, α depending on $k, n, \Omega, \|\Omega\|_{C^4}, \|f\|_{C^2}, \min f$ and $\|\varphi\|_{C^3}$. Then applying the method of continuity (see [9], Theorem 17.28), we complete the proof of Theorem 1.

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