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The Neumann problem of special Lagrangian equations with supercritical phase ☆

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Abstract

In this paper, we establish the global C^2 estimates of the Neumann problem of special Lagrangian equations with supercritical phase and the existence theorem by the method of continuity.

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Keywords: Special Lagrangian equation; Neumann problem; Supercritical phase

1. Introduction

In this paper, we study the Neumann boundary problems of a fully nonlinear elliptic equation

$$\arctan D^2u(x) = \Theta(x), \quad x \in \Omega \subset \mathbb{R}^n, \tag{1.1}$$

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where

$$\arctan D^2u =: \arctan\lambda_1 + \arctan\lambda_2 + \cdots + \arctan\lambda_n.$$

Here $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ are the eigenvalues of the Hessian matrix $D^2u = \{\frac{\partial^2 u}{\partial x_i \partial x_j}\}_{1\le i,j\le n}$. For $\Theta(x) \in (-\frac{n\pi}{2}, \frac{n\pi}{2})$, the equation (1.1) is known as the special Lagrangian equation and Θ is called the phase. In particular, $\Theta = \frac{(n-2)\pi}{2}$ is the critical phase, and if $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$, the equation (1.1) is called special Lagrangian equations with supercritical phase.

Before we recall the history for the Neumann boundary problem, we present a brief description for the Dirichlet problem of elliptic equations in $\mathbb{R}^n$. The Dirichlet problem for the Laplace equation is well studied in [6,9]. For nonlinear elliptic equations, the pioneering works have been done by Evans in [8], Krylov in [14–16], Caffarelli-Nirenberg-Spruck in [3,4] and Ivochkina in [11]. In their papers, they solved the Dirichlet problem for Monge-Ampère equations and k-Hessian equations elegantly. Since then, many interesting fully nonlinear equations with different structure conditions have been researched, such as Hessian quotient equations, which were solved by Trudinger in [25]. For more information, we refer the citations of [3]. The special Lagrangian equation (1.1) was introduced by Harvey-Lawson [10] in the study of calibrated geometries. Here Θ is a constant called the phase angle. In this case the graph $x \mapsto (x, Du(x))$ defines a calibrated, minimal submanifold of $\mathbb{R}^{2n}$. Since the work of Harvey-Lawson, special Lagrangian manifolds have gained wide interests, due in large part to their fundamental role in the Strominger-Yau-Zaslow description of mirror symmetry [23]. For the special Lagrangian equations with supercritical phase, Yuan obtained the interior C^1 estimate with Warren in [28] and the interior C^2 estimate with Wang in [27]. Recently Collins-Picard-Wu [7] obtained the existence theorem of the Dirichlet problem by adopting the classical method with some important observation about the concavity of the operator.

Meanwhile, the Neumann and oblique derivative problem of partial differential equations were widely studied. For a priori estimates and the existence theorem of Laplace equation with Neumann boundary condition, we refer to the book [9]. Also, we recommend the recent book written by Lieberman [17] for the Neumann and the oblique derivative problems of linear and quasilinear elliptic equations. In 1986, Lions-Trudinger-Urbas solved the Neumann problem of Monge-Ampère equations in the celebrated paper [19]. Recently, Ma-Qiu [20] solved the Neumann problem of k-Hessian equations. And Chen-Zhang [5] generalized the important result to the Neumann problem of Hessian quotient equations. In [12,13], Jiang-Trudinger studied the general oblique boundary value problems for augmented Hessian equations with some regular condition and concavity condition. Inspired by Urbas's work [26], Brendle and Warren [2] proved the existence of convex solution to a special Lagrangian equation with second boundary condition.

Because the second condition in [2] does not include the classical Neumann boundary condition, it is natural for us to study the corresponding Neumann problem of special Lagrangian equations. In this paper, we establish global C^1, C^2 estimates of the Neumann problem of special Lagrangian equations with supercritical phase and obtain the existence theorem as follows.

Theorem 1.1. *Suppose $\Omega \subset \mathbb{R}^n$ is a C^4 strictly convex domain and ν is outer unit normal vector of $\partial\Omega$. Let $\varphi \in C^3(\partial\Omega)$ and $\Theta(x) \in C^2(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$. Then there exists a unique solution $u \in C^{3,\alpha}(\overline{\Omega})$ to the Neumann problem of special Lagrangian equation*

$$\begin{cases} \arctan D^2u = \Theta(x), & in \quad \Omega \subset \mathbb{R}^n, \\ u_\nu = -u + \varphi(x), & on \quad \partial\Omega. \end{cases} \tag{1.2}$$

Following the classical idea (see for example [1] or [21]), we can obtain the existence theorem of the classical Neumann problem of special Lagrangian equation as below.

Theorem 1.2. *Suppose $\Omega \subset \mathbb{R}^n$ is a C^4 strictly convex domain and ν is outer unit normal vector of $\partial\Omega$. Let $\varphi \in C^3(\partial\Omega)$ and $\Theta(x) \in C^2(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$. Then there exists a unique constant β such that the Neumann problem of special Lagrangian equation*

$$\begin{cases} \arctan D^2u = \Theta(x), & in \quad \Omega \subset \mathbb{R}^n, \\ u_\nu = \beta + \varphi(x), & on \quad \partial\Omega, \end{cases} \tag{1.3}$$

has admissible solutions $u \in C^{3,\alpha}(\overline{\Omega})$, which are unique up to a constant.

Remark 1.3. For the classical Neumann problem of special Lagrangian equation (1.3), it is easy to know that a solution plus any constant is still a solution. So we can not obtain a uniform bound for the solutions of (1.3), and can not use the method of continuity to get the existence. Following the arguments in [19,21], we consider the solution u^ε of the equation

$$\begin{cases} \arctan D^2u^\varepsilon = \Theta(x), & \text{in} \quad \Omega, \\ (u^\varepsilon)_\nu = -\varepsilon u^\varepsilon + \varphi(x), & \text{on} \quad \partial\Omega, \end{cases} \tag{1.4}$$

for any small $\varepsilon > 0$. We need to establish a priori estimate of u^ε independent of ε, and the strict convexity of Ω plays an important role. By letting $\varepsilon \to 0$ and a perturbation argument, we can obtain a solution of (1.3).

The rest of the paper is organized as follows. In Section 2, we collect some properties of the special Lagrangian equation and establish the C^0 estimate for the Neumann problem of special Lagrangian equation. The C^1 and C^2 estimates are established in Section 3, Section4 respectively. Finally, we prove Theorem 1.1 and Theorem 1.2 in Section 5.

2. Some properties and C^0 estimate

In this section, we give some properties of the special Lagrangian equation with supercritical phase and establish the C^0 estimate.

2.1. Some properties

Property 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain and $\Theta(x) \in C^0(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$. Suppose $u \in C^2(\Omega)$ is a solution of special Lagrangian equation (1.1) and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ are the eigenvalues of the Hessian matrix D^2u with*

$$\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n, \tag{2.1}$$

then we have some properties:

$$\lambda_1 \geq \cdots \geq \lambda_{n-1} > 0, \tag{2.2}$$

$$|\lambda_n| \leq \lambda_{n-1}, \tag{2.3}$$

$$\sum_{i=1}^{n} \lambda_i > 0, \tag{2.4}$$

$$|\lambda_n| < C_0, \tag{2.5}$$

where $C_0 = \max\{\tan\big(\frac{(n-1)\pi}{2} - \min_{\overline{\Omega}} \Theta\big), \tan\big(\frac{\max_{\overline{\Omega}} \Theta}{n}\big)\}$.

These properties are well-known and can be found in [27,29]. We write it here for convenience.

Proof. For any $i = 1, 2, \cdots, n$, we know $\arctan \lambda_i \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then we can get

$$\begin{aligned} \arctan \lambda_{n-1} + \arctan \lambda_n &= \Theta - \arctan \lambda_1 - \arctan \lambda_2 \cdots - \arctan \lambda_{n-2} \\ &\geq \Theta - \frac{(n-2)\pi}{2} > 0. \end{aligned}$$

So $\lambda_{n-1} + \lambda_n > 0$, which implies (2.2), (2.3) and (2.4) hold.

Moreover,

$$\arctan \lambda_n = \Theta - \arctan \lambda_1 - \arctan \lambda_2 - \cdots - \arctan \lambda_{n-1} > \min_{\overline{\Omega}} \Theta - \frac{(n-1)\pi}{2},$$

$$\arctan \lambda_n < \frac{\arctan \lambda_1 + \arctan \lambda_2 + \cdots + \arctan \lambda_n}{n} \leq \frac{\max_{\overline{\Omega}} \Theta}{n},$$

so we can get

$$|\lambda_n| < \max\{\tan\Big(\frac{(n-1)\pi}{2} - \min_{\overline{\Omega}} \Theta\Big), \tan\Big(\frac{\max_{\overline{\Omega}} \Theta}{n}\Big)\}. \qquad \square$$

Property 2.2. *Suppose* $\Omega \subset \mathbb{R}^n$ *is a domain and* $\Theta(x) \in C^2(\overline{\Omega})$ *with* $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ *in* $\overline{\Omega}$. *Let* $u \in C^4(\Omega)$ *be a solution of special Lagrangian equation* (1.1). *Then for any* $\xi \in \mathbb{S}^{n-1}$, *we have*

$$\sum_{ij=1}^{n} F^{ij} u_{ij\xi\xi} \geq \Theta_{\xi\xi} - A\Theta_\xi{}^2, \qquad \text{in } \Omega, \tag{2.6}$$

where $F^{ij} = \frac{\partial \arctan D^2 u}{\partial u_{ij}}$ *and* $A = \frac{2}{\tan\left(\min\limits_{\overline{\Omega}} \Theta - \frac{(n-2)\pi}{2}\right)}$.

Proof. For any $x \in \Omega$, we can assume D^2u is diagonal with $\lambda_i = u_{ii}$, since (2.6) is invariant under rotating the coordinates. Then we have

$$F^{ij} =: \frac{\partial \arctan D^2 u}{\partial u_{ij}} = \begin{cases} \frac{1}{1+\lambda_i^2}, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

and

$$F^{ij,kl} =: \frac{\partial^2 \arctan D^2 u}{\partial u_{ij} \partial u_{kl}} = \begin{cases} -\frac{2\lambda_i}{(1+\lambda_i^2)^2}, & \text{if } i = j = k = l, \\ -\frac{\lambda_i + \lambda_j}{(1+\lambda_i^2)(1+\lambda_j^2)}, & \text{if } i = l,\ j = k,\ i \neq j, \\ 0, & \text{otherwise}\ . \end{cases}$$

From the special Lagrangian equation (1.1), we know

$$\sum_{ij=1}^{n} F^{ij} u_{ij\xi} = \Theta_\xi,$$

and

$$\begin{aligned} \sum_{ij=1}^{n} F^{ij} u_{ij\xi\xi} &= \Theta_{\xi\xi} - \sum_{ijkl=1}^{n} F^{ij,kl} u_{ij\xi} u_{kl\xi} = \Theta_{\xi\xi} - \sum_{i=1}^{n} F^{ii,ii} u_{ii\xi}^2 - \sum_{i \neq j} F^{ij,ji} u_{ij\xi}^2 \\ &\geq \Theta_{\xi\xi} - \sum_{i=1}^{n} F^{ii,ii} u_{ii\xi}^2. \end{aligned} \tag{2.7}$$

From the concavity lemma (Lemma 2.2 in [7]), we know

$$-\sum_{i=1}^{n} F^{ii,ii} u_{ii\xi}^2 \geq -\frac{2}{\tan\left(\min\limits_{\overline{\Omega}} \Theta - \frac{(n-2)\pi}{2}\right)} \Big(\sum_{i=1}^{n} F^{ii} u_{ii\xi}\Big)^2 = -\frac{2}{\tan\left(\min\limits_{\overline{\Omega}} \Theta - \frac{(n-2)\pi}{2}\right)} {\Theta_\xi}^2. \tag{2.8}$$

Hence (2.6) holds. □

2.2. C^0 estimate

The C^0 estimate is easy. For completeness, we produce a proof here following the idea of Trudinger [24].

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a C^1 bounded domain and $\varphi \in C^0(\partial\Omega)$. Suppose $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is the solution of special Lagrangian equation (1.2) and $\Theta(x) \in C^0(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$, then we have*

$$\sup_{\overline{\Omega}} |u| \le M_0, \tag{2.9}$$

where M_0 depends on n, $diam(\Omega)$, $\max_{\partial\Omega} |\varphi|$ and $\max_{\overline{\Omega}} \Theta$.

Proof. From (2.4), we know u is subharmonic. So the maximum of u is attained at a boundary point $x_0 \in \partial\Omega$. Then we can get

$$0 \le u_\nu(x_0) = -u(x_0) + \varphi(x_0). \tag{2.10}$$

Hence

$$\max_{\overline{\Omega}} u = u(x_0) \le \varphi(x_0) \le \max_{\partial\Omega} |\varphi|. \tag{2.11}$$

Without loss of generality, we assume $0 \in \Omega$, and denote $B = \frac{1}{2} \tan\left(\frac{\max_{\overline{\Omega}} \Theta}{n}\right) < +\infty$. Then we have

$$\arctan D^2 u = \Theta \le \max_{\overline{\Omega}} \Theta = \arctan D^2(B|x|^2). \tag{2.12}$$

By the comparison principle, we know $u - B|x|^2$ attains its minimum at a boundary point $x_1 \in \partial\Omega$. Then

$$\begin{aligned} 0 &\ge (u - B|x|^2)_\nu|_{x=x_1} = u_\nu(x_1) - 2Bx_1 \cdot \nu \\ &= -u(x_1) + \varphi(x_1) - 2Bx_1 \cdot \nu \\ &\ge -u(x_1) - \max_{\partial\Omega} |\varphi| - 2B\mathrm{diam}(\Omega). \end{aligned} \tag{2.13}$$

Hence

$$\begin{aligned} \min_{\overline{\Omega}} u \ge \min_{\overline{\Omega}} (u - B|x|^2) = u(x_1) - B|x_1|^2 \\ \ge -\max_{\partial\Omega} |\varphi| - 2B\mathrm{diam}(\Omega) - B\mathrm{diam}(\Omega)^2. \quad \square \end{aligned} \tag{2.14}$$

Here, following the proof of Theorem 2.3, we can easily obtain

Theorem 2.4. *Suppose $\Omega \subset \mathbb{R}^n$ is a C^1 bounded domain and $\varphi \in C^0(\partial\Omega)$. Let $\Theta(x) \in C^0(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$ and $u^\varepsilon \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be the solution of special Lagrangian equation (1.4) with $\varepsilon \in (0, 1)$, then we have*

$$\sup_{\overline{\Omega}} |\varepsilon u^\varepsilon| \le \overline{M_0}, \tag{2.15}$$

where $\overline{M_0}$ depends on n, diam(Ω), $\max_{\partial\Omega} |\varphi|$ and $\max_{\overline{\Omega}} \Theta$.

3. Global gradient estimate

In this section, we prove the global gradient estimate involving the interior gradient estimate and the near boundary gradient estimate. To state our theorems, we denote $d(x) = \text{dist}(x, \partial\Omega)$ and $\Omega_\mu = \{x \in \Omega | d(x) < \mu\}$, where μ is a small positive universal constant. In Subsection 3.1, we give the interior gradient estimate in $\Omega \setminus \Omega_\mu$ and, in Subsection 3.2, we establish the near boundary gradient estimate in Ω_μ, following the idea of Ma-Qiu [20].

3.1. Interior gradient estimate

The interior gradient estimate was established in [28]. Hence we can directly obtain the following interior gradient estimate in $\Omega \setminus \Omega_\mu$.

Theorem 3.1. *Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain and $\Theta(x) \in C^0(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$. Suppose $u \in C^2(\Omega)$ is a solution of special Lagrangian equation (1.1), then we have*

$$\sup_{\Omega\setminus\Omega_\mu} |Du| \le M_1, \tag{3.1}$$

where M_1 depends on n, μ, and $|u|_{C^0}$.

3.2. Near boundary gradient estimate

Theorem 3.2. *Suppose $\Omega \subset \mathbb{R}^n$ is a C^2 bounded domain, $\varphi \in C^2(\partial\Omega)$ and $\Theta(x) \in C^0(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$. Let $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be a solution of special Lagrangian equation (1.2), then we have*

$$\sup_{\overline{\Omega}_\mu} |Du| \le \max\{M_1, \widetilde{M}_1\}, \tag{3.2}$$

where μ is a universal positive constant depending only on Ω; M_1 depends on n, μ, and $|u|_{C^0}$; and $\widetilde{M}_1$ depends on n, μ, Ω, $|u|_{C^0}$, $\max_{\overline{\Omega}} \Theta$, $\min_{\overline{\Omega}} \Theta$ and $|\varphi|_{C^2}$.

Proof. The proof follows the idea of Ma-Qiu [20].

Since Ω is a C^2 domain, it is well known that there exists a small positive universal constant $0 < \mu < \frac{1}{10}$ such that $d(x) \in C^2(\overline{\Omega_\mu})$. As [22,17] (page 331 in [17]), we can extend ν by $\nu = -Dd$ in Ω_μ and note that ν is a $C^1(\overline{\Omega_\mu})$ vector field. As mentioned in the book [17], we also have the following formulas

$$|D\nu| \le C, \quad \text{in} \quad \overline{\Omega_\mu}, \tag{3.3}$$

$$\sum_{i=1}^n \nu^i D_j \nu^i = 0, \quad \sum_{i=1}^n \nu^i D_i \nu^j = 0, \quad |\nu| = 1, \quad \text{in} \quad \overline{\Omega_\mu}, \tag{3.4}$$

where C is depending only on n and Ω. As in [17], we define

$$c^{ij} = \delta_{ij} - \nu^i \nu^j, \quad \text{in} \quad \overline{\Omega_\mu}, \tag{3.5}$$

and for a vector $\zeta \in \mathbb{R}^n$, we write ζ' for the vector with i-th component $\sum_{j=1}^n c^{ij}\zeta^j$. Then we have

$$|(Du)'|^2 = \sum_{i,j=1}^n c^{ij} u_i u_j \quad \text{and} \quad |Du|^2 = |(Du)'|^2 + u_\nu^2. \tag{3.6}$$

We consider the auxiliary function

$$G(x) = \log |Dw|^2 + h(u) + g(d), \tag{3.7}$$

where

$$\begin{aligned} w(x) &= (1-d)u + \varphi(x)d(x), \\ h(u) &= -\log(1 + M_0 - u), \\ g(d) &= \alpha_0 d, \end{aligned}$$

with $\alpha_0 > 0$ to be determined later. Also note that here $\varphi \in C^2(\overline{\Omega})$ is an extension with universal C^2 norm.

It is easy to know $G(x)$ is well-defined in $\overline{\Omega_\mu}$. Then we assume that $G(x)$ attains its maximum at a point $x_0 \in \overline{\Omega_\mu}$. If we have $|Du|(x_0) \le 10n[|\varphi|_{C^1(\Omega)} + \sup_\Omega |u|]$, and we can get directly

$$\begin{aligned} \sup_{\Omega_\mu} \log |Du|^2 &= \sup_{\Omega_\mu} \log \frac{|Dw - dD\varphi + (\varphi - u)Dd|^2}{(1-d)^2} \\ &\le 2[\sup_{\Omega_\mu} \log |Dw|^2 + (|\varphi|_{C^1(\overline{\Omega})} + \sup_\Omega |u|)^2] \\ &\le 2[\log |Dw|^2(x_0) + \sup_{\Omega_\mu} |h(u)| + \sup_{\Omega_\mu} |g(d)| + (|\varphi|_{C^1(\overline{\Omega})} + \sup_\Omega |u|)^2] \\ &\le 2[\log |Du|^2(x_0) + \log(1 + 2M_0) + \alpha_0 + 2(|\varphi|_{C^1(\overline{\Omega})} + \sup_\Omega |u|)^2] \\ &\le 2[\log(1 + 2M_0) + \alpha_0 + (10n + 2)(|\varphi|_{C^1(\overline{\Omega})} + \sup_\Omega |u|)^2], \end{aligned} \tag{3.8}$$

so (3.2) holds.

Hence, we assume $|Du|(x_0) > 10n[|\varphi|_{C^1(\Omega)} + \sup_\Omega |u|]$ in the following. Then we have

$$\frac{1}{2}|Du|^2 \le |Dw|^2 \le 2|Du|^2, \tag{3.9}$$

since $w_i = (1-d)u_i + \varphi_i d + (\varphi - u)d_i$. Now we divide into three cases to complete the proof of Theorem 3.2.

♦ CASE I: $x_0 \in \partial\Omega_\mu \cap \Omega$.

Then $x_0 \in \Omega \setminus \Omega_\mu$, and we can use the interior gradient estimate, that is from Theorem 3.1,

$$|Du|(x_0) \le \sup_{\Omega\setminus\Omega_\mu} |Du| \le M_1, \tag{3.10}$$

then we can prove (3.2) by a calculation similar with (3.8).

♦ CASE II: $x_0 \in \partial\Omega$.

At x_0, we have $d = 0$, and

$$0 \le G_\nu = \frac{(|Dw|^2)_i \nu^i}{|Dw|^2} + h' u_\nu - g'. \tag{3.11}$$

We know from (3.6)

$$|Dw|^2 = |(Dw)'|^2 + w_\nu^2 = c^{pq} w_p w_q + w_\nu^2,$$

and by the Neumann boundary condition, we can get

$$w_\nu = (1-d)u_\nu + [u - \varphi] + \varphi_\nu d = 0.$$

Hence

$$\begin{aligned}(|Dw|^2)_i \nu^i &= [c^{pq}_{,i} w_p w_q + 2c^{pq} w_{pi} w_q + 2w_\nu D_i w_\nu]\nu^i \\ &= c^{pq}_{,i} w_p w_q \nu^i + 2c^{pq}[u_{pi} + (\varphi_p - u_p)d_i + (\varphi_i - u_i)d_p + (\varphi - u)d_{ij}]w_q \nu^i \\ &\le C_1 |Dw|^2 + 2c^{pq} u_{pi} w_q \nu^i + C_2[|Dw| + |Dw|^2],\end{aligned} \tag{3.12}$$

where $C_1 = \sum_{pq} |Dc^{pq}|$ and $C_2 = 2\sum_{pq} |c^{pq}|[2|D\varphi| + (|\varphi| + |u|)||D^2 d| + 4]$. Also by the Neumann boundary condition, we can get

$$c^{pq} D_p(u_i \nu^i) = c^{pq} D_p[-u + \varphi],$$

so

$$c^{pq} u_{pi} \nu^i = -c^{pq} u_i \nu^i_{,p} + c^{pq}(-u_p + \varphi_p).$$

Hence

$$2c^{pq} u_{pi} w_q \nu^i = -2c^{pq} u_i w_q \nu^i_{,p} + c^{pq}(-u_p + \varphi_p) w_q \le C_3[|Dw| + |Dw|^2], \tag{3.13}$$

where $C_3 = 4\sum_{pq} |c^{pq}|[|D^2 d| + |D\varphi| + 1]$. From (3.11), (3.12) and (3.13), we get

$$0 \le G_\nu \le C_1 + C_2 + C_3 + \frac{C_2 + C_3}{|Dw|} + \frac{-u + \varphi}{1 + M_0 - u} - \alpha_0. \tag{3.14}$$

We choose

$$\alpha_0 = C_1 + C_2 + C_3 + \max_{\overline{\Omega}} |u| + \max_{\overline{\Omega}} |\varphi| + 1, \tag{3.15}$$

then

$$|Dw| \le C_2 + C_3. \tag{3.16}$$

So we can prove (3.2) by a calculation similar with (3.8), or x_0 cannot be at the boundary $\partial\Omega$ by a contradiction discussion.

♦ CASE III: $x_0 \in \Omega_\mu$.

At x_0, we have $0 < d < \mu$. By rotating the coordinate $e_1, \cdots, e_n$, we can assume

$$w_1(x_0) = |Dw|(x_0) > 0, \quad \{u_{ij}(x_0)\}_{2\le i,j\le n} \text{ is diagonal.} \tag{3.17}$$

In the following, all the calculations are at x_0. So from the definition of w, we know $w_i = (1-d)u_i + [\varphi - u]d_i + \varphi_i d$, and by (3.17) we get

$$u_1 = \frac{w_1 - [\varphi - u]d_1 - \varphi_1 d}{1-d} > 0, \tag{3.18}$$

$$u_i = \frac{-[\varphi - u]d_i - \varphi_i d}{1-d}, \quad i \ge 2. \tag{3.19}$$

By the assumption $|Du|(x_0) > 10n[|\varphi|_{C^1(\Omega)} + \sup_\Omega |u|]$, we know for $i \ge 2$

$$|u_i| \le \frac{|\varphi| + |u| + |\varphi_i|}{1-d} \le \frac{1}{9n}|Du|(x_0), \tag{3.20}$$

hence

$$u_1 = \sqrt{|Du|^2 - \sum_{i=2}^n u_i^2} \ge \frac{1}{2}|Du| \ge \frac{1}{4}w_1. \tag{3.21}$$

Also we have at x_0,

$$\begin{aligned} 0 = G_1 &= \frac{(|Dw|^2)_1}{|Dw|^2} + h'u_1 + \alpha_0 d_1 \\ &= \frac{2w_{11}}{w_1} + h'u_1 + \alpha_0 d_1. \end{aligned} \tag{3.22}$$

From the definition of w, we know

$$\begin{aligned} w_{11} = (1-d)u_{11} &+ [\varphi - u]d_{11} + \varphi_{11} d \\ &+ [\varphi_1 - u_1]d_1 + [\varphi_1 - u_1]d_1. \end{aligned} \tag{3.23}$$

So we have

$$\begin{aligned}
u_{11} &= \frac{w_{11}}{1-d} - \frac{[\varphi-u]d_{11}+\varphi_{11}d+2[\varphi_1-u_1]d_1}{1-d}\\
&= \frac{-[h'u_1+\alpha_0 d_1]w_1}{2(1-d)} - \frac{[\varphi-u]d_{11}+\varphi_{11}d+2[\varphi_1-u_1]d_1}{1-d}\\
&\le \frac{-h'}{2(1-d)}u_1w_1 + \frac{\alpha_0 w_1}{2(1-d)} + \frac{(|\varphi|+|u|)|d_{11}|+|\varphi_{11}|+2|\varphi_1|}{1-d} + \frac{2u_1}{(1-d)}\\
&\le \frac{-h'}{4(1-d)}u_1w_1 \le -\frac{1}{16(1+2M_0)}w_1^2, \qquad (3.24)
\end{aligned}$$

if $w_1 \ge 8(1+2M_0)[\alpha_0+8+(|\varphi|+|u|)|D^2d|+|D^2\varphi|+2|D\varphi|] =: C_4$.

Moreover, from (2.5), we know

$$u_{11} \ge -\max\{\tan\big(\frac{(n-1)\pi}{2} - \min_{\overline{\Omega}}\Theta\big), \tan\big(\frac{\max_{\overline{\Omega}}\Theta}{n}\big)\}. \tag{3.25}$$

Hence we can get from (3.24) and (3.25)

$$w_1(x_0) \le C_5.$$

So we can prove (3.2) by a calculation similar with (3.8). $\square$

As discussed in Remark 1.3, we need to consider the equation (1.4) to prove Theorem 1.2. It is crucial to establish a global gradient estimate of u^ε independent of ε and we need the strict convexity of Ω. Following the idea of [21], we can easily obtain

Theorem 3.3. *Suppose $\Omega \subset \mathbb{R}^n$ is a C^3 strictly convex domain and $\varphi \in C^3(\partial\Omega)$. Let $\Theta(x) \in C^1(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$ and $u^\varepsilon \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be the solution of special Lagrangian equation (1.4) with $\varepsilon > 0$ sufficiently small, then we have*

$$\sup_{\overline{\Omega}} |Du^\varepsilon| \le \overline{M_1}, \tag{3.26}$$

and

$$\sup_{\overline{\Omega}} |u^\varepsilon - \frac{1}{|\Omega|}\int_\Omega u^\varepsilon| \le \overline{M_1}, \tag{3.27}$$

where $\overline{M_1}$ depends on n, Ω, $\max_{\overline{\Omega}}\Theta$, $\min_{\overline{\Omega}}\Theta$, $|\Theta|_{C^1}$ and $|\varphi|_{C^3}$.

Proof. It is easy to check that (3.27) follows (3.26). So we only need to prove (3.26).

As discussed in the proof of Theorem 3.2, the distance function $d(x) \in C^3(\overline{\Omega_\mu})$ since Ω is a C^3 domain here. Now we extend d such that $d \in C^3(\overline{\Omega})$ with a universal C^3 norm. Because Ω is a C^3 strictly convex domain, we have the defining function $\rho \in C^3(\overline{\Omega})$ such that

$$\rho = 0 \text{ on } \partial\Omega, \quad \rho < 0 \text{ in } \Omega;$$
$$|D\rho| = 1 \text{ on } \partial\Omega;$$
$$D^2\rho \geq k_0 I_n;$$

where $k_0 > 0$ depends only on Ω and I_n is the $n \times n$ identity matrix.

For sufficiently small $\varepsilon > 0$ (to be determined later), we consider the auxiliary function

$$G(x) = \log |Dw|^2 + \alpha\rho, \tag{3.28}$$

where $w(x) = (1 - \varepsilon d)u^\varepsilon + \varphi(x)d(x)$ and $\alpha > 0$ is a small number to be determined later. Also note that here $\varphi \in C^3(\overline{\Omega})$ is a extension with universal C^3 norm. We assume that $G(x)$ attains its maximum at a point $x_0 \in \overline{\Omega}$. We claim $|Dw(x_0)|^2$ is bounded or x_0 must be on the boundary $\partial\Omega$. If $|Dw(x_0)|^2$ is bounded, we can easily obtain (3.26). Otherwise, if $|Dw(x_0)|^2$ is large and $x_0 \in \Omega$, we will get a contradiction in the following.

At x_0, by rotating the coordinate $e_1, \cdots, e_n$, we can assume $\{D^2u^\varepsilon(x_0)\}$ is diagonal and all the calculations are at x_0. So we have at x_0,

$$0 = G_i = \frac{(|Dw|^2)_i}{|Dw|^2} + \alpha\rho_i, \tag{3.29}$$

and

$$\begin{aligned} 0 \geq G_{ii} &= \frac{(|Dw|^2)_{ii}}{|Dw|^2} - \frac{[(|Dw|^2)_i]^2}{[|Dw|^2]^2} + \alpha\rho_{ii} \\ &= \frac{(|Dw|^2)_{ii}}{|Dw|^2} - \alpha^2\rho_i^2 + \alpha\rho_{ii}. \end{aligned} \tag{3.30}$$

Also we have

$$F^{ij} =: \frac{\partial \arctan D^2u^\varepsilon}{\partial u^\varepsilon_{ij}} = \begin{cases} \frac{1}{1+{u^\varepsilon_{ii}}^2}, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

So we have

$$0 \geq \sum_{i=1}^n F^{ii}G_{ii} = \frac{\sum_{i=1}^n F^{ii}(|Dw|^2)_{ii}}{|Dw|^2} - \alpha^2 \sum_{i=1}^n F^{ii}\rho_i^2 + \alpha \sum_{i=1}^n F^{ii}\rho_{ii}, \tag{3.31}$$

where

$$\begin{aligned} \sum_{i=1}^n F^{ii}(|Dw|^2)_{ii} &= \sum_{i=1}^n F^{ii}[2\sum_{k=1}^n w_{ki}^2 + 2\sum_{k=1}^n w_k w_{kii}] \\ &\geq 2\sum_{ik=1}^n w_k F^{ii}[(1-\varepsilon d)u^\varepsilon_{kii} - \varepsilon(2d_i u^\varepsilon_{ki} + d_k u^\varepsilon_{ii} + 2d_{ki}u^\varepsilon_i + d_{ii}u^\varepsilon_k) + (\varphi d)_{kii}] \end{aligned}$$

$$\geq -2|Dw|\Big[(1-\varepsilon d)|D\Theta| + \varepsilon(\frac{3n}{2}|Dd| + 3|D^2 d||Du|\sum_{i=1}^{n} F^{ii})$$

$$+ |D^3(\varphi d)|\sum_{i=1}^{n} F^{ii}\Big]$$

$$\geq -C(|Dw| + \varepsilon|Dw|^2)\sum_{i=1}^{n} F^{ii}. \tag{3.32}$$

So from (3.31) and (3.32), we have

$$0 \geq \sum_{i=1}^{n} F^{ii} G_{ii} \geq -C\frac{|Dw| + \varepsilon|Dw|^2}{|Dw|^2}\sum_{i=1}^{n} F^{ii} - \alpha^2|D\rho|^2\sum_{i=1}^{n} F^{ii} + \alpha k_0\sum_{i=1}^{n} F^{ii} > 0, \tag{3.33}$$

if $\alpha = \frac{k_0}{3\max_{\overline{\Omega}}|D\rho|^2}$, $\varepsilon < \frac{\alpha k_0}{3C}$ and $|Dw(x_0)| \geq \frac{3C}{\alpha k_0}$. This is a contradiction. So $x_0 \in \partial\Omega$.

Let n denote the unit outer normal derivate and $1 \leq i \leq n-1$ denote the tangential direction on the boundary. D denotes the derivative in $\mathbb{R}^n$ and $\bigtriangledown$ denotes the derivative on the boundary. We also denote $\bigtriangledown_i(u_n) := u_{ni}$ for $1 \leq i \leq n-1$. By the boundary condition, $w_n = 0$ on $\partial\Omega$ and $w_{ni} = 0$ for $i = 1, \cdots, n-1$.

By the Gauss-Weingarten formula, we have

$$D_{kn}u = u_{nk} - h_{kj}u_j.$$

We have

$$\begin{aligned} 0 \leq G_\nu(x_0) &= \frac{(|Dw|^2)_\nu}{|Dw|^2} + \alpha\rho_\nu \\ &= \sum_{i,k=1}^{n-1}\frac{2w_k D_{kn}w}{|Dw|^2} + \alpha\rho_\nu \\ &= \sum_{k=1}^{n-1}\frac{2w_k w_{nk} - 2\sum_{i=1}^{n-1} w_k w_i h_{ik}}{|Dw|^2} + \alpha\rho_\nu \\ &= -2\frac{\sum\limits_{ik=1}^{n-1} w_k w_i h_{ik}}{|Dw|^2} + \alpha\rho_\nu \\ &\leq -2c + \alpha|D\rho|, \end{aligned} \tag{3.34}$$

where h_{ij} is the second fundamental form of $\partial\Omega$ with $\{h_{ij}\} \geq cI_{n-1} > 0$. If $\alpha < \frac{c}{\max_{\overline{\Omega}}|D\rho|}$, we get a contradiction. Hence $|Dw(x_0)|$ is bounded and (3.26) holds. $\square$

4. Global second derivatives estimate

We come now to the a priori estimates of global second derivatives and we obtain the following theorem

Theorem 4.1. *Suppose $\Omega \subset \mathbb{R}^n$ is a C^4 strictly convex domain and $\varphi \in C^3(\partial\Omega)$. Let $\Theta(x) \in C^2(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$ and $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ be a solution of special Lagrangian equation (1.2), then we have*

$$\sup_{\overline{\Omega}} |D^2 u| \le M_2, \tag{4.1}$$

where M_2 depends on n, Ω, $\max_{\overline{\Omega}} \Theta$, $\min_{\overline{\Omega}} \Theta$, $|u|_{C^1}$, $|\Theta|_{C^2}$ and $|\varphi|_{C^3}$.

Following the idea of Lions-Trudinger-Urbas [19] and Ma-Qiu [20], we divide the proof of Theorem 4.1 into two steps. We firstly reduce global second derivatives to double normal second derivatives on boundary and then we prove the estimate of double normal second derivatives on boundary.

4.1. Reduce global second derivatives to double normal second derivatives on boundary

Lemma 4.2. *Suppose $\Omega \subset \mathbb{R}^n$ is a C^4 convex domain and $\varphi \in C^3(\partial\Omega)$. Let $\Theta(x) \in C^2(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$ and $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ be a solution of special Lagrangian equation (1.2), then we have*

$$\sup_{\overline{\Omega}} |D^2 u| \le C_6 (1 + \sup_{\partial\Omega} |u_{\nu\nu}|), \tag{4.2}$$

where C_6 depends on n, Ω, $\max_{\overline{\Omega}} \Theta$, $\min_{\overline{\Omega}} \Theta$, $|u|_{C^1}$, $|\Theta|_{C^2}$ and $|\varphi|_{C^3}$.

Proof. Since Ω is a C^4 domain, it is well known that there exists a small positive universal constant $0 < \mu < \frac{1}{10}$ such that $d(x) \in C^4(\overline{\Omega_\mu})$ and $\nu = -Dd$ on $\partial\Omega$. We define $\widetilde{d} \in C^4(\overline{\Omega})$ such that $\widetilde{d} = d$ in $\overline{\Omega_\mu}$ and denote

$$\nu = -D\widetilde{d}, \quad \text{in } \Omega.$$

In fact, ν is a $C^3(\overline{\Omega})$ extension of the outer unit normal vector field on $\partial\Omega$.

We assume $0 \in \Omega$ and consider the function

$$v(x, \xi) = u_{\xi\xi} - v'(x, \xi) + K|x|^2, \tag{4.3}$$

where $v'(x, \xi) = 2(\xi \cdot \nu)\xi' \cdot (D\varphi - Du - u_l D\nu^l) = a^l u_l + b$, $\xi' = \xi - (\xi \cdot \nu)\nu$, $a^l = -2(\xi \cdot \nu)(\xi' \cdot D\nu^l) - 2(\xi \cdot \nu)(\xi')^l$, $b = 2(\xi \cdot \nu)(\xi' \cdot D\varphi)$, and $K > 0$ is to be determined later. Also note that here $\varphi \in C^3(\overline{\Omega})$ is an extension with universal C^3 norm. Denote

$$F^{ij} = \frac{\partial \arctan D^2 u}{\partial u_{ij}}.$$

For any $x \in \Omega$, we can assume D^2u is diagonal with $\lambda_i = u_{ii}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then we have

$$F^{ij} =: \frac{\partial \arctan D^2u}{\partial u_{ij}} = \begin{cases} \frac{1}{1+\lambda_i^2}, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Hence we can get from Property 2.1

$$\begin{aligned} &F^{11} \leq F^{22} \leq \cdots \leq F^{nn}; \\ &F^{nn} = \frac{1}{1+\lambda_n^2} \geq c_0 > 0; \\ &F^{ii}u_{ii} = \frac{\lambda_i}{1+\lambda_i^2} \in (-\frac{1}{2}, \frac{1}{2}); \end{aligned}$$

where $c_0 = \frac{1}{1+\max\{\tan\left(\frac{(n-1)\pi}{2} - \min\limits_{\overline{\Omega}} \Theta\right), \tan\left(\frac{\max_{\overline{\Omega}} \Theta}{n}\right)\}^2}$.

For any fixed $\xi \in \mathbb{S}^{n-1}$, we have from Property 2.2

$$\begin{aligned} \sum_{ij=1}^n F^{ij} v_{ij} &= \sum_{ij=1}^n F^{ij} u_{ij\xi\xi} - \sum_{ij=1}^n F^{ij}[a^l u_{ijl} + 2D_i a^l u_{jl} + D_{ij}a^l u_l + b_{ij}] + 2K\sum_{i=1}^n F^{ii} \\ &\geq \Theta_{\xi\xi} - \frac{2}{\tan\left(\min\limits_{\overline{\Omega}} \Theta - \frac{(n-2)\pi}{2}\right)} {\Theta_\xi}^2 - a^l \Theta_l - \sum_{i=1}^n |D_i a^i| \\ &\quad - \sum_{i=1}^n F^{ii}[D_{ii}a^l u_l + b_{ii}] + K\sum_{i=1}^n F^{ii} \\ &> 0, \end{aligned}$$

if we choose

$$K = \frac{1}{c_0}\Big[|D^2\Theta| + \frac{2}{\tan\left(\min\limits_{\overline{\Omega}} \Theta - \frac{(n-2)\pi}{2}\right)}|D\Theta|^2 + |a^l||D\Theta| + |Da^l|\Big] + |D^2a^l||Du| + |D^2b| + 1.$$

So $\max\limits_{\overline{\Omega}} v(x, \xi)$ attains its maximum on $\partial\Omega$. Hence $\max\limits_{\Omega\times\mathbb{S}^{n-1}} v(x, \xi)$ attains its maximum at some point $x_0 \in \partial\Omega$ and some direction $\xi_0 \in \mathbb{S}^{n-1}$.

Case a: ξ_0 is tangential to $\partial\Omega$ at x_0.

We directly have $\xi_0 \cdot \nu = 0$, $v'(x_0, \xi_0) = 0$, and $u_{\xi_0\xi_0}(x_0) > 0$. In the following, all the calculations are at the point x_0 and $\xi = \xi_0$.

From the boundary condition, we have

$$\begin{aligned} u_{li}\nu^l &= [c^{ij} + \nu^i\nu^j]\nu^l u_{lj} \\ &= c^{ij}[D_j(\nu^l u_l) - D_j\nu^l u_l] + \nu^i\nu^j\nu^l u_{lj} \end{aligned}$$

$$= -c^{ij}u_j + c^{ij}D_j\varphi - c^{ij}u_l D_j\nu^l + \nu^i\nu^j\nu^l u_{lj}. \tag{4.4}$$

So it follows that

$$\begin{aligned} u_{lip}\nu^l &= [c^{pq} + \nu^p\nu^q]u_{liq}\nu^l \\ &= c^{pq}[D_q(u_{li}\nu^l) - u_{li}D_q\nu^l] + \nu^p\nu^q u_{liq}\nu^l \\ &= c^{pq}D_q(-c^{ij}u_j + c^{ij}D_j\varphi - c^{ij}u_l D_j\nu^l + \nu^i\nu^j\nu^l u_{lj}) \\ &\quad - c^{pq}u_{li}D_q\nu^l + \nu^p\nu^q\nu^l u_{liq}, \end{aligned} \tag{4.5}$$

then we obtain

$$\begin{aligned} u_{\xi_0\xi_0\nu} &= \sum_{ipl=1}^{n} \xi_0^i\xi_0^p u_{lip}\nu^l \\ &= \sum_{ip=1}^{n} \xi_0^i\xi_0^p[c^{pq}D_q(-c^{ij}u_j + c^{ij}D_j\varphi - c^{ij}u_l D_j\nu^l + \nu^i\nu^j\nu^l u_{lj}) \\ &\qquad - c^{pq}u_{li}D_q\nu^l + \nu^p\nu^q\nu^l u_{liq}] \\ &= \sum_{i=1}^{n} \xi_0^i\xi_0^q[D_q(-c^{ij}u_j + c^{ij}D_j\varphi - c^{ij}u_l D_j\nu^l + \nu^i\nu^j\nu^l u_{lj}) - u_{li}D_q\nu^l] \\ &= -\xi_0^i\xi_0^q[c^{ij}u_{jq} - D_q c^{ij}u_j] + \xi_0^i\xi_0^q D_q(c^{ij}D_j\varphi) \\ &\quad - \xi_0^i\xi_0^q D_q(c^{ij}D_j\nu^l)u_l - \xi_0^j\xi_0^q u_{lq}D_j\nu^l + \xi_0^i\xi_0^q D_q\nu^i u_{\nu\nu} - \xi_0^i\xi_0^q u_{lq}D_i\nu^l \\ &\le -u_{\xi_0\xi_0} - 2\xi_0^i u_{l\xi_0}D_i\nu^l + C_7 + C_7|Du| + C_7|u_{\nu\nu}|. \end{aligned} \tag{4.6}$$

We assume $\xi_0 = e_1$, it is easy to get the bound for $u_{1i}(x_0)$ for $i > 1$ from the maximum of $v(x,\xi)$ in the ξ_0 direction. In fact, we can assume $\xi(t) = \frac{(1,t,0,\cdots,0)}{\sqrt{1+t^2}}$. Then we have

$$\begin{aligned} 0 &= \frac{dv(x_0,\xi(t))}{dt}|_{t=0} \\ &= 2u_{ij}(x_0)\frac{d\xi^i(t)}{dt}|_{t=0}\xi^j(0) - \frac{dv'(x_0,\xi(t))}{dt}|_{t=0} \\ &= 2u_{12}(x_0) - 2\nu^2(D_1\varphi - u_1 - u_l D_1\nu^l), \end{aligned} \tag{4.7}$$

so

$$|u_{12}(x_0)| = |\nu^2(D_1\varphi - u_1 - u_l D_1\nu^l)| \le C_8 + C_8|Du|. \tag{4.8}$$

Similarly, we have for all $i > 1$,

$$|u_{1i}(x_0)| \le C_8 + C_8|Du|. \tag{4.9}$$

So by $\{D_i\nu^l\} \geq 0$, we have

$$\begin{aligned} u_{\xi_0\xi_0\nu} &\leq -u_{\xi_0\xi_0} - D_1\nu^1 u_{\xi_0\xi_0} + C_9(1+|u_{\nu\nu}|) \\ &\leq -u_{\xi_0\xi_0} + C_9(1+|u_{\nu\nu}|). \end{aligned} \tag{4.10}$$

On the other hand, we have from the Hopf lemma and (4.4),

$$\begin{aligned} 0 &\leq v_\nu(x_0,\xi_0) \\ &= u_{\xi_0\xi_0\nu} - a^l u_{l\nu} - D_\nu a^l u_l - b_\nu + 2K(x\cdot\nu) \\ &\leq -u_{\xi_0\xi_0} + C_9(1+|u_{\nu\nu}|) + C_{10}. \end{aligned} \tag{4.11}$$

Then we get

$$u_{\xi_0\xi_0}(x_0) \leq (C_9 + C_{10})(1+|u_{\nu\nu}|). \tag{4.12}$$

Because u is subharmonic function and (4.12), we obtain

$$\begin{aligned} \max_{\Omega\times\mathbb{S}^{n-1}} |u_{\xi\xi}(x)| &\leq (n-1)\max_{\Omega\times\mathbb{S}^{n-1}} u_{\xi\xi}(x) \\ &\leq (n-1)[\max_{\Omega\times\mathbb{S}^{n-1}} v(x,\xi) + C_{11}] = (n-1)[v(x_0,\xi_0)+C_{11}] \\ &\leq (n-1)[u_{\xi_0\xi_0}(x_0) + 2C_{11}] \\ &\leq C_{12}(1+|u_{\nu\nu}|). \end{aligned} \tag{4.13}$$

Case b: ξ_0 is non-tangential.

We can directly have $\xi_0\cdot\nu \neq 0$. We can find a tangential vector τ such that $\xi_0 = \alpha\tau + \beta\nu$, with $\alpha = \xi_0\cdot\tau \geq 0$, $\beta = \xi_0\cdot\nu \neq 0$, $\alpha^2+\beta^2 = 1$ and $\tau\cdot\nu = 0$. Then we have

$$\begin{aligned} u_{\xi_0\xi_0}(x_0) &= \alpha^2 u_{\tau\tau}(x_0) + \beta^2 u_{\nu\nu}(x_0) + 2\alpha\beta u_{\tau\nu}(x_0) \\ &= \alpha^2 u_{\tau\tau}(x_0) + \beta^2 u_{\nu\nu}(x_0) + 2(\xi_0\cdot\nu)[\xi_0 - (\xi_0\cdot\nu)\nu][D\varphi - Du - u_l D\nu^l], \end{aligned} \tag{4.14}$$

hence

$$v(x_0,\xi_0) = \alpha^2 v(x_0,\tau) + \beta^2 v(x_0,\nu) \leq \alpha^2 v(x_0,\xi_0) + \beta^2 v(x_0,\nu). \tag{4.15}$$

From the definition of $v(x_0,\xi_0)$, we know

$$v(x_0,\xi_0) \leq v(x_0,\nu), \tag{4.16}$$

and

$$u_{\xi_0\xi_0}(x_0) \leq v(x_0,\xi_0) + C_{11} = v(x_0,\nu) + C_{11} \leq |u_{\nu\nu}| + 2C_{11}. \tag{4.17}$$

Similarly with (4.13), we can prove (4.2). □

Remark 4.3. It is noticeable that in the entire proof of Lemma 4.2, we only use the convexity of the domain in (4.10). Actually, if we consider more general type Neumann boundary problem like $u_\nu = \varphi(x, u)$, the above proof still works if $\varphi_u - \kappa_{min} < 0$ on the boundary, where κ_{min} is the smallest principal curvature of the boundary.

4.2. Estimate of double normal second derivatives on boundary

From Property 2.1, we know the smallest eigenvalue is bounded below. Hence we can directly obtain the lower estimate of double normal second derivatives on boundary as follows

Lemma 4.4. *Suppose $\Omega \subset \mathbb{R}^n$ is a domain and $\Theta(x) \in C^0(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$. Let $u \in C^2(\overline{\Omega})$ be the solution of special Lagrangian equation (1.2), then we have*

$$\min_{\partial\Omega} u_{\nu\nu} \geq -\max\{\tan\big(\frac{(n-1)\pi}{2} - \min_{\overline{\Omega}}\Theta\big), \tan\big(\frac{\max_{\overline{\Omega}}\Theta}{n}\big)\}. \tag{4.18}$$

In the following, we establish the upper estimate of double normal second derivatives on boundary.

Lemma 4.5. *Suppose $\Omega \subset \mathbb{R}^n$ is a C^3 strictly convex domain and $\varphi \in C^2(\partial\Omega)$. Let $\Theta(x) \in C^2(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$ and $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ be a solution of special Lagrangian equation (1.2), then we have*

$$\max_{\partial\Omega} u_{\nu\nu} \leq C_{13}, \tag{4.19}$$

where C_{13} depends on n, Ω, $\max_{\overline{\Omega}}\Theta$, $\min_{\overline{\Omega}}\Theta$, $|u|_{C^1}$, $|\Theta|_{C^2}$ and $|\varphi|_{C^2}$.

Proof. Because Ω is a C^3 strictly convex domain, we have the defining function $\rho \in C^3(\overline{\Omega})$ for it such that

$$\begin{aligned}
&\rho = 0 \text{ on } \partial\Omega, \quad \rho < 0 \text{ in } \Omega;\\
&|D\rho| = 1 \text{ on } \partial\Omega;\\
&D^2\rho \geq k_0 I_n;
\end{aligned}$$

where $k_0 > 0$ depending only on Ω, and I_n is the $n \times n$ identity matrix. Also, $\nu = (\nu^1, \nu^2, \cdots, \nu^n)$ is a $C^2(\overline{\Omega})$ extension of the outer unit normal vector field on $\partial\Omega$ as in Lemma 4.2.

By the classical barrier technique (see [19] or [20]), we consider the test function

$$P(x) = u_\nu + u(x) - \varphi(x) - K\rho, \tag{4.20}$$

where $K = \max\{\frac{2(1+C_0{}^2)}{k_0}(|D\Theta||\nu| + |D\nu| + \frac{n}{2}), \frac{2}{k_0}(|Du||D^2\nu| + |D^2\varphi|)\}$ and C_0 is defined in (2.5). Also note that here $\varphi \in C^2(\overline{\Omega})$ is an extension with universal C^2 norm. Denote

$$F^{ij}=\frac{\partial \arctan D^2u}{\partial u_{ij}}.$$

For any $x\in\Omega$, we can assume D^2u is diagonal with $\lambda_i=u_{ii}$ and $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n$. Then we have

$$F^{ij}=:\frac{\partial \arctan D^2u}{\partial u_{ij}}=\begin{cases}\frac{1}{1+\lambda_i^2}, & \text{if } i=j,\\ 0, & \text{if } i\neq j.\end{cases}$$

Hence we can get

$$\begin{aligned}\sum_{ij=1}^{n}F^{ij}P_{ij}&=\sum_{i=1}^{n}F^{ii}\Big\{\sum_{l}[u_{lii}\nu^l+2u_{li}(\nu^l)_i+u_l(\nu^l)_{ii}]+u_{ii}-\varphi_{ii}\Big\}-K\sum_{i=1}^{n}F^{ii}\rho_{ii}\\&=\sum_{l}\Theta_l\nu^l+2\sum_{i=1}^{n}F^{ii}u_{ii}(\nu^i)_i+\sum_{i=1}^{n}F^{ii}u_l(\nu^l)_{ii}+\sum_{i=1}^{n}F^{ii}u_{ii}-\sum_{i=1}^{n}F^{ii}\varphi_{ii}\\&\quad-K\sum_{i=1}^{n}F^{ii}\rho_{ii}\\&\leq|D\Theta||\nu|+|D\nu|+|Du||D^2\nu|\sum_{i=1}^{n}F^{ii}+\frac{n}{2}+|D^2\varphi|\sum_{i=1}^{n}F^{ii}-Kk_0\sum_{i=1}^{n}F^{ii}\\&\leq|D\Theta||\nu|+|D\nu|+\frac{n}{2}-\frac{Kk_0}{2}\frac{1}{1+{C_0}^2}\\&\quad+|Du||D^2\nu|\sum_{i=1}^{n}F^{ii}+|D^2\varphi|\sum_{i=1}^{n}F^{ii}-\frac{Kk_0}{2}\sum_{i=1}^{n}F^{ii}\\&\leq 0.\end{aligned}\tag{4.21}$$

Also, it is easy to know $P=0$ on $\partial\Omega$. Hence P attains its minimum on any boundary point. Then we can get for any $x\in\partial\Omega$,

$$\begin{aligned}0\geq P_\nu(x)&=[u_{\nu\nu}-\sum_j u_jd_{j\nu}+u_\nu-\varphi_\nu]-K\rho_\nu\\&\geq u_{\nu\nu}-|Du||D^2d|-|Du|-|D\varphi|-K,\end{aligned}\tag{4.22}$$

hence (4.19) holds. □

Remark 4.6. If we consider the Neumann boundary like $u_\nu=\varphi(x,u)$, the double normal second derivative's estimate still holds by constructing $P(x)=u_\nu-\varphi(x,u)-K\rho$.

Following the above proofs, we also can obtain the estimates of second order derivatives of u^ε in (1.4), and the strict convexity of Ω is important in reducing global second derivatives to double normal second derivatives on boundary. So we have

Theorem 4.7. *Suppose $\Omega \subset \mathbb{R}^n$ is a C^4 strictly convex domain and $\varphi \in C^3(\partial\Omega)$. Let $\Theta(x) \in C^2(\overline{\Omega})$ with $\frac{(n-2)\pi}{2} < \Theta(x) < \frac{n\pi}{2}$ in $\overline{\Omega}$ and $u \in C^4(\Omega) \cap C^3(\overline{\Omega})$ be the solution of special Lagrangian equation (1.4) with $\varepsilon > 0$ sufficiently small, then we have*

$$\sup_{\overline{\Omega}} |D^2 u^\varepsilon| \leq \overline{M_2}, \tag{4.23}$$

where $\overline{M_2}$ depends on n, Ω, $\max_{\overline{\Omega}} \Theta$, $\min_{\overline{\Omega}} \Theta$, $|\Theta|_{C^2}$ and $|\varphi|_{C^3}$.

5. Existence of the boundary problems

In this section we complete the proofs of the Theorem 1.1 and Theorem 1.2.

5.1. *Proof of Theorem 1.1*

For the Neumann problem of special Lagrangian equation (1.2), we have established the C^0, C^1 and C^2 estimates in Section 2, Section 3, Section 4, respectively. By the global C^2 priori estimate, the special Lagrangian equation (1.2) is uniformly elliptic in $\overline{\Omega}$. From the concavity Lemma (Lemma 2.2 in [7]), we know $-e^{-A \arctan D^2 u}$ is concave with respect to $D^2 u$, where A is defined in Property 2.2. Following the discussions in [18], we can get the global Hölder estimate of second derivatives,

$$|u|_{C^{2,\alpha}(\overline{\Omega})} \leq C, \tag{5.1}$$

where C and α depend on n, Ω, $\max_{\overline{\Omega}} \Theta$, $\min_{\overline{\Omega}} \Theta$, $|\Theta|_{C^2}$ and $|\varphi|_{C^3}$. From (5.1), one also obtains $C^{3,\alpha}(\overline{\Omega})$ estimates by differentiating the equation (1.2) and applies the Schauder theory for linear uniformly elliptic equations.

Applying the method of continuity (see [17]), the existence of the classical solution holds. By the standard regularity theory of uniformly elliptic partial differential equations, we can obtain the higher regularity.

5.2. *Proof of Theorem 1.2*

The proof is following the idea of [19] and [1]. By a similar proof of Theorem 1.1, we know there exists a unique solution $u^\varepsilon \in C^{3,\alpha}(\overline{\Omega})$ to (1.4) for any small $\varepsilon > 0$. Let $v^\varepsilon = u^\varepsilon - \frac{1}{|\Omega|}\int_\Omega u^\varepsilon$, and it is easy to know v^ε satisfies

$$\begin{cases} \arctan D^2 v^\varepsilon = \Theta(x), & \text{in} \quad \Omega, \\ (v^\varepsilon)_\nu = -\varepsilon v^\varepsilon - \frac{1}{|\Omega|}\int_\Omega \varepsilon u^\varepsilon + \varphi(x), & \text{on} \quad \partial\Omega. \end{cases} \tag{5.2}$$

By the global gradient estimate (3.26), it is easy to know $\varepsilon \sup_{\overline{\Omega}} |Du^\varepsilon| \to 0$. Hence there is a constant β and a function $v \in C^2(\overline{\Omega})$, such that $-\varepsilon u^\varepsilon \to \beta$, $-\varepsilon v^\varepsilon \to 0$, $-\frac{1}{|\Omega|}\int_\Omega \varepsilon u^\varepsilon \to \beta$ and $v^\varepsilon \to v$ uniformly in $C^2(\overline{\Omega})$ as $\varepsilon \to 0$. It is easy to verify that v is a solution of

$$\begin{cases} \arctan D^2 v = \Theta(x), & \text{in} \quad \Omega, \\ v_\nu = \beta + \varphi(x), & \text{on} \quad \partial\Omega. \end{cases} \tag{5.3}$$

If there is another function $v_1 \in C^2(\overline{\Omega})$ and another constant β_1 such that

$$\begin{cases} \arctan D^2 v_1 = \Theta(x), & \text{in} \quad \Omega, \\ (v_1)_\nu = \beta_1 + \varphi(x), & \text{on} \quad \partial\Omega. \end{cases} \tag{5.4}$$

Applying the maximum principle and Hopf Lemma, we can know $\beta = \beta_1$ and $v - v_1$ is a constant. By the standard regularity theory of uniformly elliptic partial differential equations, we can obtain the higher regularity.

References

[1] S.J. Altschuler, L.F. Wu, Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle, Calc. Var. Partial Differ. Equ. 2 (1994) 101–111.

[2] S. Brendle, M. Warren, A boundary value problem for minimal Lagrangian graphs, J. Differ. Geom. 84 (2010) 267–287.

[3] L. Caffarelli, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations I, Monge-Ampère equations, Commun. Pure Appl. Math. 37 (1984) 369–402.

[4] L. Caffarelli, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations III, functions of the eigenvalues of the Hessian, Acta Math. 155 (1985) 261–301.

[5] C.Q. Chen, D.K. Zhang, The Neumann problem for Hessian quotient equations, 2018, submitted for publication.

[6] Y.Z. Chen, L.C. Wu, Second Order Elliptic Equations and Elliptic Systems, Amer. Math. Soc., Providence, RI, 1998.

[7] T.C. Collins, S. Picard, X. Wu, Concavity of the Lagrangian phase operator and applications, Calc. Var. 56 (2017) 89.

[8] L.C. Evans, Classical solutions of fully nonlinear convex second-order elliptic equations, Commun. Pure Appl. Math. 35 (1982) 333–363.

[9] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, Grundlehren der Mathematischen Wissenschaften, vol. 224, Springer-Verlag, Berlin-New York, ISBN 3-540-08007-4, 1977, x+401 pp.

[10] R. Harvey, B. Lawson, Calibrated geometries, Acta Math. 148 (1) (1982) 47–157.

[11] N. Ivochkina, Solutions of the Dirichlet problem for certain equations of Monge-Ampère type (in Russian), Mat. Sb. 128 (1985) 403–415, English translation in Math. USSR Sb. 56 (1987).

[12] F. Jiang, N.S. Trudinger, Oblique boundary value problems for augmented Hessian equations I, Bull. Math. Sci. 8 (2018) 353–411.

[13] F. Jiang, N.S. Trudinger, Oblique boundary value problems for augmented Hessian equations II, Nonlinear Anal. 154 (2017) 148–173.

[14] N.V. Krylov, Boundedly inhomogeneous elliptic and parabolic equations (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. 46 (3) (1982) 487–523, 670.

[15] N.V. Krylov, Boundedly inhomogeneous elliptic and parabolic equations in a domain (Russian), Izv. Akad. Nauk SSSR, Ser. Mat. 47 (1) (1983) 75–108.

[16] N.V. Krylov, Degenerate nonlinear elliptic equations (Russian), Mat. Sb. (N.S.) 120(162) (3) (1983) 311–330, 448.

[17] G. Lieberman, Oblique Boundary Value Problems for Elliptic Equations, World Scientific Publishing, 2013.

[18] G. Lieberman, N. Trudinger, Nonlinear oblique boundary value problems for nonlinear elliptic equations, Trans. Am. Math. Soc. 295 (2) (1986) 509–546.

[19] P.L. Lions, N. Trudinger, J. Urbas, The Neumann problem for equations of Monge-Ampère type, Commun. Pure Appl. Math. 39 (1986) 539–563.

[20] X.N. Ma, G.H. Qiu, The Neumann problem for Hessian equations, Commun. Math. Phys. 366 (2019) 1–28.

[21] G.H. Qiu, C. Xia, Classical Neumann problems for Hessian equations and Alexandrov-Fenchel's inequalities, Int. Math. Res. Not. rnx296 (2018).

[22] L. Simon, J. Spruck, Existence and regularity of a capillary surface with prescribed contact angle, Arch. Ration. Mech. Anal. 61 (1976) 19–34.

[23] A. Strominger, S.T. Yau, E. Zaslow, Mirror symmetry is T-duality, Nucl. Phys. B 479 (1–2) (1996) 243–259.

[24] N.S. Trudinger, On degenerate fully nonlinear elliptic equations in balls, Bull. Aust. Math. Soc. 35 (02) (1987) 299–307.
[25] N.S. Trudinger, On the Dirichlet problem for Hessian equations, Acta Math. 175 (1995) 151–164.
[26] J. Urbas, On the second boundary value problem for equations of Monge-Ampère type, J. Reine Angew. Math. 487 (1997) 115–124.
[27] D.K. Wang, Y. Yuan, Hessian estimates for special Lagrangian equations with critical and supercritical phases in general dimensions, Am. J. Math. 136 (2014) 481–499.
[28] M. Warren, Y. Yuan, Hessian and gradient estimates for three dimensional special Lagrangian equations with large phase, Am. J. Math. 132 (2010) 751–770.
[29] Y. Yuan, Global solutions to special Lagrangian equations, Proc. Am. Math. Soc. 134 (5) (2006) 1355–1358.