THE OBATA TYPE INTEGRAL IDENTITY AND ITS APPLICATION

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ABSTRACT. In this paper we review the technique on Obata type integral identity, which was the first used by Obata in 1971, it concerned the constant scalar curvature of the conformal transformation of the standard metric on sphere, then we review the Gidas-Spruck theorem on the semilinear elliptic equation in $\mathbb{R}^n$ which was the first times to use the Obata integral identity to semilinear elliptic equation, and the Escobar theorem on the unit ball with critical exponent growth on boundary which was related to best Sobolev constant in upper half space. Lin-Ou develop the similar idea to obtain a Liouville type theorem to the harmonic function with subcritical exponent growth on boundary. At last we review the result by Ma-Ou to obtain a Liouville type theorem for the classical semilinear subcritical elliptic equation on Heisenberg group, the soul of the proof is an generalized Obata type integral identity in CR geometry found by Jerison and Lee in 1988.

1. INTRODUCTION

The Obata identity, which first appeared in Obata [21] in 1971, it concerned the constant scalar curvature of the conformal transformation $\bar{g}$ of the standard metric g on sphere, Obata proved that the new metric is obtained from the standard metric by a conformal diffeomorphissm of the sphere. This technique was used in many place in geometry partial differential equations, we shall review this technique and give some new applications.

Since the argument of Obata is quite subtle as it requires using the unknown metric $\bar{g}$ as the background metric instead of the given standard metric g on sphere, we shall give the proof via the given standard metric g on sphere. If we consider the metric $\bar{g} = u^{\frac{4}{n-2}} g$ on $\mathbb{S}^n$, a conformal metric $\bar{g}$ has constant scalar curvature $n(n-1)$ iff

$$-\frac{4}{n(n-2)}\Delta u + u = u^{\frac{n+2}{n-2}}, \quad \text{on} \quad \mathbb{S}^n. \tag{1.1}$$

In other words Obata [21] gave the following classification of the solutions for the equation (1.1)

$$u(x) = c[cosht + (sinht)x \cdot a]^{\frac{-(n-2)}{2}}, \tag{1.2}$$

where for some $c > 0, \quad t \geq 0$, and $a \in \mathbb{S}^n$.

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There are now several different proofs for this theorem. Analytically, by the stereographic projection, it is equivalent to the following equation

$$-\Delta w=\frac{n(n-2)}{4}w^{\frac{n+2}{n-2}}\quad \text{on}\quad \mathbb{R}^n \tag{1.3}$$

whose positive solutions were classified by Gidas-Ni-Nirenberg [13] and Caffarelli-Gidas-Spruck [6] using the moving plane method, and every positive solution of (1.3) assumes the form $w(x)=(4\lambda^2)^{\frac{n-2}{4}}[\lambda^2+|x-x_0|^2]^{-\frac{n-2}{2}}$, for $\lambda>0$ and some $x_0\in\mathbb{R}^n$.

Gidas-Spruck[14] first used the idea of Obata [21] to study the the semilinear elliptic equation in $\mathbb{R}^n$, and they got a Liouville type theorem to subcritical exponent for semilinear elliptic equation in $\mathbb{R}^n$. Let $u\geq 0$ satisfy

$$\Delta u+u^{\alpha}=0 \text{ in }\quad \mathbb{R}^n, \tag{1.4}$$

then $u=0$ if $1<\alpha<\frac{n+2}{n-2}$. Chen-Li [7] gave a new proof with the moving plane methods. For the p-Laplace equation with subcritical and critical exponent, Serrin-Zou [23] and Ciraolo-Figall-Roncoroni [10] obtained the similar results via the generalized Obata identity. We shall gave a proof of Gidas-Spruck[14] via Obata identity and some idea from Serrin-Zou [23].

In order to get best constant of the Sobolev inequality in upper half space, Escobar [9] s-tudied the harmonic function on the unit ball with normal derivative critical exponent growth on boundary, still he used the Obata identity and Pohozaev identity to get the classification of the solution. But Escobar's proof requires using the unknown metric $\bar{g}$ as the background metric instead of the given standard metric g on ball. We rewrite the proof via the given standard metric g on ball. Lin-Ou develop the similar idea to obtain a Liouville type theorem to the harmonic function with normal derivative subcritical exponent growth on boundary, which was obtained in [**?**].

At last we review the recent result by Ma-Ou [20]to obtain a Liouville type theorem for the classical semilinear subcritical elliptic equation on Heisenberg group. The soul of the proof is a generalized Obata identity in CR geometry founded by Jerison and Lee [18] in 1988.

The paper is organized as follows. In section 2, we introduce the Obata's theorem via the given standard metric g on sphere. In section 3 we prove the Gidas-Spruck's theorem use the Obata identity with an idea in Serrin-Zou [23]. In section 4, we prove the theorem of Escobar [9] via the given standard metric g on ball, then we mention our recent results on the subcritical exponent case. In section 5, we state a minor generalization of the Jerison-Lee's identity in CR geometry. Using this generalized identity, we prove a crucial integral estimates and the Liouville theorem in $\mathbb{H}^n$.

2. Obata Theorem on Sphere

In 1971, Obata [21] developed a new technique to get the following result, if $\bar{g}$ is the constant scalar curvature metric on S^n conformal to the standard metric g, then $\bar{g}$ is obtained from the standard metric by a conformal diffeomorphism of the sphere. In his paper the argument of Obata is quite subtle as it requires using the unknown metric $\bar{g}$ as the background metric instead of the given standard metric on sphere. In the following computation, we work under the given standard metric on sphere g . It can be written as a critical exponent elliptic equation on S^n, so the theorem of Obata stated that all the positive solutions to this elliptic equation on S^n could be classified up to parameter. There are now several different proofs for this theorem. Analytically, by the stereographic projection is equivalent to the following equation

$$-\Delta u = \frac{n(n-2)}{4} w^{\frac{n+2}{n-2}} \quad \text{on} \quad \mathbb{R}^n \tag{2.1}$$

whose positive solutions were classified by Gidas-Ni-Nirenberg [13] and Caffarelli-Gidas-Spruck [6] using the moving plane method.

Consider the metric $\bar{g} = v^{-2} g = e^{2f} g$ on $\mathbb{S}^n$, where $v > 0$ and f are smooth functions, and g is the canonical metric, let $\bar{R}$ and R denote the scalar curvature of $\bar{g}$ and g respectively, by the standard formulas we get

$$\begin{aligned}
\bar{R}_{ij} &= R_{ij} - (n-2) f_{ij} + (n-2) f_i f_j - (\Delta f + (n-2)|\nabla f|^2) g_{ij} \\
\bar{R} &= e^{-2f}[R - 2(n-1)\Delta f - (n-1)(n-2)|\nabla f|^2] \\
v^{-2} &= e^{2f}, f = -\log v, f_i = -\frac{v_i}{v}, \\
\Delta f &= -\frac{\Delta v}{v} + \frac{|\nabla v|^2}{v^2}, \\
R &= (n-1)n.
\end{aligned}$$

Combining these, we have

$$\bar{R} = v^2 [n(n-1) + \frac{2(n-1)\Delta v}{v} - n(n-1)\frac{|\nabla v|^2}{v^2}], \tag{2.2}$$

then

$$\Delta v = \frac{n}{2}\frac{|\nabla v|^2}{v} - \frac{n}{2} v + \frac{\bar{R}}{2(n-1)v} := G(v, \nabla v). \tag{2.3}$$

Recall that $R_{ij} = (n-1) g_{ij}$, so

$$\bar{R}_{ij} = (n-1) g_{ij} + \frac{(n-2) v_{ij}}{v} + [\frac{\Delta v}{v} - (n-1)\frac{|\nabla v|^2}{v^2}] g_{ij}. \tag{2.4}$$

If we define the trace free Ricci tensor with respect to the new metric $\bar{g}$, $\bar{E}_{ij} = \bar{R}_{ij} - \frac{\bar{R}}{n}\bar{g}_{ij}$, then

$$\begin{aligned}
\bar{E}_{ij} =& (n-1) g_{ij} + \frac{(n-2) v_{ij}}{v} + [\frac{\Delta v}{v} - (n-1)\frac{|\nabla v|^2}{v^2}] g_{ij} \\
& - \frac{v^{-2} g_{ij}}{n} v^2 [n(n-1) + \frac{2(n-1)\Delta v}{v} - (n-1) n \frac{|\nabla v|^2}{v^2}] \\
=& \frac{n-2}{v}[v_{ij} - \frac{\Delta v}{n} g_{ij}].
\end{aligned} \tag{2.5}$$

Now we define the trace free tensor $E_{ij} = v_{ij} - \frac{\Delta v}{n} g_{ij}$, we get the following key computation,

$$\begin{aligned}
(v^{1-n} E_{ij} v_i)_j &= v^{1-n} E_{ij} v_{ij} + v^{1-n} (E_{ij})_j v_i + (1-n) v^{-n} E_{ij} v_i v_j \\
&:= I + II + III,
\end{aligned} \tag{2.6}$$

where $I = v^{1-n} |E_{ij}|^2 \geq 0$.

By the Ricci identity, we have

$$\Delta v_i = (\Delta v)_i + R_{ij} v_j,$$

so we get

$$II = v^{1-n}(v_{ijj} - \frac{(\Delta v)_j}{n} g_{ij})v_i = v^{1-n}(G_i + R_{ij}v_j - \frac{G_i}{n})v_i$$
$$= \frac{n-1}{n} v^{1-n} G_i v_i + (n-1)v^{1-n}|\nabla v|^2.$$

And we have

$$III = (1-n)v^{-n}v_{ij}v_iv_j - (1-n)v^{-n}\frac{G}{n}|\nabla v|^2.$$

Recall that $G = \frac{n}{2}\frac{|\nabla v|^2}{v} - \frac{n}{2}v + \frac{\bar{R}}{2(n-1)v}$, if $\bar{R}$ is constant, then we have

$$G_i = n\frac{v_j v_{ji}}{v} - \frac{n}{2}\frac{|\nabla v|^2}{v^2}v_i - \frac{n}{2}v_i - \frac{\bar{R}}{2(n-1)v^2}v_i. \tag{2.7}$$

It follows that

$$\begin{aligned} II + III = &\frac{n-1}{n}v^{1-n}(n\frac{v_iv_jv_{ij}}{v} - \frac{n}{2}\frac{|\nabla v|^4}{v^2} - \frac{n}{2}|\nabla v|^2 - \frac{\bar{R}|\nabla v|^2}{2(n-1)v^2}) + (n-1)v^{1-n}|\nabla v|^2 \\ &+ (1-n)v^{-n}v_{ij}v_iv_j + \frac{n-1}{n}v^{-n}(\frac{n}{2}\frac{|\nabla v|^2}{v} - \frac{n}{2}v + \frac{\bar{R}}{2(n-1)v})|\nabla v|^2 \\ &= 0. \end{aligned} \tag{2.8}$$

So for $E_{ij} = v_{ij} - \frac{\Delta v}{n}g_{ij}$, by (2.6) and (2.8) we have

$$(v^{1-n}E_{ij}v_i)_j = v^{1-n}|E_{ij}|^2. \tag{2.9}$$

Integrate on $\mathbb{S}^n$, it gives

$$\int_{\mathbb{S}^n} v^{1-n}|E_{ij}|^2 d\sigma = \int_{\mathbb{S}^n} (v^{1-n}E_{ij}v_i)_j d\sigma = 0, \tag{2.10}$$

which means $E_{ij} = 0$.

By the standard discussion as in Obata[21] (see also [26]), we get

$$v(x) = c(cosht + (sinht)x \cdot a), \tag{2.11}$$

where for some $c > 0, \quad t \geq 0$, and $a \in \mathbb{S}^n$.

If we consider the metric $\bar{g} = u^{\frac{4}{n-2}}g$ on $\mathbb{S}^n$, A conformal metric $\bar{g}$ has constant scalar curvature $n(n-1)$ iff

$$\frac{-4}{n(n-2)}\triangle u + u = u^{\frac{n+2}{n-2}}, \quad \text{on} \quad \mathbb{S}^n. \tag{2.12}$$

From (2.11), we can get the following theorem proved by Obata[21]

Theorem 2.1. [21] *Let $u(x)$ be a positive C^2 solution for the equation (2.12) then*

$$u(x) = c[cosht + (sinht)x \cdot a]^{\frac{-(n-2)}{2}}, \tag{2.13}$$

where for some $c > 0, \quad t \geq 0$, and $a \in \mathbb{S}^n$.

It is a remarkable theorem that these are all the positive solutions of (2.12). There are now several proofs for this theorem. Analytically, by the stereographic projection (2.11) is equivalent to the following equation

$$-\triangle w = \frac{n(n-2)}{4} w^{\frac{n+2}{n-2}} \quad \text{on} \quad \mathbb{R}^n \tag{2.14}$$

whose positive solutions were classified by Gidas-Ni-Nirenberg [13] and Caffarelli-Gidas-Spruck [6] using the moving plane method, and every positive solution of (1.3) assumes the form

$$w(x) = (4\lambda^2)^{\frac{n-2}{4}} [\lambda^2 + |x - x_0|^2]^{-\frac{n-2}{2}},$$

for $\lambda > 0$ and some $x_0 \in R^n$. This Obata identity was also used to study the general semilinear elliptic equation in compact Riemannian manifold in [2] and [8].

3. Liouville theorem for subcritical semi-linear equation in R^n

In 1981, Gidas-Spruck [14] used the idea from Obata[21] to study the semilinear elliptic equation, and they proved the following famous theorem.

Theorem 3.1. [14] *Let $u(x)$ be a nonnegative C^2 solution for the equation*

$$\Delta u + u^\alpha = 0 \quad \textit{in} \quad \mathbb{R}^n, \tag{3.1}$$

then $u = 0$ if $1 < \alpha < \frac{n+2}{n-2}$.

This theorem was reproved by Chen-Li [7] with the moving plane methods. For the p-Laplace equation with subcritical exponent Serrin-Zou [23] proved a similar Liouville theorem. For the critical exponent case with some energy finite condition, Ciraolo-Figall-Roncoroni [10] obtained the similar classification results via the generalized Obata identity [23]. We shall gave a proof of the theorem by Gidas-Spruck[14] via Obata identity and some idea from Serrin-Zou [23].

Proof. By maximum principle, if u vanishes at some points, then $u \equiv 0$. Assume $u > 0$, let $u = v^{-\beta}$, $\beta = \frac{n-2}{2}$. Then v satisfies that

$$\Delta v = \frac{n}{2} \frac{|\nabla v|^2}{v} + \frac{2}{n-2} v^\gamma, \tag{3.2}$$

where

$$\gamma = \alpha(1 - \frac{n}{2}) + \frac{n}{2}, \quad \text{when} \quad 1 < \alpha < \frac{n+2}{n-2} \quad \text{then} \quad -1 < \gamma < 1. \tag{3.3}$$

We define $E_{ij} = v_{ij} - \frac{\Delta v}{n} \delta_{ij}$, and $a \in R$ will be determined later. So we get

$$(v^a E_{ij} v_i)_j = v^a E_{ij} v_{ij} + v^a (E_{ij})_j v_i + a v^{a-1} E_{ij} v_i v_j := I + II + III, \tag{3.4}$$

where

$$I = v^a |E_{ij}|^2 \geq 0, \tag{3.5}$$

and

$$II = v^a (v_{ijj} - \frac{\Delta v_j}{n} \delta_{ij}) v_i = \frac{n-1}{n} v^a v_i \Delta v_i. \tag{3.6}$$

From (3.2) we get

$$(\Delta v)_i = n\frac{v_j v_{ji}}{v} - \frac{n}{2}\frac{|\nabla v|^2}{v^2}v_i + \frac{2\gamma}{n-2}v^{\gamma-1}v_i, \tag{3.7}$$

and

$$II = -\frac{n-1}{2}v^{a-2}|\nabla v|^4 + (n-1)v^{a-1}v_{ij}v_iv_j + \frac{2(n-1)\gamma}{n(n-2)}v^{a+\gamma-1}|\nabla v|^2. \tag{3.8}$$

Still from (3.2),

$$III = av^{a-1}(v_{ij} - \frac{\Delta v}{n}\delta_{ij})v_iv_j = av^{a-1}v_{ij}v_iv_j - \frac{a}{2}v^{a-2}|\nabla v|^4 - \frac{2a}{n(n-2)}v^{a+\gamma-1}|\nabla v|^2. \tag{3.9}$$

It follow sthat

$$\begin{aligned}(v^aE_{ij}v_i)_j = v^a|E_{ij}|^2 - \frac{a-1+n}{2}v^{a-2}|\nabla v|^4 + (n-1+a)v^{a-1}v_iv_jv_{ij} \\ + [\frac{2(n-1)\gamma}{n(n-2)} - \frac{2a}{n(n-2)}]v^{a+\gamma-1}|\nabla v|^2.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2}v^{a-1}|\nabla v|^2_i v_i &= \frac{1}{2}(v^{a-1}|\nabla v|^2v_i)_i - \frac{a-1}{2}v^{a-2}|\nabla v|^4 - \frac{1}{2}v^{a-1}|\nabla v|^2\Delta v \\ &= \frac{1}{2}(v^{a-1}|\nabla v|^2v_i)_i - \frac{a-1}{2}v^{a-2}|\nabla v|^4 - \frac{n}{4}v^{a-2}|\nabla v|^4 - \frac{1}{n-2}v^{a-1+\gamma}|\nabla v|^2,\end{aligned}$$

so we obtain
(3.10)

$$\begin{aligned}(v^aE_{ij}v_i)_j &= v^a|E_{ij}|^2 + v^{a-2}|\nabla v|^4[-\frac{a-1+n}{2} - (n-1+a)(\frac{n}{4} + \frac{1}{2}(a-1)] \\ &+ \frac{1}{2}(n-1+a)(v^{a-1}|\nabla v|^2v_i)_i + [\frac{2(n-1)\gamma}{n(n-2)} - \frac{2a}{n(n-2)} - \frac{n-1+a}{n-2}]v^{a+\gamma-1}|\nabla v|^2 \\ &= v^a|E_{ij}|^2 + \frac{1}{2}(n-1+a)(v^{a-1}|\nabla v|^2v_i)_i + Av^{a-2}|\nabla v|^4 + Bv^{a+\gamma-1}|\nabla v|^2,\end{aligned}$$

where

$$A = -(n-1+a)(\frac{n}{4} + \frac{a}{2}),$$

and

$$B = \frac{2(n-1)\gamma}{n(n-2)} - \frac{2a}{n(n-2)} - \frac{n-1+a}{n-2}.$$

Notice if $\alpha = \frac{n+2}{n-2}$ then for $a = 1-n$, we get $A = 0, \quad B = 0$ and (3.10) becomes $(v^aE_{ij}v_i)_j = v^a|E_{ij}|^2$ which is the same identity with (2.9).

Now we take $a = 1-n+d$, then take $0 < d < \frac{n-2}{2}$ we have

$$A = \frac{1}{2}d(\frac{n-2}{2} - d) > 0$$

$B > 0$ is equivalent to $\frac{(n-1)\gamma}{n} - \frac{a}{n} - \frac{n-1+a}{2} > 0$, which is

$$\gamma > -1 + \frac{n+2}{2n}d,$$

from the definition of γ in (3.3)

$$\alpha(1-\frac{n}{2})+\frac{n}{2} > -1+\frac{n+2}{2n}d.$$

So for $1<\alpha<\frac{n+2}{n-2}$ if we take

$$0<d<\min\{\frac{n(n-2)}{n+2}(\frac{n+2}{n-2}-\alpha),\frac{n-2}{2}\}, \tag{3.11}$$

then we have $A>0, \quad B>0$. In the following computation, we fix the choice of d such that $A>0, \quad B>0$.

Now, we take $\zeta(x)$ be the standard cut-off function, let $\eta=\zeta(\frac{x}{R})$, then $\eta=1$ on $B_R(0)$, $\eta=0$, outside $B_{2R}(0)$, $|\nabla\eta|^2+|\nabla^2\eta|\le\frac{c(n)}{R^2}$.
For $\delta>0$ to be determined later, we times η^δ and integrate (3.10) in $\mathbb{R}^n$,

$$\begin{aligned}&-\int_{B_{2R}} v^aE_{ij}v_i(\eta^\delta)_jdx+\frac{1}{2}(n+a-1)\int_{B_{2R}} v^{a-1}v_i|\nabla v|^2(\eta^\delta)_idx\\&=\int_{B_{2R}} v^a|E_{ij}|^2\eta^\delta dx+A\int_{B_{2R}} v^{a-2}|\nabla v|^4\eta^\delta dx+B\int_{B_{2R}} v^{a+\gamma-1}|\nabla v|^2\eta^\delta dx.\end{aligned} \tag{3.12}$$

Since

$$v^av_{ij}v_i=(v^av_jv_i)_i-av^{a-1}v_j|\nabla v|^2-v^av_jv_{ii}$$

so we have

$$v^aE_{ij}v_i=(v^av_jv_i)_i-(\frac{n+1}{2}+a)v^{a-1}|\nabla v|^2v_j-\frac{2(n+1)}{n(n-2)}v^{a+\gamma}v_j,$$

it follows that

$$\begin{aligned}&\int_{B_{2R}} v^aE_{ij}v_i(\eta^\delta)_jdx+\int_{B_{2R}} v^{a-1}v_i|\nabla v|^2(\eta^\delta)_idx\\&\lesssim\frac{1}{R^2}\int_{B_{2R}} v^a|\nabla v|^2\eta^{\delta-2}dx+\frac{1}{R}\int_{B_{2R}} v^{a-1}|\nabla v|^3\eta^{\delta-1}dx+\frac{1}{R}\int_{B_{2R}} v^{a+\gamma}|\nabla v|\eta^{\delta-1}dx.\end{aligned} \tag{3.13}$$

By Young's inequality,

$$\begin{aligned}\frac{1}{R^2}\int_{B_{2R}} v^a|\nabla v|^2\eta^{\delta-2}dx&\lesssim\epsilon\int_{B_{2R}} v^{a-2}|\nabla v|^4\eta^\delta dx+\frac{1}{R^4}\int_{B_{2R}} v^{a+2}\eta^{\delta-4}dx,\\ \frac{1}{R}\int_{B_{2R}} v^{a-1}|\nabla v|^3\eta^{\delta-1}dx&\lesssim\epsilon\int_{B_{2R}} v^{a-2}|\nabla v|^4\eta^\delta dx+\frac{1}{R^4}\int_{B_{2R}} v^{a+2}\eta^{\delta-4}dx,\\ \frac{1}{R}\int_{B_{2R}} v^{a+\gamma}|\nabla v|\eta^{\delta-1}dx&\lesssim\epsilon\int_{B_{2R}} v^{a+\gamma-1}|\nabla v|^2\eta^\delta dx+\frac{1}{R^2}\int_{B_{2R}} v^{a+\gamma+1}\eta^{\delta-2}dx.\end{aligned}$$

Thus using the above inequalities with (3.12) and (3.13), we arrive the following important estimate

$$\int_{B_{2R}} v^{a-2}|\nabla v|^4\eta^\delta dx+\int_{B_{2R}} v^{a+\gamma-1}|\nabla v|^2\eta^\delta dx\lesssim\frac{1}{R^4}\int_{B_{2R}} v^{a+2}\eta^{\delta-4}dx+\frac{1}{R^2}\int_{B_{2R}} v^{a+\gamma+1}\eta^{\delta-2}dx. \tag{3.14}$$

For equation (3.2) of v, we times $v^{a+\gamma}\eta^\delta$ on both side and integral on B_{2R}, so we get

$$\int_{B_{2R}} v^{a+\gamma}\eta^\delta\Delta vdx=\int_{B_{2R}}(\frac{n}{2}v^{-1}|\nabla v|^2+\frac{2}{n-2}v^\gamma)\times v^{a+\gamma}\eta^\delta dx, \tag{3.15}$$

since

$$\int_{B_{2R}} v^{a+\gamma}\eta^\delta \Delta v dx = -\int_{B_{2R}} v^{a+\gamma}(\eta^\delta)_i v_i dx - (a+\gamma)\int_{B_{2R}} v^{a+\gamma-1}\eta^\delta |\nabla v|^2 dx$$

we get

$$\frac{2}{n-2}\int_{B_{2R}} v^{a+2\gamma}\eta^\delta dx \lesssim \int_{B_{2R}} v^{a+\gamma-1}|\nabla v|^2\eta^\delta dx + \frac{1}{R}\int_{B_{2R}} v^{a+\gamma}|\nabla v|\eta^{\delta-1}dx. \tag{3.16}$$

From Cauchy inequality we have

$$\frac{1}{R}\int_{B_{2R}} v^{a+\gamma}|\nabla v|\eta^{\delta-1}dx \lesssim \int_{B_{2R}} v^{a+\gamma-1}|\nabla v|^2\eta^\delta dx + \frac{1}{R^2}\int_{B_{2R}} v^{a+\gamma+1}\eta^{\delta-2}dx, \tag{3.17}$$

combining (3.16) and (3.17), we obtain

$$\int_{B_{2R}} v^{a+2\gamma}\eta^\delta dx \lesssim \int_{B_{2R}} v^{a+\gamma-1}|\nabla v|^2\eta^\delta dx + \frac{1}{R^2}\int_{B_{2R}} v^{a+\gamma+1}\eta^{\delta-2}dx, \tag{3.18}$$

by (3.14), we get

$$\int_{B_{2R}} v^{a+2\gamma}\eta^\delta dx \lesssim \frac{1}{R^4}\int_{B_{2R}} v^{a+2}\eta^{\delta-4}dx + \frac{1}{R^2}\int_{B_{2R}} v^{a+\gamma+1}\eta^{\delta-2}dx. \tag{3.19}$$

By Cauchy inequality

$$\frac{1}{R^2}\int_{B_{2R}} v^{a+\gamma+1}\eta^{\delta-2}dx \lesssim \epsilon\int_{B_{2R}} v^{2\gamma+a}\eta^\delta dx + \frac{1}{R^4}\int_{B_{2R}} v^{a+2}\eta^{\delta-4}dx, \tag{3.20}$$

therefore from (3.19) and (3.20) we get the following estimates

$$\int_{B_{2R}} v^{a+2\gamma}\eta^\delta dx \le \frac{C}{R^4}\int_{B_{2R}} v^{a+2}\eta^{\delta-4}dx, \tag{3.21}$$

which is

$$\int_{B_{2R}} u^{-\frac{2(a+2\gamma)}{n-2}}\eta^\delta dx \le \frac{C}{R^4}\int_{B_{2R}} u^{-\frac{2(a+2)}{n-2}}\eta^{\delta-4}dx. \tag{3.22}$$

We recall that $a = 1-n+d$, $\gamma = \alpha(1-\frac{n}{2})+\frac{n}{2} < 1$ since $1 < \alpha < \frac{n+2}{n-2}$.

$$k := -\frac{2}{n-2}(1-n+d+2\gamma) > l := -\frac{2}{n-2}(3-n+d).$$

For $n > 3$, and we can take d is sufficient small positive constant such that $-\frac{2}{n-2}(3-n+d) > 0$, then $k > l > 0$. From Young's inequality

$$\int_{B_{2R}} u^k\eta^\delta dx \le \frac{C}{R^4}\int_{B_{2R}} u^l\eta^{\delta-4}dx \lesssim \epsilon\int_{B_{2R}} (\eta^{\frac{\delta l}{k}}u^l)^{\frac{k}{l}}dx + \frac{1}{R^{4q}}\int_{B_{2R}} (\eta^{\delta-\frac{\delta l}{k}-4})^q dx.$$

We take δ sufficiently large, then we get the crucial inequality

$$\int_{B_{2R}} u^k\eta^\delta dx \lesssim R^{n-4q}, \tag{3.23}$$

where

$$\text{for} \quad p := \frac{k}{l} = \frac{1-n+d+2\gamma}{3-n+d}, \quad q := \frac{p}{p-1} = \frac{1-n+d+2\gamma}{-2+2\gamma},$$

$$n-4q = n - \frac{2(1-n+d+2\gamma)}{\gamma-1} = n - \frac{2(1+d+2\alpha(1-\frac{n}{2}))}{(1-\frac{n}{2})(\alpha-1)} < 0.$$

By taking $R \to \infty$ in (3.23) we have $u = 0$.

For $n = 3$, $1 < \alpha < 5$ by the estimates (3.22)

$$\int_{B_{2R}} u^{2(\alpha-1-d)}\eta^{\delta}dx \le \frac{C}{R^4}\int_{B_{2R}} u^{-2d}\eta^{\delta-4}dx. \tag{3.24}$$

We first recall a Harnack inequality (see Han-Lin [17] Theorem 3.2 in Chapter 4): If $-\Delta u \ge 0$, $u \ge 0$, then for all $r \in (0,3)$, we have

$$\min_{B_{2R}} u(x) \ge CR^{-\frac{3}{r}}||u||_{L^r(B_{4R})}. \tag{3.25}$$

For $q < 2$, δ is sufficient large, then by Holder inequality and (3.24)

$$\begin{aligned}\int_{B_R} u^{q(\alpha-1)}dx &\le (\int_{B_R} u^{2(\alpha-1-d)}dx)^{\frac{q}{2}}(\int_{B_R} u^{qd\times\frac{2}{2-q}}dx)^{\frac{2-q}{2}} \\ &\le C(R^{-4}\int_{B_{2R}} u^{-2d}dx)^{\frac{q}{2}}(\int_{B_R} u^{\frac{2dq}{2-q}}dx)^{\frac{2-q}{2}}.\end{aligned} \tag{3.26}$$

If $r := \frac{2dq}{2-q} < 3$, then by the Harnack inequality (3.25), in B_{2R} we have

$$u^{-2d} \le CR^{3\times\frac{2-q}{2dq}\times 2d}(\int_{B_{4R}} u^{\frac{2dq}{2-q}}dx)^{-\frac{2-q}{2dq}\times 2d}, \tag{3.27}$$

it follows that

$$\begin{aligned}\int_{B_R} u^{q(\alpha-1)}dx &\le C(R^{-4+3}\times R^{3\times\frac{2-q}{2dq}\times 2d})^{\frac{q}{2}}(\int_{B_{4R}} u^{\frac{2dq}{2-q}}dx)^{-\frac{2-q}{2dq}\times 2d\times\frac{q}{2}}(\int_{B_R} u^{\frac{2dq}{2-q}}dx)^{\frac{2-q}{2}} \\ &\le CR^{3-2q}.\end{aligned} \tag{3.28}$$

To conclude what we need are $2dq < 6 - 3q$ and $q > \frac{3}{2}$ which intersect with $q < 2$ are nonempty when d is a sufficient positive constant. If we take such q, and let $R \to \infty$ then $u = 0$.

Therefore $u = 0$, and we complete the proof of this theorem. □

In the above proof the key inequality is (3.14) which comes from Obata identity, the inequality (3.18) which comes from the equation (3.2) by integral by parts, then we get the crucial inequality (3.23). In section 5, we shall get the corresponding inequalities in semilinear subcritical elliptic equation on Heisenberg group to get the Liouviile theorem.

The above proof was generalized to p-Laplace operator with subcritical exponent in Serrin-Zou [23] and they got a similar Liouville theorem.

4. Escobar's Shapr Sobolev constant in upper half space

From the Hardy-Littlewood-Sobolev inequality on $\mathbb{S}^{n-1}$ with sharp constant, Beckner [3] derived the following family of inequalities on $\mathbb{B}^n$:

Theorem 4.1. [3] *Let* $u \in C^\infty(\overline{\mathbb{B}^n})$*, then*

$$c_n^{\frac{q-1}{q+1}}(\int_{\mathbb{S}^{n-1}} |u(\xi)|^{q+1}d\sigma(\xi))^{\frac{2}{q+1}} \le (q-1)\int_{\mathbb{B}^n} |\nabla u(x)|^2dx + \int_{\mathbb{S}^{n-1}} |u(\xi)|^2d\sigma(\xi), \tag{4.1}$$

for $1 < q < \infty$ *if* $n = 2$ *and* $1 < q \le \frac{n}{n-2}$ *if* $n \ge 3$*, where* $c_n = 2\pi^{\frac{n}{2}}/\Gamma(n/2) = |\mathbb{S}^{n-1}|$ *and* $d\sigma$ *is the standard form on* $\mathbb{S}^{n-1}$.

The critical case $q = \frac{n}{n-2}$ was also proved by Escobar [9] by a different method. In 1988, Escobar[9] also got the best constant for the Sobolev inequality in upper half space, through the conformal transformation he reduced to proof to the following uniqueness theorem.

Theorem 4.2. [9] *Let $u(x)$ be a positive smooth solution for the equation*

$$\begin{aligned} \Delta u &= 0 \quad \textit{in} \quad B_1(0) \subseteq \mathbb{R}^n, \\ u_\nu + \frac{n-2}{2}u &= \frac{n-2}{2}hu^{\frac{n}{n-2}} \quad \textit{on} \quad \partial B_1(0) = \mathbb{S}^{n-1}, \end{aligned} \tag{4.2}$$

where ν is the unit out normal and $0 < h \leq 1$ is constant. If $u > 0$ is normalized such that $\int_{S^{n-1}} d\sigma = \int_{S^{n-1}} u^{\frac{2(n-1)}{n-2}} d\sigma$, then $h = 1$.

We present the proof of the above theorem by Escobar[9] with standard metric in R^n.

Proof. Let $u = v^{-\frac{n-2}{2}}$, then v satisfies

$$\begin{aligned} \Delta v &= \frac{n}{2}\frac{|\nabla v|^2}{v} \quad \text{in } B_1(0) \subseteq \mathbb{R}^n, \\ v_\nu &= v - h \quad \text{on } \partial B_1(0). \end{aligned} \tag{4.3}$$

If we define $E_{ij} = v_{ij} - \frac{\Delta v}{n}\delta_{ij}$, then by (4.3) and $(E_{ij})_j = v_{ijj} - \frac{(\Delta v)_j}{n}\delta_{ij} = \frac{n-1}{n}\Delta v_i$, we have

$$\begin{aligned} (v^{1-n}E_{ij}v_i)_j &= (1-n)v^{-n}E_{ij}v_iv_j + v^{1-n}(E_{ij})_jv_i + v^{1-n}E_{ij}v_{ij} \\ &= (1-n)v^{-n}v_{ij}v_iv_j - (1-n)v^{1-n}\frac{\Delta v}{n}|\nabla v|^2 + \frac{n-1}{n}v^{1-n}\Delta v_iv_i + v^{1-n}|E_{ij}|^2 \\ &= v^{1-n}|E_{ij}|^2. \end{aligned} \tag{4.4}$$

It follows that

$$\begin{aligned} \int_{B_1} v^{1-n}|E_{ij}|^2dx &= \int_{B_1}(v^{1-n}E_{ij}v_i)_jdx = \int_{\partial B_1} v^{1-n}E_{ij}v_i\nu_jd\sigma \\ &= \int_{\partial B_1} v^{1-n}\Sigma_{\alpha=1}^{n-1}E_{\alpha n}v_\alpha d\sigma + \int_{\partial B_1} v^{1-n}E_{nn}v_nd\sigma \\ &= II + I, \end{aligned} \tag{4.5}$$

if we denote $\nu = e_n$, where

$$\begin{aligned} I &= \int_{\partial B_1} v^{1-n}(v_{nn} - \frac{\Delta v}{n})v_nd\sigma \\ &= \int_{\partial B_1} v^{2-n}(v_{nn} - \frac{\Delta v}{n})d\sigma - h\int_{\partial B_1} v^{1-n}(v_{nn} - \frac{\Delta v}{n})d\sigma \\ &= I_1 + I_2. \end{aligned} \tag{4.6}$$

Recall the following standard formulas (see (3.3) in [22]), we have

$$\begin{aligned} (v_n)_\alpha &= v_{n\alpha} + v_\alpha \\ \Sigma_{\alpha=1}^{n-1}v_{\alpha\alpha} &= \Delta_{\partial B_1}v + (n-1)v_n, \end{aligned} \tag{4.7}$$

so we get

$$v_{nn} - \frac{\Delta v}{n} = \Delta v - \Sigma_{\alpha=1}^{n-1} v_{\alpha\alpha} - \frac{\Delta v}{n} = \frac{n-1}{n}\Delta v - \Sigma_{\alpha=1}^{n-1} v_{\alpha\alpha}$$
$$= \frac{n-1}{2}\frac{|\nabla v|^2}{v} - \Delta_{\partial B_1} v - (n-1)v_n. \tag{4.8}$$

From (4.7), we have

$$II = \int_{\partial B_1} v^{1-n}\Sigma_{\alpha=1}^{n-1} E_{\alpha n} v_\alpha d\sigma = \int_{\partial B_1} v^{1-n}\Sigma_{\alpha=1}^{n-1} v_{\alpha n} v_\alpha d\sigma$$
$$= \int_{\partial B_1} v^{1-n}\Sigma_{\alpha=1}^{n-1} v^\alpha[(v_n)_\alpha - v_\alpha] d\sigma = 0. \tag{4.9}$$

By divergence theorem and (4.3) we have

$$\int_{\partial B_1} v^{1-n} v_n d\sigma = \int_{B_1} div(v^{1-n}\nabla v) dx$$
$$= \int_{B_1} [(1-n)v^{-n}|\nabla v|^2 + v^{1-n}\Delta v] dx$$
$$= \frac{2-n}{2}\int_{B_1} |\nabla v|^2 v^{-n} dx, \tag{4.10}$$

and

$$\int_{\partial B_1} v^{1-n}\Delta_{\partial B_1} v d\sigma = (n-1)\int_{\partial B_1} v^{-n}|\nabla_\Sigma v|^2 d\sigma. \tag{4.11}$$

Now by (4.8), (4.10)-(4.11) we get

(4.12)

$$-\frac{1}{h} I_2 = \int_{\partial B_1} v^{1-n}(v_{nn} - \frac{\Delta v}{n}) d\sigma$$
$$= \frac{n-1}{2}\int_{\partial B_1} v^{-n}|\nabla v|^2 d\sigma - \int_{\partial B_1} v^{1-n}\Delta_{\partial B_1} v d\sigma - (n-1)\int_{\partial B_1} v^{1-n} v_n d\sigma$$
$$= \frac{n-1}{2}\int_{\partial B_1} v^{-n}|\nabla v|^2 d\sigma - (n-1)\int_{\partial B_1} v^{-n}|\nabla_\Sigma v|^2 d\sigma - (n-1)(\frac{n}{2} + 1 - n)\int_{B_1} v^{-n}|\nabla v|^2 dx$$
$$= \frac{n-1}{2}[\int_{\partial B_1} v^{-n}|\nabla v|^2 d\sigma - 2\int_{\partial B_1} v^{-n}|\nabla_\Sigma v|^2 d\sigma + (n-2)\int_{B_1} v^{-n}|\nabla v|^2 dx],$$

where $\nabla_\Sigma v$ is the tangential gradient of v on ∂B_1.

Now we derive a Pohozaev identity from which we obtain $I_2 = 0$. First we use the equation (4.3) to get

(4.13)

$$\frac{n}{2} v^{-n-1}|\nabla v|^2 v_i x_i = v^{-n} v_i x_i \Delta v$$
$$= (v^{-n} v_i x_i v_l)_l - (v^{-n} v_i x_i)_l v_l$$
$$= (v^{-n} v_i x_i v_l)_l + n v^{-n-1} v_i x_i |\nabla v|^2 - v^{-n}|\nabla v|^2 - v^{-n} x_i v_l v_{il}$$
$$= (v^{-n} x_i v_i v_l)_l - \frac{1}{2}(v^{-n} x_l |\nabla v|^2)_l + \frac{n}{2} v^{-n-1} x_i v_i |\nabla v|^2 + (\frac{n}{2} - 1) v^{-n}|\nabla v|^2.$$

Integrate on B_1 in the above formula (4.13), we obtain the following Pohozaev identity

$$(n-2)\int_{B_1} v^{-n}|\nabla v|^2dx = -2\int_{\partial B_1} v^{-n}v_n^2 d\sigma + \int_{\partial B_1} v^{-n}|\nabla v|^2 d\sigma$$
$$= 2\int_{\partial B_1} v^{-n}|\nabla_\Sigma v|^2 - \int_{\partial B_1} v^{-n}|\nabla v|^2 d\sigma. \tag{4.14}$$

Then from (4.12) and (4.14) we obtain $I_2 = 0$.

From $II = 0, \quad I_2 = 0$ and (4.5) we get

$$\int_{B_1} v^{1-n}|E_{ij}|^2dx = \int_{\partial B} v^{2-n}[v_{nn} - \frac{\Delta v}{n}]d\sigma = I_1. \tag{4.15}$$

From (4.8) we have

$$2I_1 = 2\int_{\partial B_1} v^{2-n}[\frac{n-1}{2}\frac{|\nabla v|^2}{v} - \Delta_{\partial B_1}v - (n-1)v_n]d\sigma$$
$$= \int_{\partial B_1}[(n-1)v^{1-n}|\nabla_\Sigma v|^2 + (n-1)v^{1-n}v_n^2 - 2(n-2)v^{1-n}|\nabla_\Sigma v|^2 - 2(n-1)v^{2-n}v_n]d\sigma$$
$$= \int_{\partial B_1}[(3-n)v^{1-n}|\nabla_\Sigma v|^2 + (n-1)v^{1-n}(v-h)^2 - 2(n-1)v^{2-n}(v-h)]d\sigma$$
$$= \int_{\partial B_1}[(3-n)v^{1-n}|\nabla_\Sigma v|^2 - (n-1)v^{3-n}]d\sigma + (n-1)h^2\int_{\partial B_1} v^{1-n}d\sigma.$$

Then we have

$$2\int_{B_1}|E_{ij}|^2dx = \int_{\partial B_1}[(3-n)v^{1-n}|\nabla_\Sigma v|^2 - (n-1)v^{3-n}]d\sigma + (n-1)h^2\int_{\partial B_1} v^{1-n}d\sigma. \tag{4.16}$$

For $n = 3$, then

$$2\int_{\mathbb{S}^2} v^{-2}|E_{ij}|^2 d\sigma = 2h^2\int_{\mathbb{S}^2} v^{-2} - 2\int_{\mathbb{S}^2} d\sigma.$$

If u is normalized such that

$$\int_{\mathbb{S}^2} v^{-2}d\sigma = 4\pi = \int_{\mathbb{S}^2} d\sigma,$$

then

$$2\int_{\mathbb{S}^2} v^{-2}|E_{ij}|^2 d\sigma = 8\pi^2(h^2-1) \geq 0,$$

so we have $h = 1$ since $0 < h \leq 1$.

For $n \geq 4$, we normalize u such that

$$\int_{\mathbb{S}^{n-1}} v^{1-n}d\sigma = \int_{\mathbb{S}^{n-1}} d\sigma = \int_{\mathbb{S}^{n-1}} u^{\frac{2(n-1)}{n-2}}d\sigma.$$

We first state a Sobolev inequality on S^{n-1} (for example see Aubin [1]):

(4.17)
$$\int_{\mathbb{S}^{n-1}}|\nabla_\Sigma \phi|^2 + \frac{(n-1)(n-3)}{4}\int_{\mathbb{S}^{n-1}}\phi^2 d\sigma \geq \frac{(n-1)(n-3)}{4}(\int_{\mathbb{S}^{n-1}}|\phi|^{\frac{2(n-1)}{n-3}}d\sigma)^{\frac{n-3}{n-1}}|\mathbb{S}^{n-1}|^{\frac{2}{n-1}}.$$

Now we take $\phi = v^{\frac{3-n}{2}}$ and use (4.17), it follows that

$$\begin{aligned}
\text{Right hand side of}\,(4.16) &= (n-1)h^2\int_{\mathbb{S}^{n-1}}\phi^{\frac{2(n-1)}{n-3}}d\sigma - \frac{4}{n-3}\int_{\mathbb{S}^{n-1}}|\nabla\phi|^2 d\sigma - (n-1)\int_{\mathbb{S}^{n-1}}\phi^2 d\sigma \\
&= (n-1)h^2|\mathbb{S}^{n-1}| - \frac{4}{n-3}[\int_{\mathbb{S}^{n-1}}|\nabla\phi|^2 d\sigma + \frac{(n-1)(n-3)}{4}\int_{\mathbb{S}^{n-1}}\phi^2 d\sigma] \\
&\leq (n-1)h^2|\mathbb{S}^{n-1}| - \frac{4}{n-3}\frac{(n-1)(n-3)}{4}|\mathbb{S}^{n-1}|^{\frac{n-3}{n-1}}|\mathbb{S}^{n-1}|^{\frac{2}{n-1}} \\
&= (n-1)(h^2-1)|\mathbb{S}^{n-1}|.
\end{aligned}$$

Then $0 < h \leq 1$ implies $h = 1$. □

The above (Escobar [9]) theorem gave the classification of all positive solutions of equation (4.2) using the integral method and hence proved inequality (4.1) for $q = \frac{n}{n-2}$. The inequality for $1 < q < \frac{n}{n-2}$ would also follow in the same way from the following

Conjecture 1. [15] *If $u \in C^\infty(\overline{\mathbb{B}^n})$ is a positive solution of the following equation*

$$\begin{aligned}
\Delta u &= 0 \quad \text{on } \mathbb{B}^n, \\
u_\nu + \lambda u &= u^q \quad \text{on } \mathbb{S}^{n-1},
\end{aligned} \tag{4.18}$$

then u is constant provided $1 < q < \frac{n}{n-2}$ and $0 < \lambda \leq \frac{1}{q-1}$.

In Guo-Hang -Wang [16], they proved it in two dimension case via moving plane methods. In higher dimension case, Guo-Wang [15] proved the following theorem.

Theorem 4.3. [15] *For $n \geq 3$, if $u \in C^\infty(\overline{\mathbb{B}^n})$ is positive and satisfies the equation (1.2), then u is constant, provided $1 < q < \frac{n}{n-2}$ and $0 < \lambda \leq \frac{n-2}{2}$.*

Using the similar Obata identity technique, Bidaut-Véron [2] got following Sobolev inequalities,

Lemma 4.4. [2] $\int_{\mathbb{S}^{n-1}}[|\nabla u|^2 + su^2]\,d\sigma \geq s(\int_{\mathbb{S}^{n-1}}|u|^{p+1}d\sigma)^{\frac{2}{p+1}}|S^{n-1}|^{\frac{p-1}{p+1}}$ *where $1 < p \leq \frac{n+1}{n-3}$ and $(p-1)s \leq n-1$.*

From these Sobolev inequalities, Obata identity and Pohozaev identity, we study the above conjecture in a short note [**?**] . The main result is the following partial results in dimensions $n \geq 9$.

Theorem 4.5. [**?**] *If $u \in C^\infty(\overline{\mathbb{B}^n})$ is a positive solution of (1.2) when $n \geq 9$ and $1 < q \leq 1+\frac{2}{3n-5}$, then there is a $\lambda_0 \in (0, \frac{1}{q-1})$ depending on n and q implicitly, such that u is constant if $\lambda \in (0, \lambda_0)$. More precisely, $\lambda_0 = O(\frac{1}{\sqrt{q-1}})$, as $q \to 1^+$.*

5. Liuouville theorem for subcritical semi-linear equation in $\mathbb{H}^n$

In this section, we study the following equation

$$-\Delta_{\mathbb{H}^n}u = 2n^2u^q \quad \text{in} \quad \mathbb{H}^n, \tag{5.1}$$

where u is a smooth, nonnegative real function defined in H^n, while $\Delta_{\mathbb{H}^n}u = \sum_{\alpha=1}^n[u_{\alpha\overline{\alpha}} + u_{\overline{\alpha}\alpha}]$ is the Heisenberg Laplacian of u which will be defined latter. Let $Q = 2n+2$ be the homogeneous dimension of $\mathbb{H}^n$. Denote $q^* = \frac{Q+2}{Q-2}$, we shall give the sketch to prove the following Liouville type theorem.

Theorem 5.1. [20] *Let $1 < q < q^*$, then the equation (5.1) has no positive solution, namely, any nonnegative entire solution of (5.1) must be the trivial one.*

As in section 3, the soul of the proofs of Theorem 5.1 is an integral estimate. In fact we shall prove the following estimates. Let $1 < q < q^*$, $B_{4r}(\xi_0) \subset \Omega$ be any ball centered at ξ_0 with radio $4r$. Then any positive solution u of (5.1) satisfies:

$$\int_{B_r(\xi_0)} u^{3q-q^*} \leq C\, r^{Q-2\times\frac{3q-q^*}{q-1}}, \tag{5.2}$$

with some positive constant C depending only on n and q. For $1 < q < q^*$, we have $Q - 2 \times \frac{3q-q^*}{q-1} < 0$. So if u is a positive solution of (5.1), taking $r \to +\infty$ in (5.2) we get

$$\int_{\mathbb{H}^n} u^{3q-q^*} \longrightarrow 0. \tag{5.3}$$

This contradiction signifies directly the conclusion of Theorem 5.1. This integral estimate is similar to the integral estimate (3.23) in $\mathbb{R}^n$.

The equation (5.1) studied intensively by many authors in decades is connected to the CR Yamabe problem on $\mathbb{H}^n$. The number $\frac{2Q}{Q-2}$ is the CR Sobolev embedding exponent [11]. For the equation (5.1) with $q = q^* = \frac{Q+2}{Q-2}$, by the remarkable work Jerison-Lee [18], there is a nontrivial solution as follows

$$u(z,t) = C\left|t + \sqrt{-1}z \cdot \overline{z} + z \cdot \mu + \lambda\right|^{-n} \tag{5.4}$$

for some $C > 0$, $\lambda \in \mathbf{C}$, Im(λ)$>$ $|\mu|^2/4$, and $\mu \in \mathbf{C}^n$, which is the only extremals of the CR Sobolev inequality on $\mathbb{H}^n$.

In fact, for equation (5.1) with $q = \frac{Q+2}{Q-2}$, Jerison-Lee [18] obtained the uniqueness of the solution on the case of finite volume, i.e., $u \in L^{\frac{2Q}{Q-2}}(\mathbb{H}^n)$. Garofalo-Vassilev [12] also got a uniqueness result under the assumption of cylindrically symmetry on groups of Heisenberg type. For the subcritical case $1 < q \leq \frac{Q}{Q-2}$, Birindelli-Dolcetta-Cutri [4] proved that the only nonnegative entire solution of (5.1) is the trivial one. There are some partial results for the subcritical case $\frac{Q}{Q-2} < q < q^*$, for example the solutions are cylindrical symmetry or decay at infinity in [5], and as $n > 1$, $\frac{Q}{Q-2} < q \leq q^* - \frac{1}{(Q-2)(Q-1)^2}$ in [27].

In CR conformal geometry, Jerison-Lee [18] had found a magic identity which involved the derivative of torsion in the divergence term, and they got an Obata type theorem in CR geometry: if θ is a contact form associated with the standard CR structure on the sphere which has constant pseudohermitian scalar curvature, then θ is obtained from a constant multiple of the standard form $\hat{\theta}$ by a CR automorphism of the sphere. In the same paper, Jerison-Lee [18] also got the related identity to obtain the extremal function of the Sobolev inequality in Heisenberg group.

In the paper [20], based on a new observation, it gave a generalization of the Jerison-Lee's identity on Heisenberg group (see (4.2) for example in [18]) with a transparent proof, so that we can deduce a Liouville theorem for the subcritical case of the equation (5.1).

5.1. **A Generalization of Jerison-Lee's identity.** We discuss a minor generalization of the remarkable Jerison-Lee's identity ((4.2) in [18]) on Heissenberg group $\mathbb{H}^n$ to our equation (5.1). We adopt notations as in [18].

We shall first give a brief introduction to the Heissenberg group $\mathbb{H}^n$ and some notations. We consider $\mathbb{H}^n$ as the set $\mathbb{C}^n \times \mathbb{R}$ with coordinates (z, t) and group law $\circ$:

$$(z,t)\circ(w,s) = \big(z+w,\, t+s+2\mathbf{Im} z^\alpha \overline{w}^\alpha\big) \quad \text{for } \xi=(z,t),\, \zeta=(w,s)\in\mathbb{C}^n\times\mathbb{R},$$

where and in the sequel, we shall use the Einstein sum with the convention: the Greek indices $1 \le \alpha, \beta, \gamma, \delta \le n$. If $\xi = (z,t) = (z_1, z_2, ..., z_n, t) \in \mathbb{C}^n \times \mathbb{R}$ is an element of $\mathbb{H}^n$, then we set $|\xi|^4 = |(z,t)|^4 = |z|^4 + t^2$, associated with this norm is a distance function $d(\xi,\zeta) = |\zeta^{-1}\xi|$. We will use the notation $B(\xi, r)$ for the metric ball centered in $\xi = (z,t)$ with the radius $r > 0$. The Heisenberg group is a dilation group and the associated homogeneous dimension $Q = 2n+2$ such that the volume $|B(\xi, r)| \approx r^Q$.

The CR structure of $\mathbb{H}^n$ is given by the bundle $\mathcal{H}$ spanned by the left-invariant vector fields $Z_\alpha = \partial/\partial z^\alpha + \sqrt{-1}\bar{z}^\alpha \partial/\partial t$ and $Z_{\bar\alpha} = \partial/\partial \bar{z}^\alpha - \sqrt{-1} z^\alpha \partial/\partial t$, $\alpha = 1, \cdots, n$. The standard (left-invariant) contact form on $\mathbb{H}^n$ is $\mathbf{\Theta} = dt + \sqrt{-1}(z^\alpha d\overline{z}^\alpha - \overline{z}^\alpha dz^\alpha)$. With respect to the standard holomorphic frame $\{Z_\alpha\}$ and dual admissible coframe $\{dz^\alpha\}$, the $Levi\ forms$ $h_{\alpha\overline{\beta}} = 2\delta_{\alpha\overline{\beta}}$. Accordingly, for a smooth function f on $\mathbb{H}^n$, denote its derivatives by $f_\alpha = Z_\alpha f$, $f_{\alpha\overline{\beta}} = Z_{\overline{\beta}}(Z_\alpha f)$, $f_0 = \frac{\partial f}{\partial t}$, $f_{0\alpha} = Z_\alpha(\frac{\partial f}{\partial t})$, etc. We would also indicate the derivatives of functions or vector fields with indices preceded by a comma, to avoid confusion. Then as in [18] we have the following commutative formulae:

$$f_{\alpha\beta} - f_{\beta\alpha} = 0, \quad f_{\alpha\overline{\beta}} - f_{\overline{\beta}\alpha} = 2\sqrt{-1}\delta_{\alpha\overline{\beta}} f_0, \quad f_{0\alpha} - f_{\alpha 0} = 0,$$
$$f_{\alpha\beta 0} - f_{\alpha 0\beta} = 0, \qquad f_{\alpha\beta\overline{\gamma}} - f_{\alpha\overline{\gamma}\beta} = 2\sqrt{-1}\delta_{\beta\overline{\gamma}} f_{\alpha 0}, \ \cdots .$$

Now we are at the point to give the generalized identity for positive solution of the equation (5.1). Let $u > 0$ be the solution of (5.1). Take $e^f = u^{\frac{1}{n}}$ and $q = q^* + \frac{p}{n}$, then the subcritical exponent $1 < q < q^*$ is corresponding to $-2 < p < 0$. It follows that f satisfies the following equation

$$\mathbf{Re} f_{\alpha\overline{\alpha}} = -n|\partial f|^2 - n e^{(2+p)f}, \tag{5.5}$$

where $|\partial f|^2 = f_\alpha f_{\overline{\alpha}}$.

Define the tensors

$$\begin{aligned} D_{\alpha\beta} &= f_{\alpha\beta} - 2 f_\alpha f_\beta, & D_\alpha &= D_{\alpha\beta} f_{\overline{\beta}}, \\ E_{\alpha\overline{\beta}} &= f_{\alpha\overline{\beta}} - \frac{1}{n} f_{\gamma\overline{\gamma}}\delta_{\alpha\overline{\beta}}, & E_\alpha &= E_{\alpha\overline{\beta}} f_\beta, \\ G_\alpha &= \sqrt{-1} f_{0\alpha} - \sqrt{-1} f_0 f_\alpha + e^{(2+p)f} f_\alpha + |\partial f|^2 f_\alpha. \end{aligned} \tag{5.6}$$

As in [18], the above Jerison-Lee's tensors play key roles in our proof, one can see [18] the reason to introduce them. Let $g = |\partial f|^2 + e^{(2+p)f} - \sqrt{-1}f_0$, then we can rewrite equation (5.5) as

$$f_{\alpha\overline{\alpha}} = -ng. \tag{5.7}$$

Moreover, we observe that

$$\begin{aligned} E_{\alpha\overline{\beta}} &= f_{\alpha\overline{\beta}} + g\delta_{\alpha\overline{\beta}}, & E_\alpha &= f_{\alpha\overline{\beta}}f_\beta + gf_\alpha, \\ D_\alpha &= f_{\alpha\beta}f_{\overline{\beta}} - 2|\partial f|^2 f_\alpha, & G_\alpha &= \sqrt{-1}f_{0\alpha} + gf_\alpha. \end{aligned} \tag{5.8}$$

We can get

$$\begin{aligned} g_\alpha =& |\partial f|^2_\alpha + (e^{(2+p)f})_\alpha - \sqrt{-1}f_{0\alpha} \\ =& D_\alpha + E_\alpha - G_\alpha + 2gf_\alpha + pf_\alpha e^{(2+p)f}. \end{aligned} \tag{5.9}$$

Through similar computation, it follows that

$$g_{\overline{\alpha}} = D_{\overline{\alpha}} + E_{\overline{\alpha}} + G_{\overline{\alpha}} + pf_{\overline{\alpha}}e^{(2+p)f}, \tag{5.10}$$

and

$$\bar{g}_\alpha = D_\alpha + E_\alpha + G_\alpha + pf_\alpha e^{(2+p)f}. \tag{5.11}$$

In view of the above observations, now we give the crucial identity as follows

Proposition 5.2.

$$\begin{aligned} &\mathbf{Re}Z_{\overline{\alpha}}\Big\{e^{2(n-1)f}\Big[\big(g + 3\sqrt{-1}f_0\big)E_\alpha \\ &\qquad\qquad + \big(g - \sqrt{-1}f_0\big)D_\alpha - 3\sqrt{-1}f_0G_\alpha - \frac{p}{4}f_\alpha|\partial f|^4\Big]\Big\} \\ =& e^{(2n+p)f}\big(|E_{\alpha\overline{\beta}}|^2 + |D_{\alpha\beta}|^2\big) \\ &+ e^{2(n-1)f}\big(|G_\alpha|^2 + |G_\alpha + D_\alpha|^2 + |G_\alpha - E_\alpha|^2 + |D_{\alpha\beta}f_{\overline{\gamma}} + E_{\alpha\overline{\gamma}}f_\beta|^2\big) \\ &+ e^{(2n-2)f}\mathbf{Re}\big(D_\alpha + E_\alpha\big)f_{\overline{\alpha}}\big(pe^{(2+p)f} - \frac{p}{2}|\partial f|^2\big) \\ &- p(2n-1)|\partial f|^2 e^{2(n+1+p)f} - \frac{p}{4}(7n-6)|\partial f|^4 e^{(2n+p)f} \\ &- \frac{p}{4}n|\partial f|^6 e^{2(n-1)f} - 3np|f_0|^2 e^{(2n+p)f}. \end{aligned} \tag{5.12}$$

Remark 5.3. Note that for $p = 0$, (5.12) is exactly the key identity founded by Jerison and Lee (see (4.2) in [18]). For $-2 < p < 0$, the subcritical case, we will show by elementary computations that the right hand side of (5.12) is also nonnegative. The term $-\frac{p}{4}f_\alpha|\partial f|^4$ in the left of (5.12) is important for our proof in $n = 1$ case.

5.2. **Proof of Theorem 5.1.** Let f satisfy the equation (5.7) and hence the identity (5.12). Then by $q = q^* + \frac{p}{n}$, the subcritical exponent $1 < q < q^*$ is corresponding to $-2 < p < 0$. In order to complete the proof of (5.2) and hence Theorem 5.1, we only need to prove the following inequality

$$\int_{B_r(\xi_0)} e^{(2n+4+3p)f} \leq \ C r^{2n+2-2\times\frac{2n+4+3p}{2+p}}. \tag{5.13}$$

Note that (5.12) can be rewritten as

$$\begin{aligned}\mathcal{M} = \mathbf{Re} Z_{\overline{\alpha}}\Big\{\Big[&\big(D_\alpha + E_\alpha\big)(|\partial f|^2 + e^{(2+p)f}) \\ &- \sqrt{-1} f_0\big(2D_\alpha - 2E_\alpha + 3G_\alpha\big) - \frac{p}{4} f_\alpha |\partial f|^4\Big] e^{2(n-1)f}\Big\},\end{aligned} \tag{5.14}$$

we take $0 < s_0 = \frac{1}{2} + \frac{p}{4n} < 1$, then

$$\begin{aligned}\mathcal{M} =& \big(|E_{\alpha\overline{\beta}}|^2 + |D_{\alpha\beta}|^2\big) e^{(2n+p)f} + \Big(|G_\alpha|^2 + |D_{\alpha\beta} f_{\overline{\gamma}} + E_{\alpha\overline{\gamma}} f_\beta|^2\Big) e^{2(n-1)f} \\ &+ s_0\Big(|G_\alpha + D_\alpha|^2 + |G_\alpha - E_\alpha|^2\Big) e^{2(n-1)f} \\ &+ e^{2(n-1)f}\Big|\sqrt{1-s_0}(G_\alpha + D_\alpha) + \frac{p}{2\sqrt{1-s_0}} f_\alpha\big(e^{(2+p)f} - \frac{1}{2}|\partial f|^2\big)\Big|^2 \\ &+ e^{2(n-1)f}\Big|\sqrt{1-s_0}(E_\alpha - G_\alpha) + \frac{p}{2\sqrt{1-s_0}} f_\alpha\big(e^{(2+p)f} - \frac{1}{2}|\partial f|^2\big)\Big|^2 \\ &- p\frac{n(2n+p)}{4(2n-p)} |\partial f|^6 e^{2(n-1)f} - \frac{p}{4}\big[7n - 6 - \frac{8np}{2n-p}\big] |\partial f|^4 e^{(2n+p)f} \\ &- p\frac{4n^2 - 2n + p}{2n-p} |\partial f|^2 e^{2(n+1+p)f} - 3np|f_0|^2 e^{(2n+p)f},\end{aligned} \tag{5.15}$$

and clearly all the coefficients in the above are positive for $-2 < p < 0$ hence $\mathcal{M} \geq 0$.

Since $B_{4r} \subset \Omega$, we can take a real smooth cut off function η such that

$$\begin{cases} \eta \equiv 1 & in\ B_r, \\ 0 \leq \eta \leq 1 & in\ B_{2r}, \\ \eta \equiv 0 & in\ \mathbb{H}^n \backslash B_{2r}, \\ |\partial \eta| \lesssim \frac{1}{r} & in\ \mathbb{H}^n, \end{cases} \tag{5.16}$$

where we use "$\lesssim$" , "$\cong$" to replace "$\leq$" and "$=$" respectively, to drop out some positive constants independent of r and f.

Take a real $s > 0$ big enough. Multiply both sides of (5.14) by η^s and integrate on $\mathbb{H}^n$ we have

$$\begin{aligned}&\int_{\mathbb{H}^n}\eta^s\mathcal{M}\\
&=\int_{\mathbb{H}^n}\eta^s\mathbf{Re}Z_{\overline{\alpha}}\Big\{\big[\big(D_\alpha+E_\alpha)(|\partial f|^2+e^{(2+p)f})\\
&\qquad-\sqrt{-1}f_0\big(2D_\alpha-2E_\alpha+3G_\alpha\big)-\frac{p}{4}f_\alpha|\partial f|^4\big]e^{2(n-1)f}\Big\}.\end{aligned}\tag{5.17}$$

Integrating by parts and using (5.16) we get

$$\begin{aligned}&\int_{B_{2r}}\eta^s\mathcal{M}\\
&=-s\int_{B_{2r}}\eta^{s-1}\mathbf{Re}\eta_{\overline{\alpha}}\Big\{\big[\big(D_\alpha+E_\alpha)(|\partial f|^2+e^{(2+p)f})\\
&\qquad-\sqrt{-1}f_0\big(2D_\alpha-2E_\alpha+3G_\alpha\big)-\frac{p}{4}f_\alpha|\partial f|^4\big]e^{2(n-1)f}\Big\}.\end{aligned}\tag{5.18}$$

Using the Young's inequality we obtain

$$\begin{aligned}\int_{B_{2r}}\eta^s\mathcal{M}\lesssim&\frac{1}{r^2}\int_{B_{2r}}\eta^{s-2}\big(|\partial f|^4+e^{2(2+p)f}+|f_0|^2\big)e^{2(n-1)f}\\
&+\frac{1}{r}\int_{B_{2r}}\eta^{s-1}|\partial f|^5e^{2(n-1)f}.\end{aligned}\tag{5.19}$$

To go forward, we need the following Lemma 5.4, which will be proved at the end of this section.

Lemma 5.4.

$$\begin{aligned}\int_{B_{2r}}\eta^{s-2}|f_0|^2e^{2(n-1)f}\lesssim&\epsilon r^2\int_{B_{2r}}\eta^s\mathcal{M}+\int_{B_{2r}}\eta^{s-2}|\partial f|^4e^{2(n-1)f}\\
&+\int_{B_{2r}}\eta^{s-2}|\partial f|^2e^{(2n+p)f}+\frac{1}{r^2}\int_{B_{2r}}\eta^{s-4}|\partial f|^2e^{2(n-1)f}.\end{aligned}\tag{5.20}$$

Now plugging (5.20) into (5.19) with small ϵ we get

$$\begin{aligned}\int_{B_{2r}}\eta^s\mathcal{M}\lesssim&\frac{1}{r^2}\int_{B_{2r}}\eta^{s-2}e^{2(n+1+p)f}\\
&+\frac{1}{r^2}\int_{B_{2r}}\eta^{s-2}|\partial f|^4e^{2(n-1)f}+\frac{1}{r^2}\int_{B_{2r}}\eta^{s-2}|\partial f|^2e^{(2n+p)f}\\
&+\frac{1}{r^4}\int_{B_{2r}}\eta^{s-4}|\partial f|^2e^{2(n-1)f}+\frac{1}{r}\int_{B_{2r}}\eta^{s-1}|\partial f|^5e^{2(n-1)f}.\end{aligned}\tag{5.21}$$

For the last term in above, using Young's inequality one gets

$$\frac{1}{r}\int_{B_{2r}}\eta^{s-1}|\partial f|^5e^{2(n-1)f}\lesssim\epsilon\int_{B_{2r}}\eta^s|\partial f|^6e^{2(n-1)f}+\frac{1}{r^6}\int_{B_{2r}}\eta^{s-6}e^{2(n-1)f}.\tag{5.22}$$

Similarly, one has

$$\frac{1}{r^2}\int_{B_{2r}}\eta^{s-2}|\partial f|^4 e^{2(n-1)f}\lesssim\epsilon\int_{B_{2r}}\eta^{s}|\partial f|^6 e^{2(n-1)f}+\frac{1}{r^6}\int_{B_{2r}}\eta^{s-6}e^{2(n-1)f}, \tag{5.23}$$

$$\frac{1}{r^2}\int_{B_{2r}}\eta^{s-2}|\partial f|^2 e^{(2n+p)f}\lesssim\epsilon\int_{B_{2r}a}\eta^{s}|\partial f|^4 e^{(2n+p)f}+\frac{1}{r^4}\int_{B_{2r}}\eta^{s-4}e^{(2n+p)f}, \tag{5.24}$$

and

$$\frac{1}{r^4}\int_{B_{2r}}\eta^{s-4}|\partial f|^2 e^{2(n-1)f}\lesssim\epsilon\int_{B_{2r}}\eta^{s}|\partial f|^6 e^{2(n-1)f}+\frac{1}{r^6}\int_{B_{2r}}\eta^{s-6}e^{2(n-1)f}. \tag{5.25}$$

Inserting these into (5.21) and taking ϵ small yields

$$\begin{aligned}\int_{B_{2r}}\eta^{s}\mathcal{M}\lesssim&\frac{1}{r^2}\int_{B_{2r}}\eta^{s-2}e^{2(n+1+p)f}\\&+\frac{1}{r^4}\int_{B_{2r}}\eta^{s-4}e^{(2n+p)f}+\frac{1}{r^6}\int_{B_{2r}}\eta^{s-6}e^{2(n-1)f}.\end{aligned} \tag{5.26}$$

We note that (5.26) is similar to the formula (3.14)

In order to complete this proof, we state another lemma, it is similar to formula (3.18), it will also be proved at the end of this section.

Lemma 5.5.

$$\int_{B_{2r}}\eta^{s}e^{(2n+4+3p)f}\lesssim\int_{B_{2r}}\eta^{s}|\partial f|^2 e^{2(n+1+p)f}+\frac{1}{r^2}\int_{B_{2r}}\eta^{s-2}e^{(2n+2+2p)f}. \tag{5.27}$$

Now since all the coefficients in (5.15) are positive for $-2<p<0$, it follows that

$$\int_{B_{2r}}\eta^{s}|\partial f|^2 e^{2(n+1+p)f}\le c(n,p)\int_{B_{2r}}\eta^{s}\mathcal{M}. \tag{5.28}$$

Combining (5.28) with (5.27) and (5.26), we have

$$\begin{aligned}&\int_{B_{2r}}\eta^{s}e^{(2n+4+3p)f}\\\lesssim&\int_{B_{2r}}\eta^{s}|\partial f|^2 e^{2(n+1+p)f}+\frac{1}{r^2}\int_{B_{2r}}\eta^{s-2}e^{(2n+2+2p)f}\\\lesssim&\int_{B_{2r}}\eta^{s}\mathcal{M}+\frac{1}{r^2}\int_{B_{2r}}\eta^{s-2}e^{(2n+2+2p)f}\\\lesssim&\frac{1}{r^2}\int_{B_{2r}}\eta^{s-2}e^{2(n+1+p)f}\\&+\frac{1}{r^4}\int_{B_{2r}}\eta^{s-4}e^{(2n+p)f}+\frac{1}{r^6}\int_{B_{2r}}\eta^{s-6}e^{2(n-1)f}\\\lesssim&\epsilon\int_{B_{2r}}\eta^{s}e^{(2n+4+3p)f}+r^{-2\times\frac{2n+4+3p}{2+p}}\int_{B_{2r}}\eta^{s-2\times\frac{2n+4+3p}{2+p}},\end{aligned} \tag{5.29}$$

where in the last step, the Young's inequality has been used three times with different exponent pairs. Note that $0\le\eta\le 1$ in $B_{2r}(\xi_0)\subset\Omega$ and $\eta=1$ in $B_r(\xi_0)$. Therefore, by choosing $s>0$ big enough and ϵ small, we finally obtain

$$\int_{B_r(\xi_0)} e^{(2n+4+3p)f} \lesssim r^{2n+2-2\times\frac{2n+4+3p}{2+p}}. \tag{5.30}$$

This is (5.13), and hence Theorem 5.1 is proved. □

To complete this section, now we give the proofs of Lemma 5.4 and Lemma 5.5.

Proof of Lemma 5.4 :
Since f satisfies equation (5.7), a straight calculation shows

$$e^{-kf}\mathbf{Re}Z_{\overline{\alpha}}\Big(\sqrt{-1}f_0f_\alpha e^{kf}\Big) = -\mathbf{Re}G_{\overline{\alpha}}f_\alpha - n|f_0|^2 + |\partial f|^4 + |\partial f|^2 e^{(2+p)f}. \tag{5.31}$$

Multiply both sides of (5.31) by $\eta^{s-2}e^{kf}$ with $k = 2(n-1)$ and integrate we have

$$\begin{aligned}
&\int_{B_{2r}} \eta^{s-2}\mathbf{Re}Z_{\overline{\alpha}}\Big(\sqrt{-1}f_0f_\alpha e^{2(n-1)f}\Big)\\
=&\int_{B_{2r}} \eta^{s-2}\Big(-\mathbf{Re}G_{\overline{\alpha}}f_\alpha - n|f_0|^2 + |\partial f|^4 + |\partial f|^2 e^{(2+p)f}\Big)e^{2(n-1)f}.
\end{aligned} \tag{5.32}$$

Integrating by parts, using (5.16) and arranging the terms yields

$$\begin{aligned}
n\int_{B_{2r}} \eta^{s-2}|f_0|^2 e^{2(n-1)f} &= \int_{B_{2r}} \eta^{s-2}\big(|\partial f|^4 + |\partial f|^2 e^{(2+p)f}\big)e^{2(n-1)f}\\
&\quad - \int_{B_{2r}} \eta^{s-2}\mathbf{Re}G_{\overline{\alpha}}f_\alpha e^{2(n-1)f}\\
&\quad + (s-2)\int_{B_{2r}} \eta^{s-3}\mathbf{Re}\eta_{\overline{\alpha}}\Big(\sqrt{-1}f_0f_\alpha e^{2(n-1)f}\Big)\\
&\lesssim \int_{B_{2r}} \eta^{s-2}\big(|\partial f|^4 + |\partial f|^2 e^{(2+p)f}\big)e^{2(n-1)f}\\
&\quad + \int_{B_{2r}} \eta^{s-2}|G_{\overline{\alpha}}||\partial f|e^{2(n-1)f}\\
&\quad + \frac{1}{r}\int_{B_{2r}} \eta^{s-3}|f_0||\partial f|e^{2(n-1)f}.
\end{aligned} \tag{5.33}$$

For the above last two terms, Young's inequality implies

$$\begin{aligned}
&\int_{B_{2r}} \eta^{s-2}|G_{\overline{\alpha}}||\partial f|e^{2(n-1)f} + \frac{1}{r}\int_{B_{2r}} \eta^{s-3}|f_0||\partial f|e^{2(n-1)f}\\
\le\,&\epsilon r^2\int_{B_{2r}} \eta^{s}|G_\alpha|^2 e^{2(n-1)f} + \epsilon\int_{B_{2r}} \eta^{s-2}|f_0|^2 e^{2(n-1)f}\\
&\qquad + \frac{C}{\epsilon r^2}\int_{B_{2r}} \eta^{s-4}|\partial f|^2 e^{2(n-1)f}.
\end{aligned} \tag{5.34}$$

Submitting this into (5.33) with small ϵ we get

$$
\begin{aligned}
\int_{B_{2r}} \eta^{s-2}|f_0|^2 e^{2(n-1)f} \lesssim & \epsilon r^2 \int_{B_{2r}} \eta^s \mathcal{M} + \int_{B_{2r}} \eta^{s-2}|\partial f|^4 e^{2(n-1)f} \\
& + \int_{B_{2r}} \eta^{s-2}|\partial f|^2 e^{(2n+p)f} + \frac{1}{r^2}\int_{B_{2r}} \eta^{s-4}|\partial f|^2 e^{2(n-1)f}.
\end{aligned} \tag{5.35}
$$

This is just (5.20). □

The proof of Lemma 5.5 is similar to that of Lemma 5.4.

Proof of Lemma 5.5 :

Multiply both sides of the equation (5.7) by $-\eta^s e^{2(n+1+p)f}$ and integrate we have

$$
\begin{aligned}
n\int_{B_{2r}} \eta^s g e^{2(n+1+p)f} =& -\int_{B_{2r}} \eta^s f_{\alpha\overline{\alpha}} e^{2(n+1+p)f} \\
=& 2(n+1+p)\int_{B_{2r}} \eta^s |\partial f|^2 e^{2(n+1+p)f} \\
&+ s\int_{B_{2r}} \eta^{s-1} f_\alpha \eta_{\overline{\alpha}} e^{2(n+1+p)f}.
\end{aligned} \tag{5.36}
$$

Take the conjugate in (5.36), we have

$$
\begin{aligned}
n\int_{B_{2r}} \eta^s \bar{g} e^{2(n+1+p)f} =& -\int_{B_{2r}} \eta^s f_{\bar{\alpha}\alpha} e^{2(n+1+p)f} \\
=& 2(n+1+p)\int_{B_{2r}} \eta^s |\partial f|^2 e^{2(n+1+p)f} \\
&+ s\int_{B_{2r}} \eta^{s-1} f_{\bar{\alpha}} \eta_{\alpha} e^{2(n+1+p)f}.
\end{aligned} \tag{5.37}
$$

Add (5.36) and (5.37), it follows that

$$
\begin{aligned}
n\int_{B_{2r}} \eta^s [|\partial f|^2 + e^{(2+p)f}] e^{2(n+1+p)f} =& 2(n+1+p)\int_{B_{2r}} \eta^s |\partial f|^2 e^{2(n+1+p)f} \\
&+ s\int_{B_{2r}} \eta^{s-1} \mathbf{Re} f_{\bar{\alpha}} \eta_{\alpha} e^{2(n+1+p)f}.
\end{aligned} \tag{5.38}
$$

Using (5.16) and arranging the terms yields

$$
\begin{aligned}
\int_{B_{2r}} \eta^s e^{(2n+4+3p)f} \lesssim & \int_{B_{2r}} \eta^s |\partial f|^2 e^{2(n+1+p)f} + \frac{1}{r}\int_{B_{2r}} \eta^{s-1}|\partial f| e^{2(n+1+p)f} \\
\lesssim & \int_{B_{2r}} \eta^s |\partial f|^2 e^{2(n+1+p)f} + \frac{1}{r^2}\int_{B_{2r}} \eta^{s-2} e^{2(n+1+p)f},
\end{aligned} \tag{5.39}
$$

where in the last step, the Cauchy-Schwarz inequality has been used, and this is (5.27) as desired. □

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