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Calculus of Variations

Superharmonicity of curvature function for the convex level sets of harmonic functions

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Abstract
We prove that the combination of the norm of gradient and the Gaussian curvature for the convex level sets of harmonic function is superharmonic.

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1 Introduction

The study of the convexity of level sets of solutions to elliptic partial differential equations has a long history. For instance, about one hundred years ago people knew that the level curves of the Green's function of a convex domain in $\mathbb{R}^2$ are convex (see the book by Ahlfors [1]). In 1957, Gabriel [8] obtained the analogues in three dimensions. Later, these results were extended to high dimensions and more general elliptic PDEs such as p-harmonic functions and some semilinear elliptic equations by Lewis [15], Caffarelli and Spruck [5]. For more related topics on convexity of level sets in qualitative nature, please see such as these papers [2,3,9,13,27].

Note that convexity of solutions imply convexity of level sets. To obtain the convexity of level sets, one may attempt to prove the convexity of the solution u itself, or at least the convexity of $g(u)$. Here g is a suitable monotone function of one variable. In this direction, we refer the readers to [4,7,10–12,14,18,19,22], and the references therein.

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In the following, we will turn to the quantitative convexity estimates of the curvature of level sets. In 1983, Talenti [25] got the following result for harmonic functions of two variables. Let $\Omega \subset \mathbb{R}^2$ be a domain and u be a harmonic function with no critical points in Ω. Then the function $\kappa/|\nabla u|$ is harmonic in Ω. Here

$$\kappa = \frac{2u_1u_2u_{12} - u_1^2u_{22} - u_2^2u_{11}}{|\nabla u|^3}$$

is the curvature of the level sets of u. Throughout the paper we use subscripts to represent the derivatives with respect to any orthonormal frames. Similar results can also be seen in [16,23]. By maximum principle, one can easily obtain the convexity estimates of level sets of 2-dimensional harmonic functions. Recently, Ma, Ou and Zhang [17] generalized the above results to n-dimensional harmonic functions ($n \geq 2$) and obtain the convexity estimates of their level sets. More precisely, they proved the following theorem.

Theorem 1.1 (Ma, Ou and Zhang [17]). *Let Ω be a domain in $\mathbb{R}^n$ ($n \geq 2$) and u be a harmonic function defined on Ω. Assume that there are no critical points of u in Ω and all the level sets of u are strictly convex. Let K be the Gaussian curvature of the level sets of u. Set*

$$\psi_0 = |\nabla u|^{n-3}K.$$

Then the function ψ_0 satisfies the differential inequality

$$\Delta\psi_0 + \sum_{\alpha=1}^{n} b_\alpha \psi_{0,\alpha} \leq 0 \quad in\ \Omega. \tag{1.1}$$

Here $b_1, b_2, \ldots, b_n$ are smooth functions.

For more extensions on convexity estimate of level sets of harmonic functions and solutions to other elliptic PDEs, please see [6,20,21,24,26,28–30].

The proof of inequality (1.1) in [17] is much more involved. In this paper, by considering a certain power of the function ψ_0, we prove this new curvature function is superharmonic. Now we state our main theorem.

Theorem 1.2 *Let Ω be a domain in $\mathbb{R}^n$ ($n \geq 2$) and u be a harmonic function defined on Ω. Assume that there are no critical points of u in Ω and all the level sets of u are strictly convex. Let K be the Gaussian curvature of the level sets of u. Set*

$$\psi = \left(|\nabla u|^{n-3}K\right)^{\frac{1}{n-1}}.$$

Then the function ψ is superharmonic in Ω. Namely, the function ψ perfectly satisfies the differential inequality

$$\Delta\psi \leq 0 \quad in\ \Omega.$$

Note that the function ψ in Theorem 1.2 was first introduced in the paper Ma and Zhang [20], where by the method of support function they studied the concavity of the Gaussian curvature of level sets of harmonic functions with respect to the height of the functions.

Remark 1.3 For $n = 2$, the assumption of strictly convex level sets can be dropped and we recover Talenti's result. For $n = 3$, under the conditions of Theorem 1.2, in fact we proved that the function $\sqrt{K}$ is superharmonic in Ω.

As a consequence, we can give a brief proof for the convexity estimates of level sets of harmonic functions.

Corollary 1.4 (Ma, Ou and Zhang [17]). *Let u satisfy*

$$\begin{cases} \Delta u = 0 \ \ in \ \ \Omega = \Omega_0 \backslash \bar{\Omega}_1, \\ \ \ u = 0 \ on \ \partial\Omega_0, \\ \ \ u = 1 \ on \ \partial\Omega_1, \end{cases}$$

where Ω_0 and Ω_1 are bounded smooth strictly convex domains in $\mathbb{R}^n$, $n \geq 3$, $\bar{\Omega}_1 \subset \Omega_0$. Let K be the Gaussian curvature of the level sets. Then we have the following estimate

$$\min_{\bar{\Omega}} K \geq \min_{\partial\Omega} K \left(\frac{\min_{\partial\Omega_0} |\nabla u|}{\max_{\partial\Omega_1} |\nabla u|} \right)^{n-3}.$$

We will focus on the proof of Theorem 1.2 in Sect. 2. The main technique in the proof consists of regrouping terms involving third order derivatives and maximizing them in each group.

2 Proof of Theorem 1.2

In this section, we will prove Theorem 1.2. Let $\Omega \subset \mathbb{R}^n$ be a domain and u be a harmonic function with no critical points in Ω. Assume that all the level sets of u are strictly convex with respect to the upward normal direction $\mathbf{n} = \frac{|u_n|}{|\nabla u| u_n}(u_1, u_2, \ldots, u_{n-1}, u_n)$, see for example [2,17], it is well known that the Gaussian curvature K of the level sets of u can be expressed as

$$(-1)^{n-1} K = |\nabla u|^{-(n+1)} \sum_{\alpha\beta=1}^{n} \frac{\partial \det D^2 u}{\partial u_{\alpha\beta}} u_\alpha u_\beta. \tag{2.1}$$

In this paper, we make the convention that the Latin indices i, j, k run from 1 to $n-1$, and the Greek indices $\alpha, \beta, \gamma, \delta, \mu, \nu, \xi, \eta$ run form 1 to n.

Set

$$\psi = \left(|\nabla u|^a K\right)^b,$$

where K is the Gaussian curvature of the level sets and a, b are real numbers which will be determined later. For a suitable choice of a and b, we shall prove that ψ is superharmonic in Ω, i.e. the function ψ satisfies

$$\Delta \psi \leq 0 \quad \text{in } \Omega. \tag{2.2}$$

Later, we will see that the perfect candidates for a and b are

$$a = n - 3, \quad b = \frac{1}{n-1}.$$

In order to prove inequality (2.2) at an arbitrary point $x_0 \in \Omega$, we shall choose the principal coordinates such that

$$u_i(x_0) = 0 \ (1 \leq i \leq n-1), \quad u_n(x_0) = |\nabla u|(x_0) > 0, \tag{2.3}$$

$$u_{ij}(x_0) = 0 \ (1 \leq i, j \leq n-1, \ i \neq j), \quad u_{ii}(x_0) < 0 \ (1 \leq i \leq n-1). \tag{2.4}$$

For the sake of simplicity, we introduce the following notations

$$\sigma_1(\lambda) = \sum_i \lambda_i, \quad \sigma_2(\lambda) = \sum_{i<j} \lambda_i \lambda_j, \quad \sigma_{n-1}(\lambda) = \prod_i \lambda_i, \tag{2.5}$$

where $\lambda_i = u_{ii}(x_0)$ $(1 \le i \le n-1)$. Before proving Theorem 1.2, we summarize the first and second derivatives of the Hessian determinant of u in the following lemma.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^n$ be a domain and $u \in C^2(\Omega)$ be a function. For each $x_0 \in \Omega$ fixed, we choose the principal coordinates as above. Then at the point x_0, there hold*

$$\begin{cases} \frac{\partial \det D^2u}{\partial u_{ii}} = -\sigma_{n-1}(\lambda)\frac{1}{\lambda_i}\left(\sum_k \frac{1}{\lambda_k}u_{nk}^2 - \frac{1}{\lambda_i}u_{ni}^2 - u_{nn}\right), & \text{for } 1 \le i \le n-1; \\ \frac{\partial \det D^2u}{\partial u_{ij}} = \sigma_{n-1}(\lambda)\frac{1}{\lambda_i\lambda_j}u_{ni}u_{nj}, & \text{for } 1 \le i, j \le n-1, i \ne j; \\ \frac{\partial \det D^2u}{\partial u_{in}} = \frac{\partial \det D^2u}{\partial u_{ni}} = -\sigma_{n-1}(\lambda)\frac{1}{\lambda_i}u_{ni}, & \text{for } 1 \le i \le n-1; \\ \frac{\partial \det D^2u}{\partial u_{nn}} = \sigma_{n-1}(\lambda), \end{cases}$$

and

$$\begin{cases} \frac{\partial^2 \det D^2u}{\partial u_{kn}\partial u_{ii}} = -\sigma_{n-1}(\lambda)\frac{1}{\lambda_k\lambda_i}u_{nk}, & \text{for } 1 \le k, i \le n-1,\ k \ne i; \\ \frac{\partial^2 \det D^2u}{\partial u_{kn}\partial u_{ik}} = \sigma_{n-1}(\lambda)\frac{1}{\lambda_k\lambda_i}u_{ni}, & \text{for } 1 \le k, i \le n-1,\ k \ne i; \\ \frac{\partial^2 \det D^2u}{\partial u_{kn}\partial u_{nk}} = -\sigma_{n-1}(\lambda)\frac{1}{\lambda_k}, & \text{for } 1 \le k \le n-1; \\ \frac{\partial^2 \det D^2u}{\partial u_{nn}\partial u_{ii}} = \sigma_{n-1}(\lambda)\frac{1}{\lambda_i}, & \text{for } 1 \le i \le n-1; \\ \frac{\partial^2 \det D^2u}{\partial u_{\alpha n}\partial u_{\mu\nu}} = 0, & \text{otherwise}. \end{cases}$$

From now on, all the calculations will be done at the fixed point x_0 unless otherwise specified. By taking first derivatives of ψ, we have

$$\psi_\gamma = ab|\nabla u|^{ab-2}K^b \sum_\delta u_\delta u_{\delta\gamma} + b|\nabla u|^{ab}K^{b-1}K_\gamma. \tag{2.6}$$

Differentiating (2.6) once more, we have

$$\begin{aligned} \psi_{\gamma\gamma} = {} & ab(ab-2)|\nabla u|^{ab-4}K^b \sum_\mu u_\mu u_{\mu\gamma} \sum_\delta u_\delta u_{\delta\gamma} + ab|\nabla u|^{ab-2}K^b \sum_\delta u_{\delta\gamma}^2 \\ & + ab|\nabla u|^{ab-2}K^b \sum_\delta u_\delta u_{\delta\gamma\gamma} \\ & + 2ab^2|\nabla u|^{ab-2}K^{b-1}\sum_\delta u_\delta u_{\delta\gamma}K_\gamma \\ & + b(b-1)|\nabla u|^{ab}K^{b-2}K_\gamma^2 + b|\nabla u|^{ab}K^{b-1}K_{\gamma\gamma}; \end{aligned}$$

hence

$$\begin{aligned} \Delta\psi = {} & ab(ab-2)|\nabla u|^{ab-4}K^b \sum_{\gamma\mu} u_\mu u_{\mu\gamma} \sum_\delta u_\delta u_{\delta\gamma} + ab|\nabla u|^{ab-2}K^b \sum_{\gamma\delta} u_{\delta\gamma}^2 \\ & + ab|\nabla u|^{ab-2}K^b \sum_{\gamma\delta} u_\delta u_{\delta\gamma\gamma} \\ & + 2ab^2|\nabla u|^{ab-2}K^{b-1}\sum_{\gamma\delta} u_\delta u_{\delta\gamma}K_\gamma \\ & + b(b-1)|\nabla u|^{ab}K^{b-2}|\nabla K|^2 + b|\nabla u|^{ab}K^{b-1}\Delta K. \end{aligned}$$

Equivalently,

$$\frac{|\nabla u|^2}{b\psi}\Delta\psi = a(ab-2)|\nabla u|^{-2}\sum_{\gamma\mu} u_\mu u_{\mu\gamma}\sum_{\delta} u_\delta u_{\delta\gamma} + a\sum_{\gamma\delta} u_{\delta\gamma}^2 + a\sum_{\gamma\delta} u_\delta u_{\delta\gamma\gamma}$$
$$+ 2abK^{-1}\sum_{\gamma\delta} u_\delta u_{\delta\gamma} K_\gamma + (b-1)|\nabla u|^2 K^{-2}|\nabla K|^2 + |\nabla u|^2 K^{-1}\Delta K.$$

Note that u is harmonic. By (2.3)–(2.5), we have

$$\begin{aligned}\frac{|\nabla u|^2}{b\psi}\Delta\psi =\ & a(ab-2)\sum_{\gamma} u_{n\gamma}^2 + a\sum_{\gamma\delta} u_{\delta\gamma}^2 + 2abu_n K^{-1}\sum_{\gamma} u_{n\gamma}K_\gamma \\ & + (b-1)u_n^2 K^{-2}|\nabla K|^2 + u_n^2 K^{-1}\Delta K \\ =\ & a^2 b\sum_k u_{nk}^2 + a^2 b\sigma_1(\lambda)^2 - 2a\sigma_2(\lambda) + 2abu_n K^{-1}\sum_{\gamma} u_{n\gamma}K_\gamma \\ & + (b-1)u_n^2 K^{-2}|\nabla K|^2 + u_n^2 K^{-1}\Delta K \\ \triangleq\ & \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4,\end{aligned} \tag{2.7}$$

where

$$\mathrm{I}_1 = a^2 b\sum_k u_{nk}^2 + a^2 b\sigma_1(\lambda)^2 - 2a\sigma_2(\lambda), \qquad \mathrm{I}_2 = 2abu_n K^{-1}\sum_{\gamma} u_{n\gamma}K_\gamma,$$
$$\mathrm{I}_3 = (b-1)u_n^2 K^{-2}|\nabla K|^2, \qquad \mathrm{I}_4 = u_n^2 K^{-1}\Delta K.$$

Let us compute the terms I_2, I_3 and I_4, respectively. Taking derivatives on both sides of formula (2.1), we obtain

$$\begin{aligned}(-1)^{n-1}K_\gamma =& -(n+1)|\nabla u|^{-(n+3)}\sum_{\delta} u_\delta u_{\delta\gamma}\sum_{\alpha\beta}\frac{\partial \det D^2u}{\partial u_{\alpha\beta}}u_\alpha u_\beta \\ & + |\nabla u|^{-(n+1)}\sum_{\alpha\beta\mu\nu}\frac{\partial^2 \det D^2u}{\partial u_{\alpha\beta}\partial u_{\mu\nu}}u_{\mu\nu\gamma}u_\alpha u_\beta \\ & + 2|\nabla u|^{-(n+1)}\sum_{\alpha\beta}\frac{\partial \det D^2u}{\partial u_{\alpha\beta}}u_{\alpha\gamma}u_\beta.\end{aligned} \tag{2.8}$$

Recalling the relations (2.3)–(2.4) and applying Lemma 2.1, we have

$$\begin{aligned}(-1)^{n-1}K &= u_n^{-(n-1)}\sigma_{n-1}(\lambda), \\ (-1)^{n-1}K_k &= -(n+1)u_n^{-n}\sigma_{n-1}(\lambda)u_{nk} + u_n^{-(n-1)}\sigma_{n-1}(\lambda)\sum_i\frac{1}{\lambda_i}u_{iik}, \\ &\quad \text{for } k = 1, 2, \dots, n-1, \\ (-1)^{n-1}K_n &= (n-1)u_n^{-n}\sigma_{n-1}(\lambda)\sigma_1(\lambda) + u_n^{-(n-1)}\sigma_{n-1}(\lambda)\sum_i\frac{1}{\lambda_i}u_{iin} \\ &\quad - 2u_n^{-n}\sigma_{n-1}(\lambda)\sum_i\frac{1}{\lambda_i}u_{ni}^2.\end{aligned}$$

It follows that

$$u_n K^{-1} K_k = -(n+1)u_{nk} + u_n \sum_i \frac{1}{\lambda_i} u_{iik}, \quad \text{for } k = 1, 2, \ldots, n-1,$$

$$u_n K^{-1} K_n = (n-1)\sigma_1(\lambda) + u_n \sum_i \frac{1}{\lambda_i} u_{iin} - 2\sum_i \frac{1}{\lambda_i} u_{ni}^2.$$

Hence we obtain

$$\begin{aligned} \mathrm{I}_2 = & \, 2abu_n \sum_{ik} \frac{1}{\lambda_i} u_{iik} u_{nk} - 2abu_n \sigma_1(\lambda) \sum_i \frac{1}{\lambda_i} u_{iin} \\ & - 2(n+1)ab \sum_k u_{nk}^2 - 2(n-1)ab\sigma_1(\lambda)^2 + 4ab\sigma_1(\lambda) \sum_k \frac{1}{\lambda_k} u_{nk}^2, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \mathrm{I}_3 = & \, (b-1)u_n^2 \sum_k \left(\sum_i \frac{1}{\lambda_i} u_{iik} \right)^2 - 2(n+1)(b-1)u_n \sum_{ik} \frac{1}{\lambda_i} u_{iik} u_{nk} \\ & + (n+1)^2 (b-1) \sum_k u_{nk}^2 \\ & + (b-1)u_n^2 \left(\sum_i \frac{1}{\lambda_i} u_{iin} \right)^2 + 2(n-1)(b-1)u_n \sigma_1(\lambda) \sum_i \frac{1}{\lambda_i} u_{iin} \\ & + (n-1)^2 (b-1)\sigma_1(\lambda)^2 \\ & - 4(n-1)(b-1)\sigma_1(\lambda) \sum_i \frac{1}{\lambda_i} u_{ni}^2 - 4(b-1)u_n \sum_{ij} \frac{1}{\lambda_i \lambda_j} u_{iin} u_{nj}^2 \\ & + 4(b-1) \left(\sum_i \frac{1}{\lambda_i} u_{ni}^2 \right)^2. \end{aligned} \tag{2.10}$$

To compute the term I_4, we differentiate (2.8) once more. Summing with respect to γ and using the harmonicity of u, we are led to

$$\begin{aligned} (-1)^{n-1} \Delta K = & \, (n+1)(n+3)|\nabla u|^{-(n+5)} \sum_{\gamma\mu} u_\mu u_{\mu\gamma} \sum_\delta u_\delta u_{\delta\gamma} \sum_{\alpha\beta} \frac{\partial \det D^2 u}{\partial u_{\alpha\beta}} u_\alpha u_\beta \\ & - (n+1)|\nabla u|^{-(n+3)} \sum_{\gamma\delta} u_{\delta\gamma}^2 \sum_{\alpha\beta} \frac{\partial \det D^2 u}{\partial u_{\alpha\beta}} u_\alpha u_\beta \\ & - 2(n+1)|\nabla u|^{-(n+3)} \sum_{\gamma\delta} u_\delta u_{\delta\gamma} \sum_{\alpha\beta\mu\nu} \frac{\partial^2 \det D^2 u}{\partial u_{\alpha\beta} \partial u_{\mu\nu}} u_{\mu\nu\gamma} u_\alpha u_\beta \\ & - 4(n+1)|\nabla u|^{-(n+3)} \sum_{\gamma\delta} u_\delta u_{\delta\gamma} \sum_{\alpha\beta} \frac{\partial \det D^2 u}{\partial u_{\alpha\beta}} u_{\alpha\gamma} u_\beta \\ & + |\nabla u|^{-(n+1)} \sum_{\gamma\alpha\beta\mu\nu\xi\eta} \frac{\partial^3 \det D^2 u}{\partial u_{\alpha\beta} \partial u_{\mu\nu} \partial u_{\xi\eta}} u_{\mu\nu\gamma} u_{\xi\eta\gamma} u_\alpha u_\beta \end{aligned}$$

$$+4|\nabla u|^{-(n+1)}\sum_{\gamma\alpha\beta\mu\nu}\frac{\partial^2\det D^2u}{\partial u_{\alpha\beta}\partial u_{\mu\nu}}u_{\mu\nu\gamma}u_{\alpha\gamma}u_{\beta}$$
$$+2|\nabla u|^{-(n+1)}\sum_{\gamma\alpha\beta}\frac{\partial\det D^2u}{\partial u_{\alpha\beta}}u_{\alpha\gamma}u_{\beta\gamma}. \tag{2.11}$$

Keeping (2.3) in mind, combining (2.1) and (2.11), we have

$$\begin{aligned}
\mathrm{I}_4 =&\ (n+1)(n+3)\sum_{\gamma}u_{n\gamma}^2-(n+1)\sum_{\gamma\delta}u_{\delta\gamma}^2-\frac{2(n+1)}{\sigma_{n-1}(\lambda)}u_n\sum_{\gamma}u_{n\gamma}\sum_{\mu\nu}\frac{\partial^2\det D^2u}{\partial u_{nn}\partial u_{\mu\nu}}u_{\mu\nu\gamma}\\
&-\frac{4(n+1)}{\sigma_{n-1}(\lambda)}\sum_{\gamma}u_{n\gamma}\sum_{\alpha}\frac{\partial\det D^2u}{\partial u_{\alpha n}}u_{\alpha\gamma}+\frac{1}{\sigma_{n-1}(\lambda)}u_n^2\sum_{\gamma\mu\nu\xi\eta}\frac{\partial^3\det D^2u}{\partial u_{nn}\partial u_{\mu\nu}\partial u_{\xi\eta}}u_{\mu\nu\gamma}u_{\xi\eta\gamma}\\
&+\frac{4}{\sigma_{n-1}(\lambda)}u_n\sum_{\gamma\alpha\mu\nu}\frac{\partial^2\det D^2u}{\partial u_{\alpha n}\partial u_{\mu\nu}}u_{\mu\nu\gamma}u_{\alpha\gamma}+\frac{2}{\sigma_{n-1}(\lambda)}\sum_{\gamma\alpha\beta}\frac{\partial\det D^2u}{\partial u_{\alpha\beta}}u_{\alpha\gamma}u_{\beta\gamma}\\
\triangleq&\ \mathrm{I}_{41}+\mathrm{I}_{42}+\mathrm{I}_{43}+\mathrm{I}_{44}+\mathrm{I}_{45}+\mathrm{I}_{46}+\mathrm{I}_{47}.
\end{aligned}\tag{2.12}$$

In the following, we will deal with the above seven terms consecutively. First of all, it is clear that

$$\begin{aligned}
\mathrm{I}_{41}+\mathrm{I}_{42}=&\ (n+1)(n+3)\left(\sum_k u_{nk}^2+\sigma_1(\lambda)^2\right)-2(n+1)\left(\sum_k u_{nk}^2+\sigma_1(\lambda)^2-\sigma_2(\lambda)\right)\\
=&\ (n+1)^2\sum_k u_{nk}^2+(n+1)^2\sigma_1(\lambda)^2+2(n+1)\sigma_2(\lambda).
\end{aligned}\tag{2.13}$$

Then by Lemma 2.1, we get

$$\mathrm{I}_{43}=-2(n+1)u_n\sum_{ik}\frac{1}{\lambda_i}u_{iik}u_{nk}+2(n+1)u_n\sigma_1(\lambda)\sum_i\frac{1}{\lambda_i}u_{iin}, \tag{2.14}$$
$$\mathrm{I}_{44}=-4(n+1)\sigma_1(\lambda)^2-4(n+1)\sigma_1(\lambda)\sum_k\frac{1}{\lambda_k}u_{nk}^2. \tag{2.15}$$

For the term I_{45}, we simply have

$$\begin{aligned}
\mathrm{I}_{45}=&\ u_n^2\sum_k\sum_{i\neq j}\frac{1}{\lambda_i\lambda_j}u_{iik}u_{jjk}+u_n^2\sum_{i\neq j}\frac{1}{\lambda_i\lambda_j}u_{iin}u_{jjn}-2u_n^2\sum_{i\neq k}\frac{1}{\lambda_i\lambda_k}u_{iik}^2\\
&-u_n^2\sum_{i\neq j,j\neq k,k\neq i}\frac{1}{\lambda_i\lambda_j}u_{ijk}^2-u_n^2\sum_{i\neq j}\frac{1}{\lambda_i\lambda_j}u_{ijn}^2.
\end{aligned}\tag{2.16}$$

Again by Lemma 2.1, direct calculation gives

$$\begin{aligned}
\mathrm{I}_{46}=&-4u_n\sum_{\gamma}\sum_{k\neq i}\frac{1}{\lambda_k\lambda_i}u_{ii\gamma}u_{nk}u_{k\gamma}+4u_n\sum_{\gamma}\sum_{k\neq i}\frac{1}{\lambda_k\lambda_i}u_{ik\gamma}u_{ni}u_{k\gamma}\\
&-4u_n\sum_{\gamma}\sum_{k}\frac{1}{\lambda_k}u_{nk\gamma}u_{k\gamma}+4u_n\sum_{\gamma}\sum_{i}\frac{1}{\lambda_i}u_{ii\gamma}u_{n\gamma}\\
=&\ 8u_n\sum_{ki}\frac{1}{\lambda_k}u_{iik}u_{nk}-4u_n\sum_{k\neq i}\frac{1}{\lambda_k\lambda_i}u_{iin}u_{nk}^2+4u_n\sum_{k\neq i}\frac{1}{\lambda_k\lambda_i}u_{ikn}u_{ni}u_{nk}
\end{aligned}$$

$$- 4u_n \sum_i u_{iin} - 4u_n \sigma_1(\lambda) \sum_i \frac{1}{\lambda_i} u_{iin}. \tag{2.17}$$

Here we have used the relation

$$u_{nnk} = -\sum_i u_{iik}, \quad \text{for } k = 1, 2, \ldots, n-1.$$

For the last term I_{47}, with Lemma 2.1 in hand, after some trivial simplification we finally obtain

$$\begin{aligned} \mathrm{I}_{47} = \frac{2}{\sigma_{n-1}(\lambda)} \Bigg(& \sum_i \frac{\partial \det D^2u}{\partial u_{ii}} \lambda_i^2 + 2\sum_i \frac{\partial \det D^2u}{\partial u_{in}} \lambda_i u_{ni} + \sum_{ij} \frac{\partial \det D^2u}{\partial u_{ij}} u_{ni} u_{nj} \\ & + \frac{\partial \det D^2u}{\partial u_{nn}} \sum_i u_{ni}^2 + 2u_{nn} \sum_i \frac{\partial \det D^2u}{\partial u_{in}} u_{in} + \frac{\partial \det D^2u}{\partial u_{nn}} u_{nn}^2 \Bigg) \\ = 0. & \end{aligned} \tag{2.18}$$

Inserting (2.13)–(2.18) into (2.12), we have

$$\begin{aligned} \mathrm{I}_4 = & u_n^2 \sum_k \sum_{i\neq j} \frac{1}{\lambda_i\lambda_j} u_{iik}u_{jjk} + u_n^2 \sum_{i\neq j} \frac{1}{\lambda_i\lambda_j} u_{iin}u_{jjn} - 2u_n^2 \sum_{i\neq k} \frac{1}{\lambda_i\lambda_k} u_{iik}^2 - u_n^2 \sum_{i\neq j, j\neq k, k\neq i} \frac{1}{\lambda_i\lambda_j} u_{ijk}^2 \\ & - u_n^2 \sum_{i\neq j} \frac{1}{\lambda_i\lambda_j} u_{ijn}^2 - 2(n+1)u_n \sum_{ik} \frac{1}{\lambda_i} u_{iik}u_{nk} + 2(n+1)u_n\sigma_1(\lambda) \sum_i \frac{1}{\lambda_i} u_{iin} \\ & + 8u_n \sum_{ki} \frac{1}{\lambda_k} u_{iik}u_{nk} - 4u_n \sum_{k\neq i} \frac{1}{\lambda_k\lambda_i} u_{iin}u_{nk}^2 + 4u_n \sum_{k\neq i} \frac{1}{\lambda_k\lambda_i} u_{ikn}u_{ni}u_{nk} - 4u_n \sum_i u_{iin} \\ & - 4u_n\sigma_1(\lambda) \sum_i \frac{1}{\lambda_i} u_{iin} + (n+1)^2 \sum_k u_{nk}^2 + (n+1)(n-3)\sigma_1(\lambda)^2 + 2(n+1)\sigma_2(\lambda) \\ & - 4(n+1)\sigma_1(\lambda) \sum_k \frac{1}{\lambda_k} u_{nk}^2. \end{aligned} \tag{2.19}$$

Let us put (2.9), (2.10) and (2.19) into (2.7). Then we regroup the terms in (2.7) as follows: II_1, the terms involving u_{iik} $(1 \le i, k \le n-1)$; II_2, the terms involving u_{iin} $(1 \le i \le n-1)$; II_3, the terms involving u_{ijn} $(1 \le i, j \le n-1, i \neq j)$ and u_{ijk} $(1 \le i, j, k \le n-1, i \neq j, j \neq k, k \neq i)$; II_4, all the rest of the terms. More precisely, we have

$$\begin{aligned} \mathrm{II}_1 = \sum_k \Bigg\{ & (b-1)\cdot\left(u_n \frac{1}{\lambda_k} u_{kkk}\right)^2 + \sum_{i\neq k} \left(b - 1 - \frac{2\lambda_i}{\lambda_k}\right)\cdot\left(u_n \frac{1}{\lambda_i} u_{iik}\right)^2 \\ & + 2b \sum_{i<j} \left(u_n \frac{1}{\lambda_i} u_{iik}\right)\cdot\left(u_n \frac{1}{\lambda_j} u_{jjk}\right) + 2\sum_i \left[(a-n-1)b + \frac{4\lambda_i}{\lambda_k}\right] u_{nk} \cdot \left(u_n \frac{1}{\lambda_i} u_{iik}\right) \Bigg\}, \end{aligned}$$

$$\begin{aligned} \mathrm{II}_2 = & (b-1) \sum_i \left(u_n \frac{1}{\lambda_i} u_{iin}\right)^2 + 2b \sum_{i<j} \left(u_n \frac{1}{\lambda_i} u_{iin}\right)\cdot\left(u_n \frac{1}{\lambda_j} u_{jjn}\right) \\ & + 2b\left[(-a+n-1)\sigma_1(\lambda) - 2\sum_k \frac{1}{\lambda_k} u_{nk}^2\right] \sum_i \left(u_n \frac{1}{\lambda_i} u_{iin}\right) - 4\sum_i \left(\lambda_i - \frac{1}{\lambda_i} u_{ni}^2\right)\cdot\left(u_n \frac{1}{\lambda_i} u_{iin}\right), \end{aligned}$$

$$\mathrm{II}_3 = -\sum_{i\neq j} \left(u_n \frac{1}{\sqrt{\lambda_i\lambda_j}} u_{ijn}\right)^2 + 4\sum_{i\neq j} \left(u_n \frac{1}{\sqrt{\lambda_i\lambda_j}} u_{ijn}\right)\cdot\left(\frac{1}{\sqrt{\lambda_i\lambda_j}} u_{ni}u_{nj}\right) - u_n^2 \sum_{i\neq j, j\neq k, k\neq i} \frac{1}{\lambda_i\lambda_j} u_{ijk}^2,$$

and the remaining terms are

$$\mathrm{II}_4 = (a-n-1)^2 b \sum_k u_{nk}^2 + \left[(a-n+1)^2 b - 4\right] \sigma_1(\lambda)^2 - 2(a-n-1)\sigma_2(\lambda)$$
$$+ 4\left[(a-n+1)b - 2\right] \sigma_1(\lambda) \sum_k \frac{1}{\lambda_k} u_{nk}^2 + 4(b-1)\left(\sum_k \frac{1}{\lambda_k} u_{nk}^2\right)^2 .$$

It follows that

$$\frac{|\nabla u|^2}{b\psi} \Delta\psi = \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3 + \mathrm{II}_4. \tag{2.20}$$

To maximize the terms II_1 and II_2, we need the following elementary lemma.

Lemma 2.2 *Let $n \geq 2$ be an integer, and $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be negative numbers. Let $a = n-3$, $b = \frac{1}{n-1}$, and $c_1, c_2, \ldots, c_{n-1}$ be constants. Define two quadratic polynomials*

$$f_1(x_1, x_2, \ldots, x_{n-1}) = (b-1)x_1^2 + \sum_{i=2}^{n-1}\left(b - 1 - \frac{2\lambda_i}{\lambda_1}\right) x_i^2 + 2b \sum_{1 \leq i < j \leq n-1} x_i x_j$$
$$+ 2c_1 \sum_{i=1}^{n-1}\left[(a-n-1)\, b + \frac{4\lambda_i}{\lambda_1}\right] x_i ,$$

and

$$\begin{aligned} &g(x_1, x_2, \ldots, x_{n-1}) \\ &= (b-1)\sum_{i=1}^{n-1} x_i^2 + 2b \sum_{1 \leq i < j \leq n-1} x_i x_j \\ &\quad + 2b\left[(-a+n-1)\sum_{k=1}^{n-1}\lambda_k - 2\sum_{k=1}^{n-1}\frac{1}{\lambda_k}c_k^2\right]\sum_{i=1}^{n-1} x_i - 4\sum_{i=1}^{n-1}\left(\lambda_i - \frac{1}{\lambda_i}c_i^2\right) x_i . \end{aligned}$$

Then we have

$$f_1(x_1, x_2, \ldots, x_{n-1}) \leq 8\left(1 - \frac{2}{n-1} + \frac{1}{\lambda_1}\sum_{k=1}^{n-1}\lambda_k\right) c_1^2,$$
$$g(x_1, x_2, \ldots, x_{n-1}) \leq \frac{4}{n-1} \sum_{1 \leq i < j \leq n-1}\left[\left(\lambda_i - \frac{1}{\lambda_i}c_i^2\right) - \left(\lambda_j - \frac{1}{\lambda_j}c_j^2\right)\right]^2 .$$

Proof It is easy to see that the $(n-1) \times (n-1)$ matrix

$$\begin{pmatrix} b-1 & b & \cdots & b \\ b & b-1-\frac{2\lambda_2}{\lambda_1} & \cdots & b \\ \cdots & \cdots & \cdots & \cdots \\ b & b & \cdots & b-1-\frac{2\lambda_{n-1}}{\lambda_1} \end{pmatrix}$$

is negative definite. So the polynomial $f_1(x_1, x_2, \ldots, x_{n-1})$ has a unique maximum point. At this maximum point, there hold

$$2(b-1)x_1 + 2b\sum_{i=2}^{n-1} x_i + 2c_1\left[(a-n-1)b + 4\right] = 0,$$

$$2\left(b-1-\frac{2\lambda_k}{\lambda_1}\right)x_k+2b\sum_{1\le i\neq k\le n-1}x_i+2c_1\left[(a-n-1)b+\frac{4\lambda_k}{\lambda_1}\right]=0,$$

for each $k=2,3,\ldots,n-1$. Equivalently, we have

$$x_1=6c_1,\ x_2=2c_1,\ldots,x_{n-1}=2c_1;$$

hence

$$\begin{aligned}f_1(x_1,x_2,\ldots,x_{n-1})&\le f_1(6c_1,2c_1,\ldots,2c_1)\\&=8\left(1-\frac{2}{n-1}+\frac{1}{\lambda_1}\sum_{k=1}^{n-1}\lambda_k\right)c_1^2.\end{aligned}$$

For the polynomial $g(x_1,x_2,\ldots,x_{n-1})$, we directly have

$$\begin{aligned}g(x_1,x_2,\ldots,x_{n-1})=&-\frac{1}{n-1}\sum_{1\le i<j\le n-1}(x_i-x_j)^2\\&+\frac{4}{n-1}\sum_{1\le i<j\le n-1}\left[\left(\lambda_i-\frac{1}{\lambda_i}c_i^2\right)-\left(\lambda_j-\frac{1}{\lambda_j}c_j^2\right)\right]\cdot(x_i-x_j)\\&\le\frac{4}{n-1}\sum_{1\le i<j\le n-1}\left[\left(\lambda_i-\frac{1}{\lambda_i}c_i^2\right)-\left(\lambda_j-\frac{1}{\lambda_j}c_j^2\right)\right]^2.\end{aligned}$$

We finished the proof of the lemma. □

Let us return to the proof of Theorem 1.2. From now on, we fix $a=n-3$ and $b=\frac{1}{n-1}$. Firstly, we will analyze the term II_1. Without loss of generality, we consider the case $k=1$. For $i=1,2,\ldots,n-1$, set $x_i=u_n\frac{1}{\lambda_i}u_{ii1}$, $c_1=u_{n1}$. By Lemma 2.2, we have

$$\begin{aligned}&(b-1)\cdot\left(u_n\frac{1}{\lambda_1}u_{111}\right)^2+\sum_{i\neq 1}\left(b-1-\frac{2\lambda_i}{\lambda_1}\right)\cdot\left(u_n\frac{1}{\lambda_i}u_{ii1}\right)^2\\&\quad+2b\sum_{i<j}\left(u_n\frac{1}{\lambda_i}u_{ii1}\right)\cdot\left(u_n\frac{1}{\lambda_j}u_{jj1}\right)+2\sum_i\left[(a-n-1)b+\frac{4\lambda_i}{\lambda_1}\right]u_{n1}\\&\quad\cdot\left(u_n\frac{1}{\lambda_i}u_{ii1}\right)\\&\le\ 8\left(1-\frac{2}{n-1}+\frac{1}{\lambda_1}\sigma_1(\lambda)\right)u_{n1}^2.\end{aligned}$$

In a similar way, we can obtain

$$\mathrm{II}_1\le 8\sum_k\left(1-\frac{2}{n-1}+\frac{1}{\lambda_k}\sigma_1(\lambda)\right)u_{nk}^2.\tag{2.21}$$

Secondly, for the term II_2, we set $x_i=u_n\frac{1}{\lambda_i}u_{iin}$ and $c_i=u_{ni}$ for each $i=1,2,\ldots,n-1$. Then Lemma 2.2 implies

$$\mathrm{II}_2\le\frac{4}{n-1}\sum_{i<j}\left[\left(\lambda_i-\frac{1}{\lambda_i}u_{ni}^2\right)-\left(\lambda_j-\frac{1}{\lambda_j}u_{nj}^2\right)\right]^2.\tag{2.22}$$

Thirdly, it is obvious that

$$\begin{aligned} II_3 &\le -\sum_{i\neq j}\left(u_n\frac{1}{\sqrt{\lambda_i\lambda_j}}u_{ijn}\right)^2+4\sum_{i\neq j}\left(u_n\frac{1}{\sqrt{\lambda_i\lambda_j}}u_{ijn}\right)\cdot\left(\frac{1}{\sqrt{\lambda_i\lambda_j}}u_{ni}u_{nj}\right)\\ &\le 4\sum_{i\neq j}\frac{1}{\lambda_i\lambda_j}u_{ni}^2u_{nj}^2, \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} II_4 =& \frac{16}{n-1}\sum_k u_{nk}^2+4\left(\frac{1}{n-1}-1\right)\sigma_1(\lambda)^2+8\sigma_2(\lambda)\\ &-8\left(\frac{1}{n-1}+1\right)\sum_k\frac{1}{\lambda_k}u_{nk}^2+4\left(\frac{1}{n-1}-1\right)\left(\sum_k\frac{1}{\lambda_k}u_{nk}^2\right)^2. \end{aligned} \tag{2.24}$$

Summarizing (2.20)–(2.24), we achieve

$$\begin{aligned} \frac{|\nabla u|^2}{b\psi}\Delta\psi \le& 8\sum_k\left(1-\frac{2}{n-1}+\frac{1}{\lambda_k}\sigma_1(\lambda)\right)u_{nk}^2\\ &+\frac{4}{n-1}\sum_{i<j}\left[\left(\lambda_i-\frac{1}{\lambda_i}u_{ni}^2\right)-\left(\lambda_j-\frac{1}{\lambda_j}u_{nj}^2\right)\right]^2\\ &+4\sum_{i\neq j}\frac{1}{\lambda_i\lambda_j}u_{ni}^2u_{nj}^2+\frac{16}{n-1}\sum_k u_{nk}^2+4\left(\frac{1}{n-1}-1\right)\sigma_1(\lambda)^2+8\sigma_2(\lambda)\\ &-8\left(\frac{1}{n-1}+1\right)\sum_k\frac{1}{\lambda_k}u_{nk}^2+4\left(\frac{1}{n-1}-1\right)\left(\sum_k\frac{1}{\lambda_k}u_{nk}^2\right)^2\\ =& 0. \end{aligned}$$

We complete the proof of Theorem 1.2. □

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