THE EXTERIOR DIRICHLET PROBLEM FOR THE HOMOGENEOUS k-HESSIAN EQUATION

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Abstract. We study the exterior Dirichlet problem for the homogeneous k-Hessian equation. The prescribed asymptotic behavior at infinity of the solution is zero if $k < \frac{n}{2}$, it is $\log|x| + O(1)$ if $k = \frac{n}{2}$ and it is $|x|^{\frac{2k-n}{n}} + O(1)$ if $k > \frac{n}{2}$. By constructing smooth solutions of approximating non-degenerate k-Hessian equations with uniform $C^{1,1}$-estimates, we prove the existence part. The uniqueness follows from the comparison theorem and thus the $C^{1,1}$ regularity of the solution of the homogeneous k-Hessian equation in the exterior domain is proved. We also prove a uniform positive lower bound of the gradient. As an application of the $C^{1,1}$ estimates, we derive an almost monotonicity formula along the level set of the approximating solution. In particular, we get an weighted geometric inequality which is a natural generalization of the $k = 1$ case.

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Contents

1. Introduction

Let u be a C^2 function and $\lambda = (\lambda_1, \cdots, \lambda_n)$ be the eigenvalues of D^2u, the k-Hessian operator is defined by

$$S_k(D^2u) := S_k(\lambda) = \sum_{1\leq i_1<\cdots i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k}, \tag{1.1}$$

where $1 \leq k \leq n$. When $k = 1$, $S_1(D^2u) = \Delta u$. When $k = n$, $S_n(D^2u) = \det D^2u$.

Let Ω be a bounded smooth domain in $\mathbb{R}^n$, the Dirichlet problem for the k-Hessian equation is as follows

$$\begin{cases} S_k(D^2u) = f & \text{in} \quad \Omega, \\ u = \varphi & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where f and φ are given smooth functions. When $k = 1$, the k-Hessian equation is the Poisson equation. When $k = n$, it is the well known Monge-Ampère equation.

1.1. Some known results.

We briefly give some known results of the Dirichlet problem for the k-Hessian equation in the nondegenerate case i.e. $f > 0$ and in the degenerate cases i.e. $f \geq 0$. In general, the k-Hessian equation is a fully nonlinear equation.

1.1.1. *Results on bounded domains.* If $f > 0$, Caffarelli-Nirenberg-Spruck [7] solved (1.2) in a bounded $(k-1)$-convex domain. Guan [14] solved (1.2) by only assuming the existence of a subsolution. The advantage of Guan's result is that there are no geometric restriction on the domain.

The Dirichlet problem in bounded domains of degenerate fully nonlinear equations has been studied extensively. For the Dirichlet problem of degenerate Monge-Ampère equation in bounded convex domain, Caffarelli-Nirenberg-Spruck [8] show the $C^{1,1}$ regularity for the homogeneous case i.e. $f \equiv 0$. If f satisfies $f^{\frac{1}{n-1}} \in C^{1,1}$, Guan-Trudinger-Wang [20] proved the $C^{1,1}$ regularity, which is optimal by Wang's counterexample [33]. The $C^{1,1}$ regularity problem of degenerate k-Hessian equation with Dirichlet boundary value in bounded $(k-1)$-convex domain was solved by Krylov [22, 23] and Ivochina-Trudinger-Wang [21] (PDE's proof) with the assumption $f^{\frac{1}{k}} \in C^{1,1}$. Dong [12] studied the mixed Hessian equations.

1.1.2. *Results on unbounded domains.* The exterior Dirichlet problem for viscosity solutions of nondegenerate fully nonlinear equations has been studied extensively. The C^0 viscosity solution for the Monge-Ampère equation: $\det D^2u = 1$ with prescribed asymptotic behavior at infinity was solved by Caffarelli-Li [6]. The related problem for the k-Hessian equation : $S_k(D^2u) = 1$ was proved by Bao-Li-Li [4]. For the related results on other type nondegenerate fully nonlinear equations, one can see [3, 25, 26, 28]. Note that in these cases the regularity are only continuous.

Li-Wang [27] proved the global $C^{k+2,\alpha}$ regularity of the homogeneous Monge-Ampère equation: $\det(u_{ij}) = 0$ in a strip region: $\mathbb{R}^{n-1} \times [0, 1]$ by assuming that the boundary functions are locally uniformly convex and $C^{k,\alpha}$. Moreover, they gave a counterexample to show the necessity of the uniform convexity of the boundary functions.

1.2. **Motivation.** The motivation of this paper arises from proving geometric inequality by establishing certain monotonicity formula on the level set of solutions in exterior domains. Another one comes from studying the regularity of extremal function of the complex Monge-Ampère operator.

1.2.1. *Geometric inequalities.* One motivation for us to consider the exterior Dirichlet problem for the homogeneous k-Hessian equation comes from the following geometric inequalities:

$$\left(\frac{V_{n-l}(\Omega)}{V_{n-1-l}(B)}\right)^{\frac{1}{n-l}} \leq \left(\frac{V_{n-k}(\Omega)}{V_{n-1-k}(B)}\right)^{\frac{1}{n-k}}, \tag{1.3}$$

where $0 \leq l < k \leq n$, $V_{n-k}(\Omega) = \int_{\partial\Omega} H_{k-1}(\kappa)dA$, $V_{-1} := |\Omega|$ and H_k is the k-Hessian operator of the principal curvature $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ of $\partial\Omega$. (1.3) are called Alexandrov-Fenchel inequalities. An open question is whether (1.3) holds for $(k-1)$-convex domain Ω i.e. $H_m > 0$ for $1 \leq m \leq k-1$.

When Ω is $(k-1)$-convex and starshaped, Guan-Li [19] proved (1.3) by the method of inverse curvature flows. If Ω is k-convex, Chang-Wang [9], Qiu [29] proved the above inequalities when $l = 0$ by the optimal transport method.

Very recently, by considering the exterior Dirichlet problem of the Laplace equation, Agostiniani-Mazzieri [2] proved several geometric inequalities such as the Willmore inequality. By studying the the exterior Dirichlet problem of the p-Laplacian equation, Fogagnolo and Mazzieri and Pinamonti [13] showed the volumetric Minkowski inequality i.e. the Alexandrov-Fenchel inequality with $l = 0$ and $k = 2$ for smooth convex domains. Later, Agostiniani-Fogagnolo-Mazzieri [1] removed the convexity assumption for the domain. The key point for them is to prove a monotonicity formula along the level set of the solution of the exterior Dirichlet problem for the p-Laplace equation.

1.2.2. *Regularity problems of extremal functions.* P. F. Guan. [17, 18] proved the $C^{1,1}$ regularity of the homogeneous complex Monge-Ampère equation in $U := V_0 \setminus V$ with $V = \cup_{i=1}^N V_i$, where V_0 and V_i are strongly pseudoconvex and bounded smooth domains in a complex manifold M^n, V is holomorphically convex subset of Ω_1. Then he solved a conjecture of Chern-Levine-Nirenberg on the extended intrinsic norms. B. Guan [15] proved the $C^{1,1}$ regularity of solutions of the exterior Dirichlet problem for the homogeneous complex Monge-Ampère equation in $\mathbb{C}^n \setminus \bar{V}$ with $V = (\cup_{i=1}^N V_i)$, where V_i are strongly pseudoconvex and bounded smooth domains and V is a holomorphically convex subset of V_0. If V is strictly convex and smooth (analytic), the smooth (analytic) regularity of this problem was proved by Lempert [24]

1.3. **Our main results.** In this paper, we consider the following exterior Dirichlet problem for the k-Hessian equation. For convenience, we always assume $0 \in \Omega$ and there exists positive constants r_0, R_0 such that $B_{r_0} \subset\subset \Omega \subset B_{\frac{R_0}{2}}$, where B_r and $B_{\frac{R_0}{2}}$ are balls centered at 0 with radius r and $\frac{R_0}{2}$ respectively.

1.3.1. **Case1:** $1 \le k < \frac{n}{2}$. Since the Green function in this case is $-|x|^{\frac{2k-n}{k}}$, we consider the k-Hessian equation when $k < \frac{n}{2}$ as follows

$$\begin{cases} S_k(D^2u) = 0 & \text{in} \quad \Omega^c := \mathbb{R}^n \setminus \Omega, \\ u = -1 & \text{on } \partial\Omega, \\ \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{1.4}$$

Theorem 1.1. *Assume $1 \le k < \frac{n}{2}$. Let Ω be a smoothly convex domain in $\mathbb{R}^n$ and strictly $(k-1)$-convex. There exists a unique k-convex solution $u \in C^{1,1}(\overline{\Omega^c})$ of the equation (1.4). Moreover, there exists uniform constant C such that for any $x \in \Omega^c$ the following holds*

$$\begin{cases} C^{-1}|x|^{-\frac{n-2k}{k}} \le -u(x) \le C|x|^{-\frac{n-2k}{k}}, \\ C^{-1}|x|^{-\frac{n-k}{k}} \le |Du|(x) \le C|x|^{-\frac{n-k}{k}}, \\ |D^2u|(x) \le C|x|^{-\frac{n}{k}}, \end{cases} \tag{1.5}$$

where the k-convex solution is defined in Section 2 and we use the notation $\overline{\Omega}^c := \mathbb{R}^n \setminus \Omega$.

1.3.2. **Case2:** $k > \frac{n}{2}$. Since the Green function in this case is $|x|^{\frac{2k-n}{k}}$, we consider the k-Hessian equation when $k > \frac{n}{2}$ as follows

$$\begin{cases} S_k(D^2u) = 0 & \text{in} \quad \Omega^c, \\ u = 1 & \text{on} \quad \partial\Omega, \\ u(x) = |x|^{\frac{2k-n}{k}} + O(1) \text{ as } |x| \to \infty. \end{cases} \tag{1.6}$$

Theorem 1.2. *Assume $k > \frac{n}{2}$. Let Ω be a smoothly convex domain in $\mathbb{R}^n$ and strictly $(k-1)$-convex. There exists a unique k-convex solution $u \in C^{1,1}(\overline{\Omega^c})$ of the equation (1.6).*

Moreover, there exists uniform constant C such that for any $x \in \Omega^c$ the following holds

$$\begin{cases} |u(x) - |x|^{\frac{2k-n}{k}}| \leq C, \\ C^{-1}|x|^{\frac{k-n}{k}} \leq |Du|(x) \leq C|x|^{\frac{k-n}{k}}, \\ |D^2u|(x) \leq C|x|^{-\frac{n}{k}}. \end{cases} \tag{1.7}$$

1.3.3. **Case3:** $k = \frac{n}{2}$. Since the Green function in this case is $\log|x|$, we consider the k-Hessian equation when $k = \frac{n}{2}$ as follows

$$\begin{cases} S_{\frac{n}{2}}(D^2u) = 0 & \text{in } \Omega^c, \\ u = 0 & \text{on } \partial\Omega, \\ u(x) = \log|x| + O(1) \text{ as } |x| \to \infty. \end{cases} \tag{1.8}$$

Theorem 1.3. *Assume $k = \frac{n}{2}$. Let Ω be a smoothly convex domain in $\mathbb{R}^n$ and strictly $(k-1)$-convex. There exists a unique k-convex solution $u \in C^{1,1}(\overline{\Omega^c})$ of the equation (1.8). Moreover, there exists uniform constant C such that for any $x \in \Omega^c$ the following holds*

$$\begin{cases} |u(x) - \log|x|| \leq C, \\ C^{-1}|x|^{-1} \leq |Du|(x) \leq C|x|^{-1}, \\ |D^2u|(x) \leq C|x|^{-2}. \end{cases} \tag{1.9}$$

To solve the above problems, we consider the following approximating equation

$$\begin{cases} S_k(u^\varepsilon) = f^\epsilon \text{ in } \Omega^c, \\ u^\varepsilon = -1 \text{ if } k < \dfrac{n}{2}, u^\varepsilon = 1 \text{ if } k > \dfrac{n}{2}, u^\varepsilon = 0, \text{ if } k = \dfrac{n}{2} \text{ on } \partial\Omega, \\ u^\varepsilon(x) \to 0 \text{ if } k < \dfrac{n}{2}, u^\varepsilon(x) = |x|^{\frac{2k-n}{k}} + O(1) \text{ if } k > \dfrac{n}{2}, u^\varepsilon(x) = \log|x| + O(1) \text{ if } k = \dfrac{n}{2}, |x| \to \infty. \end{cases}$$

where $f^\varepsilon = c_{n,k}\varepsilon^2(|x|^2 + \varepsilon^2)^{-\frac{n}{2}-1}$ (see the precise value of $c_{n,k}$ in Section 4).

u^ε will be obtained by approximating solutions $u^{\varepsilon,R}$ defined on bounded domains: $\Omega_R := B_R \setminus \overline{\Omega}$(see Section 4 for precise definition of $u^{\varepsilon,R}$). The existence and uniqueness of the k-convex solution of $u^{\varepsilon,R}$ follows from B. Guan [14] if we can construct a subsolution, which can be constructed since we assume Ω is convex.

The key point is to establish the uniform C^2 estimates for $u^{\varepsilon,R}$.

1.4. **Applications.** As an application of our C^2 estimates, we can prove an almost monotonicity formula along the level set of u^ε (see Section 6). Consequently, we get geometric inequalities of $\partial\Omega$ as follows.

Theorem 1.4. *Let Ω be a smoothly convex domain in $\mathbb{R}^n$ and strictly $(k-1)$-convex.*

(i) Assume $1 \leq k < \frac{n}{2}$ and $b \geq \frac{k(n-k-1)}{n-k}$. Let u be the unique $C^{1,1}$ solution in Theorem 1.1. We have

$$\int_{\partial\Omega} |Du|^{b+1} H_{k-1} \leq \frac{n-2k}{n-k} \int_{\partial\Omega} |Du|^b H_k, \tag{1.10}$$

where H_m is the m-Hessian operator of the principal curvature $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ of $\partial\Omega$.

(ii) Assume $k = \frac{n}{2}$ and $b > \frac{n}{2} - 1$. Let u be the unique $C^{1,1}$ solution in Theorem 1.3. We have

$$\int_{\partial\Omega} |Du|^{b+1} H_{k-1} \leq \int_{\partial\Omega} |Du|^b H_k. \tag{1.11}$$

Remark 1.5. *When $k = 1$, (1.10) was proved by Agostiniani- Mazzieri [2].*

In section 2, we give some preliminaries. In section 3, we solve the Dirichlet problem of degenerate k-Hessian equation in a ring domain. Section 4 is the main part of this paper. We show uniform $C^{1,1}$ estimate of the solution which is the limit of the solutions of nondegenerate k-Hessian equation. The key ingredient is to establish uniform gradient estimates and uniform second order estimates. We use the idea of Chow-Wang [11] to establish the uniform second order estimate. Theorem 1.1, Theorem 1.2 and Theorem 1.3 will be proved in Section 5. In section 6, we prove an almost monotonicity formula along the level set of the approximating solution and thus prove Theorem 1.4.

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Very recently (July 12th, 2022), when $k < \frac{n}{2}$, Xiao [35] solved the exterior Dirichlet problem for the homogenous k-Hessian equations in which Xiao assumed the domain is strictly $(k-1)$-convex and starshaped. For the case of $k < \frac{n}{2}$, our proof is different from Xiao's. We directly prove the uniform C^2 decay estimates for the approximating solutions.

2. Preliminaries

2.1. *k*-convex solutions.

In this section we give the definition of k-convex functions and the definition of k-convex solutions.

The Γ_k-cone is defined by

$$\Gamma_k := \{\lambda \in \mathbb{R}^n | S_i(\lambda) > 0, 1 \leq i \leq k\} \tag{2.1}$$

Recall $S_k(\lambda) := \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1} \cdots \lambda_{i_k}$.

One can find the concavity property of $S_k^{\frac{1}{k}}$ in [7].

Lemma 2.1. *$S_k^{\frac{1}{k}}$ is a concave function in Γ_k. In particular, $\log S_k$ is concave in Γ_k.*

For more properties of the k-Hessian operator, one can see the Lecture notes by Wang [34]. We following the definition by Trudinger-Wang [31] to give the definition of k-convex functions.

Definition 2.2. *Let U be a domain in $\mathbb{R}^n$.*

*(1). A function $u \in C^2(U)$ is called k-*convex *(*strictly k-convex*) if $\lambda(D^2u) \in \overline{\Gamma}_k$ ($\lambda(D^2u) \in \Gamma_k$).*

(2). A function $u \in C^0(U)$ is called k-convex in U if there exists a sequence of functions $\{u_i\} \subset C^2(U)$ such that in any bounded subdomain $V \subset\subset U$, u_i is k-convex and converges uniformly to u.

Definition 2.3. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $\varphi \in C^0(\partial\Omega)$. A function $u \in C^0(\Omega^c)$ is called a k-convex solution of the homogeneous k-Hessian equation*

$$\begin{cases} S_k(D^2u) = 0 & in \quad \Omega^c := R^n \setminus \overline{\Omega}, \\ u = \varphi & on \quad \partial\Omega, \end{cases} \tag{2.2}$$

if there exists a sequence of k-convex functions $\{u_m\} \subset C^2(\Omega^c)$ converging in $C^0(\Omega^c)$ to u with $\{S_k(D^2u)\}$ converging in $L^1_{loc}(\Omega^c)$ to 0 and $u = \varphi$ on $\partial\Omega$.

We need the following comparison principle by Wang-Trudinger [31] (see also [30,32]) to prove the uniqueness of our equations.

Lemma 2.4. *Let u, v be k-convex functions in a bounded smooth domain U in $\mathbb{R}^n$ satisfying*

$$\begin{cases} S_k(D^2u) \geq S_k(D^2v) & in\ U, \\ u \leq v & on\ \partial U, \end{cases} \tag{2.3}$$

in the viscosity sense. Then $u \leq v$ in U.

2.2. **The existence of the subsolution.**

Definition 2.5. *A C^2 domain U is called $(k-1)$-convex (strictly $(k-1)$-convex) if for any $x \in \partial U$, the principal curvature $\kappa := (\kappa_1, \cdots, \kappa)$ of ∂U at $x \in \partial U$ satisfies $\kappa \in \overline{\Gamma}_k$ ($\kappa \in \Gamma_k$).*

Note that a C^2 domain U is $(n-1)$-convex if and only if U is convex.

Definition 2.6. *Let U be a smoothly bounded domain in $\mathbb{R}^n$. Φ is called a defining function of U if $U = \{x : \Phi(x) < 0\}$, $\Phi|_{\partial U} = 0$ and $|D\Phi||_{\partial U} = 1$.*

Caffarelli-Nirenberg-Spruck [7] proved the following.

Lemma 2.7. *Let U be a smoothly and strictly $(k-1)$-convex bounded domain in $\mathbb{R}^n$. There exists a smoothly and strictly k-convex defining function Φ on $\overline{U}$.*

We need the following lemma by Guan [17] to construct the subsolution of the k-Hessian equation in a ring.

Lemma 2.8. *Suppose that U is a bounded smooth domain in $\mathbb{R}^n$. For $h, g \in C^m(U)$, $m \geq 2$, for all $\delta > 0$, there is an $H \in C^m(U)$ such that*
(1) $H \geq \max\{h, g\}$ and

$$H(x) = \begin{cases} h(x), & \text{if } h(x) - g(x) > \delta, \\ g(x), & \text{if } g(x) - h(x) > \delta; \end{cases}$$

(2) There exists $|t(x)| \leq 1$ such that

$$\left\{H_{ij}(x)\right\} \geq \left\{\frac{1 + t(x)}{2} g_{ij} + \frac{1 - t(x)}{2} h_{ij}\right\}, \text{ for all } x \in \{|g - h| < \delta\}.$$

By Lemma 2.1, we see H is k-convex if f and g are both k-convex.

Lemma 2.9. *Let Ω_0 and Ω_1 be smoothly and strictly $(k-1)$-convex domain in $\mathbb{R}^n$ with $\Omega_0 \subset\subset \Omega_1$. Assume that Ω_0 is convex. Then there exists a strictly k-convex function $\underline{u} \in C^\infty(\overline{U})$ with $U := \Omega_1 \setminus \overline{\Omega}_0$ satisfying*

$$\begin{cases} S_k(D^2\underline{u}) \geq \epsilon_0, & \text{in } \overline{U}, \\ \underline{u} = \tau_0\Phi^0, & \text{near } \partial\Omega_0, \\ \underline{u} = 1 + K_1\Phi^1, & \text{near } \partial\Omega_1, \end{cases} \tag{2.4}$$

where Φ^i is the defining function of Ω_i, τ_0 and K_1 are uniform constants.

Proof. If Ω_0 is $(k-1)$-convex and smooth, Caffarelli-Nirenberg-Spruck [7] constructed a strictly k-convex defining function $\Phi_0 \in C^\infty(\overline{\Omega}_0)$ satisfying

$$\begin{cases} S_k(D^2\Phi^0) \geq \epsilon_0 & \text{on } \overline{\Omega}_0, \\ \Phi^0 = t_0^{-1}\left(e^{-t_0 d(x)} - 1\right) & \text{near } \partial\Omega_0, \end{cases} \tag{2.5}$$

where ϵ_0, t_0 are positive constants and $d(x)$ is the distance function from x to $\partial\Omega_0$. Since we also assume that Ω_0 is convex, $d(x)$ is smooth in Ω_0^c. Then we can take $\Phi^0(x) = t_0^{-1}\left(e^{t_0 d(x)} - 1\right)$ for any $x \in \Omega_0^c$ and we still have

$$S_k(D^2\Phi^0) \geq \epsilon_0 \text{ in } \Omega_0^c. \tag{2.6}$$

Let $g = \tau_0\Phi^0$, $h = 1 + K_1\Phi^1$. By Lemma 2.8 ($\delta = \frac{1}{2}$), for $K_1 > 0$ sufficiently large, there exists a smooth function $\underline{u}$ satisfying (2.4). Indeed, define $\Omega_{t_1} = \{x \in \Omega_1 : \Phi^1(x) < -t_1\}$ with $t_1 > 0$. Then for t_1 small enough, $\Omega_0 \subset\subset \Omega_{t_1}$ and $\text{dist}(\partial\Omega_{t_1}, \partial\Omega_0) > \frac{1}{2}\text{dist}(\partial\Omega_1, \partial\Omega_0)$. Let $\Omega_{\frac{t_1}{8}} = \{x \in \Omega_1 : \Phi^1(x) < -\frac{t_1}{8}\}$.

For any $x \in \overline{\Omega}_{t_1} \setminus \Omega_0$, by choosing $K_1 = 2t_1^{-1}$ large enough, we have

$$g(x) - h(x) \geq -h(x) \geq -1 + K_1 t_1 = 1 > \frac{1}{2}.$$

Then $\underline{u} = g = \tau_0\Phi^0$ in $\overline{\Omega}_{t_1} \setminus \Omega_0$.

For any $x \in \overline{\Omega}_1 \setminus \Omega_{\frac{t_1}{8}}$, by choosing τ_0 small enough, we have

$$h - g \geq 1 - \frac{1}{4} - \tau_0|\Phi^0(x)| > \frac{1}{2}.$$

Then $\underline{u} = h = 1 + 2t_1^{-1}\Phi^1$ in $\overline{\Omega}_1 \setminus \Omega_{\frac{t_1}{8}}$. Moreover, by Lemma 2.8, $\underline{u}$ is strictly k-convex. □

2.3. **Level sets.** For any function u on a domain U, we define the level set of u with height t as follows

$$S_t := \{x \in U : u(x) = t\}. \tag{2.7}$$

Let $H_m(x)$ be the m-Hessian operator of the principal curvature $\kappa(x) = (\kappa_1, \cdots, \kappa_{n-1})$ of $x \in S_t$. We have the following useful formula which can be founded in [5].

Lemma 2.10. *Let $u \in C^2(U)$ and $|Du| \neq 0$. Then on S_t, for $1 \leq m \leq n$, we have*

$$\begin{aligned} H_{m-1} =& \frac{S_m^{ij}(D^2u)u_iu_j}{|Du|^{m+1}}, \\ S_m(D^2u) =& H_m|Du|^m + S_m^{ij}u_iu_lu_{lj}|Du|^{-2}, \end{aligned}$$

where $S_m^{ij} := \frac{\partial S_k(D^2u)}{\partial u_{ij}}$ and the curvature is defined with respect to the upward normal as in [5]. In particular, if u is k-convex (strictly k-convex), the level set S_t is $(k-1)$-convex (strictly k-convex).

3. The Dirichlet problem for the homogeneous k-Hessian equations in the ring

In this section, we prove the existence of the Dirichlet problem of degenerate k-Hessian equation in a smooth ring.

$$\begin{cases} S_k(D^2u) = 0, & \text{in} \quad U := \Omega_1 \setminus \overline{\Omega}_0, \\ u = 0, & \text{on} \quad \partial\Omega_0, \\ u = 1, & \text{on} \quad \partial\Omega_1. \end{cases} \tag{3.1}$$

We assume that Ω_1 is smoothly and strictly $(k-1)$-convex domain and Ω_0 is a smoothly strictly $(k-1)$-convex and convex domain. Using Lemma 2.9, there exists a smoothly and strictly k-convex subsolution $\underline{u}$ satisfying

$$\begin{cases} S_k(D^2\underline{u}) \geq \epsilon_0, & \text{in } \overline{U}, \\ \underline{u} = \tau_0\Phi^0, & \text{near } \partial\Omega_0, \\ \underline{u} = 1 + K_1\Phi^1, & \text{near } \partial\Omega_1, \end{cases} \tag{3.2}$$

where τ_0, K_1 are positive constants and Φ^i are defining functions of Ω_i.

Theorem 3.1. *Let Ω_0, Ω_1 be smooth $(k-1)$-convex domain and assume that Ω_0 is convex. There exists a unique solution $u \in C^{1,1}(\overline{U})$ of the equation (3.1).*

The uniqueness follows from the classical comparison theorem for k-convex solutions of k-Hessian equations. Next, we prove the existence and regularity of k-convex solution by approximation. Indeed, for every $0 < \epsilon < \epsilon_0$, we consider the following problem

$$\begin{cases} S_k(D^2u^\epsilon) = \epsilon, & \text{in} \quad \Omega, \\ u^\epsilon = 0, & \text{on} \quad \partial\Omega_0, \\ u^\epsilon = 1, & \text{on} \quad \partial\Omega_1. \end{cases} \tag{3.3}$$

Since u^ϵ is a subsolution, by Guan [14], the above problem has a unique smooth solution u^ϵ.

Next, we want to show the $C^{1,1}$ estimates are independent of ϵ. Firstly, by maximum principal, $u^{\epsilon_1} \geq u^{\epsilon_2}$ for any $\epsilon_1 \leq \epsilon_2$. Thus $u^0 := \lim\limits_{\epsilon\to\infty} u^\epsilon$ exists. If we could prove uniform $C^{1,1}$ estimates, then u^0 is the $C^{1,1}$ solution of equation (3.1).

Theorem 3.2. *Let u^ϵ be the smooth k-convex solution of* (3.3). *Then there exists a uniform constant C independent of ε such that*

$$|u^\varepsilon|_{C^{1,1}(\overline{U})} \leq C.$$

In the following subsections, for simplicity, we use u instead of u^ϵ.

3.1. C^1-estimates.

Lemma 3.3. *There exists a uniform constant C such that*

$$|u|_{C^1(\overline{U})} \leq C. \tag{3.4}$$

Proof. Let h be the unique solution of the problem

$$\begin{cases} \Delta h = 0, & \text{in} \quad U, \\ h = 0, & \text{on} \quad \partial\Omega_0, \\ h = 1, & \text{on} \quad \partial\Omega_1. \end{cases} \tag{3.5}$$

By the maximal principle, we have $\underline{u} \leq u \leq h$. This gives the uniform C^0 estimates.

Let $F^{ij} := \frac{\partial}{\partial u_{ij}} \log S_k(D^2u)$. Since $F^{ij}(u_\xi)_{ij} = 0$ for any unit constant vector ξ, we have $\max\limits_{\overline{U}} |Du| = \max\limits_{\partial U} |Du|$. Since $\underline{u} \leq u^\varepsilon \leq h$ in U and $\underline{u} = u^\varepsilon = h$ on ∂U, we have

$$h_\nu \leq u_\nu \leq \underline{u}_\nu < 0, \text{ on } \partial\Omega_0$$
$$h_\nu \geq u_\nu \geq \underline{u}_\nu > 0, \text{ on } \partial\Omega_1,$$

where ν is the unit normal vector of ∂U (inner normal vector of $\partial\Omega_0$). Thus we have

$$\max_{\overline{U}} |Du| = \max_{\partial U} |Du| \leq C. \tag{3.6}$$

□

3.2. **Second order estimates.**

Lemma 3.4. *There exists a uniform constant C such that*

$$\max_{\overline{U}} |D^2u| \le C. \tag{3.7}$$

Proof. Since $F^{ij}u_{\xi\xi ij} = -F^{ij,kl}u_{\xi ij}u^{\varepsilon}_{\xi kl} \ge 0$, we have $\max_{\overline{\Omega}} u_{\xi\xi} \le \max_{\partial\Omega} u_{\xi\xi}$. Thus we need to prove the second order estimate on the boundary ∂U. Here we use the method by B. Guan [14] and P. F. Guan [17] (see also [16]).

Tangential derivative estimates on ∂U

For any fixed $x_0 \in \partial U$, we choose the coordinate such that $x_0 = 0$, $\partial U \bigcap B_\delta(x_0) = (x', \rho(x'))$, $\rho(0) = 0$ and $\nabla\rho(0) = 0$. Since $u(x', \rho(x')) = 0$, we have

$$\begin{aligned} 0 =& u_\alpha(x', \rho(x')) + u_n(x', \rho(x'))\rho_\alpha(x'), \\ 0 =& u_{\alpha\beta}(0) + u_{\alpha n}(0)\rho_\beta(0) + u_{n\beta}(0)\rho_\alpha(0) + u_{nn}(0)\rho_\alpha(0)\rho_\beta(0) + u_{n\beta}(0)\rho_{\alpha\beta}(0) \\ =& u_{\alpha\beta}(0) + u_n(0)\rho_{\alpha\beta}(0). \end{aligned}$$

Then we have $|u_{\alpha\beta}(0)| \le C \max_{\partial\Omega} |Du| \le C$.

Tangential-normal derivative estimates on ∂U.

We use Guan's method [14] (see slao [16]). Our barrier function here is simpler than before since u is constant on the boundary and the right hand side of the approximating equation is a sufficiently small constant ϵ.

For any fixed $x_0 \in \partial U$, we choose the coordinate such that $x_0 = 0$, $\partial U \bigcap B_\delta(x_0) = (x', \rho(x'))$,$\nabla\rho(0) = 0$ and $\rho(x') = \sum_{\alpha<n} \kappa_\alpha |x_\alpha|^2 + O(|x'|^3)$. Consider the tangential operator $T_\alpha = \partial_\alpha + \kappa_\alpha(x_\alpha\partial_n - x_n\partial_\alpha)$.

We will prove $w = A_1(u - \underline{u}) + A_2|x|^2 \pm T_\alpha u \ge 0$ in $U_\delta := B_\delta(0) \cap U$.

Since $u - \underline{u} = 0$ and $|T_\alpha u| \le C|x'|^2$ on $\partial U \cap B_\delta(0)$, we have

$$w|_{\partial\Omega\cap B_\delta(0)} = A_2|x|^2 - C|x|^2 \ge 0,$$

where we require $A_2 > C$. Since $|T_\alpha u| \le C$ and $u \ge \underline{u}$, on $U \cap \partial B_\delta(0)$, we have

$$w|_{\Omega\cap\partial B_\delta(0)} = A_2\delta^2 - C > 0,$$

where $A_2 > 2C\delta^{-2}$. Thus we have $w \ge 0$ on ∂U_δ.

Next we show $F^{ij}w_{ij} < 0$ in U_δ. Indeed, recall $\underline{u}$ is k-convex and $S_k(D^2\underline{u}) \ge \epsilon_0 > 0$, there exits $\tau_0 > 0$ sufficiently small depending only on ϵ_0 and $|\underline{u}|_{C^2}$ such that $\underline{\tilde{u}} := \underline{u} - \tau_0|x|^2$ is k-convex and $S_k(D^2\underline{\tilde{u}}) \ge \frac{\epsilon_0}{2}$.

By the concavity of $\log S_k$, we have

$$\begin{aligned}F^{ij}(u_{ij}-\tilde{\underline{u}}_{ij}) \le & F(D^2u)-F(D^2\underline{u})\\ =&\log\epsilon-\log S_k(D^2\tilde{\underline{u}})\\ \le&\log\epsilon-\log\frac{\epsilon_0}{2}\\ <&0,\end{aligned}$$

where we take $2\epsilon<\epsilon_0$. Thus we have

$$\begin{aligned}F^{ij}(u-\underline{u})_{ij}=&F^{ij}(u_{ij}-\tilde{\underline{u}}_{ij})-2\tau_0\mathcal{F}\\ <&-2\tau_0\mathcal{F},\end{aligned}$$

where $\mathcal{F}=\sum_{i=1}^n F^{ii}$. Then we have

$$\begin{aligned}F^{ij}w_{ij}=&F^{ij}(A_1(u-\underline{u})+A_2|x|^2+T_\alpha u)_{ij}\\ =&A_1F^{ij}(u-\underline{u})_{ij}+2A_2\mathcal{F}\\ \le&-2A_1\tau_0\mathcal{F}+2A_2\mathcal{F}\\ <&0,\end{aligned}$$

where we use $F^{ij}(T_\alpha u)_{ij}=0$ and we take $A_1=\frac{A_2}{2\tau_0}$. Then we obtain $w\ge 0$ in U_δ and $w(0)=0$. Namely we have

$$|T_\alpha u|\le A(u-\underline{u})+A_2|x|^2 \text{ in } U_\delta, \tag{3.8}$$

$$(T_\alpha u)(0)=0. \tag{3.9}$$

This gives $|u_{\alpha n}(0)|\le C$.

Double normal derivative estimates on ∂U

For any fixed $x_0\in\partial U$, we choose the coordinate such that $x_0=0$, $\partial U\bigcap B_r(x_0)=(x',\rho(x'))$ and $\nabla\rho(0)=0$.

Case 1: $x_0\in\partial\Omega_1$

We have

$$u_{\alpha\beta}(0)=-u_n(0)\rho_{\alpha\beta}(0)=|Du|(0)\rho_{\alpha\beta}(0).$$

Since $|Du|(0)\ge c>0$ on $\partial\Omega_1$ and Ω_1 is $(k-1)$-convex, we have

$$S_k(u_{\alpha\beta}(0))\ge c^kS_k(\rho_{\alpha\beta}(0))\ge c_1>0. \tag{3.10}$$

Case 2: $x_0\in\partial\Omega_0$

Since $\frac{\partial u}{\partial\nu}\le\frac{\partial\underline{u}}{\partial\nu}=-|D\underline{u}|<0$ on $\partial\Omega_0$, we have $|Du|>|D\underline{u}|>a_0>0$ and then there exists a smooth function g such that $u=g\underline{u}$ near $\partial\Omega_0$. Since $u\ge\underline{u}>0$ in U, we have $g\ge 1$ near $\partial\Omega_0$. On the other hand, since $\underline{u}=0$ on $\partial\Omega_0$, we have for any $1\le\alpha,\beta\le n-1$,

$u_\alpha(0) = \underline{u}_\alpha(0) = 0$. Thus

$$\begin{aligned} u_{\alpha\beta}(0) =& g_{\alpha\beta}(0)\underline{u}(0) + g_\alpha(0)\underline{u}_\beta(0) + g_\beta(0)\underline{u}_\alpha(0) + g(0)\underline{u}_{\alpha\beta}(0) \\ =& g(0)\tau\Phi^0_{\alpha\beta}(0), \end{aligned}$$

where we have used $\underline{u} = \tau_0\Phi^0$ near $\partial\Omega_0$. Therefore

(3.11)

$$S_{k-1}(u_{\alpha\beta}(0)) = g^{k-1}(0)\tau_0^{k-1}S_{k-1}\left(\Phi^0_{\alpha\beta}(0)\right) \geq \tau_0^{k-1}C_n^{k-1}C_n^{\frac{k-1}{k}}\min_{\partial\Omega_0} S_k^{\frac{k-1}{k}}\left(D^2\Phi^0\right) := c_2 > 0.$$

Let $c_0 = \min\{c_1, c_2\}$ (see (3.10) and (3.11)), we have

$$\begin{aligned} u_{nn}c_0 \leq u_{nn}(0)S_{k-1}(u_{\alpha\beta}(0)) =& S_k(D^2u(0)) - S_k(u_{\alpha\beta}(0)) + \sum_{i=1}^{n-1} u_{in}^2 S_{k-2}(u_{\alpha\beta}) \\ \leq& C. \end{aligned}$$

Then we obtain

$$u_{nn}(0) \leq C,$$

where C is a uniform constant. On the other hand, $u_{nn}(0) \geq \sum_{i=1}^{n-1} u_{\alpha\alpha}(0) \geq -C$. In conclusion, we have $|u_{nn}(0)| \leq C$.

In conclusion, we get the uniform C^2 estimate. □

3.3. **Proof of Theorem 3.1.** The uniqueness follows from the comparison principal for k-convex solutions of k-Hessian equations in Lemma 2.4 by Wang-Trudinger [31] (see also [30, 32])..

For the existence part, since u^ϵ is increasing on ϵ, $u^0 := \lim_{\epsilon\to 0} u^\epsilon$ exits. Since $|u^\epsilon|_{C^2(\overline{U})} \leq C$, there exists a subsequence u^{ϵ_i} converges to u^0 in $C^{1,\alpha}$ on $\overline{U}$ and $u^0 \in C^{1,1}(\overline{U})$.

4. Solving the approximating equation in $\Omega_R := B_R \setminus \Omega$.

We always assume Ω is a smoothly convex domain and strictly $(k-1)$-convex. Recall that we always assume $B_r \subset\subset \Omega \subset\subset B_{\frac{R_0}{2}}$.

4.1. **Case 1: $k < \frac{n}{2}$.** Since the Green function in this case is $-|x|^{\frac{2k-n}{k}}$, we want to solve the following k-Hessian equation .

$$\begin{cases} S_k(D^2u) = 0 & \text{in} \quad \Omega^c := \mathbb{R}^n \setminus \Omega, \\ \qquad u = -1 & \text{on} \quad \partial\Omega, \\ \lim_{x\to\infty} u(x) = 0. \end{cases} \tag{4.1}$$

Define $w^{1,\varepsilon} := -\left(R_0^2+\varepsilon^2\right)^{\frac{n-2k}{2k}}\left(|x|^2+\varepsilon^2\right)^{-\frac{n-2k}{2k}}$. We have

$$f^{1,\varepsilon} := S_k(D^2 w^{1,\varepsilon}) = C_n^k\left(\frac{k}{n-2k}\right)^k (R_0^2+\varepsilon^2)^{\frac{n-2k}{2}}(|x|^2+\varepsilon^2)^{-\frac{n}{2}-1}\varepsilon^2.$$

For ε small enough, we can construct a smoothly strictly k-convex function $\underline{u}^{1,\varepsilon}$ as follows

Lemma 4.1. *For any $\varepsilon \in (0, \frac{R_0}{3})$ small enough, there exists a strictly k-convex function $\underline{u}^{1,\epsilon} \in C^\infty(\mathbb{R}^n \setminus \Omega)$ satisfying*

$$\underline{u}^{1,\varepsilon} = \begin{cases} \tau_0(e^{d(x)}-1)-1, & \text{in } B_{\frac{2}{3}R_0}\setminus\Omega, \\ w^{1,\varepsilon} & \text{in } B^c_{2R_0}, \end{cases} \tag{4.2}$$
$$\underline{u}^{1,\varepsilon} \geq \max\left\{w^{1,\varepsilon}, \tau_0(e^{d(x)}-1)-1\right\} \quad \text{in } B_{2R_0}\setminus B_{\frac{2}{3}R_0},$$
$$S_k(D^2\underline{u}^{1,\varepsilon}) \geq f^{1,\varepsilon}, \quad \text{in } \Omega^c,$$

where $\tau_0 = 2^{-\frac{n}{2k}}\frac{2^{\frac{n-2k}{2k}}-1}{e^{3R_0}-1} > 0$ since $k < \frac{n}{2}$.

Proof. We apply Lemma 2.8 by taking $U = B_{3R_0}\setminus\overline{\Omega}$, $h = w^{1,\varepsilon}$, $g = \tau_0(e^{d(x)}-1)-1$ and $\delta = 2^{-\frac{n}{2k}}(2^{\frac{n-2k}{2k}}-1)$ to get a function $u^{1,\varepsilon} \in C^\infty(\overline{U})$ which is strictly k-convex and satisfies

$$\underline{u}^{1,\varepsilon} \geq \max\left\{w^{1,\varepsilon}, \tau_0(e^{d(x)}-1)-1\right\} \quad \text{in } B_{2R_0}\setminus B_{\frac{2}{3}R_0}$$
$$S_k(D^2\underline{u}^{1,\varepsilon}) \geq f^{1,\varepsilon} \text{ in } U.$$

Next we prove (4.2). When $x \in B_{3R_0}\setminus B_{2R_0}$, for $\varepsilon < R_0$,

$$\begin{aligned} h(x)-g(x) =& w^{1,\varepsilon}(x) - \tau_0(e^{d(x)}-1)+1 \\ \geq& -\left(R_0^2+\varepsilon^2\right)^{\frac{n-2k}{2k}}\left(4R_0^2+\varepsilon^2\right)^{-\frac{n-2k}{2k}} - \tau_0(e^{3R_0}-1)+1 \\ >& 1-2^{-\frac{n-2k}{2k}} - \tau_0(e^{3R_0}-1) \\ >& \frac{1}{2}(1-2^{-\frac{n-2k}{2k}}) \\ =:& \delta > 0, \end{aligned}$$

where $\delta > 0$ since $k < \frac{n}{2}$.
When $x \in B_{\frac{2}{3}R_0}\setminus\Omega$, since $\varepsilon < \frac{R_0}{3}$, we have

$$\begin{aligned} g(x)-h(x) \geq& -w^{1,\varepsilon}(x)-1 \\ \geq& \left(R_0^2+\varepsilon^2\right)^{\frac{n-2k}{2k}}\left(\frac{4}{9}R_0^2+\varepsilon^2\right)^{-\frac{n-2k}{2k}} - 1 \\ \geq& 2^{\frac{n-2k}{2k}} - 1 > \delta. \end{aligned}$$

We can finish the proof by extending the domain of $u^{1,\epsilon}$ to Ω^c by taking $u^{1,\epsilon} = w^{1,\epsilon}$ in $B^c_{3R_0}$. □

Now for any $\varepsilon \in (0, \varepsilon_0)$ and $R \in (K_0(R_0 + 1), +\infty)$ with ε_0 small enough and K_0 large enough, we consider the approximating equation

$$\begin{cases} S_k(D^2u) = f^{1,\varepsilon} & \text{in} \quad \Omega_R = B_R \setminus \Omega, \\ u = -1 & \text{on} \quad \partial\Omega, \\ u(x) = \underline{u}^{1,\varepsilon} & \text{on} \quad \partial B_R. \end{cases} \tag{4.3}$$

Since $u^{1,\varepsilon}$ is a subsolution, by Guan [14], equation (4.3) has a strictly k-convex solution $u^{\varepsilon,R} \in C^\infty(\overline{\Omega}_R)$. Our goal is to establish uniform C^2 estimates of $u^{\varepsilon,R}$, which are independent of ε and R. We prove the following

Theorem 4.2. *Assume $1 \le k < \frac{n}{2}$. For every sufficiently small ε and sufficiently large R, $u^{\varepsilon,R}$ satisfies*

$$\begin{cases} C^{-1}|x|^{-\frac{n-2k}{k}} \le -u^{\varepsilon,R}(x) \le C|x|^{-\frac{n-2k}{k}}, \\ C^{-1}|x|^{-\frac{n-k}{k}} \le |Du^{\varepsilon,R}|(x) \le C|x|^{-\frac{n-k}{k}}, \\ |D^2u^{\varepsilon,R}|(x) \le C|x|^{-\frac{n}{k}}, \end{cases}$$

where C is a uniform constant which is independent of ε and R.

4.2. **Case 2: $k > \frac{n}{2}$.** Since the Green function in this case is $|x|^{\frac{2k-n}{k}}$, we want to solve the k-Hessian equation as follows

$$\begin{cases} S_k(D^2u) = 0 & \text{in} \quad \Omega^c, \\ u = 1 & \text{on } \partial\Omega, \\ u(x) = |x|^{\frac{2k-n}{k}} + O(1) & \text{as } x \to \infty. \end{cases} \tag{4.4}$$

4.2.1. *The approximating equation.* Define $w^{2,\varepsilon} := \left(|x|^2 + \varepsilon^2\right)^{\frac{2k-n}{2k}} - \left(R_0^2 + \varepsilon^2\right)^{\frac{2k-n}{2k}} + 1$ and we have

$$f^{2,\varepsilon} := S_k(D^2w^{2,\varepsilon}) = C_n^k\left(\frac{k}{2k-n}\right)^k (|x|^2 + \varepsilon^2)^{-\frac{n}{2}-1}\varepsilon^2.$$

We construct a smoothly and strictly k-convex function $\underline{u}^{2,\varepsilon}$ as follows

Lemma 4.3. *For any $\varepsilon \in (0, \frac{R_0}{3})$ small enough, there exists a strictly k-convex function $\underline{u}^{2,\varepsilon} \in C^\infty(\mathbb{R}^n \setminus \Omega)$ satisfying*

$$\underline{u}^{2,\epsilon} = \begin{cases} \tau_0(e^{d(x)} - 1) + 1, & \text{in } B_{\frac{2}{3}R_0} \setminus \overline{\Omega}, \\ w^{2,\varepsilon} & \text{in } B^c_{2R_0}, \end{cases} \tag{4.5}$$
$$\underline{u}^{2,\varepsilon} \geq \max\left\{w^{2,\varepsilon}, \tau_0(e^{d(x)} - 1) + 1\right\} \quad \text{in } B_{2R_0} \setminus B_{\frac{2}{3}R_0},$$
$$S_k(D^2\underline{u}^{2,\epsilon}) \geq f^{2,\varepsilon}, \quad \text{in } \Omega^c,$$

where $\tau_0 = \frac{1}{2}R_0^{\frac{2k-n}{k}} \frac{2^{\frac{2k-n}{2k}}-1}{e^{3R_0}-1} > 0$ since $k > \frac{n}{2}$.

Proof. We apply Lemma 2.8 by taking $U = B_{3R_0} \setminus \overline{\Omega}$, $h = w^{2,\varepsilon}$, $g = \tau_0(e^{d(x)} - 1) - 1$ and $\delta = \frac{1}{2}R_0^{\frac{2k-n}{k}}(2^{\frac{2k-n}{2k}} - 1)$ to get a function $u^{2,\varepsilon} \in C^\infty(\overline{U})$ which is strictly k-convex and satisfies

$$\underline{u}^{2,\varepsilon} \geq \max\left\{w^{2,\varepsilon}, \tau_0(e^{d(x)} - 1) - 1\right\} \quad \text{in } B_{2R_0} \setminus B_{\frac{2}{3}R_0}$$
$$S_k(D^2\underline{u}^{2,\varepsilon}) \geq f^{2,\varepsilon} \text{ in } U.$$

Next we prove (4.5). When $x \in B_{3R_0} \setminus B_{2R_0}$, for $\varepsilon < R_0$,

$$\begin{aligned} h(x) - g(x) =& w^{2,\varepsilon}(x) - \tau_0(e^{d(x)} - 1) - 1 \\ \geq& \left(4R_0^2 + \varepsilon^2\right)^{\frac{2k-n}{2k}} - \left(R_0^2 + \varepsilon^2\right)^{\frac{n-2k}{2k}} - \tau_0(e^{3R_0} - 1) \\ >& R_0^{\frac{2k-n}{k}}(2^{\frac{2k-n}{2k}} - 1) - \tau_0(e^{3R_0} - 1) \\ >& \frac{1}{2}R_0^{\frac{2k-n}{k}}(2^{\frac{2k-n}{2k}} - 1) \\ =:& \delta > 0, \end{aligned}$$

where we choose $\tau_0 = \frac{1}{2}R_0^{\frac{2k-n}{k}} \frac{2^{\frac{2k-n}{2k}}-1}{e^{3R_0}-1} > 0$ and $\delta > 0$ since $k > \frac{n}{2}$.
When $x \in B_{\frac{2}{3}R_0} \setminus \Omega$,

$$\begin{aligned} g(x) - h(x) \geq& -w^{2,\varepsilon}(x) - 1 \\ \geq& (R_0^2 + \varepsilon^2)^{\frac{2k-n}{2k}} - (\frac{4}{9}R_0^2 + \varepsilon^2)^{\frac{2k-n}{2k}} \\ \geq& 2^{-\frac{2k-n}{2k}} R_0^{\frac{2k-n}{k}}(2^{\frac{2k-n}{2k}} - 1) > \delta. \end{aligned}$$

Then we finish the proof by extending the domain of $u^{2,\epsilon}$ to Ω^c by taking $u^{2,\epsilon} = w^{2,\epsilon}$ in $B^c_{3R_0}$. □

We consider the approximating equation as follows

$$\begin{cases} S_k(D^2u) = f^{2,\varepsilon}, \text{ in } B_R \setminus \overline{\Omega}, \\ u = 1, \text{ on } \partial\Omega, \\ u = \underline{u}^{2,\varepsilon}, \text{ on } \partial B_R. \end{cases} \tag{4.6}$$

Since $u^{2,\varepsilon}$ is a subsolution, by Guan [14], equation (4.6) has a strictly k-convex solution $u^{\varepsilon,R} \in C^\infty(\overline{\Omega}_R)$. Our goal is to establish uniform C^2 estimates of $u^{\varepsilon,R}$, which are independent of ε and R.

We prove the following

Theorem 4.4. *Assume $k > \frac{n}{2}$. For every sufficiently small ε and sufficiently large R, $u^{\varepsilon,R}$ satisfying*

$$\begin{cases} |u^{\varepsilon,R}(x) - |x|^{\frac{2k-n}{k}}| \leq C, \\ C^{-1}|x|^{\frac{k-n}{k}} \leq |Du^{\varepsilon,R}|(x) \leq C|x|^{\frac{k-n}{k}}, \\ |D^2u^{\varepsilon,R}|(x) \leq C|x|^{-\frac{n}{k}}, \end{cases}$$

where C is a uniform constant which is independent of ε and R.

4.3. **Case 3: $k = \frac{n}{2}$.** Since the Green function in this case is $\log|x|$, we want to solve the k-Hessian equation as follows

$$\begin{cases} S_{\frac{n}{2}}(D^2u) = 0 \quad \text{in} \quad \Omega^c, \\ u = 0 \quad \text{on } \partial\Omega, \\ u(x) = \log|x| + O(1) \text{ as } x \to \infty. \end{cases} \tag{4.7}$$

4.3.1. *The approximating equation.* Define $w^{3,\varepsilon} := \frac{1}{2}\log\frac{|x|^2+\varepsilon^2}{R_0^2+\varepsilon^2}$ and we have

$$f^{3,\varepsilon} := S_k(D^2w^{3,\varepsilon}) = 2^{k+1}C_{n-1}^{\frac{n}{2}-1}\varepsilon^2(|x|^2+\varepsilon^2)^{-\frac{n}{2}-1} \tag{4.8}$$

We construct a smoothly and strictly k-convex function $\underline{u}^{3,\varepsilon}$ as follows

Lemma 4.5. *For any $\varepsilon \in (0, \frac{R_0}{3})$, there exists a strictly k-convex function $\underline{u}^{3,\varepsilon} \in C^\infty(\mathbb{R}^n \setminus \Omega)$ satisfying*

$$\underline{u}^{3,\varepsilon} = \begin{cases} \tau_0(e^{d(x)} - 1) & \text{in } B_{\frac{2}{3}R_0} \setminus \Omega, \\ w^{3,\varepsilon} & \text{in } B^c_{2R_0}, \end{cases} \tag{4.9}$$

$$\underline{u}^{3,\varepsilon} \geq \max\left\{w^{3,\varepsilon}, \tau_0(e^{d(x)} - 1)\right\} \quad \text{in } B_{2R_0} \setminus B_{\frac{2}{3}R_0},$$

$$S_k(D^2\underline{u}^{3,\varepsilon}) \geq f^{3,\varepsilon} \quad \text{in } \Omega^c,$$

where $\tau_0 = \frac{1}{4}\frac{\log 2}{e^{3R_0}-1}$.

Proof. We apply Lemma 2.8 by taking $U = B_{3R_0} \setminus \overline{\Omega}$, $h = w^{2,\varepsilon}$, $g = \tau_0(e^{d(x)} - 1)$ and $\delta = \frac{1}{4}\log 2$ to get a function $u^{2,\varepsilon} \in C^\infty(\overline{U})$ which is strictly k-convex and satisfies

$$\underline{u}^{3,\varepsilon} \geq \max\left\{w^{3,\varepsilon}, \tau_0(e^{d(x)} - 1)\right\} \quad \text{in } B_{2R_0} \setminus B_{\frac{2}{3}R_0}$$
$$S_k(D^2 u^{3,\varepsilon}) \geq f^{3,\varepsilon} \text{ in } U.$$

Next we prove (4.9). When $x \in B_{3R_0} \setminus B_{2R_0}$,

$$\begin{aligned} h(x) - g(x) =& w^{3,\varepsilon}(x) - \tau_0(e^{d(x)} - 1) \\ \geq& \frac{1}{2}\log\frac{4R_0^2 + \varepsilon^2}{R_0^2 + \varepsilon^2} - \tau_0(e^{3R_0} - 1) \\ >& \frac{1}{2}\log 2 - \tau_0(e^{3R_0} - 1) \\ >& \frac{1}{4}\log 2 =: \delta, \end{aligned}$$

where we use $\varepsilon < R_0$ and we choose $\tau_0 = \frac{1}{4}\frac{\log 2}{e^{3R_0} - 1}$.
When $x \in B_{\frac{2}{3}R_0} \setminus \Omega$, since $\varepsilon < \frac{1}{3}R_0$, we have

$$\begin{aligned} g(x) - h(x) \geq& - w^{2,\varepsilon}(x) \\ \geq& \frac{1}{2}\log\frac{R_0^2 + \varepsilon^2}{\frac{4}{9}R_0^2 + \varepsilon^2} \\ \geq& \frac{1}{2}\log 2 > \delta. \end{aligned}$$

Then we finish the proof by extending the domain of $u^{2,\epsilon}$ to Ω^c by taking $u^{2,\epsilon} = w^{2,\epsilon}$ in $B^c_{3R_0}$. □

We consider the approximating equation as follows

$$\left\{ \begin{array}{c} S_k(D^2 u) = f^{3,\varepsilon}, \text{ in } B_R \setminus \overline{\Omega}, \\ u = 0, \text{ on } \partial\Omega, \\ u = \underline{u}^{3,\varepsilon}, \text{ on } \partial B_R. \end{array} \right. \tag{4.10}$$

Since $u^{3,\varepsilon}$ is a subsolution, by Guan [14], equation (4.6) has a strictly k-convex solution $u^{\varepsilon,R} \in C^\infty(\overline{\Omega}_R)$.

We prove the following

Theorem 4.6. *Assume $k = \frac{n}{2}$. For every sufficiently small ε and sufficiently large R, $u^{\varepsilon,R}$ satisfies*

$$\begin{cases} |u^{\varepsilon,R}(x) - \log|x|| \le C, \\ C^{-1}|x|^{-1} \le |Du^{\varepsilon,R}|(x) \le C|x|^{-1}, \\ |D^2u^{\varepsilon,R}|(x) \le C|x|^{-2}, \end{cases}$$

where C is a uniform constant which is independent of ε and R.

In the next subsections, we will prove uniform C^2-estimates of solutions of equations (4.3), (4.6) and (4.10). The key point is that these estimates are independent of ε and R.

4.4. C^0 estimates.

4.4.1. *Case 1: $k < \frac{n}{2}$.* We fisrt prove $u^{\epsilon,R}$ is increasing with R. Indeed, since for any $\widetilde{R} > R \ge 100(R_0 + 1)$, we have

$$\begin{cases} S_k(D^2u^{\varepsilon,R}) = S_k(D^2u^{\varepsilon,\widetilde{R}}) = f^{1,\varepsilon} \text{ in } \Omega_R \\ u^{\varepsilon,R} = u^{\varepsilon,\widetilde{R}} = -1 \text{ on } \partial\Omega, \\ u^{\varepsilon,R} = \underline{u}^{\varepsilon} \le u^{\varepsilon,\widetilde{R}} \text{ on } \partial B_R \end{cases}$$

Applying the maximum principle in Ω_R, we have

$$u^{\varepsilon,R} \le u^{\varepsilon,\tilde{R}}. \tag{4.11}$$

Since

$$\begin{cases} S_k(D^2u^{\varepsilon,R}) = f^{1,\varepsilon} > 0 = S_k\left(D^2\left(r_0^{\frac{n-2k}{k}}|x|^{-\frac{n-2k}{k}}\right)\right), \text{ in } \Omega_{\tilde{R}} \\ u^{\varepsilon,\tilde{R}} = -1 < -r_0^{\frac{n-2k}{k}}|x|^{-\frac{n-2k}{k}}, \text{ on } \partial\Omega \text{ (}\mathbf{For}\ B_{r_0} \subset\subset \Omega\text{)}, \\ u^{\varepsilon,\tilde{R}} = -\left(\dfrac{R_0^2 + \varepsilon^2}{\widetilde{R}^2 + \varepsilon^2}\right)^{\frac{n-2k}{2k}} < -r_0^{\frac{n-2k}{k}}\widetilde{R}^{-\frac{n-2k}{k}}, \text{ on } \partial B_{\tilde{R}} \end{cases}$$

Applying the maximum principle in $\Omega_{\widetilde{R}}$, we have

$$u^{\varepsilon,\tilde{R}} < -r_0^{\frac{n-2k}{k}}|x|^{-\frac{n-2k}{k}} \quad \text{in} \quad \Omega_{\widetilde{R}}. \tag{4.12}$$

Then by (4.11) and (4.12), for any $\widetilde{R} > R \ge 100(R_0 + 1)$,

$$u^{\varepsilon,R} \le u^{\varepsilon,\tilde{R}} < -r_0^{\frac{n-2k}{k}}|x|^{-\frac{n-2k}{k}} \quad \text{in} \quad \Omega_R. \tag{4.13}$$

On the other hand, for any $x \in \Omega_R$, we have $u^{\varepsilon,R}(x) \ge \underline{u}^{1,\varepsilon}(x) \ge w^{1,\varepsilon} = -(R_0^2 + \varepsilon^2)^{\frac{n-2k}{2k}}(|x|^2 + \varepsilon^2)^{-\frac{n-2k}{2k}}$. Thus when $k < \frac{n}{2}$, for any $x \in \Omega_R$,

$$r_0^{\frac{n-2k}{k}}|x|^{-\frac{n-2k}{k}} \le -u^{\varepsilon,R}(x) \le R_0^{\frac{n-2k}{k}}|x|^{-\frac{n-2k}{k}}.$$

4.4.2. *Case 2: $k > \frac{n}{2}$.* Firstly, we have for any $x \in \Omega_R$, $u^{\varepsilon,R}(x) \geq \underline{u}^{2,\varepsilon} \geq w^{2,\varepsilon} = (|x|^2 + \varepsilon^2)^{\frac{2k-n}{2k}} - (R_0^2 + \varepsilon^2)^{\frac{2k-n}{2k}} + 1$ and this gives the lower bound of $u^{\varepsilon,R}$.

Since $|x|^{\frac{2k-n}{k}} - r_0^{\frac{2k-n}{k}} + 1$ is the upper barrier of $u^{\varepsilon,R}$ in Ω_R, we get $u^{\varepsilon,R} - |x|^{\frac{2k-n}{k}} \leq C$.

4.4.3. *Case 3: $k = \frac{n}{2}$.* The proof is similar as that in Case 2.

4.5. **Gradient estimates.** In this subsection, we prove the global gradient estimate. The key point is that the estimate here does not depend on ε and R. We also prove that the positive lower bound of the gradient of the solution and thus the level set of the solution is compact.

4.5.1. *Reducing global gradient estimates to boundary gradient estimates* This part is the key part of gradient estimates. The point in here is that the gradient estimate is independent of the approximating process. This estimates is motivated by B. Guan [15].

Theorem 4.7. *Let u be the solution of the approximating equation (4.3), (4.6) or (4.10). Denote by*

$$P = \begin{cases} |Du|^2 e^{2u}, & k = \frac{n}{2}, \\ |Du|^2 u^{\frac{2(n-k)}{2k-n}}, & k > \frac{n}{2}, \\ |Du|^2 (-u)^{-\frac{2(n-k)}{n-2k}}, & k < \frac{n}{2}. \end{cases} \tag{4.14}$$

then we have the following gradient estimate

$$\max_{B_R \setminus \Omega} P \leq \begin{cases} \max\left\{ \max\limits_{B_R\setminus\Omega}(e^{2u}|D\log f^{3,\varepsilon}|^2), \max\limits_{\Gamma_R} P \right\}, & k = \frac{n}{2}, \\ \max\left\{ \left[\frac{2k-n}{k(n+1-k)}\right]^2 \max\limits_{B_R\setminus\Omega}(u^{\frac{2k}{2k-n}}|D\log f^{2,\varepsilon}|^2), \max\limits_{\Gamma_R} P \right\}, & k > \frac{n}{2}, \\ \max\left\{ \left[\frac{n-2k}{k(n+1-k)}\right]^2 \max\limits_{B_R\setminus\Omega}\left[(-u)^{-\frac{2k}{n-2k}}|D\log f^{1,\varepsilon}|^2\right], \max\limits_{\Gamma_R} P \right\}, & k < \frac{n}{2}, \end{cases} \tag{4.15}$$

where $\Gamma_R := \partial(B_R \setminus \Omega)$.

Proof. For simplicity, we use f instead of $f^{1,\varepsilon}$, $f^{2,\varepsilon}$ or $f^{3,\varepsilon}$ during the proof. Consider the function $G = \log|Du|^2 + g(u)$.

$$\begin{aligned} 0 = G_i =& \frac{|Du|_i^2}{|Du|^2} + g'u_i = \frac{2u_k u_{ki}}{u_1^2} + g'u_i \\ =& \frac{2u_{1i}}{u_1} + g'u_i. \end{aligned}$$

Then we have

$$u_{1i} = 0, i \geq 2, \lambda_1 = u_{11} = -\frac{g'}{2}u_1^2.$$

In the following, we will take g in three cases

$$g(u) = \begin{cases} 2u, & k = \frac{n}{2}, \\ \frac{2(n-k)}{(2k-n)} \log u, & k > \frac{n}{2}, \\ -\frac{2(n-k)}{(n-2k)} \log(-u), & k < \frac{n}{2}. \end{cases} \tag{4.16}$$

In these three cases, we always have $g' > 0$. This implies $\lambda_1 < 0$ and thus $(\lambda|1) \in \Gamma_k$ which is crucial during the proof.

Thus u_{ij} is diagonal at x_0 and

$$\begin{aligned}
0 \geq F^{ii}G_{ii} =& \frac{F^{ii}|Du|^2_{ii}}{|Du|^2} - F^{ii}\left||Du|^2_i\right|^2 + g''F^{ii}u_i^2 + g'F^{ii}u_{ii} \\
=& \frac{2F^{ii}u_{ki}^2 + 2F^{ii}u_k u_{kii}}{|Du|^2} - F^{ii}(g')^2u_i^2 + g''u_i^2 + g'u_{ii} \\
=& \frac{2F^{ii}u_{ii}^2 + 2F^{ii}u_1u_{1ii}}{u_1^2} - (g')^2F^{11}u_1^2 + g''F^{11}u_1^2 + g'F^{ii}u_{ii} \\
=& 2u_1^{-2}\Big(S_1(\lambda)f - (k+1)S_{k+1}(\lambda) + u_1f_1\Big) + \Big(g'' - (g')^2\Big)S_{k-1}(\lambda|1)u_1^2 + kg'f \\
=& 2u_1^{-2}\left(S_1(\lambda)f - (k+1)S_{k+1}(\lambda) + u_1f_1 + \frac{1}{2}\Big(g'' - (g')^2\Big)S_{k-1}(\lambda|1)u_1^4 + \frac{k}{2}g'u_1^2f\right) \\
=& 2u_1^{-2}\left(S_1(\lambda)f - (k+1)S_{k+1}(\lambda) + u_1f_1 + 2\Big(\frac{g''}{(g')^2} - 1\Big)S_{k-1}(\lambda|1)\lambda_1^2 - kf\lambda_1\right),
\end{aligned}$$

where we use $\lambda_1 = -\frac{g'}{2}u_1^2$.
Therefore

$$\begin{aligned}
0 \geq \frac{1}{2}u_1^2F^{ii}G_{ii} &= S_1(\lambda)f - (k+1)S_{k+1}(\lambda) + u_1f_1 + 2\Big(\frac{g''}{(g')^2} - 1\Big)S_{k-1}(\lambda|1)\lambda_1^2 - kf\lambda_1 \\
&= S_1(\lambda|1)f - (k-1)f\lambda_1 - (k+1)S_{k+1}(\lambda) + 2\Big(\frac{g''}{(g')^2} - 1\Big)S_{k-1}(\lambda|1)\lambda_1^2 + u_1f_1.
\end{aligned}$$

Sine $\lambda_1 S_{k-1}(\lambda|1) + S_k(\lambda|1) = f$, we have $\lambda_1 = \frac{f}{S_{k-1}(\lambda|1)} - \frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)}$. We first manipulate the term $-(k-1)f\lambda_1$.

$$\begin{aligned}
-(k-1)f\lambda_1 =& -(k-1)f\Big(\frac{f}{S_{k-1}(\lambda|1)} - \frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)}\Big) \\
=& -(k-1)\frac{f^2}{S_{k-1}(\lambda|1)} + (k-1)f\frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)}.
\end{aligned} \tag{4.17}$$

Next we manipulate $-(k+1)S_{k+1}(\lambda)$.

$$
\begin{aligned}
-(k+1)S_{k+1}(\lambda) =& -(k+1)\lambda_1 S_k(\lambda|1) - (k+1)S_{k+1}(\lambda|1)\\
=& -(k+1)S_k(\lambda|1)\Big(\frac{f}{S_{k-1}(\lambda|1)} - \frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)}\Big) - (k+1)S_{k+1}(\lambda|1)\\
=& -(k+1)f\frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)} + (k+1)\frac{S_k^2(\lambda|1)}{S_{k-1}(\lambda|1)} - (k+1)S_{k+1}(\lambda|1) \qquad (4.18)
\end{aligned}
$$

At last, we manipulate the trouble term $2\Big(\frac{g''}{(g')^2}-1\Big)S_{k-1}(\lambda|1)\lambda_1^2$.

$$
\begin{aligned}
&2\Big(\frac{g''}{(g')^2}-1\Big)S_{k-1}(\lambda|1)\lambda_1^2\\
&= 2\Big(\frac{g''}{(g')^2}-1\Big)S_{k-1}(\lambda|1)\Big(\frac{f}{S_{k-1}(\lambda|1)} - \frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)}\Big)^2\\
&=2\Big(\frac{g''}{(g')^2}-1\Big)\frac{f^2}{S_{k-1}(\lambda|1)} - 4\Big(\frac{g''}{(g')^2}-1\Big)f\frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)} + 2\Big(\frac{g''}{(g')^2}-1\Big)\frac{S_k^2(\lambda|1)}{S_{k-1}(\lambda|1)}. \qquad (4.19)
\end{aligned}
$$

Substitute the above three equality into the original terms, we have

$$
\begin{aligned}
0 \geq& \frac{1}{2}u_1^2F^{ii}G_{ii}\\
=&\Big(2\frac{g''}{(g')^2}+k-1\Big)\frac{S_k^2(\lambda|1)}{S_{k-1}(\lambda|1)} - (k+1)S_{k+1}(\lambda|1) + 2\Big(1-\frac{2g''}{(g')^2}\Big)f\frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)}\\
&+\Big(2\frac{g''}{(g')^2}-k-1\Big)\frac{f^2}{S_{k-1}(\lambda|1)} + S_1(\lambda|1)f + u_1f_1\\
\geq&\Big(2\frac{g''}{(g')^2}+k-1-\frac{k(n-k-1)}{n-k}\Big)\frac{S_k^2(\lambda|1)}{S_{k-1}(\lambda|1)} + 2\Big(1-\frac{2g''}{(g')^2}\Big)f\frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)}\\
&+\Big(2\frac{g''}{(g')^2}-k-1\Big)\frac{f^2}{S_{k-1}(\lambda|1)} + S_1(\lambda|1)f + u_1f_1, \qquad (4.20)
\end{aligned}
$$

where in the last inequality we use the Maclaurin inequality:

$$
\frac{S_{k+1}(\lambda|1)/C_{n-1}^{k+1}}{S_k(\lambda|1)/C_{n-1}^k} \leq \frac{S_k(\lambda|1)/C_{n-1}^k}{S_{k-1}(\lambda|1)/C_{n-1}^{k-1}}.
$$

Case1: $k<\frac{n}{2}$,
Since the foundamental solution is $-|x|^{2-\frac{n}{k}}$ and its gradient is $\sim |x|^{1-\frac{n}{k}}$. We take $g(u) = a\log(-u)$, where $a=-\frac{2(n-k)}{n-2k}<0$. $2\frac{g''}{(g')^2} = -\frac{2}{a} = \frac{n-2k}{n-k}$. Substituting it into (4.21), we have

$$
\begin{aligned}
0 \geq & \frac{1}{2}u_1^2 F^{ii}G_{ii} \geq \Big(2\frac{g''}{(g')^2} + k - 1 - \frac{k(n-k-1)}{n-k}\Big)\frac{S_k^2(\lambda|1)}{S_{k-1}(\lambda|1)} \\
& + 2\Big(1 - \frac{2g''}{(g')^2}\Big) f \frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)} + \Big(2\frac{g''}{(g')^2} - k - 1\Big)\frac{f^2}{S_{k-1}(\lambda|1)} + S_1(\lambda|1) f + u_1 f_1 \\
= & \frac{2k}{n-k} f \frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)} - k\frac{n+1-k}{n-k}\frac{f^2}{S_{k-1}(\lambda|1)} + S_1(\lambda|1) f + u_1 f_1 \\
\geq & f\Big(\frac{k}{n-1} S_1(\lambda|1) - k\frac{n+1-k}{n-k}\lambda_1\Big) + u_1 f_1 (*) \\
\geq & \frac{a}{2}k\frac{n+1-k}{n-k} = \frac{k(n+1-k)}{n-2k} f \frac{u_1^2}{-u} + u_1 f_1,
\end{aligned} \tag{4.21}
$$

where in (*), we use the same argument as the $k > \frac{n}{2}$ case.
From this, we have $u_1 \leq \frac{n-2k}{k(n+1-k)}(-u)|D\log f|$. Then we have

$$
|Du|^2(-u)^a \leq u_1^2(-u)^a \leq \Big(\frac{n-2k}{k(n+1-k)}\Big)^2 (-u)^{\frac{2k}{2k-n}} |D\log f|^2.
$$

Case2: $k > \frac{n}{2}$**.**
We take $g = a\log u$, where $a = \frac{2(n-k)}{2k-n}$ (since the foudament solution is $|x|^{2-\frac{n}{k}}$). By (4.21), we have

$$
\begin{aligned}
0 \geq & \frac{1}{2}u_1^2 F^{ii}G_{ii} \\
= & 2\left[1 - \frac{2g''}{(g')^2}\right] f \frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)} + \left[2\frac{g''}{(g')^2} - k - 1\right]\frac{f^2}{S_{k-1}(\lambda|1)} + S_1(\lambda|1) f + u_1 f_1 \\
= & 2(1 + \frac{2}{a}) f \frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)} - (1 + \frac{2}{a} + k)\frac{f^2}{S_{k-1}(\lambda|1)} + S_1(\lambda|1) f + u_1 f_1 \\
= & \frac{2k}{n-k} f \frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)} - k\frac{n+1-k}{n-k}\frac{f^2}{S_{k-1}(\lambda|1)} + S_1(\lambda|1) f + u_1 f_1 \\
= & \frac{2k}{n-k} f \frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)} - k\frac{n+1-k}{n-k} f \left[\lambda_1 + \frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)}\right] + S_1(\lambda|1) f + u_1 f_1 \\
= & f\left[S_1(\lambda|1) + \frac{k(k+1-n)}{n-k}\frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)} - k\frac{n+1-k}{n-k}\lambda_1\right] + u_1 f_1 \\
\geq & f\left[\frac{k}{n-1} S_1(\lambda|1) - k\frac{n+1-k}{n-k}\lambda_1\right] + u_1 f_1 \\
\geq & c_{n,k} f \frac{u_1^2}{u} + u_1 f_1,
\end{aligned}
$$

where $c_{n,k} = \frac{a}{2}k\frac{n+1-k}{n-k} = \frac{k(n+1-k)}{2k-n} > 0$.

This gives

$$u_1 \leq \frac{2k-n}{k(n+1-k)} u|D\log f|.$$

Therefore we have

$$|Du|^2 u^a = u_1^2 u^a \leq \Big(\frac{2k-n}{k(n+1-k)}\Big)^2 u^{\frac{2k}{2k-n}} |D\log f|^2. \tag{4.22}$$

Case3: $k = \frac{n}{2}$,
In this case, we must choose $g(u) = 2u$ (it is uniquely determined by the foundamental solution $\log|x|$), then from (4.20), we have

$$\begin{aligned}
0 \geq& \frac{1}{2} u_1^2 F^{ii} G_{ii} \geq 2f \frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)} - (\frac{n}{2}+1)\frac{f^2}{S_{k-1}(\lambda|1)} + S_1(\lambda|1) f + u_1 f_1 \\
=& f\Big(S_1(\lambda|1) + 2\frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)} - (\frac{n}{2}+1)\frac{f}{S_{k-1}(\lambda|1)}\Big) + u_1 f_1 \\
=& f\Big(S_1(\lambda|1) + 2\frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)} - (\frac{n}{2}+1)\Big(\lambda_1 + \frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)}\Big)\Big) + u_1 f_1 \\
=& f\Big(-(\frac{n}{2}+1)\lambda_1 + S_1(\lambda|1) - (\frac{n}{2}-1)\frac{S_k(\lambda|1)}{S_{k-1}(\lambda|1)}\Big) + u_1 f_1 \\
\geq& f\Big((\frac{n}{2}+1)\lambda_1 + \frac{n-2}{2(n-1)} S_1(\lambda|1)\Big) + u_1 f_1 \\
\geq& f(\frac{n}{2}+1) u_1^2 + u_1 f_1.
\end{aligned}$$

This implies $u_1 \leq -(\log f)_1 \leq |D\log f|$. Thus we have

$$|Du|^2 e^{2u} \leq u_1^2 e^{2u} \leq e^{2u} |D\log f|^2.$$

□

4.5.2. Boundary gradient estimates

We always assume $R >> 100(1+R_0)$. To prove the boundary gradient estimates, we will construct upper barriers on $\partial\Omega$ and ∂B_R respectively.

Case1: $k < \frac{n}{2}$

Let $h \in C^\infty(\overline{\Omega}_{R_0})$ be the unique solution of

$$\begin{cases}
\Delta h = 0, \text{ in } \Omega_{R_0}, \\
h = -1, \text{ on } \partial\Omega, \\
h = -r_0^{\frac{n-2k}{k}} R_0^{-\frac{n-2k}{k}}, \text{ on } \partial B_{R_0}.
\end{cases}$$

By maximum principle, $\underline{u}^{1,\varepsilon} \leq u \leq h^R$ in $\overline{\Omega}_{R_0}$. Then for any $x \in \partial\Omega$

$$0 > -\tau_0 = \tau_0 \Phi_\nu(x) = \underline{u}^{1,\varepsilon}{}_\nu(x) \geq u_\nu(x) \geq h_\nu(x),$$

where ν is the outward normal of $\partial\Omega_R$ (inward normal of $\partial\Omega$). Then

$$0 < \tau_0 \le \tau_0 \max_{\partial\Omega} |Du| = \max_{\partial\Omega}(-u_\nu) \le \max_{\partial\Omega} |h_\nu| \le C. \tag{4.23}$$

This proves that P is uniformly bounded on $\partial\Omega$.

Next we show P is uniformly bounded on ∂B_R. Indeed, we consider

$$\rho^{\varepsilon,R} = -a^{\varepsilon,R} r_0^{\frac{n-2k}{k}} |x|^{-\frac{n-2k}{k}} + a^{\varepsilon,R} - 1, \tag{4.24}$$

where $a^{\varepsilon,R}$ is defined as follows

$$a^{\varepsilon,R} = \left(1 - \left(\frac{r_0}{R_0}\right)^{\frac{n-2k}{k}}\right)^{-1} \left(\left(\frac{R_0^2+\varepsilon^2}{R^2+\varepsilon^2}\right)^{\frac{n-2k}{2k}} + 1\right) > 1. \tag{4.25}$$

Then we have $\rho^{\varepsilon,R} = u$ on ∂B_R and $\underline{u}^{1,\varepsilon} \le u \le \rho^{\varepsilon,R}$ in Ω_R. Then for any $x \in \partial B_R$

$$CR^{-\frac{n-k}{k}} \ge {\underline{u}^{1,\varepsilon}}_\nu(x) \ge u_\nu(x) \ge \rho_\nu^{\varepsilon,R}(x) = a^{\varepsilon,R}\frac{n-2k}{k} r_0^{\frac{n-2k}{k}} R^{-\frac{n-k}{k}} \ge cR^{-\frac{n-k}{k}}, \tag{4.26}$$

where C and c are uniformly positive constants. Thus

$$cR^{-\frac{n-k}{k}} \le \max_{\partial B_R} |Du| = \max_{\partial B_R} u_\nu \le CR^{-\frac{n-k}{k}}. \tag{4.27}$$

Combing (4.23) with (4.27), we have

$$c|x|^{1-\frac{n}{k}} \le |Du| \le C|x|^{1-\frac{n}{k}} \text{ on } \partial\Omega_R \tag{4.28}$$

This implies P is uniformly bounded on ∂B_R.

In conclusion, when $k < \frac{n}{2}$, P is uniformly bounded in $\overline{\Omega}_R$.

Case 2: $k > \frac{n}{2}$

Let $h \in C^\infty(\overline{\Omega}_{R_0})$ be the unique solution of

$$\begin{cases} \Delta h = 0, \text{ in } \Omega_{R_0}, \\ h = 1, \text{ on } \partial\Omega, \\ h = R_0^{\frac{2k-n}{k}} - r_0^{\frac{2k-n}{k}} + 1, \text{ on } \partial B_{R_0}. \end{cases}$$

By maximum principle, $\underline{u}^{2,\varepsilon} \le u \le h$ in $\overline{\Omega}_{R_0}$. Then

$$0 < c \le |Du| \le C, \text{ on } \partial\Omega \tag{4.29}$$

Thus we have P is uniformly bounded on $\partial\Omega$.

We construct the upper barrier of u in Ω_R as follows

$$\rho^{\varepsilon,R} = a^{\varepsilon,R}\left(|x|^{\frac{2k-n}{k}} - r_0^{\frac{2k-n}{k}}\right) + 1, \tag{4.30}$$

where $a^{\varepsilon,R}$ is defined by

$$a^{\varepsilon,R} = \left(R^{\frac{2k-n}{k}} - r_0^{\frac{2k-n}{k}}\right)^{-1} \left((R^2+\varepsilon^2)^{1-\frac{n}{2k}} - (R_0^2+\varepsilon^2)^{1-\frac{n}{2k}}\right) > a_0 > 0, \tag{4.31}$$

where $a_0 > 0$ is independent of ε and R. Then we have $\rho^{\varepsilon,R} = u$ on ∂B_R and $\underline{u}^{2,\varepsilon} \le u \le \rho^{\varepsilon,R}$ in Ω_R. Thus

$$cR^{-\frac{n-k}{k}} \le \max_{\partial B_R} |Du| = \max_{\partial B_R} u_\nu \le CR^{-\frac{n-k}{k}}. \tag{4.32}$$

Combing (4.29) with (4.32), we have

$$c|x|^{1-\frac{n}{k}} \le |Du| \le C|x|^{1-\frac{n}{k}} \text{ on } \partial\Omega_R \tag{4.33}$$

This implies P is uniformly bounded on ∂B_R. In conclusion, when $k > \frac{n}{2}$, P is uniformly bounded in $\overline{\Omega}_R$.

Case 3: $k = \frac{n}{2}$

Let $h \in C^\infty(\overline{\Omega}_{R_0})$ be the unique solution of

$$\begin{cases} \Delta h = 0, \text{ in } \Omega_{R_0}, \\ \ \ h = 0, \text{ on } \partial\Omega, \\ \ \ h = \log R_0 - \log r_0, \text{ on } \partial B_{R_0}. \end{cases}$$

By maximum principle, $\underline{u}^{3,\varepsilon} \le u \le h$ in $\overline{\Omega}_{R_0}$. Then

$$0 < c \le |Du| \le C, \text{ on } \partial\Omega \tag{4.34}$$

Thus P is uniformly bounded on $\partial\Omega$.

We construct the upper barrier of u in Ω_R as follows

$$\rho^{\varepsilon,R} = a^{\varepsilon,R}\left(\log|x| - \log r_0\right),$$

where $a^{\varepsilon,R}$ is defined by

$$a^{\varepsilon,R} = \left(\log R - \log r_0\right)^{-1} \frac{1}{2} \log \frac{R^2 + \varepsilon^2}{R_0^2 + \varepsilon^2} > a_0 > 0, \tag{4.35}$$

where $a_0 > 0$ is independent of ε and R. Then we have $\rho^{\varepsilon,R} = u$ on ∂B_R and $\underline{u}^{3,\varepsilon} \le u \le \rho^{\varepsilon,R}$ in Ω_R. Thus

$$cR^{-1} \le \max_{\partial B_R} |Du| = \max_{\partial B_R} u_\nu \le CR^{-1}. \tag{4.36}$$

Combing (4.34) with (4.36), we have

$$c|x|^{-1} \le |Du| \le C|x|^{-1} \text{ on } \partial\Omega_R \tag{4.37}$$

This implies P is uniformly bounded on $\partial\Omega_R$. In conclusion, when $k = \frac{n}{2}$, P is uniformly bounded in $\overline{\Omega}_R$.

4.5.3. ***Positive lower bound of*** $|Du|$.

Lemma 4.8. *Let u be the k-convex solution of the approximating equation* (4.3)*,* (4.6) *or* (4.10)*. For sufficiently large R and sufficiently small ε, there exists a uniform constant c_0 such that for any $x \in \overline{\Omega}_R$*

$$x \cdot Du(x) \geq \begin{cases} c_0|x|^{2-\frac{n}{k}}, & \text{if } k < \frac{n}{2} \text{ or } k > \frac{n}{2}, \\ c_0, & \text{if } k = \frac{n}{2}. \end{cases} \tag{4.38}$$

In particular,

$$|Du(x)| \geq \begin{cases} c_0|x|^{1-\frac{n}{k}}, & \text{if } k < \frac{n}{2} \text{ or } k > \frac{n}{2}, \\ c_0|x|^{-1}, & \text{if } k = \frac{n}{2}. \end{cases} \tag{4.39}$$

Proof. Case 1: $k < \frac{n}{2}$

We consider the function $H := x \cdot Du(x) + b_1 u$. Recall $F^{ij} = \frac{\partial}{\partial u_{ij}}(\log S_k(D^2u))$.

Direct manipulation gives

$$F^{ij}H_{ij} = (2 + b_1)k + x_m\tilde{f}_m, \tag{4.40}$$

Recall $\tilde{f} = \log f$, we have

$$x_m\tilde{f}_m = -(n+2)|x|^2(|x|^2 + \varepsilon^2)^{-1}. \tag{4.41}$$

Then if $b_1 < k^{-1}$, we have

$$F^{ij}H_{ij} = (|x|^2 + \varepsilon^2)^{-1}\Big(\big((2 + b_1)k - (n+2)\big)|x|^2 + (2 + b_1)k\varepsilon^2\Big) < 0, \tag{4.42}$$

By maximum principle, $H \geq \min_{\partial\Omega_R} H$. By choosing b_1 sufficiently small, we can prove $\min_{\partial\Omega_R} H > 0$.

Indeed, for any $x \in \partial\Omega$, since Ω is convex, we have

$$\begin{aligned} H(x) =& x \cdot Du(x) - b_1 = (x \cdot \nu(x))|Du(x)| - b_1 \\ \geq& \min_{\partial\Omega}(x \cdot \nu(x))c - b_1 > 0, \end{aligned} \tag{4.43}$$

where the last term is positive if we choose $b_1 < \min_{\partial\Omega}(x \cdot \nu(x))c$ and $\nu(x)$ is the outward unit normal vector of Ω at $x \in \partial\Omega$.

Indeed, for any $x \in \partial B_R$, we have

$$\begin{aligned} H(x) =& x \cdot Du(x) - b_1(R_0^2 + \varepsilon^2)^{\frac{n}{k}-2}(R^2 + \varepsilon^2)^{2-\frac{n}{k}} \\ =& Ru_\nu - b_1(R_0^2 + \varepsilon^2)^{\frac{n}{k}-2}(R^2 + \varepsilon^2)^{\frac{2k-n}{2k}} \\ \geq& cR^{\frac{2k-n}{k}} - C_1 b_1 R^{\frac{2k-n}{k}} \\ >& 0, \end{aligned} \tag{4.44}$$

where we require $b_1 < \frac{c}{2C_1}$. In conclusion, we prove $H > 0$ in $\overline{\Omega}_R$ and thus we prove (4.38).

Case 2: $k > \frac{n}{2}$

We consider the function $H := x \cdot Du(x) - b_2(u-1) - a_2$, where b_2 and a_2 are positive constants to be determined later with $a_2 < \frac{b_2}{2}$

For any $x \in \partial\Omega$, since $u(x) = 1$, we have $H(x) = x \cdot Du(x) - a_2 \geq c(x \cdot \nu(x)) - a_2 > 0$ if $a_2 < \min_{\partial\Omega}(x \cdot \nu(x))c$ is small enough.

For any $x \in \partial B_R$, recall the upper barrier $\rho^{\varepsilon,R}$ in (4.30) and $a^{\varepsilon,R}$ in (4.31), we have

$$\begin{aligned}
H(x) =& Ru_\nu - b_2\left(\left(R^2+\varepsilon^2\right)^{\frac{2k-n}{2k}} - \left(R_0^2+\varepsilon^2\right)^{\frac{2k-n}{2k}} - 1\right) - a_2 \\
\geq& R\rho_\nu^{\varepsilon,R} - b_2\left(R^2+\varepsilon^2\right)^{\frac{2k-n}{2k}} \\
=& \frac{2k-n}{k} a^{\varepsilon,R} R^{\frac{2k-n}{k}} - b_2\left(R^2+\varepsilon^2\right)^{\frac{2k-n}{2k}} \\
=& R^{\frac{2k-n}{k}}\left(\frac{2k-n}{k} a^{\varepsilon,R} - b_2\left(1+\varepsilon^2 R^{-2}\right)^{\frac{2k-n}{2k}}\right).
\end{aligned}$$

If we take $b_2 = \frac{2k-n}{k} - \frac{1}{2k} > 0$ (since $2k - n \geq 1$), the above is positive since $a^{\varepsilon,R}$ is close to 1 for R sufficiently large. For such b_2, we have

$$\begin{aligned}
F^{ij}H_{ij} =& (|x|^2+\varepsilon^2)^{-1}\left(\left((2-b_2)k-(n+2)\right)|x|^2 + (2-b_2)k\varepsilon^2\right) \\
\leq& (|x|^2+\varepsilon^2)^{-1}\left(-|x|^2 + (n+1)\varepsilon^2\right) \\
<& 0, \qquad (4.45)
\end{aligned}$$

where we assume ε small enough (note that $|x| \geq r_0$ for $x \in \Omega^c$).

By maximum principle, we have $H > \min_{\partial\Omega_R} H > 0$. Thus we get for any $x \in \overline{\Omega}_R$,

$$\begin{aligned}
x \cdot Du(x) \geq& b_2(u-1) + a_2 \\
\geq& \frac{a_2}{2}u \geq \frac{a_2}{4(1+C)}|x|^{\frac{2k-n}{k}},
\end{aligned}$$

where we use $u \geq \max\{|x|^{\frac{2k-n}{k}} - C, 1\}$.

Case 3: $k = \frac{n}{2}$

We consider $H = x \cdot Du(x) - b_3$ which is positive on the boundary of $\overline{\Omega}_R$ if we take b_3 small enough. Since $F^{ij}H_{ij} = (|x|^2+\varepsilon^2)^{-1}(-2|x|^2 + n\varepsilon^2) < 0$ for ε small enough, we have $H = x \cdot Du(x) - b_3 > 0$ in $\overline{\Omega}_R$ and we can get the desired estimate.

□

4.6. **Second order estimates.** We will prove the second order estimate of the approximating equations.

4.6.1. ***The global second order estimate can be reduced to the boundary second order estimate***.

Theorem 4.9. *Let u be the k-convex solution of (4.3) or (4.6) or (4.10) and consider $\tilde{G} = u_{\xi\xi}\varphi(P)h(u)$, then we have*

$$\max_{B_R\setminus\Omega} \tilde{G} \le C + \max_{\Gamma_R} \tilde{G}. \tag{4.46}$$

where $\Gamma_R := \partial(B_R \setminus \Omega)$, $\varphi(t)$ and h are defined by

$$\varphi(t) = \begin{cases} (M-t)^{-\tau}, k < n, \\ \quad 1, \qquad k = n, \end{cases} \tag{4.47}$$

where $M := 2\max P + 1$ and τ is a uniform positive constant determined in (4.74) (if $k \neq \frac{n}{2}$) and (4.75) (if $k = \frac{n}{2}$). h is defined by

$$h(u) = \begin{cases} e^{2u}, & k = \frac{n}{2}, \\ u^{\frac{n}{2k-n}}, & k > \frac{n}{2}, \\ (-u)^{-\frac{n}{n-2k}}, & k < \frac{n}{2}. \end{cases}$$

Proof. For simplicity, we write f instead of $f^{1,\varepsilon}$, $f^{2,\varepsilon}$ or $f^{3,\varepsilon}$ during the proof.
We rewrite the equation as

$$F(D^2u) = \log S_k(D^2u) = \tilde{f},$$

where $\tilde{f} = \log f$ satisfying

$$|D\tilde{f}|^2 + |D^2\tilde{f}| \le C|x|^{-2}. \tag{4.48}$$

Now we are ready to prove the second order estimate, we first recall

$$P = |Du|^2\tilde{g}(u) := \begin{cases} |Du|^2 e^{2u}, & k = \frac{n}{2}, \\ |Du|^2 u^{\frac{2(n-k)}{2k-n}}, & k > \frac{n}{2}, \\ |Du|^2(-u)^{-\frac{2(n-k)}{n-2k}}, & k < \frac{n}{2}. \end{cases}$$

Direct manipulation shows that

$$(\log h)' = \begin{cases} 2, & k = \frac{n}{2}, \\ \frac{n}{2k-n}u^{-1}, & k > \frac{n}{2}, \\ \frac{n}{n-2k}(-u)^{-1}, & k < \frac{n}{2}, \end{cases}$$

$$h((\log h)')^{-1} \le C|x|^2, \tag{4.49}$$

moreover,

$$\tilde{g} = c_{n,k}h(\log h)', \tag{4.50}$$

where $c_{n,k} = 2$ when $k = \frac{n}{2}$ and $c_{n,k} = \frac{n}{|n-2k|}$ when $k \neq \frac{n}{2}$.

Assume $\tilde{G}$ attains its maximum at $x_0 \in \Omega_R$ along the direction ξ_0. We choose the coordinate at $x_0 \in \Omega_R$ such that $D^2u(x_0) = \{\lambda_i\delta_{ij}\}$. Then one can check $\xi_0 = (1, 0\cdots, 0)$.

Then $G = \log u_{11} + \log \varphi(P) + \log h(u)$ attains its maximum at x_0.
Our goal is to prove the uniform upper bound of $\lambda_1 h$.

All the calculations are at x_0, fisrtly, we have

$$\begin{aligned} 0 = G_i =& \frac{u_{11i}}{u_{11}} + \frac{\varphi_i}{\varphi} + \frac{h_i}{h} \\ =& \frac{u_{11i}}{u_{11}} + \frac{\varphi'}{\varphi} P_i + \frac{h'}{h} u_i. \end{aligned}$$

From the above, we have

$$\frac{u_{11i}}{u_{11}} = -\frac{\varphi'}{\varphi} P_i - \frac{h'}{h} u_i, \tag{4.51}$$

$$\frac{h'}{h} u_i = -\frac{u_{11i}}{u_{11}} - \frac{\varphi'}{\varphi} P_i. \tag{4.52}$$

Differentiating G twice, we have

$$G_{ii} = \frac{u_{11ii}}{u_{11}} - \left(\frac{u_{11i}}{u_{11}}\right)^2 + \frac{\varphi'}{\varphi} P_{ii} + \left(\frac{\varphi''}{\varphi} - \left(\frac{\varphi'}{\varphi}\right)^2\right) |P_i|^2 + \frac{h''}{h} u_{ii} + \left(\frac{h''}{h} - \left(\frac{h'}{h}\right)^2\right) |u_i|^2. \tag{4.53}$$

Then we have

$$\begin{aligned} 0 \geq F^{ii} G_{ii} =& \frac{F^{ii} u_{11ii}}{u_{11}} - F^{ii} \left(\frac{u_{11i}}{u_{11}}\right)^2 + \frac{\varphi'}{\varphi} F^{ii} P_{ii} + \left(\frac{\varphi''}{\varphi} - \left(\frac{\varphi'}{\varphi}\right)^2\right) F^{ii} |P_i|^2 + \frac{h''}{h} F^{ii} u_{ii} + \left(\frac{h''}{h} - \left(\frac{h'}{h}\right)^2\right) F^{ii} |u_i|^2 \\ =& \frac{(\tilde{f})_{11} - F^{ij,rs} u_{ij1} u_{rs1}}{u_{11}} - \sum_{i=2}^n F^{ii} \left(\frac{u_{11i}}{u_{11}}\right)^2 - F^{11} \left(\frac{\varphi'}{\varphi} P_1 + \frac{h'}{h} u_1\right)^2 + \left(\frac{\varphi''}{\varphi} - \left(\frac{\varphi'}{\varphi}\right)^2\right) F^{ii} |P_i|^2 \\ &+ \frac{\varphi'}{\varphi} F^{ii} P_{ii} + \left(\frac{h''}{h} - \left(\frac{h'}{h}\right)^2\right) F^{ii} |u_i|^2 + \frac{h'}{h} F^{ii} u_{ii} \\ \geq& 2\lambda_1^{-1} f^{-1} \sum_{i=2}^n S_{k-2}(\lambda|1i) |u_{11i}|^2 - \sum_{i=2}^n F^{ii} \left(\frac{u_{11i}}{u_{11}}\right)^2 \\ &+ \left(\frac{\varphi''}{\varphi} - \left(\frac{\varphi'}{\varphi}\right)^2\right) \sum_{i=2}^n F^{ii} |P_i|^2 + \left(\frac{h''}{h} - \left(\frac{h'}{h}\right)^2\right) \sum_{i=2}^n F^{ii} |u_i|^2 \\ &+ \left(\frac{\varphi''}{\varphi} - 3\left(\frac{\varphi'}{\varphi}\right)^2\right) F^{11} |P_1|^2 + \left(\frac{h''}{h} - 3\left(\frac{h'}{h}\right)^2\right) F^{11} |u_1|^2 \\ &+ \frac{\varphi'}{\varphi} F^{ii} P_{ii} + \frac{kh'}{h} - \lambda_1^{-1} (\tilde{f})_{11}, \end{aligned} \tag{4.54}$$

where we use the concavity property of $\log S_k$.

We fist deal with the easy case: $k = n$. Note that in this case $\varphi = 1$, $h = u$ and $F^{ii} = \lambda_i^{-1}$. From (4.54), we have

$$0 \geq F^{ii}G_{ii} \geq nu^{-1} - 2\lambda_1^{-1}u^{-2}u_1^2 - \lambda_1^{-1}(\tilde{f})_{11}. \tag{4.55}$$

Multiply $\lambda_1 u^2$ in the above inequality, we have

$$\lambda_1 u \leq 2u_1^2 + u^2(\tilde{f})_{11} \leq C, \tag{4.56}$$

where we use $|Du| \leq C$, $u \leq C|x|$ and $|D^2\tilde{f}| \leq C|x|^{-2}$ (see (4.48)). Thus we finish the proof when $k = n$.

In the remaining proof, we always assume $k < n$.

We manipulate P_{ii} directly.

$$\begin{aligned} P_i =& 2\tilde{g}\sum_{l=1}^n u_l u_{li} + |Du|^2\tilde{g}'u_i, \\ P_{ii} =& 2\tilde{g}\sum_{l=1}^n u_l u_{lii} + 2\tilde{g}u_{ii}^2 + 4\tilde{g}'u_{ii}u_i^2 + |Du|^2\tilde{g}''u_i^2 + |Du|^2\tilde{g}'u_{ii} \end{aligned}$$

Thus we have

$$\begin{aligned} F^{ii}P_{ii} =& 2\tilde{g}\sum_{i=1}^n F^{ii}u_{ii}^2 + 4g'\sum_{i=1}^n F^{ii}u_{ii}u_i^2 + |Du|^2\tilde{g}''\sum_{i=1}^n F^{ii}u_i^2 + k\tilde{g}'|Du|^2 + 2\tilde{g}\sum_{i=1}^n u_i\tilde{f}_i \\ \geq& \tilde{g}\sum_{i=1}^n F^{ii}u_{ii}^2 - 4\tilde{g}^{-1}(\tilde{g}')^2\sum_{i=1}^n F^{ii}u_i^4 + |Du|^2|\tilde{g}''|\sum_{i=1}^n F^{ii}u_i^2 - 2\tilde{g}|Du||D\tilde{f}| \\ \geq& \tilde{g}\sum_{i=1}^n F^{ii}u_{ii}^2 - C((\log h)')^2|Du|^2\tilde{g}\sum_{i=1}^n F^{ii}u_i^2 - 2c_{n,k}h(\log h)'|Du||D\tilde{f}| \\ \geq& c_{n,k}h(\log h)'\sum_{i=1}^n F^{ii}u_{ii}^2 - C((\log h)')^2 P\sum_{i=1}^n F^{ii}u_i^2 - CP^{\frac{1}{2}}(\log h)', \end{aligned} \tag{4.57}$$

where we have used $P = |Du|^2\tilde{g}$, $\tilde{g} = c_{n,k}h(\log h)'$ and $\tilde{g}^{\frac{1}{2}}|D\tilde{f}| \leq C(\log h)'$ which follows from $|D\tilde{f}| \leq C|x|^{-1}$ (see (4.48)) and $h((\log h)')^{-1} \leq C|x|^2$(see (4.49)).

Similar as Chou-Wang [CPAM, 2001], we now divide two cases to obtain the upper bound of $\lambda_1 h(u)$.

Case1: $\lambda_k \geq \delta\lambda_1$,

Since $\lambda_k \geq \delta\lambda_1$, there exists a constant θ such that $S_{k-1}(\lambda|k) \geq \theta S_{k-1}(\lambda)$, we have

$$\begin{aligned} \sum_{i=1}^n F^{ii}u_{ii}^2 \geq& F^{kk}u_{kk}^2 \geq \theta f^{-1}S_{k-1}(\lambda)u_{kk}^2 \\ \geq& \delta^2\theta f^{-1}S_{k-1}(\lambda)\lambda_1^2 = \tilde{\theta}f^{-1}S_{k-1}(\lambda)\lambda_1^2. \end{aligned}$$

Then by (4.57), we have

$$
\begin{aligned}
F^{ii}P_{ii} \geq & c_{n,k}h(\log h)'\tilde{\theta}f^{-1}S_{k-1}(\lambda)\lambda_1^2 - C((\log h)')^2PS_{k-1}(\lambda)|Du|^2 - CP^{\frac{1}{2}}(\log h)' \\
= & \frac{1}{2}c_{n,k}h(\log h)'\tilde{\theta}f^{-1}S_{k-1}(\lambda)\lambda_1^2 - C((\log h)')^2PS_{k-1}(\lambda)|Du|^2 \\
& + \frac{1}{2}c_{n,k}h(\log h)'\tilde{\theta}f^{-1}S_{k-1}(\lambda)\lambda_1^2 - CP^{\frac{1}{2}}(\log h)' \\
\geq & h^{-1}(\log h)'f^{-1}S_{k-1}(\lambda)(\frac{1}{2}c_{n,k}\tilde{\theta}(h\lambda_1)^2 - CP^2) \\
& + (\log h)'(\frac{1}{2}c_{n,k}\tilde{\theta}f^{-1}S_{k-1}\lambda_1(h\lambda_1) - CP^{\frac{1}{2}}) \\
\geq & (\log h)'(\frac{1}{2}c_{n,k}\tilde{\theta}f^{-1}S_{k-1}\lambda_1(h\lambda_1) - CP^{\frac{1}{2}}) \\
\geq & (\log h)'\frac{1}{4n}c_{n,k}\tilde{\theta}f^{-1}S_{k-1}\lambda_1(h\lambda_1) \qquad (4.58) \\
= & : c_0(\log h)'f^{-1}S_{k-1}\lambda_1(h\lambda_1),
\end{aligned}
$$

where we use the following Maclaurin inequality in (4.58)

$$
\frac{S_k(\lambda)}{S_{k-1}(\lambda)} \leq \frac{n-k+1}{nk}S_1(\lambda),
$$

and we have assumed

$$
(\lambda_1 h)^2 > 4c_{n,k}^{-1}\tilde{\theta}^{-1}CP^2 =: C_1, \qquad (4.59)
$$

$$
\lambda_1 h \geq 8c_{n,k}^{-1}\tilde{\theta}^{-1}CP^{\frac{1}{2}} := C_2^{\frac{1}{2}}. \qquad (4.60)
$$

In this case, inserting (4.51) into (4.54), we have

$$
\begin{aligned}
0 \geq F^{ii}G_{ii} \geq & \Big(\frac{\varphi''}{\varphi} - 3\Big(\frac{\varphi'}{\varphi}\Big)^2\Big)\sum_{i=1}^n F^{ii}|P_i|^2 + \Big(\frac{h''}{h} - 3\Big(\frac{h'}{h}\Big)^2\Big)\sum_{i=1}^n F^{ii}|u_i|^2 \\
& + \frac{\varphi'}{\varphi}F^{ii}P_{ii} + \frac{kh'}{h} - \lambda_1^{-1}(\tilde{f})_{11} \\
\geq & \Big(\frac{h''}{h} - 3\Big(\frac{h'}{h}\Big)^2\Big)\sum_{i=1}^n F^{ii}|u_i|^2 + \frac{\varphi'}{\varphi}F^{ii}P_{ii} + \frac{kh'}{h} - \lambda_1^{-1}(\tilde{f})_{11}. \qquad (4.61)
\end{aligned}
$$

Combining (4.61) with (4.58) and note that $\left|\frac{h''}{h}-3(\frac{h'}{h})^2\right|\le C((\log h)')^2$, we have

$$
\begin{aligned}
0 \ge& F^{ii}G_{ii} \ge \left(\frac{h''}{h}-3\left(\frac{h'}{h}\right)^2\right)\sum_{i=1}^n F^{ii}|u_i|^2 + c_0\frac{\varphi'}{\varphi}(\log h)' f^{-1}S_{k-1}\lambda_1(h\lambda_1) + k(\log h)' - \lambda_1^{-1}(\tilde f)_{11}\\
\ge& -Cf^{-1}S_{k-1}(\lambda)|Du|^2((\log h)')^2 + c_0\frac{\tau}{M-P}(\log h)' f^{-1}S_{k-1}\lambda_1(\lambda_1 h) - \lambda_1^{-1}\tilde f_{11}\\
=& h^{-1}(\log h)' f^{-1}S_{k-1}(\lambda)\left(\frac{c_0\tau}{M-P}(\lambda_1 h)^2 - CP\right) - \lambda_1^{-1}\tilde f_{11}\\
\ge& \frac12 h^{-1}(\log h)' f^{-1}S_{k-1}(\lambda)c_0\frac{\tau}{M-P}(\lambda_1 h)^2 - \lambda_1^{-1}\tilde f_{11},
\end{aligned}
\tag{4.62}
$$

where we have assumed

$$
(\lambda_1 h)^2 \ge 2c_0^{-1}\tau^{-1}(M-P)CP =: C_3. \tag{4.63}
$$

Since $\frac{S_k(\lambda)}{S_{k-1}(\lambda)} \le \frac{n-k+1}{nk}S_1(\lambda)$, by (4.62), we have

$$
(\lambda_1 h)^2 \le Ch((\log h)')^{-1}\tilde f_{11}. \tag{4.64}
$$

In conclusion, by (4.59), (4.60), (4.63) and (4.64), we obtain an upper bound of $\lambda_1 u^b$ as follows

$$
(\lambda_1 u^b)^2 \le C_1 + C_2 + C_3 + Ch((\log h)')^{-1}\tilde f_{11} \le \tilde C, \tag{4.65}
$$

where we use $|D^2\tilde f| \le C|x|^{-2}$ (see (4.48)) and $h((\log h)')^{-1} \le C|x|^2$ (see (4.49)).

Case2: $\lambda_k \le \delta\lambda_1$,

Since $\lambda_k + \lambda_{k+1} + \cdots \lambda_n = S_1(\lambda_1|12\ldots,k-1) > 0$, we have $-\lambda_n \le (n-k)\lambda_k < n\delta\lambda_1$, thus $|\lambda_i| < n\delta\lambda_1, i = k+1,\ldots,n.$

Inserting (4.57) into (4.54), we obtain

$$
\begin{aligned}
0 \ge& F^{ii}G_{ii} \ge 2\lambda_1^{-1}f^{-1}\sum_{i=2}^n S_{k-2}(\lambda|1i)|u_{11i}|^2 - \sum_{i=2}^n F^{ii}\left(\frac{u_{11i}}{u_{11}}\right)^2\\
&+\left(\frac{\varphi''}{\varphi}-\left(\frac{\varphi'}{\varphi}\right)^2\right)\sum_{i=2}^n F^{ii}|P_i|^2 + \left(\frac{h''}{h}-\left(\frac{h'}{h}\right)^2\right)\sum_{i=2}^n F^{ii}|u_i|^2\\
&+\left(\frac{\varphi''}{\varphi}-3\left(\frac{\varphi'}{\varphi}\right)^2\right)F^{11}|P_1|^2 + \left(\frac{h''}{h}-3\left(\frac{h'}{h}\right)^2\right)F^{11}|u_1|^2\\
&+\frac{\varphi'}{\varphi}c_{n,k}h(\log h)'\sum_{i=1}^n F^{ii}u_{ii}^2 - C\frac{\varphi'}{\varphi}((\log h)')^2 P\sum_{i=1}^n F^{ii}u_i^2 - C\frac{\varphi'}{\varphi}P^{\frac12}(\log h)' - \lambda_1^{-1}(\tilde f)_{11}.
\end{aligned}
\tag{4.66}
$$

Claim: *We claim that for sufficiently small* $\tau > 0$*, there exists sufficiently small* $\delta > 0$ *such that the following holds*

$$\begin{aligned}(*) := {} & 2\lambda_1^{-1} f^{-1} \sum_{i=2}^{n} S_{k-2}(\lambda|1i)|u_{11i}|^2 - \sum_{i=2}^{n} F^{ii}\left(\frac{u_{11i}}{u_{11}}\right)^2 + \left(\frac{\varphi''}{\varphi} - \left(\frac{\varphi'}{\varphi}\right)^2\right)\sum_{i=2}^{n} F^{ii}|P_i|^2 \\ & + (\log h)'' \sum_{i=2}^{n} F^{ii}|u_i|^2 - C\frac{\varphi'}{\varphi}((\log h)')^2 P \sum_{i=2}^{n} F^{ii}u_i^2 \\ \geq {} & 0.\end{aligned} \tag{4.67}$$

We will prove the estimate of $\lambda_1 h$ based on the claim above. Indeed, by (4.67) and (4.66), we have

$$\begin{aligned}0 \geq {} & F^{ii}P_{ii} \geq (\log\varphi)' c_{n,k} h(\log h)' \sum_{i=1}^{n} F^{ii}u_{ii}^2 \\ & - C(\log\varphi)'((\log h)')^2 F^{11}u_1^2 - C(\log\varphi)'(\log h)' - \lambda_1^{-1}\tilde{f}_{11} \\ \geq {} & h^{-1}(\log\varphi)'(\log h)' F^{11}\left(\frac{c_{n,k}}{2}(\lambda_1 h)^2 - Ch(\log h)' u_1^2\right) \\ & + (\log\varphi)'\frac{c_{n,k}}{2} h(\log h)' \sum_{i=1}^{n} F^{ii}u_{ii}^2 - C(\log\varphi)'(\log h)' - \lambda_1^{-1}\tilde{f}_{11} \\ \geq {} & h^{-1}(\log\varphi)'(\log h)' F^{11}\left(\frac{c_{n,k}}{2}(\lambda_1 h)^2 - CP\right) \\ & + c_0(\log\varphi)'(\log h)'(\lambda_1 h) - C(\log\varphi)'(\log h)' - \lambda_1^{-1}\tilde{f}_{11} \\ \geq {} & \frac{1}{2}c_0(\log\varphi)'(\log h)'(\lambda_1 h) - \lambda_1^{-1}\tilde{f}_{11},\end{aligned} \tag{4.68}$$

where we use the Maclaurin inequality and assume $\lambda_1 h$ is sufficiently large. From (4.68), we obtain

$$(\lambda_1 h)^2 \leq Ch((\log h)')^{-1}\tilde{f}_{11} \leq \tilde{C}, \tag{4.69}$$

where we use $|D^2\tilde{f}| \leq C|x|^{-2}$ (see (4.48)) and $|h((\log h)')^{-1} \leq x|^2$ (see (4.49)).

Now we prove the Claim (4.67).

Proof of the Claim (4.67)*:* By Page1037 (3.5) in Chou-Wang [11], for any sufficiently small $\epsilon_0 > 0$, there exists $\delta > 0$ such that

$$2\lambda_1 S_{k-2}(\lambda|1i) - (2-\epsilon_0)S_{k-1}(\lambda|i) > 0. \tag{4.70}$$

Now we use (4.70) to prove the claim by choosing appropriate ϵ_0 and δ. Firstly, by (4.52), we have

$$\begin{aligned}\sum_{i=2}^{n} F^{ii}|(\log h)_i|^2 =& -\sum_{i=2}^{n}|\frac{u_{11i}}{u_{11}} + (\log\varphi)_i|^2\\ \geq& -2\sum_{i=2}^{n}|\frac{u_{11i}}{u_{11}}|^2 - 2\sum_{i=2}^{n}|(\log\varphi)_i|^2.\end{aligned} \tag{4.71}$$

Subcase 1: $k < \frac{n}{2}$ or $\frac{n}{2} < k < n$.

We deal with the term $(\log h)'' \sum\limits_{i=2}^{n} F^{ii}u_i^2$ as follows

$$\begin{aligned}(\log h)'' \sum_{i=2}^{n} F^{ii}u_i^2 =& (\log h)''((\log h)')^{-2}\sum_{i=2}^{n}|\frac{u_{11i}}{u_{11}} + (\log\varphi)_i|^2 = -\frac{|n-2k|}{n}\sum_{i=2}^{n}|\frac{u_{11i}}{u_{11}} + (\log\varphi)_i|^2\\ \geq& -(1+\tau_1)\frac{|n-2k|}{n}\sum_{i=2}^{n}|\frac{u_{11i}}{u_{11}}|^2 - (1+\tau_1^{-1})\frac{|n-2k|}{n}\sum_{i=2}^{n}|(\log\varphi)_i|^2,\end{aligned} \tag{4.72}$$

where $\tau_1 > 0$ is a sufficiently small constant.
Inserting (4.71) and (4.72) into (4.67), we have

$$\begin{aligned}(*) \geq& \left(1 - \epsilon_0 - (1+\tau_1)\frac{|n-2k|}{n} - 2\tau CP(M-P)^{-1}\right)|\frac{u_{11i}}{u_{11}}|^2\\ &+ \left(\frac{\varphi''}{\varphi} - \left(1 + (1+\tau_1^{-1})\frac{|n-2k|}{n} + 2\tau CP(M-P)^{-1}\right)((\log\varphi)')^2\right)F^{ii}|P_i|^2\\ \geq& \left(1 - \epsilon_0 - (1+\tau_1)\frac{|n-2k|}{n} - 2C\tau\right)|\frac{u_{11i}}{u_{11}}|^2\\ &+ \left(\frac{\varphi''}{\varphi} - \left(1 + (1+\tau_1^{-1})\frac{|n-2k|}{n} + 2C\tau\right)((\log\varphi)')^2\right)F^{ii}|P_i|^2.\end{aligned} \tag{4.73}$$

Since $|n-2k| < n$, we can choose τ_1 such that $b = (1+\tau_1)\frac{|n-2k|}{n} < 1$ and then we choose $\epsilon_0 = \frac{1-b}{2}$ and $M > \frac{2C_1}{1-b}$ such that the first term of (4.73) is nonnegative. At last, since $\frac{\varphi''}{\varphi} = (1+\tau^{-1})((\log\varphi)')^2$, the second term of (4.73) is nonnegative if we choose positive small constant τ as follows

$$\tau < \min\left\{\frac{1-b}{4C}, \frac{n}{(1+\tau_1^{-1})|n-2k| + 2nC}\right\}. \tag{4.74}$$

Subcase 2: $k = \frac{n}{2}$.
In this case $(\log h)'' = 0$, inserting (4.71) into (4.67) and using (4.70) with $\epsilon_0 = \frac{1}{2}$, we have

$$(*) \geq \left(\frac{1}{2} - 2\tau CP(M-P)^{-1}\right)|\frac{u_{11i}}{u_{11}}|^2 + \left(\frac{\varphi''}{\varphi} - (1 + 2\tau CP(M-P)^{-1})((\log\varphi)')^2\right)F^{ii}|P_i|^2$$
$$\geq 0,$$

where we choose the positive small constant τ satisfying

$$\tau < \frac{1}{4C+4}. \tag{4.75}$$

Then we finish the proof of the Claim.

Combining (4.65) with (4.69), we get the estimate of $\lambda_1 h$.

□

4.6.2. ***Second order estimate on the boundary*** $\partial\Omega_R$. .

Step1: tangential derivative estimates.

We first prove the tangential derivative estimate on $\partial\Omega$. For any $x_0 \in \partial\Omega$, we choose the coordinate such that $x_0 = 0$, $\partial\Omega \bigcap B_\delta(0) = (x', \rho(x'))$, $\rho(0) = 0$ and $\nabla\rho(0) = 0$. Since $u(x', \rho(x'))$ is constant, we have

$$|u_{\alpha\beta}(0)| = |u_n\rho_{\alpha\beta}(0)| \leq C|Du|(0) \leq C.$$

Next we can prove $S_{k-1}(u_{\alpha\beta}(0)) \geq c_1 > 0$ as that in Section 3 (see (3.11))

For any $x_0 \in \partial B_R$, we choose the coordinate such that $x_0 = (0, \cdots, 0, -R)$, then near x_0, ∂B_R is locally represented by $x_n = -(R^2 - |x'|^2)^{\frac{1}{2}}$.

Since $u|_{\partial B_R} = constant$, we have

$$\begin{aligned} u_{\alpha\beta}(x_0) =& - u_n(x_0)\frac{\partial^2 x_n}{\partial x_\alpha \partial x_\beta}(x_0) = -R^{-1}u_n(x_0)\delta_{\alpha\beta} \\ =& R^{-1}u_\nu(x_0)\delta_{\alpha\beta}. \end{aligned} \tag{4.76}$$

Since we have the boundary gradient estimate on ∂B_R (see (4.26), (4.32) and (4.36)),

$$CR^{-\frac{n-k}{k}} \geq u_\nu(x) \geq cR^{-\frac{n-k}{k}},$$

then by (4.76), we have

$$|u_{\alpha\beta}(x_0)| \leq CR^{-\frac{n}{k}} \tag{4.77}$$

$$\{u_{\alpha\beta}(x_0)\} \geq cR^{-\frac{n}{k}}\{\delta_{\alpha\beta}\}. \tag{4.78}$$

Step2: tangential-normal derivative estimates $\partial\Omega_R$

For any $x_0 \in \partial B_R$, choose the coordinate such that $y_0 = (0, \cdots, 0, -R)$, $\partial B_R \cap B_{\frac{1}{2}R}(y_0)$ is represented by

$$x_n = \rho(x') = -(R^2 - |x'|^2)^{\frac{1}{2}},$$

Consider the tangential operator $T_\alpha = (x_\alpha\partial_n - x_n\partial_\alpha), 1 \le \alpha \le n-1$. Since $u(x', \rho(x'))$ is constant, we have

$$0 = u_\alpha + u_n\rho_\alpha = u_\alpha - x_\alpha\rho^{-1}u_n$$

Then on $\partial B_1 \cap B_\delta(y_0)$, we have

$$T_\alpha u = x_\alpha u_n - \rho u_\alpha = 0.$$

We consider the function

$$w = A_1(R^{\frac{n-2k}{k}}(u - u|_{\partial B_R}) - 2v^R) + A_1R^{-2}|x - x_0|^2 \pm R^{\frac{n-2k}{k}}T_\alpha u \text{ in } B_R \cap B_{\frac{R}{2}}(x_0),$$

where $v^R(x) = \frac{|x|^2}{R^2} - 1$. Obviously, $w(x_0) = 0$. Since $T_\alpha u = 0$ on $B_R \setminus B_{\frac{1}{2}}(x_0)$, we have $w|_{B_R\setminus B_{\frac{1}{2}}(x_0)} \ge 0$.

Since $R^{\frac{n-2k}{k}}|T_\alpha u| \le C_1R^{\frac{n-2k}{k}}|x||Du| \le C$, choosing $A_1 > 4C$, we have

$$w \ge \frac{1}{4}A_1 - C > 0 \quad \text{on} \quad \partial(B_1 \cap B_{\frac{1}{2}R}(y_0)) \tag{4.79}$$

Next we show $F^{ij}w_{ij} < 0$ if we choose ε is small enough. Indeed, firstly, recall $\tilde{f} = \log f = -(\frac{n}{2} + 1)\log(|x|^2 + \varepsilon) + C$, then we have

$$F^{ij}(T_\alpha u)_{ij} = x_\alpha\tilde{f}_n - x_n\tilde{f}_\alpha = 0. \tag{4.80}$$

By the concavity of $\log S_k$, we have

$$\begin{aligned} F^{ij}\left(R^{\frac{n-2k}{k}}u - v^R\right)_{ij} \le& F(R^{\frac{n-2k}{k}}D^2u) - F(D^2v^R) \\ \le& 2\log\varepsilon - (2k+1)\log R + C + 2k\log R \\ =& 2\log\varepsilon - k\log R + C \\ <& 0, \end{aligned}$$

where we require ε small enough. Thus we have

$$\begin{aligned} F^{ij}w_{ij} =& A_1F^{ij}\left(R^{\frac{n-2k}{k}}u - v^R\right)_{ij} - A_1F^{ij}v^R_{ij} + A_1R^{-2}F^{ij}(|x - x_0|^2)_{ij} \\ =& A_1F^{ij}\left(R^{\frac{n-2k}{k}}u - v^R\right)_{ij} \\ <& 0. \end{aligned}$$

By maximum principle, w attains its minimum 0 at x_0. Then we have

$$0 \ge w_n(x_0) = A_1(R^{\frac{n-2k}{k}}u_n(x_0) + 4R^{-1}) \pm R^{\frac{n-k}{k}}u_{\alpha n}(x_0).$$

Then $|u_{\alpha n}(x_0)| \le A_1R^{-1}|Du(x_0)| \le CR^{-\frac{n}{k}}$ and thus we have the uniform tangential-normal derivative estimates on ∂B_R.

For any x_0, since $S_k(D^2\underline{u}^{i,\varepsilon}) \ge \epsilon_0 > 0$ near $\partial\Omega$, we can prove the tangential-normal derivative estimates on $\partial\Omega$ similar as that in Section 3.

Step3: double normal derivative estimates $\partial\Omega_R$
We can choose the coordinate at x_0 such that $u_n(x_0) = |Du|$ and $\{u_{\alpha\beta}(x_0)\}_{1\le\alpha,\beta\le n-1}$ is diagonal.

If $x_0 \in \partial B_R$, we have

$$\begin{aligned} u_{nn}c_0R^{-\frac{n(k-1)}{k}} \le u_{nn}(x_0)S_{k-1}(u_{\alpha\beta}(x_0)) =& S_k(D^2u(x_0)) - S_k(u_{\alpha\beta}(x_0)) + \sum_{i=1}^{n-1} u_{in}^2 S_{k-2}(u_{\alpha\beta}) \\ \le& f + C_{n-2}^{k-2}M_{21}^{k-2}M_{22}^2R^{-n} \\ \le& CR^{-n}. \end{aligned}$$

This gives $u_{nn} \le CR^{-\frac{n}{k}}$. On the other hand, $u_{nn} \ge -\sum_{i=1}^{n-1} u_{ii} \ge -(n-1)M_{21}R^{-\frac{n}{k}}$. Then we have $|u_{nn}(x_0)| \le CR^{-\frac{n}{k}}$.

If $x_0 \in \partial\Omega$, since $S_{k-1}(u_{\alpha\beta}(x_0)) \ge c_1$ which can be proved similar as that in Section 3, then we have $|u_{nn}(x_0)| \le C$.

In conclusion, we obtain $|D^2u(x)| \le C|x|^{-\frac{n}{k}}$ on the boundary $\partial\Omega_R$ and thus $|D^2u|(x) \le C|x|^{-\frac{n}{k}}$ for any $x \in \overline{\Omega}_R$.

5. Proof of Theorem 1.1, Theorem 1.2 and Theorem 1.3

5.1. **Uniqueness.** The uniqueness follows from the comparison principle for k-convex solutions of the k-Hessian equation in bounded domains in Lemma 2.4 by Wang-Trudinger [31] (see also [30, 32]).

Case1: $k < \frac{n}{2}$
Let u_1, u_2 be solutions of the k-Hessian equation. For any $x_0 \in \Omega^c$, we want to prove $u_1(x_0) \ge u_2(x_0)$. Indeed, since $\lim_{|x|\to\infty} u_i = 0$, for any $\epsilon > 0$, there esists sufficiently large R such that $x_0 \in B_R(0)$ and $u_1 \ge u_2 - \epsilon$ on $\partial B_R(0)$. Note we also have $u_1 = u_2 = 0$ on $\partial\Omega$, by comparison theorem in Ω_R, we then have $u_1 \ge u_2 - \epsilon$ in Ω_R. Let ϵ go to 0, we have $u_1(x_0) \ge u_2(x_0)$ and thus $u_1 \ge u_2$ in Ω^c. Similarly, we can prove $u_2 \ge u_1$ in Ω^c. Then we have $u_1 = u_2$ in Ω^c and thus prove the uniqueness part.

Case 2: $k > \frac{n}{2}$
For any $x_0 \in \Omega^c$, we want to prove $u_1(x_0) \ge u_2(x_0)$. Indeed, for any $t \in (0, 1)$, since $u_1 - tu_2 = (1-t)|x|^{\frac{2k-n}{k}} + O(1)$ when $|x| \to \infty$, there exists sufficiently large R such that

$$x_0 \in B_R(0) \quad \text{and} \quad u_1 > tu_2 \quad \text{on} \quad \partial B_R(0) \tag{5.1}$$

Note that we also have $u_1 = 1 > tu_2$ on $\partial\Omega$. By comparison theorem, we then have $u_1 \ge tu_2$ in Ω_R. In particular $u_1(x_0) \ge tu_2(x_0)$. Let t tend to 1, we have $u_1(x_0) \ge u_2(x_0)$ and thus $u_1 \ge u_2$ in Ω^c.

Similarly, we have $u_2 \geq u_1$. Then we have $u_1 = u_2$ in Ω^c and thus prove the uniqueness part.

Case 3: $k = \frac{n}{2}$
Let $x_0 \in \Omega^c$. For any $t \in (0,1)$, since $u_1 - tu_2 = (1-t)\log|x| + O(1)$ when $|x| \to \infty$, there exists sufficiently large R such that

$$x_0 \in B_R(0) \quad \text{and} \quad u_1 > tu_2 \quad \text{on} \quad \partial B_R(0) \tag{5.2}$$

Since $u_1 = tu_2 = 0$ on $\partial\Omega$, by comparison theorem, we then have $u_1 \geq tu_2$ in $B_R(0)$. In particular $u_1(x_0) \geq tu_2(x_0)$. Let t tend to 1, we have $u_1(x_0) \geq u_2(x_0)$ and thus $u_1 \geq u_2$ in Ω^c.

Similarly, we have $u_2 \geq u_1$. Then we have $u_1 = u_2$ in Ω^c and thus prove the uniqueness part.

5.2. **Existence and $C^{1,1}$-estimates.** The existence follows from the uniform C^2-estimates for $u^{\varepsilon,R}$.

Case 1: $k < \frac{n}{2}$
For any fixed sufficiently small $\varepsilon > 0$, by Guan [14], $|u^{\varepsilon,R}|_{C^m(\Omega_{K_0})} \leq C(\epsilon, K_0, m)$ for any $K_0 > R_0$ and $m \geq 0$ (we always assume $\Omega \subset\subset B_{\frac{R_0}{2}}$). Then there exists a subsequence u^{ε,R_i} converging smoothly to a strictly k-convex u^ε in K and $u^\infty \in C^\infty(\Omega^c)$ satisfies

$$\begin{cases} S_k(D^2u^\varepsilon) = f^{1,\varepsilon} & \text{in } \Omega^c, \\ u^\varepsilon = -1, & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

Moreover, by Theorem 4.2, we get

$$\begin{cases} C^{-1}|x|^{-\frac{n-2k}{k}} \leq -u^\varepsilon(x) \leq C|x|^{-\frac{n-2k}{k}}, \\ C^{-1}|x|^{-\frac{n-k}{k}} \leq |Du^\varepsilon|(x) \leq C|x|^{-\frac{n-k}{k}}, \\ |D^2u^\varepsilon|(x) \leq C|x|^{-\frac{n}{k}}, \end{cases}$$

Thus there exits a subsequence u^{ϵ_i} converges to u in $C^{1,\alpha}_{loc}$ such that $u \in C^{1,1}(\Omega^c)$ is the k-convex solution of the k-Hessian equation (1.4) and satisfies the estimates (1.5).

Case 2: $k > \frac{n}{2}$
For any fixed sufficiently small $\varepsilon > 0$, by Guan [14], $|u^{\varepsilon,R}|_{C^m(\Omega_{K_0})} \leq C(\epsilon, K_0, m)$ for any $K_0 > R_0$ and $m \geq 0$. Then there exists a subsequence u^{ε,R_i} converging smoothly to a strictly k-convex u^ε in K and $u^\varepsilon \in C^\infty(\Omega^c)$ satisfies

$$\begin{cases} S_k(D^2u^\varepsilon) = f^{2,\varepsilon} & \text{in } \Omega^c, \\ u^\varepsilon = 1, & \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

Moreover, by Theorem 4.4, we get

$$\begin{cases} |u^\varepsilon(x) - |x|^{\frac{2k-n}{k}}| \leq C, \\ C^{-1}|x|^{-\frac{n-k}{k}} |Du^\varepsilon|(x) \leq C|x|^{-\frac{n-k}{k}}, \\ |D^2u^\varepsilon|(x) \leq C|x|^{-\frac{n}{k}}, \end{cases}$$

Thus there exits a subsequence u^{ϵ_i} converges to u in $C^{1,\alpha}_{loc}$ such that $u \in C^{1,1}(\Omega^c)$ is the k-convex solution of the k-Hessian equation (1.6) and satisfies the estimates (1.7).

Case 3: $k = \frac{n}{2}$

For any fixed sufficiently small $\varepsilon > 0$, by Guan [14], $|u^{\varepsilon,R}|_{C^m(\Omega_{K_0})} \leq C(\epsilon, K_0, m)$ for any $K_0 > R_0$ and $m \geq 0$. Then there exists a subsequence u^{ε,R_i} converging smoothly to a strictly k-convex u^ε in K and $u^\varepsilon \in C^\infty(\Omega^c)$ satisfies

$$\begin{cases} S_k(D^2u^\varepsilon) = f^{3,\varepsilon} & \text{in } \Omega^c, \\ u^\varepsilon = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.5}$$

Moreover, by Theorem 4.6, we get

$$\begin{cases} |u^\varepsilon(x) - \log|x|| \leq C, \\ C^{-1}|x|^{-1} \leq |Du^\varepsilon|(x) \leq C|x|^{-1}, \\ |D^2u^\varepsilon|(x) \leq C|x|^{-2}, \end{cases}$$

Thus there exits a subsequence u^{ϵ_i} converges to u in $C^{1,\alpha}_{loc}$ such that $u \in C^{1,1}(\Omega^c)$ is the k-convex solution of the k-Hessian equation (1.8) and satisfies the estimates (1.9).

6. Almost monotonicity formula along the level set of the approximating solution

Agostiniani-Mazzieri [2] proved an monotonicity formula along the level set of the solution of the following problem

$$\begin{cases} \Delta u = 0 \text{ in } \Omega^c \\ u = -1 \text{ on } \partial\Omega \\ \lim_{|x|\to\infty} u(x) = 0. \end{cases} \tag{6.1}$$

In our setting, note that u is only $C^{1,1}$, we consider similar quantity on the level set of u^ε since u^ε is smooth and $|Du^\varepsilon| \equiv |x|^{1-\frac{n}{k}}$.

Firstly, as an application of the C^0 estimates of u^ϵ, we prove the following property.

Lemma 6.1. *Assume $k < \frac{n}{2}$.*

$$\lim_{\epsilon\to 0} \int_{R^n\setminus\bar\Omega} S_k^{ij}(D^2u^\epsilon)u_i^\epsilon u_j^\epsilon = \int_{\partial\Omega} |Du|^k H_{k-1}(\kappa) dA, \tag{6.2}$$

Remark 6.2. *We may call $\int_{\partial\Omega} |Du|^k S_{k-1}(\kappa)dA$ as the k-Capacity of Ω, since when $k = 1$, $Cap(\Omega) = \int_{\partial\Omega} |Du| dA$. The left hand side may be ∞ when $k \geq \frac{n}{2}$.*

Proof. Let $\Omega_t := \{x \in \Omega^c : u^\varepsilon < t\}$ and $S_t = \{x \in \Omega^c : u^\varepsilon(x) = t\}$ with $t \in [-1, 0)$.

Since $|Du^\varepsilon| > 0$, the level set $S_t := \{u^\varepsilon = t\}$ with $t \in [-1, 0)$ is a smooth closed hypersurface. Let ν be the outward unit normal vector of S_t and then we have $\nu = \frac{Du}{|Du|}$. By the divergence free property of the k-Hessian operator: $D_i S^{ij} = 0$, we have,

$$
\begin{aligned}
&\int_{\partial\Omega_t} S_k^{ij}(D^2u^\varepsilon)u_j^\varepsilon \nu_i - \int_{\partial\Omega} S_k^{ij}(D^2u^\varepsilon)u_j^\varepsilon \nu_i \\
&= \int_{\Omega_t\setminus\bar\Omega} D_i\left(S_k^{ij}(D^2u^\varepsilon)u_j^\varepsilon\right)dx = \int_{\Omega_t\setminus\bar\Omega} S_k^{ij}(D^2u^\varepsilon)u_{ij}^\varepsilon dx \\
&= \int_{S_t\setminus\bar\Omega} kS_k(D^2u^\varepsilon)dx = kc_{n,k}\varepsilon^2 \int_{\Omega_t\setminus\bar\Omega} \left(|x|^2+\varepsilon^2\right)^{-\frac{n}{2}-1}.
\end{aligned}
\tag{6.3}
$$

Since $C^{-1}|x|^{-\frac{n-2k}{k}} \le |u^\varepsilon| \le C|x|^{-\frac{n-2k}{k}}$ in Ω^c, for any $t \in [-1, 0)$,

$$
B_{C^{-1}t^{-\frac{k}{n-2k}}} \subset \Omega_t \subset B_{Ct^{-\frac{k}{n-2k}}},
\tag{6.4}
$$

then we have

$$
\left|\int_{S_t} S_k^{ij}(D^2u)u_j^\varepsilon \nu_i - \int_{\partial\Omega} S_k^{ij}(D^2u^\varepsilon)u_j^\varepsilon \nu_i\right| \le C\varepsilon^2(r_0^{-2} - |t|^{\frac{2k}{n-2k}}).
\tag{6.5}
$$

Thus for any $t \in [-1, 0)$, we have

$$
\int_{S_t} S_k^{ij}(D^2u^\varepsilon)u_j^\varepsilon \nu_i = \int_{\partial\Omega} S_k^{ij}(D^2u^\varepsilon)u_j^\varepsilon \nu_i + O(\varepsilon^2)(r_0^{-2} - |t|^{\frac{2k}{n-2k}}).
\tag{6.6}
$$

Then by the coarea formula, we have

$$
\begin{aligned}
&\int_{R^n\setminus\bar\Omega} S_k^{ij}(D^2u^\varepsilon)u_i^\varepsilon u_j^\varepsilon \\
&= \int_{-1}^{0}\int_{S_t} S_k^{ij}(D^2u^\varepsilon)u_j^\varepsilon \frac{u_i^\varepsilon}{|Du^\epsilon|}dA(t)dt \ \ \textit{(Coarea Formula)} \\
&= \int_{\{u=-1\}=\partial\Omega} S_k^{ij}(D^2u^\varepsilon)u_j^\varepsilon \frac{u_i^\varepsilon}{|Du^\varepsilon|}dA + O(\varepsilon^2)(\textbf{\textit{By}}\ (6.6)) \\
&= \int_{\partial\Omega} |Du^\varepsilon|^k H_{k-1}(\kappa)dA + O(\varepsilon^2),
\end{aligned}
\tag{6.7}
$$

where we use $H_{k-1}(\kappa) = |Du^\varepsilon|^{-k-1}S_k^{ij}(D^2u^\varepsilon)u_i^\varepsilon u_j^\varepsilon$.

Let ε tend to 0 and note that $|Du^\varepsilon|$ tends to $|Du|$, we have

$$
\lim_{\varepsilon\to 0}\int_{R^n\setminus\bar\Omega} S_k^{ij}(D^2u^\varepsilon)u_i^\varepsilon u_j^\epsilon = \int_{\partial\Omega} |Du|^k H_{k-1}(\kappa)dA.
$$

□

By the uniform C^2 estimates and positive lower bound of u^ϵ, we can estimate $|S_t|$, where $S_t = \{x \in \mathbb{R}^n \setminus \Omega : u(x) = t\}$.

Lemma 6.3. *There exits uniform constant C such that*

$$|S_t| \leq \begin{cases} C|t|^{-\frac{k(n-1)}{n-2k}} \quad \text{for any} \quad t \in (0,-1] \text{ if } k < \frac{n}{2}, \\ C|t|^{\frac{k(n-1)}{2k-n}} \quad \text{for any} \quad t \in [1,\infty) \text{ if } k > \frac{n}{2}, \\ Ce^{(n-1)t} \text{ for any} \quad t \in [0,\infty) \text{ if } k = \frac{n}{2}. \end{cases} \tag{6.8}$$

Proof. Since

$$|S_t| - |\partial\Omega| = \int_{\Omega_t} \mathrm{div}\Big(\frac{u_i^\varepsilon}{|Du^\varepsilon|}\Big) dx$$

By the uniform C^2-estimates and the uniform lower bound of u^ε, we finish the proof. □

We define the following quantity

$$I_{a,b,k}(t) := \int_{S_t} g^a(u^\varepsilon)|Du^\varepsilon|^{b-k} S_k^{ij}(D^2u^\varepsilon)u_i^\varepsilon u_j^\varepsilon, \tag{6.9}$$

where $g(u)$ s defined by

$$g(u) = \begin{cases} (-u)^{\frac{n-k}{2k-n}}, k < \frac{n}{2}, \\ u^{\frac{n-k}{2k-n}}, \quad k > \frac{n}{2}, \\ e^u, \quad k = \frac{n}{2}. \end{cases} \tag{6.10}$$

We choose $a = b - k + 1$ and one can see that $I_{a,b,k}(t)$ is uniformly bounded from the C^2 estimates of u^ε and the lower bound of $|Du^\varepsilon|$.
When $k = 1$ and $a = b$, $I_{a,b,k}(t)$ is exactly the one in [2].
We define

$$J_{a+a_0,b,k}(t,t_0) := -g^{a_0}(t)I'_{a,b,k}(t) + g^{a_0}(t_0)I'_{a,b,k}(t_0) \tag{6.11}$$

. We prove the following useful inequalities along the level set of u^ε.

Lemma 6.4. *Let u^ε be the solution of the approximating k-Hessian equation with $a = b - k + 1$. We have the following inequalities*

$$\begin{aligned} J_{a+a_0,b,k}(t,t_0) \geq & -ba \int_t^{t_0} \int_{S_s} g^{a+a_0}|Du^\varepsilon|^{b-k-1} \frac{H_k}{H_{k-1}} S_k dA ds - (b+1) \int_{S_t} \Big(g^{a+a_0}|Du^\varepsilon|^{b-k} S_k\Big) dA \\ & + a \int_t^{t_0} \int_{S_s} g^{a+a_0}|Du^\varepsilon|^{b-1} H_{k-1}^{-1}\Big(c_{n,k}H_k^2 - (k+1)H_{k-1}H_{k+1}\Big) dA ds \\ & + a \int_t^{t_0} \int_{S_s} g^{a+a_0}|Du^\varepsilon|^{b-1} \mathcal{L}. \end{aligned} \tag{6.12}$$

where $a_0, b, c_{n,k} = \frac{k(n-k-1)}{n-k}$ *and the functions* $\mathcal{L}$ *are choosing as follows*

(i) If $1 \leq k < \frac{n}{2}$, *we require* $-1 \leq t < t_0 < 0$, $a_0 = -2\frac{n-2k}{n-k}, b \geq c_{n,k}$ *and* $\mathcal{L} = (b - c_{n,k})\Big(\frac{n-k}{n-2k}|D\log u^\varepsilon| - \frac{H_k}{H_{k-1}}\Big)^2$

(ii) If $k = \frac{n}{2}$, *we require* $0 \leq t < t_0 < \infty$, $a_0 = 0$ $b \geq \frac{n}{2} - 1$ *and* $\mathcal{L} = a\Big(|Du^\varepsilon| - \frac{H_k}{H_{k-1}}\Big)^2$.

(iii) If $n > k > \frac{n}{2}$, *we require* $1 \leq t < t_0 < \infty$ *and* $a_0 = 2\frac{2k-n}{n-k}$, $b \geq k-1$ *and* $\mathcal{L} = (b - c_{n,k})\Big(\frac{n-k}{n-2k}|D\log u^\varepsilon| - \frac{H_k}{H_{k-1}}\Big)^2$.

Proof. **For simplicity, we use u instead of u^ε and S_k intead of $S_k(D^2u^\varepsilon)$ during the proof.**

By the divergence theorem and the divergence free property of the k-Hessian operator i.e. $\sum_{j=1}^{n} D_j S_k^{ij} = 0$, we have

$$\begin{aligned}
I_{a,b,k}(t_0) - I_{a,b,k}(t) =& \int_{\Omega_{t_0}\setminus\Omega_t} D_j\left(g^a|Du|^{b+1-k}S_k^{ij}u_i\right)\\
=& a\int_{\Omega_{t_0}\setminus\overline{\Omega}} g^{a-1}g'|Du|^{b+1-k}S_k^{ij}u_iu_j\\
&+ (b+1-k)\int_{\Omega_{t_0}\setminus\overline{\Omega}} g^a|Du|^{b-k-1}S_k^{ij}u_iu_lu_{ij} + k\int_{\Omega_{t_0}\setminus\overline{\Omega}} g^a|Du|^{b+1-k}S_k\\
=& a\int_{\Omega_{t_0}\setminus\overline{\Omega}} g^{a-1}g'|Du|^{b+1-k}S_k^{ij}u_iu_j\\
&- (b+1-k)\int_{\Omega_{t_0}\setminus\overline{\Omega}} g^a|Du|^{b-k-1}S_{k+1}^{ij}u_iu_j + (b+1)\int_{\Omega_{t_0}\setminus\overline{\Omega}} g^a|Du|^{b+1-k}S_k\\
=& a\int_t^{t_0}\int_{S_s} g^{a-1}g'|Du|^{b-k}S_k^{ij}u_iu_j - (b+1-k)\int_t^{t_0}\int_{S_s} g^a|Du|^{b-k-2}S_{k+1}^{ij}u_iu_j\\
&+ (b+1)\int_t^{t_0}\int_{S_s} g^a|Du|^{b-k}S_k,
\end{aligned} \tag{6.13}$$

where we use $S_k^{ij}u_iu_lu_{lj} = |Du|^2S_k - S_{k+1}^{ij}u_iu_j$ and the coarea formula.

Then

The derivative of $I_{a,b,k}(t)$ is

$$\begin{aligned}
I'_{a,b,k}(t) =& a\int_{S_t} g^{a-1}g'|Du|^{b-k}S_k^{ij}u_iu_j\\
&- (b+1-k)\int_{S_t} g^a|Du|^{b-k-2}S_{k+1}^{ij}u_iu_j + E_{a,b,k}(t),
\end{aligned} \tag{6.14}$$

where $E_{a,b,k}(t) = (b+1)\int_{S_t} g^a|Du|^{b-k}S_k$.

Then we have

$$J_{a,b,k}(t,t_0) := -g^{a_0}(t)I'_{a,b,k}(t) + g^{a_0}(t_0)I'_{a,b,k}(t_0)$$
$$= a\int_{\Omega_{t_0}\setminus\Omega_t} D_j\Big(g^{a+a_0-1}g'|Du|^{b-k+1}S_k^{ij}u_i\Big)$$
$$- (b-k+1)(I_{a+a_0,b-1,k+1}(t_0) - I_{a+a_0,b-1,k+1}(t)) + E_{a+a_0,b,k}(t_0) - E_{a+a_0,b,k}(t) \tag{6.15}$$

Firstly we have

$$\int_{\Omega_{t_0}\setminus\overline{\Omega}_t} D_j\Big(g^{a+a_0-1}g'|Du|^{b-k+1}S_k^{ij}u_i\Big)dx$$
$$= \int_t^{t_0}\int_{S_s}\Big((g^{a+a_0-1}g')\Big)'|Du|^{b-k}S_k^{ij}u_iu_jdAds$$
$$+ (b-k+1)\int_t^{t_0}\int_{S_s}\Big(g^{a+a_0-1}g'|Du|^{b-k-2}S_k^{ij}u_iu_lu_{lj}\Big)dAds + k\int_t^{t_0}\int_{S_s}\Big(g^{a+a_0-1}g'|Du|^{b-k}S_k\Big)dAds$$
$$= \int_t^{t_0}\int_{S_s}\Big((g^{a+a_0-1}g')'|Du|^{b-k}S_k^{ij}u_iu_j\Big)dAds$$
$$- (b-k+1)\int_t^{t_0}\int_{S_s}\Big(g^{a+a_0-1}g'|Du|^{b-k-2}S_{k+1}^{ij}u_iu_j\Big)dAds + (b+1)\int_t^{t_0}\int_{S_s}\Big(g^{a+a_0-1}g'|Du|^{b-k}S_k\Big)dAds$$
$$= \int_t^{t_0}\int_{S_s}\Big((g^{a+a_0-1}g')'|Du|^{b+1}H_{k-1}\Big)dAds - (b-k+1)\int_t^{t_0}\int_{S_s}\Big(g^{a+a_0-1}g'|Du|^{b}H_k\Big)dAds$$
$$+ (b+1)\int_t^{t_0}\int_{S_s}\Big(g^{a+a_0-1}g'|Du|^{b-k}S_k\Big)dAds, \tag{6.16}$$

where we use the identity $H_{m-1}|Du|^{m+1} = S_m^{ij}u^iu^j$ for $m \in \{1,2,\cdots,n\}$(see Lemma 2.10).

For the term $I_{a+a_0,b-1,k+1}(t_0) - I_{a+a_0,b-1,k+1}(t)$, similar as the calculation of (6.13), we have

$$I_{a+a_0,b-1,k+1}(t_0) - I_{a+a_0,b-1,k+1}(t)$$
$$= (a+a_0)\int_t^{t_0}\int_{S_s} g^{a+a_0-1}g'|Du|^{b-k-2}S_{k+1}^{ij}u_iu_jdAds$$
$$- (b-1-k)\int_t^{t_0}\int_{S_s} g^{a+a_0}|Du|^{b-k-4}S_{k+2}^{ij}u_iu_j$$
$$+ b\int_t^{t_0}\int_{S_s} g^{a+a_0}|Du|^{b-k-2}S_{k+1}. \tag{6.17}$$

Next we deal with the term involving S_{k+1}. Choose the coordinate such that $u_n(x_0) = |Du|(x_0)$ and $\{u_{ij}(x_0))\}_{1\le i,j\le n-1} = \{\tilde{\lambda}_i\delta_{ij}\}_{1\le i,j\le n-1}$ is diagonal, we have

$$\begin{aligned} S_{k+1} =& u_{nn}S_k(\tilde{\lambda}) + S_{k+1}(\tilde{\lambda}) - \sum_{i=1}^{n-1} S_{k-1}(\tilde{\lambda}|i)u_{ni}^2 \\ S_k =& u_{nn}S_{k-1}(\tilde{\lambda}) + S_k(\tilde{\lambda}) - \sum_{i=1}^{n-1} S_{k-2}(\tilde{\lambda}|i)u_{ni}^2, \end{aligned}$$

where $\tilde{\lambda} = (\tilde{\lambda}_1, \cdots, \tilde{\lambda}_{n-1})$ and recall we use the notation $S_k = S_k(D^2u)$. Then we get

$$\begin{aligned} S_{k+1} =& \frac{S_k(\widetilde{\lambda})}{S_{k-1}(\widetilde{\lambda})}S_k - \frac{S_k^2(\widetilde{\lambda})}{S_{k-1}(\tilde{\lambda})} + \sum_{i=1}^{n-1} u_{ni}^2 \frac{S_k(\tilde{\lambda}|i)S_{k-2}(\tilde{\lambda}|i) - S_{k-1}^2(\tilde{\lambda}|i)}{S_{k-1}(\tilde{\lambda})} + S_{k+1}(\tilde{\lambda}) \\ \le& \frac{S_k(\tilde{\lambda})}{S_{k-1}(\widetilde{\lambda})}S_k - \frac{S_k^2(\widetilde{\lambda})}{S_{k-1}(\tilde{\lambda})} + S_{k+1}(\tilde{\lambda}), \end{aligned} \tag{6.18}$$

where we use the Newton's inequality (one can see the proof in [10]).
Inserting (6.18) into (6.19) and noting that $S_m(\tilde{\lambda}) = |Du|^{-2}S_{m+1}^{ij}u_iu_j = H_m|Du|^m$ is a global defined function, then we have

$$\begin{aligned} &I_{a+a_0,b-1,k+1}(t_0) - I_{a+a_0,b-1,k+1}(t) \\ &\le (a+a_0)\int_t^{t_0}\int_{S_s} g^{a+a_0-1}g'|Du|^bH_k dAds \\ &\quad + (k+1)\int_t^{t_0}\int_{S_s} g^{a+a_0}|Du|^{b-1}H_{k+1}dAds \\ &\quad - b\int_t^{t_0}\int_{S_s} g^{a+a_0}|Du|^{b-1}\frac{H_k^2}{H_{k-1}}dAds + b\int_t^{t_0}\int_{S_s} g^{a+a_0}|Du|^{b-k-1}\frac{H_k}{H_{k-1}}S_k dAds. \end{aligned} \tag{6.19}$$

Inserting (6.16) and (6.19) into (6.15), if $a = b-k+1 \ge 0$, we obtain

$$\begin{aligned} J_{a+a_0,b,k}(t,t_0) \ge& -ba\int_t^{t_0}\int_{S_s} g^{a+a_0}|Du|^{b-k-1}\frac{H_k}{H_{k-1}}S_k dAds - (b+1)\int_{S_t}\Big(g^{a+a_0}|Du|^{b-k}S_k\Big)dA \\ &+ a\int_t^{t_0}\int_{S_s} g^{a+a_0}|Du|^{b-1}H_{k-1}^{-1}\Big(c_{n,k}H_k^2 - (k+1)H_{k-1}H_{k+1}\Big)dAds \\ &+ a\int_t^{t_0}\int_{S_s} g^{a+a_0}|Du|^{b-1}H_{k-1}\mathcal{L}dAds, \end{aligned} \tag{6.20}$$

where the function $\mathcal{L}$ is defined by

$$\begin{aligned}\mathcal{L} =&(b-c_{n,k})\Big(\frac{H_k}{H_{k-1}}\Big)^2-(2a+a_0)(\log g)'|Du|\frac{H_k}{H_{k-1}}\\ &+\Big((\log g)''+(a+a_0)((\log g)')^2\Big)|Du|^2.\end{aligned}\tag{6.21}$$

Now we divide two cases to prove the $\mathcal{L}\geq 0$ under some restrictions on a and b.

Case1: $k<\frac{n}{2}$ **and** $\frac{n}{2}<k<n$**.**

We choose $c_{n,k}=\frac{k(n-k-1)}{n-k}$.

Note that $\log g=\frac{n-k}{2k-n}\log(-u)$, then

$$\begin{aligned}(\log g)''+(a+a_0)((\log g)')^2 =&\frac{n-k}{n-2k}u^{-2}+(a+a_0)(\frac{n-k}{n-2k})^2u^{-2}\\ =&(\frac{n-k}{n-2k})^2u^{-2}(\frac{n-2k}{n-k}+a+a_0)\\ =&(b-c_{n,k})(\frac{n-k}{n-2k})^2u^{-2},\end{aligned}\tag{6.22}$$

where we choose $a_0=-2\frac{n-2k}{n-k}$ and we use $a=b-k+1$. We also have

$$-(2a+a_0)(\log g)'=2\frac{n-k}{n-2k}(b-c_{n,k})u^{-1}.\tag{6.23}$$

By direct manipulation, we have

$$\mathcal{L}=(b-c_{n,k})\Big(\frac{n-k}{n-2k}|D\log u|-\frac{H_k}{H_{k-1}}\Big)^2.\tag{6.24}$$

Consequently, we obtain

$$\begin{aligned}J_{a+a_0,b,k}(t,t_0)\geq&-ba\int_t^{t_0}\int_{S_s}g^{a+a_0}|Du|^{b-k-1}\frac{H_k}{H_{k-1}}S_k dAds-\int_{S_t}\Big(g^{a+a_0}|Du|^{b-k}S_k\Big)dA\\ &+a\int_t^{t_0}\int_{S_s}g^{a+a_0}|Du|^{b-1}H_{k-1}^{-1}\Big(c_{n,k}H_k^2-(k+1)H_{k-1}H_{k+1}\Big)dAds\\ &+a(b-c_{n,k})\int_t^{t_0}\int_{S_s}g^{a+a_0}|Du|^{b-1}\Big(\frac{n-k}{n-2k}|D\log u|-\frac{H_k}{H_{k-1}}\Big)^2\end{aligned}\tag{6.25}$$

Case 2: $k=\frac{n}{2}$**.**

We have $c_{n,k}=\frac{n}{2}-1>0$. We require $b\geq\frac{n}{2}-1$, $a=b-\frac{n}{2}+1=b-c_{n,k}\geq 0$ and $a_0=0$. Since $g=e^u$ and thus $(a+a_0)^{-1}(g^{a+a_0})''=(a+a_0)g^{a+a_0}$. We obtain

$$\mathcal{L}=a\Big(|Du|-\frac{H_k}{H_{k-1}}\Big)^2.\tag{6.26}$$

□

From the above formula, we have the following almost monotonicity formula along the level set of u^ε and we prove the first part of Theorem 1.4.

Lemma 6.5. *Let u^ε be the solution of the approximating k-Hessian equation. Assume $k < \frac{n}{2}$ and $b \geq \frac{k(n-k-1)}{n-k}$, then for any $t \in [-1, 0)$, we have*

$$\frac{d}{dt} I_{a,b,k}(t) \leq C\varepsilon^2 |t|^{\frac{2k}{n-2k}-1}. \tag{6.27}$$

Consequently, for any $-1 \leq t \leq s < 0$,

$$I_{a,b,k}(s) - I_{a,b,k}(t) \leq C\varepsilon^2. \tag{6.28}$$

In particular, we have the following weighted inequality

$$\int_{\partial\Omega} |Du|^{b+1} H_{k-1} \leq \frac{n-2k}{n-k} \int_{\partial\Omega} |Du|^b H_k, \tag{6.29}$$

where u is the unique $C^{1,1}$ solution of the homogeneous k-Hessian equation (1.4).

Remark 6.6. *When $k = 1$,* (6.29) *was proved by Agostiniani- Mazzieri [2].*

Proof. By the Lemma 6.4, for any $-1 \leq t < t_0 < 0$, we have

$$\begin{aligned} -t^2 I'_{a,b,k}(t) + t_0^2 I'_{a,b,k}(t_0) & \\ \geq - ab \int_{\Omega_{t_0}\setminus\overline{\Omega}_t} & (-u^\varepsilon)^{a\frac{n-k}{2k-n}+2} |Du^\varepsilon|^{a-1} \frac{H_k}{H_{k-1}} S_k \\ - (b+1) \int_{S_t} & (-u^\varepsilon)^{a\frac{n-k}{2k-n}+2} |Du^\varepsilon|^{a-1} S_k. \end{aligned} \tag{6.30}$$

By the MacLaurin inequality: $\frac{H_k}{H_{k-1}} \leq \frac{C_{n-1}^k}{C_{n-1}^{k-1}} \left(\frac{H_{k-1}}{C_{n-1}^{k-1}}\right)^{\frac{1}{k-1}}$ and the uniform C^2-estimates of u^ε (we also use $|Du^\varepsilon| \geq c|x|^{1-\frac{n}{k}}$), for any $x \in \Omega_t^c$, we have

$$\begin{aligned} (-u^\varepsilon)^{a\frac{n-k}{2k-n}+2} |Du^\varepsilon|^{a-1} \frac{H_k}{H_{k-1}} S_k \leq & C(-u^\varepsilon)^{a\frac{n-k}{2k-n}+2} |Du^\varepsilon|^{a-1} H_{k-1}^{\frac{1}{k-1}} |x|^{-n-2} \\ \leq & C|x|^{a\frac{n-k}{k}+2\frac{2k-n}{k}} |x|^{(a-1)\frac{k-n}{k}} |x|^{-1} |x|^{-n-2} \\ = & C|x|^{-n-\frac{n}{k}}, \end{aligned}$$

then

$$\int_{\Omega_{t_0}\setminus\overline{\Omega}_t} (-u^\varepsilon)^{a\frac{n-k}{2k-n}+2} |Du^\varepsilon|^{a-2} \frac{H_k}{H_{k-1}} S_k \leq C\varepsilon^2 |t|^{\frac{n}{n-2k}}.$$

Similarly, we have

$$\int_{S_t} (-u^\varepsilon)^{a\frac{n-k}{2k-n}+2} |Du^\varepsilon|^{a-1} S_k \leq C\varepsilon^2 |t|^{\frac{n}{n-2k}},$$

where we use $|S_t| \leq C|t|^{-\frac{k(n-1)}{n-2k}}$ (see Lemma 6.3).

Thus we get

$$-t^2 I'_{a,b,k}(t) + t_0^2 I'_{a,b,k}(t_0) \geq -C\varepsilon^2 |t|^{\frac{n}{n-2k}}. \tag{6.31}$$

By the uniform C^2 estimates for u^ε and $|Du^\varepsilon| \geq c|x|^{1-\frac{n}{k}}$, we have for any $t_0 \in [-1, 0)$

$$t_0^2 |I'_{a,b,k}(t_0)\Big| \leq C|t_0|. \tag{6.32}$$

Let t_0 tend to 0 in (6.31), we have

$$I'_{a,b,k}(t) \leq C\varepsilon^2 |t|^{\frac{2k}{n-2k}-1}. \tag{6.33}$$

In particular, taking $t = -1$, we have

$$I'_{a,b,k}(-1) \leq C\varepsilon^2. \tag{6.34}$$

On the other hand, by (6.14), we have

$$I_{a,b,k}(-1) \geq a\frac{n-k}{n-2k}\int_{\partial\Omega} |Du^\varepsilon|^{b+1} H_{k-1} - a\int_{\partial\Omega} |Du^\varepsilon|^b H_k. \tag{6.35}$$

Consequently, we get

$$\frac{n-k}{n-2k}\int_{\partial\Omega} |Du^\varepsilon|^{b+1} H_{k-1} - \int_{\partial\Omega} |Du^\varepsilon|^b H_k \leq C\varepsilon^2 \tag{6.36}$$

Since $|Du^\varepsilon|$ converges to $|Du|$ on $\partial\Omega$, we finish the proof of (6.29) by taking $\varepsilon \to 0$ in (6.36). □

Next we prove the second part of Theorem 1.4.

Lemma 6.7. *Assume $k = \frac{n}{2}$ and $b > \frac{n}{2} - 1$. We have*

$$I'_{a,b,k}(t) \leq C\varepsilon^2 e^{-2t}, \tag{6.37}$$

In particular, we have

$$\int_{\partial\Omega} |Du|^{b+1} H_{k-1} \leq \int_{\partial\Omega} |Du|^b H_k, \tag{6.38}$$

where u is the unique $C^{1,1}$ solution the homogeneous k-Hessian equation (1.8).

Proof. By Lemma 6.4 and similar as the proof in the above lemma, for any $0 \leq t < t_0$, we have

$$I'_{a,b,k}(t_0) \geq I'_{a,b,k}(t) - C\varepsilon^2 e^{-2t}.$$

By integrating the above form t to t_0, we have

$$I_{a,b,k}(t_0) - I_{a,b,k}(t) \geq (I'_{a,b,k}(t) - C\varepsilon^2 e^{-2t})(t_0 - t),$$

Since $I_{a,b,k}(t)$ is uniformly bounded which follows from the C^2-estimates of u^ε and $|Du^\varepsilon| \geq c|x|^{1-\frac{n}{k}}$, we have

$$\Big(I'_{a,b,k}(t) - C\varepsilon^2 e^{-2t}\Big)(1 - tt_0^{-1}) \leq t_0^{-1}(I_{a,b,k}(t_0) - I_{a,b,k}(t)) \leq Ct_0^{-1}.$$

Let t_0 tend to 0, we obtain

$$I'_{a,b,k}(t) \leq C\varepsilon^2 e^{-2t}.$$

On the other hand, we have

$$I'_{a,b,k}(0) \geq a\int_{\partial\Omega} |Du^\varepsilon|^{b+1} H_{k-1} - a\int_{\partial\Omega} |Du^\varepsilon|^b H_k.$$

Combining the above two inequalities and noting that $|Du^\varepsilon| \to |Du|$, we get

$$\int_{\partial\Omega} |Du|^{b+1} H_{k-1} \leq \int_{\partial\Omega} |Du|^b H_k \tag{6.39}$$

□

When $\frac{n}{2} < k < n$, we have the following inequality.

Lemma 6.8. *Let u^ε be the solution of the approximating k-Hessian equation. Assume $k > \frac{n}{2}$, and $b \geq -k+1$, then for any $1 \leq t \leq t_0 < \infty$, we have*

$$t^2 I'_{a,b,k}(t) - t_0^2 I'_{a,b,k}(t_0) \leq C\varepsilon^2 |t|^{\frac{2k}{n-2k}-1}. \tag{6.40}$$

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