THE DIRICHLET PROBLEM OF THE HOMOGENEOUS k-HESSIAN EQUATION IN A PUNCTURED DOMAIN

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Abstract. In this paper, we consider the Dirichlet problem for the homogeneous k-Hessian equation with prescribed asymptotic behavior at $0 \in \Omega$ where Ω is a $(k-1)$-convex bounded domain in the Euclidean space. The prescribed asymptotic behavior at 0 of the solution is zero if $k > \frac{n}{2}$, it is $\log|x| + O(1)$ if $k = \frac{n}{2}$ and $-|x|^{\frac{2k-n}{n}} + O(1)$ if $k < \frac{n}{2}$. To solve this problem, we consider the Dirichlet problem of the approximating k-Hessian equation in $\Omega \setminus \overline{B_r(0)}$ with r small. We firstly construct the subsolution of the approximating k-Hessian equation. Then we derive the pointwise C^2-estimates of the approximating equation based on new gradient and second order estimates established previously by the second author and the third author. In addition, we prove a uniform positive lower bound of the gradient if the domain is starshaped with respect to 0. As an application, we prove an identity along the level set of the approximating solution and obtain a nearly monotonicity formula. In particular, we get a weighted geometric inequality for smoothly and strictly $(k-1)$-convex starshaped closed hypersurface in $\mathbb{R}^n$ with $\frac{n}{2} \le k < n$.

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Contents

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ and $u \in C^2(\Omega)$. The k-Hessian operator $F_k[u]$ is defined by

$$F_k[u] := S_k(D^2u), \tag{1.1}$$

where $S_k(D^2u)$ is the sum of all principal $k \times k$ minors of D^2u. If $\lambda = (\lambda_1, \cdots, \lambda_n)$ are the eigenvalues of D^2u, one can see that $S_k(D^2u) = \sum_{1\le i_1<\cdots<i_k\le n} \lambda_{i_1} \cdots \lambda_{i_k}$.

Caffarelli-Nirenberg-Spruck [6] solved the following Dirichlet problem for the k-Hessian equation

$$\begin{cases} S_k(D^2u) = f & \text{in} \quad \Omega, \\ u = \varphi & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where $f > 0$ and φ are given smooth functions. By assuming the existence of a subsolution, Guan [10, 13] solved (1.2).

For the degenerate case i.e. $f \ge 0$, Wang [30] solved the Dirichlet problem: $S_k(D^2u) = f(x,u)$ in Ω, $u = 0$ on $\partial\Omega$ and proved the Sobolev-type inequality for the related functional $\int_\Omega uS_k(D^2u)dx$.

Wang-Chou [8] used the parabolic method to prove the existence of k-convex solutions u to the problem $S_k(D^2u) = f(x,u)$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is strictly $(k-1)$-convex. In [8], Wang-Chou established the important Pogorelov type second order estimate for the k-Hessian equation.

Krylov [17, 18] proved the $C^{1,1}$ regularity of the problem: $S_k(D^2u) = f(x)$ in Ω and $u = \varphi$ on $\partial\Omega$ by assuming $f^{\frac{1}{k}} \in C^{1,1}$, $\varphi \in C^2$ and $(k-1)$-convexity of Ω. Ivochina-Trudinger-Wang [15] gave a new and simple proof. Li-Luc [21] studied the existence and uniqueness of the Green's function for the nonlinear Yamabe equation.

In the seminal papers [26–28], Trudinger-Wang studied systematically the Hessian measure for the k-convex function in $\mathbb{R}^n$ where they only assume that the function was continuous, locally bounded and locally integrable respectively. Labutin [19] continued to study the potential theory of the k-Hessian measure.

The fundamental solutions of the k-Hessian equation are as follows

$$G_k(x) = \begin{cases} -|x|^{2-\frac{n}{k}} & \text{if} \quad k < \dfrac{n}{2}, \\ \log|x| & \text{if} \quad k = \dfrac{n}{2}, \\ |x|^{2-\frac{n}{k}} & \text{if} \quad k > \dfrac{n}{2}. \end{cases} \tag{1.3}$$

In this paper, we want to study the regularity problem for the homogeneous k-Hessian equation in $\Omega \setminus 0$.

In the complex Euclidean space, Klimek [16] introduced the extremal fucntion

$$g_\Omega(z, z_0) = \sup\{v \in \mathcal{PSH}(\Omega) : v < 0,\ v(z) \le \log|z - z_0| + O(1)\}.$$

$g_\Omega(z,\xi)$ is called the pluricomplex Green function on $\Omega \subset \mathbb{C}^n$ with a logarithminc pole at z_0. If Ω is hyperconvex, Demailly [9] showed that $u(z) = g_\Omega(z, z_0)$ is continuous and solves uniquely the following homogeneous complex Monge-Ampère equation

$$\begin{cases} (dd^c u)^n = 0 & \text{in } \Omega \setminus \{z_0\}, \\ u = 0 & \text{on } \partial\Omega, \\ u(z) = \log|z - z_0| + O(1) & \text{as } z \to z_0. \end{cases} \tag{1.4}$$

If Ω is strictly convex with smooth boundary, Lempert [20] proved the solution is smooth. For the strongly pseudonconvex case, B. Guan [11] proved $C^{1,\alpha}$ regularity and later, Błocki [4] showed the $C^{1,1}$ regularity. The $C^{1,1}$ regularity is optimal by the counterexamples by Bedford-Demailly [2], .

P. Guan [14] established the $C^{1,1}$ regularity of the extremal function associated to intrinsic norms of Chen-Levine-Nirenberg [7] and Beford-Taylor [3] where the extremal function solves

$$\begin{cases} (dd^c u)^n = 0 & \text{in } \Omega_0 \setminus (\cup_{i=1}^m \Omega_i), \\ u = 0 & \text{on } \partial\Omega_i,\ i = 1, \cdots, n \\ u = 1 & \text{on } \partial\Omega_0. \end{cases}$$

1.1. **Our main results.** Motivated by Labutin's work [19] and Guan's work [11], we consider the following Dirichlet problem for the homogeneous k-Hessian equation with interior isolated singularities. For convenience, we assume the singularity is $0 \in \Omega$ and there exists positive constants r_0, R_0 such that $B_{r_0} \subset\subset \Omega \subset\subset B_{R_0}$, where B_r and B_{R_0} are balls centered at 0 with radius r and R_0 respectively.

We divide three cases to state our main results.

1.1.1. ***Case1:*** $k > \frac{n}{2}$. In this case, since the fundamental solution of the homogeneous k-Hessian equation is $|x|^{2-\frac{n}{k}}$ which tends to 0 as $x \to 0$, we consider the following problem

$$\begin{cases} S_k(D^2u) = 0 & \text{in} \quad \Omega \setminus \{0\}, \\ u = 1 & \text{on} \quad \partial\Omega, \\ \lim_{|x|\to 0} u(x) = 0. \end{cases} \tag{1.5}$$

We prove the following uniqueness and existence result.

Theorem 1.1. *Assume $k > \frac{n}{2}$. Let Ω be a smoothly convex domain in $\mathbb{R}^n$ and strictly $(k-1)$-convex. There exists a unique k-convex solution $u \in C^{1,1}(\overline{\Omega^c})$ of the equation (1.5). Moreover, there exists uniform constant C such that for any $x \in \Omega^c$ the following holds*

$$\begin{cases} C^{-1}|x|^{\frac{2k-n}{k}} \le u(x) \le C|x|^{\frac{2k-n}{k}}, \\ |Du|(x) \le C|x|^{\frac{k-n}{k}}, \\ |D^2u|(x) \le C|x|^{-\frac{n}{k}}. \end{cases} \tag{1.6}$$

1.1.2. ***Case2:*** $1 \le k < \frac{n}{2}$. We consider the following problem

$$\begin{cases} S_k(D^2u) = 0 & \text{in} \quad \Omega \setminus \{0\}, \\ u = -1 & \text{on } \partial\Omega, \\ u(x) = -|x|^{2-\frac{n}{k}} + O(1) & \text{as } x \to 0. \end{cases} \tag{1.7}$$

If we prescribe $u = -C_0|x|^{2-\frac{n}{k}} + O(1)$ as $x \to 0$ for some positive constant C_0, then $\tilde{u} = C_0^{-1}u + C_0^{-1} - 1$ solves (1.7).

Theorem 1.2. *Assume $1 \le k < \frac{n}{2}$. Let Ω be a smoothly, strictly $(k-1)$-convex domain in $\mathbb{R}^n$. There exists a unique k-convex solution $u \in C^{1,1}(\overline{\Omega} \setminus \{0\})$ of the equation (1.7). Moreover, there exists uniform constant C such that for any $x \in \overline{\Omega} \setminus \{0\}$, the following holds*

$$\begin{cases} \left| u(x) - |x|^{-\frac{n-2k}{k}} \right| \le C, \\ |Du|(x) \le C|x|^{-\frac{n-k}{k}}, \\ |D^2u|(x) \le C|x|^{-\frac{n}{k}}. \end{cases} \tag{1.8}$$

Remark 1.3. *Assume $1 \le k \le \frac{n}{2}$. Labutin [19] proved if u is k-convex solving $S_k(D^2u) = 0$ in $B_R \setminus \{0\}$, $u < 0$ and 0 is the singular point of u, there exists a positive constant C_0 such that $u(x) = C_0G_k(x) + O(1)$ as $x \to 0$. This is the reason why we prescribe the above asymptotic behavior in (1.7).*

1.1.3. ***Case3:*** $k = \frac{n}{2}$. Since the Green function in this case is $\log|x|$, we consider the k-Hessian equation when $k = \frac{n}{2}$ as follows

$$\begin{cases} S_{\frac{n}{2}}(D^2u) = 0 & \text{in } \Omega \setminus \{0\}, \\ u = 0 & \text{on } \partial\Omega, \\ u(x) = \log|x| + O(1) & \text{as } |x| \to 0. \end{cases} \tag{1.9}$$

If we prescribe $u = C_0 \log|x| + O(1)$ as $x \to 0$ for some positive constant C_0, then $\tilde{u} = C_0^{-1}u$ solves (1.9).

Theorem 1.4. *Assume $k = \frac{n}{2}$. Let Ω be a smoothly and strictly $(k-1)$-convex domain in $\mathbb{R}^n$. There exists a unique k-convex solution $u \in C^{1,1}(\overline{\Omega} \setminus \{0\})$ of the equation (1.9). Moreover, there exists uniform constant C such that for any $x \in \overline{\Omega} \setminus \{0\}$ the following holds*

$$\begin{cases} |u(x) - \log|x|| \le C, \\ |Du|(x) \le C|x|^{-1}, \\ |D^2u|(x) \le C|x|^{-2}. \end{cases} \tag{1.10}$$

To solve the above problems, for example when $k > \frac{n}{2}$ we will prove there exists a smooth k-convex function u^ε solving

$$\begin{cases} S_k(u^\varepsilon) = \varepsilon & \text{in} \quad \Omega \setminus \{0\}, \\ u^\varepsilon = 1 & \text{on} \quad \partial\Omega, \\ \lim_{|x|\to 0} u^\varepsilon(x) \to 0. \end{cases}$$

Note that the right hand side of the above approximating equation is ε which is different from the exterior Dirichlet problem case. To solve the above approximating equation, we consider the approximating k-Hessian equation in $\Omega_r := \Omega \setminus \overline{B}_r$ and we will prove the uniform $C^{1,1}$-estimates. We firstly construct a subsolution of the approximating k-Hessian equation in Ω_r. This follows from a key lemma due to P. F. Guan [14] by the $(k-1)$-convexity of the domain. Note that the second and third author have proved the global gradient and second order estimate in [22]. Thus we only need to prove the boundary estimates.

1.2. **Applications to the starshaped $(k-1)$ convex domain.** As an application of our C^2 estimates for the approximating equation, we can prove an almost monotonicity formula along the level set of u^ε when Ω is additionally starshaped. Consequently, we get some weighted geometric inequalities of $\partial\Omega$ when $\frac{n}{2} \le k < n$.

Theorem 1.5. *Let Ω be a bounded smooth starshaped domain with respect to 0 in $\mathbb{R}^n$ and strictly $(k-1)$-convex.*

(i) Assume $\frac{n}{2} < k < n$. Assume $b \ge \frac{k(n-k-1)}{n-k}$. Let u be the unique $C^{1,1}$ solution in Theorem 1.1. We have

$$\int_{\partial\Omega} |Du|^{b+1} H_{k-1} \ge \frac{2k-n}{n-k} \int_{\partial\Omega} |Du|^b H_k, \tag{1.11}$$

where H_m is the m-Hessian operator of the principal curvature $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$ of $\partial\Omega$.

(ii) Assume $k = \frac{n}{2}$ and $b \ge \frac{n}{2} - 1$. Let u be the unique $C^{1,1}$ solution in Theorem 1.4. We have

$$\int_{\partial\Omega} |Du|^{b+1} H_{k-1} \ge \int_{\partial\Omega} |Du|^b H_k. \tag{1.12}$$

Remark 1.6. *If we assume Ω is starshaped with respect to $x_0 \in \Omega$, the above inequality still holds for u which solves the homogeneous k-Hessian equation in $\Omega \setminus \{x_0\}$.*

Organization of this paper. In section 2, we firstly construct a subsolution for the approximating equation by a lemma due to P. F. Guan [14]. Based on the new gradient and second order estimates in [22], we show uniform $C^{1,1}$ estimate of the approximating solution. The positive lower bound of the gradient of the approximating solution is proved if we also assume Ω is starshaped. Theorem 1.1, Theorem 1.2 and Theorem 1.4 will be proved in Section 4. In section 5, we prove an almost monotonicity formula along the level set of the approximating solution and then we show Theorem 1.5.

2. Solving the approximating equation in $\Omega_r := \Omega \setminus B_r$.

We need the following lemma by P. F. Guan [14] to construct the existence of the subsolution of the k-Hessian equation in $\Omega \setminus \overline{B_r}$.

Lemma 2.1. *Suppose that U is a bounded smooth domain in $\mathbb{R}^n$. For $h, g \in C^m(U)$, $m \geq 2$, for all $\delta > 0$, there is an $H \in C^m(U)$ such that*

(1) $H \geq \max\{h, g\}$ and

$$H(x) = \begin{cases} h(x), & \text{if } h(x) - g(x) > \delta, \\ g(x), & \text{if } g(x) - h(x) > \delta; \end{cases}$$

(2) There exists $|t(x)| \leq 1$ such that

$$\left\{H_{ij}(x)\right\} \geq \left\{\frac{1 + t(x)}{2} g_{ij} + \frac{1 - t(x)}{2} h_{ij}\right\}, \text{ for all } x \in \{|g - h| < \delta\}.$$

By the convacity of $S^{\frac{1}{k}}$, we can prove that H is k-convex if f and g are both k-convex.

Recall that we always assume $B_{r_0} \subset\subset \Omega \subset\subset B_{(1-\tau_0)R_0}$ for some $\tau_0 \in (0, \frac{1}{2})$. Firstly we state a useful fact for the strictly $(k-1)$-convex domain, which can be found in [6, Section 3].

Lemma 2.2. *Let Ω be a smoothly and strictly $(k-1)$-convex bounded domain. There exists $\mu_0 > 0$ small such that $\Omega_{2\mu_0} := \{x \in \Omega : d(x) < 2\mu_0\}$ is close to $\partial\Omega$,$B_{r_0} \subset\subset \{x \in \Omega : d(x) > 2\mu_0\}$ and $d(x)$ is smooth in $\overline{\Omega}_{2\mu_0}$. Moreover, $\Phi^0 := t_0^{-1}(e^{-t_0 d(x)} - 1)$ is smooth and strictly k-convex and $S_k(D^2(\Phi^0)) \geq \epsilon_0$ in $\overline{\Omega}_{2\mu_0}$ for some uniform positive constants t_0 and ϵ_0.*

2.1. Case 1: $k > \frac{n}{2}$.

Since the Green function in this case is $|x|^{\frac{2k-n}{k}}$, we want to solve the k-Hessian equation as follows

$$\begin{cases} S_k(D^2 u) = 0 & \text{in } \mathring{\Omega}, \\ u = 1 & \text{on } \partial\Omega, \\ \lim_{x \to 0} u(x) = 0. \end{cases} \tag{2.1}$$

2.1.1. *The approximating equation.* We will use the solution of a sequence of nondegenetare equations in Ω_r to approximate the solution of the homogeneous k-Hessian equation. The existence of the approximating solution can be obtained if we can construct a smooth subsolution. We use the $(k-1)$-convexity of $\partial\Omega$ and the Lemma 2.1 by P. F. Guan [14] to prove the existence of the subsolution.

Denote $w := \frac{1}{2}\Big(\frac{|x|}{R_0}\Big)^{2-\frac{n}{k}} + \frac{|x|^2}{2R_0^2}$. By the concavity of $S_k^{\frac{1}{k}}$,

$$S_k^{\frac{1}{k}}(D^2w) = S_k^{\frac{1}{k}}\Big(\frac{1}{2}D^2(\frac{|x|}{R_0})^{2-\frac{n}{2}} + \frac{1}{2R_0^2}D^2|x|^2\Big) \geq S_k^{\frac{1}{k}}(\frac{1}{R_0^2}I).$$

Then we have

$$S_k(D^2w) \geq C_n^k R_0^{-2k}.$$

Then we construct a smoothly and strictly k-convex function $\underline{u}$ by lemma (2.1) as follows.

Lemma 2.3. *There exists a strictly k-convex function $\underline{u} \in C^\infty(\overline{\Omega}_r)$ satisfying*

$$\underline{u} = \begin{cases} K_0\Phi^0 + 1 & \text{if } d(x) \leq \dfrac{\mu_0}{M_0}, \\ w & \text{if } d(x) > \mu_0, \end{cases} \tag{2.2}$$

$$\underline{u} \geq \max\Big\{w, K_0\Phi^0 + 1\Big\} \quad \text{if } \frac{\mu_0}{M_0} \leq d(x) \leq \mu_0,$$

$$S_k(D^2\underline{u}) \geq \epsilon_1 := \min\{C_n^k R_0^{-2k}, K_0^k\epsilon_0\} \quad \text{in } \Omega,$$

where $K_0 = \frac{2t_0}{1-e^{-\mu_0 t_0}}$ and M_0 is determined by $K_0(1 - e^{-t_0\frac{\mu_0}{M_0}}) = t_0\delta$ with $\delta := \frac{1}{2}(1-\tau_0)^{2-\frac{n}{k}}$.

Remark 2.4. *This lemma tells us that $\underline{u}$ is $K_0\Phi^0 + 1$ near $\partial\Omega$ and $\underline{u}$ is w outside $\Omega_{2\mu_0}$. Moreover, $\underline{u}$ is smooth and strictly k-convex. Although this lemma is elementary, it is crucial for the proof of $C^{1,1}$ estimates.*

Proof. Applying Guan's lemma for $U = \Omega_{2\mu_0} := \{x \in \Omega : d(x) < 2\mu_0\}$, $g = K_0\Phi^0 + 1$, $h = w$ and $\delta = \frac{1}{2}(1-\tau_0)^{2-\frac{n}{k}}$, we get a strictly and smoothly k-convex function $\underline{u}$ in $\Omega_{2\mu_0}$. In the following, we prove (2.2).

For any $x \in \overline{\Omega}_{2\mu_0} \setminus \Omega_{\mu_0} := \{x \in \overline{\Omega} : \mu_0 \leq d(x) \leq 2\mu_0\}$, since $K_0 = \frac{2t_0}{1-e^{-t_0\mu_0}}$, we have

$$g(x) \leq -1.$$

Then

$$h - g \geq -g \geq 1 > \delta \quad \text{in } \overline{\Omega}_{2\mu_0} \setminus \Omega_{\mu_0}. \tag{2.3}$$

This implies $\underline{u} = w$ in $\overline{\Omega}_{2\mu_0} \setminus \Omega_{\mu_0}$.

For any $x \in \overline{\Omega_{\frac{\mu_0}{M_0}}} := \{x \in \overline{\Omega} : d(x) \leq \frac{\mu_0}{M}\}$, since $\Omega \subset\subset B_{(1-\tau_0)R_0}$, we have

$$\begin{aligned} g - h =& t_0^{-1}K_0(e^{-t_0 d(x)} - 1) + 1 - \frac{1}{2}\Big(\frac{|x|}{R_0}\Big)^{\frac{2k-n}{k}} - \frac{|x|^2}{2R_0^2} \\ \geq& t_0^{-1}K_0(e^{-t_0\frac{\mu_0}{M}} - 1) + 1 - (1-\tau_0)^{2-\frac{n}{k}} \\ \geq& \frac{1}{2}(1 - (1-\tau_0)^{2-\frac{n}{k}}) = \delta, \end{aligned} \tag{2.4}$$

where M_0 is defined by $K_0(1-e^{-t_0\frac{\mu_0}{M_0}}) = t_0\delta$. This implies $\underline{u} = K_0\Phi^0 + 1$ in $\Omega_{\frac{\mu_0}{M_0}}$.

At last, we define $\underline{u} = w$ in $\Omega_r \setminus \Omega_{2\mu_0}$. In $\Omega_{\frac{\mu_0}{M_0}}$, by Lemma 2.2, $S_k(D^2\underline{u}) = S_k(K_0\Phi^0) \geq K_0^k\epsilon_0$. In $\Omega_r\backslash\Omega_{2\mu_0}$, $S_k(D^2\underline{u}) = S_k(D^2w) \geq C_n^k R_0^{-2k}$. In $\Omega_{2\mu_0}\backslash\Omega_{\frac{\mu_0}{M_0}}$, by the concavity of $S_k^{\frac{1}{k}}$, $S_k^{\frac{1}{k}}(D^2\underline{u}) \geq \frac{1+t(x)}{2}S_k^{\frac{1}{k}}(D^2w) + \frac{1-t(x)}{2}S_k^{\frac{1}{k}}(K_0D^2\Phi^0)$. The proof is complete. □

Now we consider the following approximating equation

$$\begin{cases} S_k(D^2u) = \varepsilon & \text{in } \Omega \setminus \overline{B_r}, \\ u = 1 & \text{on } \partial\Omega, \\ u = \underline{u} = \dfrac{1}{2}\Big(\dfrac{r}{R_0}\Big)^{2-\frac{n}{k}} + \dfrac{r^2}{2R_0^2} & \text{on } \partial B_r. \end{cases} \tag{2.5}$$

If $\varepsilon < \epsilon_1$, $\underline{u}$ is a subsolution by the above lemma. By B. Guan [10] (see also [13]), equation (2.5) has a strictly k-convex solution $u^{\varepsilon,r} \in C^\infty(\overline{\Omega}_r)$. Our goal is to establish uniform C^2 estimates of $u^{\varepsilon,r}$, which are independent of ε and r.

We can check that $\bar{u} := \Big(\frac{|x|}{r_0}\Big)^{2-\frac{n}{k}}$ is a supersolution of the above approximating equation. Indeed, $\bar{u}$ is smooth in Ω_r and $S_k(D^2\bar{u}) = 0$.
On ∂B_r, we have

$$u^{\varepsilon,r} = \frac{1}{2}\Big(\frac{r}{R_0}\Big)^{2-\frac{n}{k}} + \frac{r^2}{2R_0^2} \leq \Big(\frac{r}{R_0}\Big)^{2-\frac{n}{k}} < \Big(\frac{r}{r_0}\Big)^{2-\frac{n}{k}}.$$

On $\partial\Omega$, since $B_{r_0} \subset\subset \Omega$, we have

$$u^{\varepsilon,r} = 1 < \Big(\frac{|x|}{r}\Big)^{2-\frac{n}{k}} = \bar{u},$$

where we use $2k > n$. Thus by comparison principal, we have $u < \bar{u}$ in $\overline{\Omega}_r$.
Our goal is to prove the following estimates.

Theorem 2.5. *Assume $k > \frac{n}{2}$. For sufficiently small ε and r, $u^{\varepsilon,r}$ satisfies*

$$\begin{cases} C^{-1}|x|^{\frac{2k-n}{k}} \leq u^{\varepsilon,r}(x) \leq C|x|^{\frac{2k-n}{k}}, \\ |Du^{\varepsilon,r}|(x) \leq C|x|^{\frac{k-n}{k}}, \\ |D^2u^{\varepsilon,r}|(x) \leq C|x|^{-\frac{n}{k}}, \end{cases}$$

where C is a uniform constant independent of ε and r.

2.2. **Case 2: $k < \frac{n}{2}$.** Since the Green function in this case is $-|x|^{\frac{2k-n}{k}}$, we want to solve the following k-Hessian equation .

$$\begin{cases} S_k(D^2u) = 0 & \text{in} \quad \Omega \setminus \{0\}, \\ u = -1 & \text{on} \quad \partial\Omega \\ u = -|x|^{\frac{2k-n}{k}} + O(1) \text{ as } x \to 0. \end{cases} \tag{2.6}$$

Denote $w := -|x|^{2-\frac{n}{k}} + R_0{}^{2-\frac{n}{k}} - 1 + a_0\frac{|x|^2}{2R_0^2}$. We choose $a_0 = \Big((1-\tau_0)^{2-\frac{n}{k}} - 1\Big)R_0^{2-\frac{n}{k}}$ such that $w < -\frac{1}{2}\Big((1-\tau_0)^{2-\frac{n}{k}} - 1\Big)R_0^{2-\frac{n}{k}} - 1$ in $\overline{\Omega}$. By the concavity of $S_k^{\frac{1}{k}}$, we also have

$$S_k^{\frac{1}{k}}(D^2w) = S_k^{\frac{1}{k}}(D^2(-|x|^{2-\frac{n}{k}}) + \frac{a_0}{2R_0^2}D^2|x|^2) \geq S_k^{\frac{1}{k}}(\frac{a_0}{R_0^2}I),$$

then

$$S_k(D^2w) \geq C_n^k a_0^k R_0^{-2k}.$$

Then we construct a smoothly and strictly k-convex function $\underline{u}$ by Lemma 2.1 as follows.

Lemma 2.6. *There exists a strictly k-convex function $\underline{u} \in C^\infty(\overline{\Omega}_r)$ satisfying*

$$\underline{u} = \begin{cases} K_0\Phi^0 - 1 & \text{if } d(x) \leq \dfrac{\mu_0}{M_0}, \\ w & \text{if } d(x) > \mu_0, \end{cases} \tag{2.7}$$

$$\underline{u} \geq \max\Big\{w, K_0\Phi^0 - 1\Big\} \quad \text{if } \frac{\mu_0}{M_0} \leq d(x) \leq \mu_0,$$

$$S_k(D^2\underline{u}) \geq \epsilon_1 := \min\{C_n^k a_0^k R_0^{-2k}, K_0^k\epsilon_0\} \quad \text{in } \Omega_r,$$

where K_0 and M_0 are uniform constants.

Proof. Applying Guan's Lemma 2.1 for $U = \Omega_{2\mu_0}$, $g = K_0\Phi^0 - 1$, $h = w$ and $\delta = \frac{1}{4}\Big((1-\tau_0)^{2-\frac{n}{k}} - 1\Big)R_0^{2-\frac{n}{k}}$, we get a strictly and smoothly k-convex function $\underline{u}$ in $\Omega_{2\mu_0}$. In the following, we prove (2.7).

For any $x \in \overline{\Omega}_{2\mu_0} \setminus \Omega_{\mu_0} := \{x \in \overline{\Omega} : \mu_0 \leq d(x) \leq 2\mu_0\}$, we have

$$\begin{aligned} h - g =& -|x|^{2-\frac{n}{k}} + R_0^{2-\frac{n}{k}} + a_0\frac{|x|^2}{2R_0^2} - K_0\Phi^0 \\ \geq& - r_0{}^{2-\frac{n}{k}} + R_0^{2-\frac{n}{k}} + t_0^{-1}K_0(1 - e^{-t_0\mu_0}) \\ =& R_0^{2-\frac{n}{k}}, \end{aligned} \tag{2.8}$$

where we use $K_0 = \frac{t_0 r_0{}^{2-\frac{n}{k}}}{1-e^{-t_0\mu_0}}$. This implies $\underline{u} = w$ in $\overline{\Omega}_{2\mu_0} \setminus \Omega_{\mu_0}$.

For any $x \in \overline{\Omega_{\frac{\mu_0}{M_0}}} := \{x \in \overline{\Omega} : d(x) \leq \frac{\mu_0}{M_0}\}$, since $\Omega \subset\subset B_{(1-\tau_0)R_0}$, we have

$$\begin{aligned} g - h =& |x|^{2-\frac{n}{k}} - R_0^{2-\frac{n}{k}} - a_0 \frac{|x|^2}{2R_0^2} + K_0\Phi^0 \\ \geq& \frac{1}{2}\Big((1-\tau_0)^{2-\frac{n}{k}} - 1\Big)R_0^{2-\frac{n}{k}} + t_0^{-1}K_0(1 - e^{-t_0\frac{\mu_0}{M_0}}) \\ =& \frac{1}{4}\Big((1-\tau_0)^{2-\frac{n}{k}} - 1\Big)R_0^{2-\frac{n}{k}} := \delta \end{aligned} \tag{2.9}$$

where M_0 is defined by $K_0(1 - e^{-t_0\frac{\mu_0}{M_0}}) = 2t_0\delta$. This implies $\underline{u} = K_0\Phi^0 + 1$ in $\Omega_{\frac{\mu_0}{M_0}}$.
At last, we define $\underline{u} = w$ in $\Omega_r \backslash \Omega_{2\mu_0}$. In $\Omega_{\frac{\mu_0}{M_0}}$, by Lemma 2.2, $S_k(D^2\underline{u}) = S_k(K_0\Phi^0) \geq K_0^k\epsilon_0$. In $\Omega_r \backslash \Omega_{2\mu_0}$, $S_k(D^2\underline{u}) = S_k(D^2w) \geq C_n^k a_0^k R_0^{-2k}$. In $\Omega_{2\mu_0} \backslash \Omega_{\frac{\mu_0}{M_0}}$, by the concavity of $S_k^{\frac{1}{k}}$, $S_k^{\frac{1}{k}}(D^2\underline{u}) \geq \frac{1+t(x)}{2} S_k^{\frac{1}{k}}(D^2w) + \frac{1-t(x)}{2} S_k^{\frac{1}{k}}(K_0D^2\Phi^0)$. The proof is complete. □

We consider the approximating equation

$$\begin{cases} S_k(D^2u^{\varepsilon,r}) = \varepsilon & \text{in} \quad \Omega_r, \\ \qquad u^{\varepsilon,r} = \underline{u} & \text{on} \quad \partial\Omega_r. \end{cases} \tag{2.10}$$

Then $\underline{u}$ is a strict subsolution of the above k-Hessian equation for any ε small, by Guan [10] (see also Guan [13]), equation (2.10) has a strictly k-convex solution $u^{\varepsilon,r} \in C^\infty(\overline{\Omega}_r)$. By maximum principle and assmuing r is sufficiently small, $u^{\varepsilon,r} < -1$ in Ω_r. We want to derive uniform C^2 estimates of $u^{\varepsilon,r}$, which are independent of ε and r. We prove the following

Theorem 2.7. *Assume $1 \leq k < \frac{n}{2}$. For every sufficiently small ε and r, $u^{\varepsilon,r}$ satisfies*

$$\begin{cases} C^{-1}|x|^{-\frac{n-2k}{k}} \leq -u^{\varepsilon,r}(x) \leq C|x|^{-\frac{n-2k}{k}}, \\ \quad |Du^{\varepsilon,r}|(x) \leq C|x|^{-\frac{n-k}{k}}, \\ |D^2u^{\varepsilon,r}|(x) \leq C|x|^{-\frac{n}{k}}, \end{cases}$$

where C is a uniform constant independent of ε and r.

2.3. **Case 3: $k = \frac{n}{2}$.** Since the Green function in this case is $\log|x|$, we want to solve the k-Hessian equation as follows

$$\begin{cases} S_{\frac{n}{2}}(D^2u) = 0 \quad \text{in} \quad \mathring{\Omega}, \\ \qquad u = 0 \quad \text{on } \partial\Omega, \\ \quad u(x) = \log|x| + O(1) \text{ as } x \to 0. \end{cases} \tag{2.11}$$

2.3.1. *The approximating equation.* Denote $w := \log \frac{|x|}{R_0} + a_0 \frac{|x|^2}{2R_0^2}$ where $a_0 = \frac{1}{2}\log \frac{1}{1-\tau_0} > 0$ such that $w < \frac{1}{2}\log(1-\tau_0)$, By the concavity of $S_k^{\frac{1}{k}}$, we also have

$$S_{\frac{n}{2}}^{\frac{2}{n}}(D^2 w) = S_k^{\frac{1}{k}}(D^2 \log \frac{|x|}{R_0} + \frac{a_0}{2R_0^2} D^2|x|^2) \geq S_k^{\frac{1}{k}}(\frac{a_0}{R_0^2} I),$$

then

$$S_{\frac{n}{2}}(D^2 w) \geq C_n^{\frac{n}{2}} a_0^{\frac{n}{2}} R_0^{-n}.$$

Then we construct a smoothly and strictly k-convex function $\underline{u}$ by Lemma 2.1 as follows.

Lemma 2.8. *There exists a strictly k-convex function $\underline{u} \in C^\infty(\overline{\Omega}_r)$ satisfying*

$$\underline{u} = \begin{cases} K_0\Phi^0 & \text{if } d(x) \leq \dfrac{\mu_0}{M_0}, \\ w & \text{if } d(x) > \mu_0, \end{cases} \tag{2.12}$$

$$\underline{u} \geq \max\left\{w, K_0\Phi^0\right\} \quad \text{if } \frac{\mu_0}{M_0} \leq d(x) \leq \mu_0,$$

$$S_k(D^2\underline{u}) \geq \epsilon_1 := \min\{C_n^{\frac{n}{2}} a_0^{\frac{n}{2}} R_0^{-n}, K_0^{\frac{n}{2}}\epsilon_0\} \quad \text{in } \Omega,$$

where K_0 and M_0 are uniform constants.

Proof. Applying Guan's Lemma 2.1 for $U = \Omega_{2\mu_0}$, $g = K_0\Phi^0$, $h = w$ and $\delta = \frac{1}{4}\log\frac{1}{1-\tau_0} > 0$, we get a strictly and smoothly k-convex function $\underline{u}$ in $\Omega_{2\mu_0}$. In the following, we prove (2.12).
For any $x \in \overline{\Omega}_{2\mu_0} \setminus \Omega_{\mu_0} := \{x \in \overline{\Omega} : \mu_0 \leq d(x) \leq 2\mu_0\}$, we have

$$\begin{aligned} h - g &= \log\frac{|x|}{R_0} + a_0\frac{|x|^2}{2R_0^2} - K_0\Phi^0 \\ &\geq \log\frac{r_0}{R_0} + t_0^{-1}K_0(1 - e^{-\mu_0 t_0}) \\ &= \log\frac{R_0}{r_0} \geq \log\frac{1}{1-\tau_0} \\ &> \delta, \end{aligned} \tag{2.13}$$

where we choose $K_0 = \frac{2t_0 \log\frac{R_0}{r_0}}{1-e^{-\mu_0 t_0}}$ and we use $r_0 < (1-\tau_0)R_0$.

For any $x \in \overline{\Omega_{\frac{\mu_0}{M_0}}} := \{x \in \overline{\Omega} : d(x) \leq \frac{\mu_0}{M}\}$, since $\Omega \subset\subset B_{(1-\tau_0)R_0}$, we have

$$
\begin{aligned}
g - h =& K_0\Phi^0 - w \\
\geq& t_0^{-1}K_0(e^{-\frac{\mu_0 t_0}{M_0}} - 1) - \log\frac{|x|}{R_0} - a_0\frac{|x|^2}{2R_0^2} \\
\geq& t_0^{-1}K_0(e^{-\frac{\mu_0 t_0}{M_0}} - 1) - \log(1-\tau_0) - \frac{a_0}{2} \\
=& 2\delta > \delta,
\end{aligned} \tag{2.14}
$$

where we use $a_0 = 2\delta = \frac{1}{2}\log\frac{1}{1-\tau_0} > 0$ and M_0 is determined by $K_0(1 - e^{-\frac{\mu_0 t_0}{M_0}}) = t_0\delta$.

We finish the proof by defining $\underline{u} = w$ in $\Omega_r \setminus \Omega_{2\mu_0}$. □

Then we consider the following approximating equation

$$
\begin{cases} S_k(D^2u^{\varepsilon,r}) = \varepsilon & \text{in } \Omega_r, \\ u = \underline{u} & \text{on } \partial\Omega_r. \end{cases} \tag{2.15}
$$

We will prove the following pointwise estimates

Theorem 2.9. *Assume $k = \frac{n}{2}$. For every sufficiently small ε and r, for any $x \in \overline{\Omega}_r$ $u^{\varepsilon,r}$ satisfies*

$$
\begin{cases} |u^{\varepsilon,r}(x) - \log|x|| \leq C, \\ |Du^{\varepsilon,r}|(x) \leq C|x|^{-1}, \\ |D^2u^{\varepsilon,r}|(x) \leq C|x|^{-2}, \end{cases}
$$

where C is a uniform constant which is independent of ε and r.

In the next subsections, we will prove uniform C^2 estimates of solutions of equations (2.5), (2.10) and (2.15). The key point is that these estimates are independent of ε and r.

2.4. C^0 **estimates.** We first prove $u^{\varepsilon,r}$ is increasing with r. For any $r \geq \tilde{r}$, we have $u^{\varepsilon,\tilde{r}} \geq \underline{u}$ in $\Omega_{\tilde{r}}$ and then

$$
\begin{cases} S_k(D^2u^{\varepsilon,r}) = \varepsilon = S_k(D^2u^{\varepsilon,\tilde{r}}) & \text{in } \Omega_r, \\ u^{\varepsilon,r} = u^{\varepsilon,\tilde{r}} & \text{on } \partial\Omega, \\ u^{\varepsilon,r} = \underline{u} \leq u^{\varepsilon,\tilde{r}} & \text{on } \partial B_r. \end{cases}
$$

Applying the maximum principle in Ω_r, we have

$$
u^{\varepsilon,r} \leq u^{\varepsilon,\tilde{r}} \text{ in } \overline{\Omega}_r. \tag{2.16}
$$

Proposition 2.10. *Let $u^{\varepsilon,r}$ be the k-convex solution of the approximating equation (2.5), (2.10) or (2.15). For sufficiently small ε and r, for any $x \in \overline{\Omega}_r$, we have*

$$\begin{aligned} \frac{1}{2}R_0^{\frac{n}{k}-2}|x|^{2-\frac{n}{k}} \le u^{\varepsilon,r}(x) \le r_0^{\frac{n}{k}-2}|x|^{2-\frac{n}{k}} \quad & \textit{if } k > \frac{n}{2}, \\ |x|^{-\frac{n-2k}{k}} - r_0^{\frac{n-2k}{k}} + 1 \le -u^{\varepsilon,r}(x) \le |x|^{-\frac{n-2k}{k}} - R_0^{\frac{n-2k}{k}} + 1 \quad & \textit{if } k < \frac{n}{2}, \\ \log|x| - \log R_0 \le u^{\varepsilon,r} \le \log|x| - \log r_0 \quad & \textit{if } k = \frac{n}{2}. \end{aligned}$$

Proof. The lower bound of $u^{\varepsilon,r}$ holds since $u^{\varepsilon,r} \ge \underline{u}$.

Case 1: $k > \frac{n}{2}$

We can check that $\bar{u} := \left(\frac{|x|}{r_0}\right)^{2-\frac{n}{k}}$ is a supersolution of the above approximating equation. Indeed, $\bar{u}$ is smooth in Ω_r and $S_k(D^2\bar{u}) = 0$.
On ∂B_r, we have

$$u^{\varepsilon,r} = \frac{1}{2}\Big(\frac{r}{R_0}\Big)^{2-\frac{n}{k}} + \frac{r^2}{2R_0^2} \le \Big(\frac{r}{R_0}\Big)^{2-\frac{n}{k}} < \Big(\frac{r}{r_0}\Big)^{2-\frac{n}{k}}.$$

On $\partial\Omega$, since $B_{r_0} \subset\subset \Omega$, we have

$$u^{\varepsilon,r} = 1 < \Big(\frac{|x|}{r}\Big)^{2-\frac{k}{n}} = \bar{u},$$

where we use $2k > n$. Thus we have

$$\begin{cases} S_k(D^2u^{\varepsilon,r}) = \varepsilon > 0 = S_k\left(D^2\Big(r_0^{\frac{n-2k}{k}}|x|^{-\frac{n-2k}{k}}\Big)\right) & \text{in } \Omega_r \\ u^{\varepsilon,r} = 1 < r_0^{\frac{n-2k}{k}}|x|^{-\frac{n-2k}{k}} & \text{on } \partial\Omega\ , \\ u^{\varepsilon,r} = \underline{u} < r_0^{\frac{n-2k}{k}}|x|^{-\frac{n-2k}{k}} & \text{on } \partial B_r. \end{cases}$$

By maximum principal, we have $u^{\varepsilon,r} \le \bar{u}$ in $\overline{\Omega}_r$.

Case 2: $k < \frac{n}{2}$

One can check $\bar{u} = -|x|^{2-\frac{n}{k}} + r_0^{2-\frac{n}{k}} - 1$ is a supersolution. Indeed, we have

$$\begin{cases} S_k(D^2u^{\varepsilon,r}) = \varepsilon > 0 = S_k\left(D^2\Big(-|x|^{2-\frac{n}{k}}\Big)\right) & \text{in } \Omega_r \\ u^{\varepsilon,r} = -1 < \bar{u} & \text{on } \partial\Omega\ , \\ u^{\varepsilon,r} = \underline{u} < \bar{u} & \text{on } \partial B_r. \end{cases}$$

Applying the maximum principle in Ω_r, we have

$$u^{\varepsilon,r} \le \bar{u}. \tag{2.17}$$

Case 3: $k = \frac{n}{2}$

Since $u^{\varepsilon,r} = \underline{u} \le \log|x| - \log r_0$ on ∂B_r, $\underline{u} = 0 < \log|x| - \log r_0$ on $\partial\Omega$ and $S_k(D^2u^{\varepsilon,r}) = \varepsilon > 0 = S_k(D^2(\log|x|))$, then we have $u^{\varepsilon,r} \le \log|x| - \log r_0$. ◻

2.5. **Gradient estimates.** In this subsection, we prove the global gradient estimate based on our key estimate in [22]. If we further assume Ω is starshaped, we can prove the positive lower bound of the gradient and thus the level set of the approximating solution is compact.

Motivated by B. Guan [12] where he proved the gradient estimate for the complex Monge-Ampere equation, we proved the following gradient estimate for the k-Hessian equation in [22].

Theorem 2.11. *Let $U \subset \mathbb{R}^n$ be a domain, $u \in C^3(U) \cap C^1(\overline{U})$ be a solution of the k-Hessian equation $S_k(D^2u) = f$ in U and $u < 0$ if $k \leq \frac{n}{2}$ and $u > 0$ if $k > \frac{n}{2}$. Denote by*

$$P = \begin{cases} |Du|^2 e^{2u}, & k = \frac{n}{2}, \\ |Du|^2 u^{\frac{2(n-k)}{2k-n}}, & k > \frac{n}{2}, \\ |Du|^2 (-u)^{-\frac{2(n-k)}{n-2k}}, & k < \frac{n}{2}. \end{cases} \tag{2.18}$$

then we have the following gradient estimate

$$\max_U P \leq \begin{cases} \max\left\{\max_U (e^{2u}|D\log f|^2), \max_{\partial U} P\right\}, & k = \frac{n}{2}, \\ \max\left\{\left(\frac{2k-n}{k(n+1-k)}\right)^2 \max_U (u^{\frac{2k}{2k-n}}|D\log f|^2), \max_{\partial U} P\right\}, & k > \frac{n}{2}, \\ \max\left\{\left(\frac{n-2k}{k(n+1-k)}\right)^2 \max_U \left((-u)^{-\frac{2k}{n-2k}}|D\log f|^2\right), \max_{\partial U} P\right\}, & k < \frac{n}{2}, \end{cases} \tag{2.19}$$

Applying the above estimate in our setting i.e. we take $U = \Omega_r$ and $f = \varepsilon$, we get the following

Proposition 2.12. *Let $u^{\varepsilon,r} \in C^3(\Omega_r) \cap C^1(\overline{\Omega}_r)$ be a k-convex solution of the approximating equation (2.5), (2.10) or (2.15). For sufficiently small ε and r, we have*

$$\max_{\overline{\Omega}_r} P \leq \max_{\partial\Omega_r} P. \tag{2.20}$$

Proposition 2.13. *Let $u^{\varepsilon,r} \in C^3(\Omega_r) \cap C^1(\overline{\Omega}_r)$ be a k-convex solution of the approximating equation (2.5), (2.10) or (2.15). For sufficiently small ε and r, we have*

$$\max_{\overline{\Omega}_r} P \leq C. \tag{2.21}$$

Proof. We only need to prove boundary gradient estimates.

For simplicity, we use u instead of $u^{\varepsilon,r}$ during the proof.

We will construct upper barriers near $\partial\Omega$ and ∂B_r respectively.

<u>Case 1: $k > \frac{n}{2}$</u>

Let $h \in C^\infty(\overline{\Omega}_{r_0})$ be the unique solution of

$$\begin{cases} \Delta h = 0 & \text{in } \Omega_{r_0}, \\ h = 1 & \text{on } \partial\Omega, \\ h = \dfrac{1}{2} & \text{on } \partial B_{r_1}, \end{cases}$$

where $r_1 = 2^{-\frac{k}{2k-n}} r_0$. By maximum principle and the C^0 estimate of u, $\underline{u} \le u \le h$ in $\overline{\Omega}_{r_0}$. Then for any $x \in \partial\Omega$

$$0 < c_0 \le h_\nu \le u_\nu(x) \le \underline{u}_\nu(x) \le C,$$

where ν is the outward normal of $\partial\Omega$. Then

$$0 < c \le \max_{\partial\Omega} |Du| = \max_{\partial\Omega}(u_\nu) \le C. \tag{2.22}$$

This proves that P is uniformly bounded on $\partial\Omega$.

Next we show P is uniformly bounded on ∂B_r. We consider $\tilde{u}(y) := r^{\frac{n}{k}-2} u(x)$ and $\underline{\tilde{u}}(y) := r^{\frac{n}{k}-2} \underline{u}(x)$ for $y := \frac{x}{r} \in B_2 \setminus B_1$. $\tilde{u}$ satisfies

$$\begin{cases} S_k(D^2\tilde{u}) = r^n \varepsilon & \text{in } B_2 \setminus \bar{B}_1, \\ \tilde{u} = \tilde{w} & \text{on } \partial B_1, \end{cases} \tag{2.23}$$

where $\tilde{w}(y) = r^{\frac{n}{k}-2} w(x)$ and recall $\underline{u} = w$ in $\Omega_r \subset \Omega_r$.

By the C^0 estimate of u, we have

$$R_0{}^{\frac{n}{k}-2} |y|^{2-\frac{n}{k}} \le \tilde{u} \le r_0{}^{\frac{n}{k}-2} |y|^{2-\frac{n}{k}}.$$

Then $\tilde{u}$ is uniformly bounded in $\overline{B_2 \setminus B_1}$. Let $\tilde{h}(y)$ be the smooth function solving

$$\begin{cases} \Delta \tilde{h} = 0 & \text{in } B_2 \setminus B_1, \\ \tilde{h} = \tilde{w} = \dfrac{R_0^{\frac{n}{k}-2}}{2} + \dfrac{a_0}{2R_0^2} r^{\frac{n}{k}} & \text{on } \partial B_1, \\ \tilde{h} = r_0{}^{\frac{n}{k}-2} 2^{\frac{n}{k}-2} & \text{on } \partial B_2. \end{cases} \tag{2.24}$$

Then $\tilde{h}$ is uniformly C^2 bounded in $\overline{B_2 \setminus B_1}$. By maximum principal, we have

$$\tilde{w} \le \tilde{u} \le \tilde{h}. \tag{2.25}$$

Then

$$\tilde{w}_\nu \le \tilde{u}_\nu \le \tilde{h}_\nu \le C \text{ on } \partial B_1. \tag{2.26}$$

Note that on ∂B_1, we have

$$\tilde{w}_\nu = r^{\frac{n}{k}-1} w_{x_i} y_i > (1 - \frac{n}{2k}) R_0^{2-\frac{n}{k}} > 0.$$

where we use $k > \frac{n}{2}$. Thus we have

$$c \le |D\tilde{u}| \le C \text{ on } \partial B_1.$$

Therefore, we get

$$c|x|^{1-\frac{n}{k}} \le |Du| \le C|x|^{1-\frac{n}{k}} \text{ on } \partial B_r.$$

This implies P is uniformly bounded on ∂B_r.

In conclusion, when $k > \frac{n}{2}$, P is uniformly bounded in $\overline{\Omega}_r$.

Case2: $k < \frac{n}{2}$

The gradient estimate on $\partial\Omega$ is similar as case 1. We only prove the gradient estimate on ∂B_1. We consider $\tilde{u}(y) := r^{\frac{n}{k}-2}u(x)$ and $\tilde{w}(y) := r^{\frac{n}{k}-2}w(x)$ for $y := \frac{x}{r} \in B_2 \setminus B_1$. $\tilde{u}$ satisfies

$$\begin{cases} S_k(D^2\tilde{u}) = r^n\varepsilon & \text{in } B_2 \setminus \bar{B}_1, \\ \tilde{u} = \tilde{w} & \text{on } \partial B_1. \end{cases} \tag{2.27}$$

By the C^0 estimate of u and assuming r is small enough, we have

$$\frac{1}{2}|y|^{2-\frac{n}{k}} \le -\tilde{u} \le 2|y|^{2-\frac{n}{k}}.$$

Then $\tilde{u}$ is uniformly bounded in $\overline{B_2 \setminus B_1}$. Let $\tilde{h}(y)$ be the smooth function solving

$$\begin{cases} \Delta\tilde{h} = 0 & \text{in } B_2 \setminus B_1, \\ \tilde{h} = \tilde{w} & \text{on } \partial B_1, \\ \tilde{h} = -\frac{1}{2}2^{2-\frac{n}{k}} & \text{on } \partial B_2. \end{cases} \tag{2.28}$$

Then $\tilde{h}$ is uniformly C^2 bounded in $\overline{B_2 \setminus B_1}$. By maximum principal, we have

$$\tilde{w} \le \tilde{u} \le \tilde{h}. \tag{2.29}$$

Then

$$\tilde{w}_\nu \le \tilde{u}_\nu \le \tilde{h}, \tag{2.30}$$

where $\nu(y) = y$ is the outward normal to ∂B_1. Note that

$$\tilde{w}_\nu = \frac{n}{k} - 2 + \frac{a_0}{R_0^2}r^{\frac{n}{k}} > \frac{n}{k} - 2 > 0 \text{ on } \partial B_1,$$

where we choose r small enough and use $k < \frac{n}{2}$. Thus we have

$$c \le |D\tilde{u}| \le C \text{ on } \partial B_1. \tag{2.31}$$

Therefore, we get

$$c|x|^{1-\frac{n}{k}} \le |Du| \le C|x|^{1-\frac{n}{k}} \text{ on } \partial B_r.$$

Thus P is uniformly bounded on ∂B_r.

In conclusion, when $k < \frac{n}{2}$, P is uniformly bounded in $\overline{\Omega}_r$.

Case 3: $k = \frac{n}{2}$

The gradient estimate on $\partial\Omega$ is similar as case 1. We will prove the gradient estimate on ∂B_1. Define $\tilde{u}(y) = u(x)$ with $y = \frac{x}{r} \in \overline{B}_2 \setminus B_1$, we have

$$\begin{cases} S_k(D^2\tilde{u}) = r^n\varepsilon & \text{in } B_2 \setminus \bar{B}_1, \\ \tilde{u} = \log r & \text{on } \partial B_1. \end{cases} \tag{2.32}$$

By the C^0 estimate of u:

$$\log|y| - \log R_0 \leq \tilde{u} - \log r \leq \log|y| - \log r_0.$$

Let $\bar{h}(y)$ be the smooth function solving

$$\begin{cases} \Delta\tilde{h} = 0 & \text{in } B_2 \setminus B_1, \\ \tilde{h} = \tilde{w} & \text{on } \partial B_1, \\ \tilde{h} = \log r + \log\dfrac{2}{r_0} & \text{on } \partial B_2. \end{cases} \tag{2.33}$$

We have $|D\tilde{h}| \leq C$ in $\overline{B_2 \setminus B_1}$. By comparison, we have $\tilde{w} \leq \tilde{u} \leq \tilde{h}$. Recall $\tilde{w} = \tilde{u} = \tilde{h}$ on ∂B_1, we get

$$0 < c \leq \tilde{w}_\nu \leq \tilde{u}_\nu \leq \tilde{h}_\nu \leq C \quad \text{on } \partial B_1, \tag{2.34}$$

where c and C are uniform positive constants. Then we have

$$c \leq |D\tilde{u}| = \tilde{u}_\nu \leq C \quad \text{on } \partial B_1 \tag{2.35}$$

Therefore

$$cr^{-1} \leq |Du| = \tilde{u}_\nu \leq Cr^{-1} \quad \text{on } \partial B_r \tag{2.36}$$

Thus P is uniformly bounded on Ω_r. □

2.5.1. *Positive lower bound of* $|Du|$ *when* Ω *is strictly* $(k-1)$ *convex and starshaped.*

Lemma 2.14. *Let Ω be strictly $(k-1)$ convex and starshaped. Let u be the k-convex solution of the approximating equation (2.5), (2.10) or (2.15). For sufficiently small ε and r, there exists a uniform constant c_0 such that for any $x \in \overline{\Omega}_r$*

$$x \cdot Du(x) \geq c_0|x|^{2-\frac{n}{k}}. \tag{2.37}$$

In particular,

$$|Du(x)| \geq c_0|x|^{1-\frac{n}{k}}. \tag{2.38}$$

Proof. Recall $F^{ij} = \frac{\partial}{\partial u_{ij}}(\log S_k(D^2u))$. By Maclaurin inequality, we have

$$\mathcal{F} = (n-k+1)\frac{S_{k-1}}{S_k} \geq k(C_n^k)^{\frac{1}{k}}S_k^{-\frac{1}{k}} \geq k\varepsilon^{-\frac{1}{k}}. \tag{2.39}$$

We first prove the positive lower bound of $x \cdot Du(x)$ on $\partial\Omega_r$. In fact, since $|Du| \geq c$ on $\partial\Omega$ and Ω is starshaped, we have

$$x \cdot Du = x \cdot \nu |Du| \geq c \min_{\partial\Omega} x \cdot \nu := c_1 > 0. \tag{2.40}$$

On ∂B_r, since $Du = |Du|\nu = |Du|\frac{x}{r}$, we have

$$x \cdot Du = r|Du| \geq cr^{2-\frac{n}{k}}. \tag{2.41}$$

Then for any $x \in \partial\Omega_r$, we have

$$x \cdot Du \geq c_0|x|^{2-\frac{n}{k}}. \tag{2.42}$$

Case 1: $k < \frac{n}{2}$

We consider the function $H := x \cdot Du(x) - b_{11}u - b_{12}\frac{|x|^2}{2}$ with $b_{11} = \frac{c_0}{2}r_0^{2-\frac{n}{k}}$ and $b_{12} = \frac{c_0}{4R_0^{\frac{n}{k}}}$.

Since $u \leq r_0^{\frac{n}{k}-2}|x|^{2-\frac{n}{k}}$, by (2.42), we have

$$H \geq \frac{c_0}{2}|x|^{2-\frac{n}{k}} - b_{11}u + \frac{c_0}{2}|x|^{2-\frac{n}{k}} - b_{12}\frac{|x|^2}{2} > 0 \quad \text{on } \partial\Omega_r.$$

On the other hand, we have

$$\begin{aligned} F^{ij}H_{ij} &= (2 - b_{11})k - b_{12}\mathcal{F} \\ &\leq 2k - b_{12}k\varepsilon^{-\frac{1}{k}} < 0, \end{aligned} \tag{2.43}$$

assume $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 \leq \left(\frac{b_{12}}{2}\right)^k$.

By maximum principle,

$$H \geq \min_{\partial\Omega_r} H > 0.$$

Case 2: $k < \frac{n}{2}$. Consider the function $H := x \cdot Du(x) + b_{21}a_1u - \frac{b_{12}}{2}|x|^2$. Our goal is to show H is positive in $\overline{\Omega}_r$. Indeed, By (2.42) and $-u \leq C|x|^{2-\frac{n}{k}}$, for $b_{21} := \frac{1}{2}C^{-1}a_0$ and $b_{22} = \frac{c_0}{2R_0^{\frac{n}{k}}}$, for any $x \in \partial\Omega_r$, we have

$$\begin{aligned} H \geq& \frac{1}{2}x \cdot Du - \frac{b_{22}}{2}|x|^2 \\ \geq& \frac{1}{2}|x|^{2-\frac{n}{k}}(c_0 - b_{22}R_0^{\frac{n}{k}}) \\ \geq& \frac{c_0}{4}|x|^{2-\frac{n}{k}} > 0 \quad \text{on } \partial\Omega_r. \end{aligned} \tag{2.44}$$

On the other hand, we have

$$\begin{aligned} F^{ij}H_{ij} &= (2 + b_{11}a_1)k - b_{12}\mathcal{F} \\ &\leq (2 + b_{21})k - b_{22}k\varepsilon^{-\frac{1}{k}} < 0, \end{aligned} \tag{2.45}$$

where we use (2.39) and we assume $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 \leq \left(\frac{b_{22}}{2(1+b_{21})}\right)^k$. By maximum principle,

$$H \geq \min_{\partial\Omega_r} H > 0.$$

In conclusion, we prove $H > 0$ in $\overline{\Omega}_r$ and thus (2.37) is obtained.

By maximum principle, we have $H > \min_{\partial\Omega_r} H > 0$.

Case 3: $k = \frac{n}{2}$

We consider $H = x \cdot Du(x) - b_{31} - b_{32}\frac{|x|^2}{2}$ which is positive on the boundary of $\overline{\Omega}_r$ if we take b_{31} and b_{32} small enough. Since $F^{ij}H_{ij} \leq k\varepsilon^{-\frac{1}{k}}(\varepsilon^{\frac{1}{k}} - b_{32}) < 0$ for ε small enough, we have $H = x \cdot Du(x) - b_{31} - b_{32}\frac{|x|^2}{2} > 0$ in $\overline{\Omega}_r$ and we can get the desired estimate.

□

2.6. **Second order estimates.** By the uniform gradient estimate, we have proved that P is uniformly bounded in $\overline{\Omega}_r$. We will prove the second order estimate of the approximating equations based on the following second order estimate in [22] by the second author and the third author.

2.6.1. ***The global second order estimate can be reduced to the boundary second order estimate***.

Theorem 2.15. *Let $u \in C^4(\Omega_r) \cap C^2(\overline{\Omega}_r)$ be a k-convex solution of (2.5) or (2.10) or (2.15). Define $G = u_{\xi\xi}\varphi(P)h(u)$, then we have*

$$\max_{\Omega_r} G \leq C + \max_{\partial\Omega_r} G. \tag{2.46}$$

where h is defined by

$$h(u) = \begin{cases} u^{\frac{n}{2k-n}}, & k > \frac{n}{2}, \\ (-u)^{-\frac{n}{n-2k}}, & k < \frac{n}{2}. \\ e^{2u}, & k = \frac{n}{2}, \end{cases}$$

and φ is defined by

$$\varphi(t) = \begin{cases} (M - t)^{-\tau}, & k < n, \\ 1, & k = n, \end{cases} \tag{2.47}$$

where $M := 2\max P + 1$, τ is a uniform positive constant

2.6.2. ***Second order estimate on the boundary*** $\partial\Omega_r$. . The second order estimate on $\partial\Omega$ is the same as [6] (see also [22]). Here we only need prove the second order estimate on ∂B_r.

Tangential second derivatives estimates

For any $x_0 \in \partial B_r$, we choose the coordinate such that $x_0 = (0, \cdots, 0, r)$, then near x_0, ∂B_r is locally represented by $x_n = (r^2 - |x'|^2)^{\frac{1}{2}}$ and $\frac{\partial^2 x_n}{\partial x_\alpha \partial x_\beta}(x_0) = r^{-1}\delta_{\alpha\beta}$ with $1 \le \alpha, \beta \le n-1$. Since $u|_{\partial B_r} = constant$, we have

$$\begin{aligned} u_{\alpha\beta}(x_0) = & - u_n(x_0)\frac{\partial^2 x_n}{\partial x_\alpha \partial x_\beta}(x_0) = r^{-1}u_n(x_0)\delta_{\alpha\beta} \\ = & r^{-1}u_\nu(x_0)\delta_{\alpha\beta}. \end{aligned} \tag{2.48}$$

Since we have the boundary gradient estimate on ∂B_r,

$$Cr^{-\frac{n-k}{k}} \ge u_\nu(x) \ge cr^{-\frac{n-k}{k}},$$

then by (2.48), we have

$$|u_{\alpha\beta}(x_0)| \le Cr^{-\frac{n}{k}} \tag{2.49}$$

$$\{u_{\alpha\beta}(x_0)\} \ge cr^{-\frac{n}{k}}\{\delta_{\alpha\beta}\}. \tag{2.50}$$

Tangential-normal derivative estimates $\partial\Omega_r$

For any $x_0 \in \partial B_r$, choose the coordinate such that $x_0 = (0, \cdots, 0, r)$, $\partial B_r \cap B_{\frac{1}{2}r}(x_0)$ is represented by

$$x_n = \rho(x') = (r^2 - |x'|^2)^{\frac{1}{2}},$$

Consider the tangential operator $T_\alpha = (x_\alpha \partial_n - x_n \partial_\alpha)$,$1 \le \alpha \le n-1$. Since $u(x', \rho(x'))$ is constant, we have

$$0 = u_\alpha + u_n\rho_\alpha = u_\alpha - x_\alpha \rho^{-1} u_n$$

Then on $\partial B_r \cap B_{\frac{r}{2}}(x_0)$, we have

$$T_\alpha u = x_\alpha u_n - \rho u_\alpha = 0.$$

We consider the function

$$w = A_1(1 - r^{-1}x_n) \pm r^{\frac{n-2k}{k}} T_\alpha u \text{ in } B_r \cap B_{\frac{r}{2}}(x_0),$$

where A_1 is positive large constant. Since $x_0 = (0 \cdots, 0, r)$ and $T_\alpha u = 0$ on ∂B_r, we have $w(x_0) = 0$. Since $T_\alpha u = 0$ on $\partial B_r \cap B_{\frac{r}{2}}(x_0)$, we have $w|_{\partial B_r \cap B_{\frac{r}{2}}(x_0)} \ge 0$.
Since on $B_r \cap B_{\frac{r}{2}(x_0)}$, $r^{\frac{n-2k}{k}}|T_\alpha u| \le C_1 r^{\frac{n-2k}{k}}|x||Du| \le C$ and $x_n \le \frac{7r}{8}$, choosing $A_1 > 16C$, we have

$$w \ge \frac{1}{8}A_1 - C > C > 0 \quad \text{on} \quad B_1 \cap \partial B_{\frac{1}{2}r}(y_0). \tag{2.51}$$

Observe that $F^{ij}w_{ij} = \pm F^{ij}T_\alpha u = T_\alpha(F^{ij}u_{ij}) = 0$. By maximum principle, w attains its minimum 0 at x_0. Then we have

$$0 \ge w_n(x_0) = -A_1 r^{-1} \pm r^{\frac{n-k}{k}} u_{\alpha n}(x_0).$$

Then $|u_{\alpha n}(x_0)| \leq A_1 r^{-1}|Du(x_0)| \leq Cr^{-\frac{n}{k}}$ and thus we have the uniform tangential-normal derivative estimates on ∂B_R.

Double normal derivative estimates $\partial\Omega_r$

We can choose the coordinate at x_0 such that $u_n(x_0) = |Du|$ and $\{u_{\alpha\beta}(x_0)\}_{1\leq\alpha,\beta\leq n-1}$ is diagonal.

For any $x_0 \in \partial B_r$, by (2.50), we have

$$\begin{aligned} u_{nn}c_0 r^{-\frac{n(k-1)}{k}} \leq u_{nn}(x_0)S_{k-1}(u_{\alpha\beta}(x_0)) =& S_k(D^2u(x_0)) - S_k(u_{\alpha\beta}(x_0)) + \sum_{i=1}^{n-1} u_{in}^2 S_{k-2}(u_{\alpha\beta}) \\ \leq& \varepsilon + Cr^{-n} \leq 2Cr^{-n}. \end{aligned}$$

This gives $u_{nn} \leq Cr^{-\frac{n}{k}}$. On the other hand, $u_{nn} \geq -\sum_{i=1}^{n-1} u_{ii} \geq -cr^{-\frac{n}{k}}$. Then we have $|u_{nn}(x_0)| \leq Cr^{-\frac{n}{k}}$.

In conclusion, we obtain $|D^2u(x)| \leq C|x|^{-\frac{n}{k}}$ on the boundary $\partial\Omega_r$ and thus $|D^2u|(x) \leq C|x|^{-\frac{n}{k}}$ for any $x \in \overline{\Omega_r}$.

3. Proof of Theorem 1.1, Theorem 1.2 and Theorem 1.4

3.1. **Uniqueness.** The uniqueness follows from the comparison principle for k-convex solutions of the k-Hessian equation in bounded domains by Wang-Trudinger [26] (see also [25, 29]). See [22] for the detailed argument.

3.2. **Existence and $C^{1,1}$-estimates.** The existence follows from the uniform C^2-estimates for $u^{\varepsilon,r}$.

For any fixed sufficiently small $\varepsilon > 0$ and compact subset $K \subset\subset \Omega \setminus \{0\}$, there exist r_0 sufficiently small such that $K \subset\subset \Omega_r$, $|u^{\varepsilon,r}|_{C^2(\Omega_{r_0})} \leq C(\epsilon, K)$ for any $r < r_0$. By Evans-Krylov theory, $|u^{\varepsilon,r}|_{C^{2,\alpha}(K)} \leq C(\epsilon, K, m)$. Then there exists a subsequence u^{ε,r_i} converging in $C^{2,\beta}$-norm ($\beta < \alpha$) to a strictly k-convex u^ε in K and $u^\varepsilon \in C^{2,\alpha}(K)$ satisfies

$$\begin{cases} S_k(D^2u^\varepsilon) = \varepsilon & \text{in } \Omega \setminus \{0\}, \\ u^\varepsilon = 1 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Moreover, by Theorem 2.7, we have the following estimate

$$\begin{cases} C^{-1}|x|^{-\frac{n-2k}{k}} \leq -u^\varepsilon(x) \leq C|x|^{-\frac{n-2k}{k}}, \\ |Du^\varepsilon|(x) \leq C|x|^{-\frac{n-k}{k}}, \\ |D^2u^\varepsilon|(x) \leq C|x|^{-\frac{n}{k}}, \end{cases}$$

Thus there exits a subsequence u^{ϵ_i} converges to u in $C^{1,\alpha}_{loc}$ such that $u \in C^{1,1}(\overline{\Omega} \setminus \{0\})$ is the k-convex solution of the k-Hessian equation (1.7) and satisfies the estimates (1.8).

Case 2: $k < \frac{n}{2}$
Similar as case 1, there exists a subsequence u^{ε, r_i} converging smoothly to a strictly k-convex u^ε in K and $u^\varepsilon \in C^{2,\alpha}(\Omega \setminus \{0\})$ satisfies

$$\begin{cases} S_k(D^2 u^\varepsilon) = \varepsilon & \text{in } \Omega \setminus \{0\}, \\ u^\varepsilon = -1, & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Moreover, by Theorem 2.5, we get

$$\begin{cases} |u^\varepsilon(x) - |x|^{\frac{2k-n}{k}}| \leq C, \\ |Du^\varepsilon|(x) \leq C|x|^{-\frac{n-k}{k}}, \\ |D^2 u^\varepsilon|(x) \leq C|x|^{-\frac{n}{k}}, \end{cases}$$

Thus there exits a subsequence u^{ϵ_i} converges to u in $C^{1,\alpha}_{loc}$ such that $u \in C^{1,1}(\overline{\Omega} \setminus \{0\})$ is the k-convex solution of the k-Hessian equation (1.5) and satisfies the estimates (1.6).

Case 3: $k = \frac{n}{2}$
Similar as case 1, there exists a subsequence u^{ε, r_i} converging smoothly to a strictly k-convex u^ε in K and $u^\varepsilon \in C^{\infty}(\Omega \setminus \{0\})$ satisfies

$$\begin{cases} S_k(D^2 u^\varepsilon) = \varepsilon & \text{in } \Omega \setminus \{0\}, \\ u^\varepsilon = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Moreover, by Theorem 2.9, we get

$$\begin{cases} |u^\varepsilon(x) - \log|x|| \leq C, \\ |Du^\varepsilon|(x) \leq C|x|^{-1}, \\ |D^2 u^\varepsilon|(x) \leq C|x|^{-2}, \end{cases}$$

Thus there exits a subsequence u^{ϵ_i} converges to u in $C^{1,\alpha}_{loc}$ such that $u \in C^{1,1}(\Omega \setminus \{0\})$ is the k-convex solution of the k-Hessian equation (1.9) and satisfies the estimates (1.10).

4. A monotonicity formula along the level set of the approximating solution

Agostiniani-Mazzieri [1] proved an monotonicity formula along the level set of the solution of the following problem

$$\begin{cases} \Delta u = 0 \text{ in } \Omega^c \\ u = -1 \text{ on } \partial\Omega \\ \lim\limits_{|x| \to \infty} u(x) = 0. \end{cases} \tag{4.1}$$

Since the solution of the homogeneous k-Hessian equation is only $C^{1,1}$, we consider the level set of u^ε. In [22], we prove an monotonicity formula along the level set of the solution of the exterior Dirichlet problem of the approximating k-Hessian equation. As an application of our uniform $C^{1,1}$ estimates of u^ε and the positive lower bound of $|Du^\varepsilon|$, we prove an interior version of [22].

We firstly estimate the area of the level set $S_t = \{x \in \Omega \setminus \{0\} : u^\varepsilon(x) = t\}$.

Lemma 4.1. *There exits uniform constant C such that*

$$|S_t| \leq \begin{cases} Ct^{\frac{k(n-1)}{2k-n}} & \forall t \in (0,1] \quad \text{if } k > \frac{n}{2}, \\ C|t|^{-\frac{k(n-1)}{n-2k}} & \forall\, t \in (-\infty,-1] \quad \text{if } k < \frac{n}{2}, \\ Ce^{(n-1)t} & \forall t \in (-\infty,0] \quad \text{if } k = \frac{n}{2}. \end{cases} \tag{4.2}$$

Proof. For any fixed t, assume $r > 0$ sufficiently small, we have

$$|S_t| - |\partial B_r| = \int_{\{u<t\}\setminus B_r} \operatorname{div}\Big(\frac{Du^\varepsilon}{|Du^\varepsilon|}\Big) dx.$$

Case1: $k > \frac{n}{2}$

For any $x \in \{x : u(x) < t\}$, since $|D^2 u^\varepsilon|(x) \leq C|x|^{-\frac{n}{k}}$ and $|Du^\varepsilon| \geq c|x|^{1-\frac{n}{k}}$, we have

$$\Big|\operatorname{div}\Big(\frac{Du^\varepsilon}{|Du^\varepsilon|}\Big)\Big| = \Big|\frac{\Delta u^\varepsilon}{|Du^\varepsilon|} - \frac{u^\varepsilon_{ij}u^\varepsilon_i u^\varepsilon_j}{|Du^\varepsilon|^3}\Big| \leq C|D^2u^\varepsilon||Du^\varepsilon|^{-1} \leq C|x|^{-1}.$$

Combining the above estimate with $\{u < t\} \subset B_{Ct^{\frac{k}{2k-n}}}$, we have

$$\begin{aligned} 0 \leq |S_t| - |\partial B_r| \leq & C \int_{B_{Ct^{\frac{k}{2k-n}}}} |x|^{-1} dx \\ \leq & C \int_0^{Ct^{\frac{k}{2k-n}}} s^{n-2} ds \\ \leq & Ct^{(n-1)\frac{k}{2k-n}}. \end{aligned} \tag{4.3}$$

Taking $r \to 0$, we have

$$|S_t| \leq \ C|t|^{(n-1)\frac{k}{2k-n}}.$$

Case2: $k < \frac{n}{2}$**.** Similar argument shows that

$$\begin{aligned} |S_t| - |\partial B_r| \leq & C \int_0^{C|t|^{-\frac{k}{2n-k}}} s^{n-2} ds \\ \leq & C|t|^{-(n-1)\frac{k}{n-2k}}. \end{aligned} \tag{4.4}$$

Case3: $k = \frac{n}{2}$**.**

We have

$$
\begin{aligned}
|S_t| - |\partial B_r| &\leq C \int_0^{Ce^t} s^{n-2} ds \\
&\leq Ce^{(n-1)t}. \qquad (4.5)
\end{aligned}
$$

□

Similar as the exterior case in [22], we consider the following quantity

$$
I_{a,b,k}(t) := \int_{S_t} g^a(u^\varepsilon)|Du^\varepsilon|^{b-k} S_k^{ij}(D^2u^\varepsilon) u_i^\varepsilon u_j^\varepsilon, \tag{4.6}
$$

where $g(u^\varepsilon)$ is defined by

$$
g(u^\varepsilon) = \begin{cases} (u^\varepsilon)^{\frac{n-k}{2k-n}}, & k > \frac{n}{2}, \\ (-u^\varepsilon)^{\frac{n-k}{2k-n}}, & k < \frac{n}{2}, \\ e^{u^\varepsilon}, & k = \frac{n}{2}. \end{cases} \tag{4.7}
$$

We choose $a = b - k + 1$ and one can see that $I_{a,b,k}(t)$ is uniformly bounded due to the C^2 estimates of u^ε and the positive lower bound of $|Du^\varepsilon|$. We define

$$
J_{a+a_0,b,k}(t, t_0) := g^{a_0}(t) I'_{a,b,k}(t) - g^{a_0}(t_0) I'_{a,b,k}(t_0). \tag{4.8}
$$

We prove the following useful equality along the level set of u^ε.

Proposition 4.2. *Let u^ε be the solution of the approximating k-Hessian equation with $a = b - k + 1$. We have the following identity*

$$
\begin{aligned}
J_{a+a_0,b,k}(t, t_0) = &- ba \int_{t_0}^t \int_{S_s} \Big(g^{a+a_0}|Du^\varepsilon|^{b-k-1} \frac{H_k}{H_{k-1}} S_k \Big) dA ds + (b+1) \int_{t_0}^t \int_{S_s} \Big(g^{a+a_0-1} g' |Du^\varepsilon|^{b-k} S_k \Big) dA ds \\
&+ (b+1) \int_{S_t} \Big(g^{a+a_0}|Du^\varepsilon|^{b-k} S_k \Big) dA - (b+1) \int_{S_{t_0}} \Big(g^{a+a_0}|Du^\varepsilon|^{b-k} S_k \Big) dA \\
&+ a \int_{t_0}^t \int_{S_s} g^{a+a_0}|Du^\varepsilon|^{b-1} H_{k-1}^{-1} \Big(c_{n,k} H_k^2 - (k+1) H_{k-1} H_{k+1} \Big) dA ds \\
&+ a \int_{t_0}^t \int_{S_s} g^{a+a_0}|Du^\varepsilon|^{b-1} \mathcal{L} \, dA ds - ab \int_{t_0}^t \int_{S_s} g^{a+a_0}|Du^\varepsilon|^{b-k-2} \mathcal{M} \, dA ds,
\end{aligned} \tag{4.9}
$$

where H_m is the m-th order fundamental symmetric function of principal curvatures m-Hessian operator of the level set S_s of u^ε, $a_0, b, c_{n,k} = \frac{k(n-k-1)}{n-k}$ and the functions $\mathcal{L}$ are chosen as follows

(i) If $1 \leq k < \frac{n}{2}$, we require $-\infty < t_0 < t \leq -1$, $a_0 = -2\frac{n-2k}{n-k}$ and $\mathcal{L} = (b - c_{n,k})\Big(\frac{n-k}{n-2k}|D \log u^\varepsilon| - \frac{H_k}{H_{k-1}}\Big)^2$.

(ii) If $k = \frac{n}{2}$, *we require* $-\infty < t < t_0 \leq 0$, $a_0 = 0$ *and* $\mathcal{L} = a\Big(|Du^\varepsilon| - \frac{H_k}{H_{k-1}}\Big)^2$.

(iii) If $k > \frac{n}{2}$, *we require* $0 < t < t_0 \leq 1$, $a_0 = 2\frac{2k-n}{n-k}$, $\mathcal{L} = (b - c_{n,k})\Big(\frac{n-k}{n-2k}|D\log u^\varepsilon| - \frac{H_k}{H_{k-1}}\Big)^2$.

and

$$\mathcal{M} := S_{k+1} - \frac{H_k}{H_{k-1}}|Du^\varepsilon|S_k + \frac{H_k^2}{H_{k-1}}|Du^\varepsilon|^{k+1} - H_{k+1}|Du|^{k+1} \leq 0. \tag{4.10}$$

Proof. **For simplicity, we use u instead of u^ε and S_k intead of $S_k(D^2u^\varepsilon)$ during the proof.**

We use the notation $\Omega_t := \{x \in \Omega \setminus \{0\} : u(x) > t\}$ and we define $\Omega_{t_0t} := \Omega_{t_0} \setminus \overline{\Omega_t}$ for any $t_0 < t$.

By the divergence theorem and the divergence free property of the k-Hessian operator i.e. $\sum_{j=1}^n D_j S_k^{ij} = 0$, we have

$$\begin{aligned}
I_{a,b,k}(t) - I_{a,b,k}(t_0) =& \int_{\Omega_{t_0t}} D_j\left(g^a|Du|^{b+1-k}S_k^{ij}u_i\right)\\
=&a\int_{\Omega_{t_0t}} g^{a-1}g'|Du|^{b+1-k}S_k^{ij}u_iu_j\\
&+(b+1-k)\int_{\Omega_{t_0t}} g^a|Du|^{b-k-1}S_k^{ij}u_iu_lu_{ij} + k\int_{\Omega_{t_0t}} g^a|Du|^{b+1-k}S_k\\
=&a\int_{\Omega_{t_0t}} g^{a-1}g'|Du|^{b+1-k}S_k^{ij}u_iu_j\\
&-(b+1-k)\int_{\Omega_{t_0t}} g^a|Du|^{b-k-1}S_{k+1}^{ij}u_iu_j + (b+1)\int_{\Omega_{t_0t}} g^a|Du|^{b+1-k}S_k\\
=&a\int_{t_0}^t\int_{S_s} g^{a-1}g'|Du|^{b-k}S_k^{ij}u_iu_j - (b-k+1)\int_{t_0}^t\int_{S_s} g^a|Du|^{b-k-2}S_{k+1}^{ij}u_iu_j\\
&+(b+1)\int_{t_0}^t\int_{S_s} g^a|Du|^{b-k}S_k,
\end{aligned} \tag{4.11}$$

where we use $S_k^{ij}u_iu_lu_{lj} = |Du|^2S_k - S_{k+1}^{ij}u_iu_j$ and the coarea formula.

Then

$$\begin{aligned}
I'_{a,b,k}(t) =&a\int_{S_t} g^{a-1}g'|Du|^{b-k}S_k^{ij}u_iu_j\\
&-(b+1-k)\int_{S_t} g^a|Du|^{b-k-2}S_{k+1}^{ij}u_iu_j + E_{a,b,k}(t),
\end{aligned} \tag{4.12}$$

where $E_{a,b,k}(t) = (b+1)\int_{S_t} g^a|Du|^{b-k}S_k$.

Then we have

$$
\begin{aligned}
J_{a+a_0,b,k}(t,t_0) &= g^{a_0}(t)I'_{a,b,k}(t) - g^{a_0}(t_0)I'_{a,b,k}(t_0)\\
=&a\int_{\Omega_{t_0 t}} D_j\Big(g^{a+a_0-1}g'|Du|^{b-k+1}S_k^{ij}u_i\Big)\\
&- a\Big(I_{a+a_0,b-1,k+1}(t) - I_{a+a_0,b-1,k+1}(t_0)\Big) + E_{a+a_0,b,k}(t) - E_{a+a_0,b,k}(t_0),
\end{aligned} \tag{4.13}
$$

where we use $a = b - k + 1$. We will compute the terms in (4.13).
Firstly we have

$$
\begin{aligned}
&\int_{\Omega_{t_0 t}} D_j\Big(g^{a+a_0-1}g'|Du|^{b-k+1}S_k^{ij}u_i\Big)dx\\
=&\int_{t_0}^{t}\int_{S_s}\Big((g^{a+a_0-1}g')\Big)'|Du|^{b-k}S_k^{ij}u_iu_jdAds\\
&+(b-k+1)\int_{t_0}^{t}\int_{S_s}\Big(g^{a+a_0-1}g'|Du|^{b-k-2}S_k^{ij}u_iu_lu_{lj}\Big)dAds + k\int_{t_0}^{t}\int_{S_s}\Big(g^{a+a_0-1}g'|Du|^{b-k}S_k\Big)dAds\\
=&\int_{t_0}^{t}\int_{S_s}\Big((g^{a+a_0-1}g')'|Du|^{b-k}S_k^{ij}u_iu_j\Big)dAds\\
&-(b-k+1)\int_{t_0}^{t}\int_{S_s}\Big(g^{a+a_0-1}g'|Du|^{b-k-2}S_{k+1}^{ij}u_iu_j\Big)dAds + (b+1)\int_{t_0}^{t}\int_{S_s}\Big(g^{a+a_0-1}g'|Du|^{b-k}S_k\Big)dAds\\
=&\int_{t_0}^{t}\int_{S_s}\Big((g^{a+a_0-1}g')'|Du|^{b+1}H_{k-1}\Big)dAds - (b-k+1)\int_{t_0}^{t}\int_{S_s}\Big(g^{a+a_0-1}g'|Du|^{b}H_k\Big)dAds\\
&+(b+1)\int_{t_0}^{t}\int_{S_s}\Big(g^{a+a_0-1}g'|Du|^{b-k}S_k\Big)dAds,
\end{aligned} \tag{4.14}
$$

where we use the identity $H_{m-1}|Du|^{m+1} = S_m^{ij}u_iu_j$ for $m \in \{1,2,\cdots,n\}$ (see e.g. [5,23,24]).

For the term $I_{a+a_0,b-1,k+1}(t) - I_{a+a_0,b-1,k+1}(t_0)$, similar as the manipulation of (4.11), we have

$$
\begin{aligned}
I_{a+a_0,b-1,k+1}(t_0) - I_{a+a_0,b-1,k+1}(t)&\\
=&(a+a_0)\int_{t_0}^{t}\int_{S_s} g^{a+a_0-1}g'|Du|^{b-k-2}S_{k+1}^{ij}u_iu_jdAds\\
&-(b-1-k)\int_{t_0}^{t}\int_{S_s} g^{a+a_0}|Du|^{b-k-4}S_{k+2}^{ij}u_iu_j\\
&+b\int_{t_0}^{t}\int_{S_s} g^{a+a_0}|Du|^{b-k-2}S_{k+1}.
\end{aligned} \tag{4.15}
$$

Next we deal with the above term involving S_{k+1}. Choose the coordinate such that $u_n(x_0) = |Du|(x_0)$ and $\{u_{ij}(x_0))\}_{1\le i,j\le n-1} = \{\tilde\lambda_i\delta_{ij}\}_{1\le i,j\le n-1}$ is diagonal, we have

$$\begin{aligned} S_{k+1} =& u_{nn}S_k(\tilde\lambda) + S_{k+1}(\tilde\lambda) - \sum_{i=1}^{n-1} S_{k-1}(\tilde\lambda|i)u_{ni}^2 \\ S_k =& u_{nn}S_{k-1}(\tilde\lambda) + S_k(\tilde\lambda) - \sum_{i=1}^{n-1} S_{k-2}(\tilde\lambda|i)u_{ni}^2, \end{aligned}$$

where $\tilde\lambda = (\tilde\lambda_1, \cdots, \tilde\lambda_{n-1})$ and recall we use the notation $S_k = S_k(D^2u)$. Then we get

$$S_{k+1} = \frac{S_k(\widetilde{\lambda})}{S_{k-1}(\widetilde{\lambda})} S_k - \frac{S_k^2(\widetilde{\lambda})}{S_{k-1}(\tilde\lambda)} + \sum_{i=1}^{n-1} u_{ni}^2 \frac{S_k(\tilde\lambda|i)S_{k-2}(\tilde\lambda|i) - S_{k-1}^2(\tilde\lambda|i)}{S_{k-1}(\tilde\lambda)} + S_{k+1}(\tilde\lambda). \tag{4.16}$$

Noting that $S_m(\tilde\lambda) = |Du|^{-2}S_{m+1}^{ij}u_iu_j = H_m|Du|^m$ is globally defined, we obtain

$$S_{k+1} - \frac{H_k}{H_{k-1}}|Du^\varepsilon|S_k + \frac{H_k^2}{H_{k-1}}|Du^\varepsilon|^{k+1} - H_{k+1}|Du|^{k+1} = \sum_{i=1}^{n-1} u_{ni}^2 \frac{S_k(\tilde\lambda|i)S_{k-2}(\tilde\lambda|i) - S_{k-1}^2(\tilde\lambda|i)}{S_{k-1}(\tilde\lambda)} \le 0.$$

This proves (4.10) Inserting (4.16) into (4.15) and noting that $S_m(\tilde\lambda) = |Du|^{-2}S_{m+1}^{ij}u_iu_j = H_m|Du|^m$ is globally defined , then we have

$$\begin{aligned} & I_{a+a_0,b-1,k+1}(t) - I_{a+a_0,b-1,k+1}(t_0) \\ &\qquad = (a+a_0)\int_{t_0}^t \int_{S_s} g^{a+a_0-1}g'|Du|^b H_k dAds \\ &\qquad + (k+1)\int_{t_0}^t \int_{S_s} g^{a+a_0}|Du|^{b-1}H_{k+1}dAds \\ &\qquad - b\int_{t_0}^t \int_{S_s} g^{a+a_0}|Du|^{b-1}\frac{H_k^2}{H_{k-1}}dAds + b\int_{t_0}^t \int_{S_s} g^{a+a_0}|Du|^{b-k-1}\frac{H_k}{H_{k-1}}S_k dAds \\ &\qquad + b\int_{t_0}^t \int_{S_s} g^{a+a_0}|Du|^{b-k-2}\left(S_{k+1} - \frac{H_k}{H_{k-1}}|Du|S_k + \frac{H_k^2}{H_{k-1}}|Du|^{k+1} - H_{k+1}|Du|^{k+1}\right) \end{aligned} \tag{4.17}$$

Inserting (4.14) and (4.15) into (4.13), we obtain

$$
\begin{aligned}
J_{a+a_0,b,k}(t,t_0) =& - ba\int_{t_0}^{t}\int_{S_s} g^{a+a_0}|Du|^{b-k-1}\frac{H_k}{H_{k-1}}S_k dAds \\
&+ (b+1)\int_{S_t} g^{a+a_0}|Du|^{b-k}S_k dA - (b+1)\int_{S_{t_0}} g^{a+a_0}|Du|^{b-k}S_k dA \\
&+ a\int_{t_0}^{t}\int_{S_s} g^{a+a_0}|Du|^{b-1}H_{k-1}^{-1}\Big(c_{n,k}H_k^2 - (k+1)H_{k-1}H_{k+1}\Big)dAds \\
&+ a\int_{t_0}^{t}\int_{S_s} g^{a+a_0}|Du|^{b-1}H_{k-1}\mathcal{L} dAds \\
&- ab\int_{t_0}^{t}\int_{S_s} g^{a+a_0}|Du|^{b-k-2}\Bigg(S_{k+1} - \frac{H_k}{H_{k-1}}|Du|S_k + \frac{H_k^2}{H_{k-1}}|Du|^{k+1} - H_{k+1}|Du|^{k+1}\Bigg).
\end{aligned} \tag{4.18}
$$

where the function $\mathcal{L}$ is defined by

$$
\begin{aligned}
\mathcal{L} =&(b - c_{n,k})\Big(\frac{H_k}{H_{k-1}}\Big)^2 - (2a + a_0)(\log g)'|Du|\frac{H_k}{H_{k-1}} \\
&+ \Big((\log g)'' + (a + a_0)((\log g)')^2\Big)|Du|^2.
\end{aligned} \tag{4.19}
$$

Now we divide two cases to prove the $\mathcal{L} \geq 0$ under some restrictions on a and b.

Case1: $k < \frac{n}{2}$ **and** $\frac{n}{2} < k < n$**.**

We choose $c_{n,k} = \frac{k(n-k-1)}{n-k}$. Then we have

$$
\begin{aligned}
(\log g)'' + (a + a_0)((\log g)')^2 =&\frac{n-k}{n-2k}u^{-2} + (a + a_0)(\frac{n-k}{n-2k})^2 u^{-2} \\
=&(\frac{n-k}{n-2k})^2 u^{-2}(\frac{n-2k}{n-k} + a + a_0) \\
=&(b - c_{n,k})(\frac{n-k}{n-2k})^2 u^{-2},
\end{aligned} \tag{4.20}
$$

where we choose $a_0 = -2\frac{n-2k}{n-k}$ and we use $a = b - k + 1$. We also have

$$
-(2a + a_0)(\log g)' = 2\frac{n-k}{n-2k}(b - c_{n,k})u^{-1}. \tag{4.21}
$$

Then we have

$$
\mathcal{L} =(b - c_{n,k})\Big(\frac{n-k}{n-2k}|D\log u| - \frac{H_k}{H_{k-1}}\Big)^2. \tag{4.22}
$$

Consequently, we obtain the desired identity.

Case 2: $k = \frac{n}{2}$**.**

We have $c_{n,k} = \frac{n}{2} - 1 > 0$. We require $b \geq \frac{n}{2} - 1$, $a = b - \frac{n}{2} + 1 = b - c_{n,k} \geq 0$ and $a_0 = 0$.

Since $g = e^u$ and thus $(a+a_0)^{-1}(g^{a+a_0})'' = (a+a_0)g^{a+a_0}$. We obtain

$$\mathcal{L} = a\Big(|Du| - \frac{H_k}{H_{k-1}}\Big)^2. \tag{4.23}$$

At last we prove $\mathcal{M} := S_{k+1} - \frac{H_k}{H_{k-1}}|Du|S_k + \frac{H_k^2}{H_{k-1}}|Du|^{k+1} - H_{k+1}|Du|^{k+1}$ is non-positive similar as that in Ma-Zhang [22].

□

From the above formula, we have the following almost monotonicity formula along the level set of u^ε and we prove the first part of Theorem 1.5.

Proposition 4.3. *Let u^ε be the solution of the approximating k-Hessian equation. Assume $\frac{n}{2} < k < n$ and $b \geq c_{n,k} = \frac{k(n-k-1)}{n-k}$ and $b \neq k-1$, then for any $t \in (0,1]$, we have*

$$\frac{d}{dt}I_{a,b,k}(t)\begin{cases} \geq -C\varepsilon t^{\frac{nk}{2k-n}-1} \text{ if } a > 0, \\ \leq C\varepsilon |t|^{\frac{nk}{2k-n}-1} \text{ if } a < 0. \end{cases} \tag{4.24}$$

In particular, we have the following weighted inequality

$$\int_{\partial\Omega} |Du|^{b+1}H_{k-1} \geq \frac{2k-n}{n-k}\int_{\partial\Omega}|Du|^b H_k, \tag{4.25}$$

where u is the unique $C^{1,1}$ solution of the homogeneous k-Hessian equation (1.7).

Proof. We divide two cases.

Case1: $a > 0$

By Proposition 4.2, for any $0 < t_0 < t \leq 1$, we have

$$\begin{aligned} t^2 I'_{a,b,k}(t) - t_0^2 I'_{a,b,k}(t_0) =& J_{a+a_0,b,k}(t) - J_{a+a_0,b,k}(t_0) \\ \geq& - ab\int_{\Omega_{t_0}\setminus\overline{\Omega}_t}(u^\varepsilon)^{a\frac{n-k}{2k-n}+2}|Du^\varepsilon|^{a-1}\frac{H_k}{H_{k-1}}S_k \\ &- (b+1)\int_{S_{t_0}}(u^\varepsilon)^{a\frac{n-k}{2k-n}+2}|Du^\varepsilon|^{a-1}S_k. \end{aligned} \tag{4.26}$$

By the MacLaurin inequality: $\frac{H_k}{H_{k-1}} \leq \frac{C_{n-1}^k}{C_{n-1}^{k-1}}\Big(\frac{H_{k-1}}{C_{n-1}^{k-1}}\Big)^{\frac{1}{k-1}}$ and the uniform C^2-estimates of u^ε (we also use $|Du^\varepsilon| \geq c|x|^{1-\frac{n}{k}}$), for any $x \in \Omega_t^c$, we have

$$\begin{aligned} (u^\varepsilon)^{a\frac{n-k}{2k-n}+2}|Du^\varepsilon|^{a-1}\frac{H_k}{H_{k-1}} \leq& C(u^\varepsilon)^{a\frac{n-k}{2k-n}+2}|Du^\varepsilon|^{a-1}H_{k-1}^{\frac{1}{k-1}} \\ \leq& C|x|^{a\frac{n-k}{k}+2\frac{2k-n}{k}}|x|^{(a-1)\frac{k-n}{k}}|x|^{-1} \\ =& C|x|^{2-\frac{n}{k}} \leq Ct, \end{aligned}$$

then

$$\int_{\Omega_{t_0}\setminus\overline{\Omega}_t} (u^\varepsilon)^{a\frac{n-k}{2k-n}+2}|Du^\varepsilon|^{a-1}\frac{H_k}{H_{k-1}}S_k \le C\varepsilon t^{\frac{n(k-1)+2k}{2k-n}}.$$

Similarly, we have

$$\int_{S_{t_0}} (u^\varepsilon)^{a\frac{n-k}{2k-n}+2}|Du^\varepsilon|^{a-1}S_k \le C\varepsilon t_0^{\frac{n(k-1)+2k}{2k-n}},$$

where we use $|S_{t_0}| \le Ct_0^{\frac{k(n-1)}{2k-n}}$ (see Lemma 4.1).

Thus we get

$$t^2 I'_{a,b,k}(t) - t_0^2 I'_{a,b,k}(t_0) \ge -C\varepsilon t^{\frac{n(k-1)+2k}{2k-n}} - C\varepsilon t_0^{\frac{n(k-1)+2k}{2k-n}}. \tag{4.27}$$

By the uniform C^2 estimates for u^ε and $|Du^\varepsilon| \ge c|x|^{1-\frac{n}{k}}$, we have for any $t_0 \in (0,1]$

$$t_0^2\left|I'_{a,b,k}(t_0)\right| \le Ct_0. \tag{4.28}$$

Let t_0 tend to 0 in (4.27), we have

$$I'_{a,b,k}(t) \ge -C\varepsilon t^{\frac{nk}{2k-n}-1}. \tag{4.29}$$

In particular, taking $t = 1$ we have

$$I'_{a,b,k}(1) \ge -C\varepsilon. \tag{4.30}$$

On the other hand, by (4.12), we have

$$I'_{a,b,k}(1) \le a\frac{n-k}{n-2k}\int_{\partial\Omega}|Du^\varepsilon|^{b+1}H_{k-1} - a\int_{\partial\Omega}|Du^\varepsilon|^b H_k + C\varepsilon. \tag{4.31}$$

Consequently, we get

$$\frac{n-k}{2k-n}\int_{\partial\Omega}|Du^\varepsilon|^{b+1}H_{k-1} - \int_{\partial\Omega}|Du^\varepsilon|^b H_k \ge -a^{-1}C\varepsilon \tag{4.32}$$

Since $|Du^\varepsilon|$ converges to $|Du|$ on $\partial\Omega$, we finish the proof of (4.25) by taking $\varepsilon \to 0$ in (4.32).

Case2: $a < 0$

Similar as case 1, we have

$$I'_{a,b,k}(t) \le C\varepsilon t^{\frac{nk}{2k-n}-1}. \tag{4.33}$$

On the other hand, we have

$$I'_{a,b,k}(t) \ge a\frac{n-k}{n-2k}\int_{\partial\Omega}|Du^\varepsilon|^{b+1}H_{k-1} - a\int_{\partial\Omega}|Du^\varepsilon|^b H_k. \tag{4.34}$$

Then the desired inequality follows. □

Next we prove the second part of Theorem 1.5.

Lemma 4.4. *Assume $k = \frac{n}{2}$ and $b \geq \frac{n}{2} - 1$. We have*

$$I'_{a,b,k}(t) \geq -C\varepsilon e^{nt}, \tag{4.35}$$

In particular, we have

$$\int_{\partial\Omega} |Du|^{b+1} H_{k-1} \geq \int_{\partial\Omega} |Du|^b H_k, \tag{4.36}$$

where u is the unique $C^{1,1}$ solution the homogeneous k-Hessian equation (1.9).

Proof. By Proposition 4.2 and similar as the argument in Proposition 4.3, for any $-\infty < t_0 \leq s < t \leq 0$, we have

$$I'_{a,b,k}(t) - I'_{a,b,k}(s) \geq -C\varepsilon e^{nt}.$$

Integrating the above from t_0 to t, we have

$$I_{a,b,k}(t) - I_{a,b,k}(t_0) \leq \Big(I'_{a,b,k}(t) + C\varepsilon e^{nt}\Big)(t - t_0),$$

Then

$$\Big(I'_{a,b,k}(t) + C\varepsilon e^{nt}\Big)(-tt_0^{-1} + 1) \geq -t_0^{-1}(I_{a,b,k}(t) - I_{a,b,k}(t_0)) \geq Ct_0^{-1}.$$

let t_0 tend to 0 and note that $I_{a,b,k}(t)$ is uniformly bounded which follows from the C^2-estimates of u^ε and $|Du^\varepsilon| \geq c|x|^{1-\frac{n}{k}}$, we obtain

$$I'_{a,b,k}(t) \geq -C\varepsilon e^{nt}.$$

On the other hand, we have

$$I'_{a,b,k}(0) \leq a \int_{\partial\Omega} |Du^\varepsilon|^{b+1} H_{k-1} - a \int_{\partial\Omega} |Du^\varepsilon|^b H_k + C\varepsilon.$$

Combining the above two inequalities and noting that $|Du^\varepsilon| \to |Du|$, we get

$$\int_{\partial\Omega} |Du|^{b+1} H_{k-1} \geq \int_{\partial\Omega} |Du|^b H_k. \tag{4.37}$$

□

When $k < \frac{n}{2}$, we have the following inequality.

Lemma 4.5. *Let u^ε be the solution of the approximating k-Hessian equation. Assume $k < \frac{n}{2}$, and $b \geq c_{n,k}$, then for any $-\infty < t_0 \leq t \leq -1$, we have*

$$t^2 I'_{a,b,k}(t) - t_0^2 I'_{a,b,k}(t_0) \geq -C\varepsilon |t|^{-\frac{nk}{n-2k}-1}. \tag{4.38}$$

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