THE DIRICHLET PROBLEM OF HOMOGENEOUS COMPLEX *k*-HESSIAN EQUATION IN A (*k* – 1)-PSEUDOCONVEX DOMAIN WITH ISOLATED SINGULARITY

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ABSTRACT. In this paper, we consider the homogeneous complex k-Hessian equation in $\Omega \setminus \{0\}$. We prove the existence and uniqueness of the $C^{1,\alpha}$ solution by constructing approximating solutions. The key point for us is to construct the subsolution for approximating problem and establish uniform gradient estimates and complex Hessian estimates which is independent of the approximation.

1. INTRODUCTION

Let Ω be a smooth bounded domain of \mathbb{C}^n and k be an integer such that $1 \le k \le n$. We consider the homogeneous complex *k*-Hessian equations

$$(dd^{c}u)^{k} \wedge \omega^{n-k} = 0 \quad \text{in } \Omega \setminus \{0\}.$$

Let *u* be a real C^2 function in \mathbb{C}^n and $\lambda = (\lambda_1, \dots, \lambda_n)$ be the eigenvalues of the complex Hessian $(\frac{\partial^2 u}{\partial z_i \partial \overline{z}_i})$, the complex *k*-Hessian operator is defined by

$$H_k[u] := \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}$$

where $1 \le k \le n$. Using the operators $d = \partial + \overline{\partial}$ and $d^c = \sqrt{-1}(\overline{\partial} - \partial)$, such that $dd^c = 2\sqrt{-1}\partial\overline{\partial}$, one gets

$$(dd^{c}u)^{k} \wedge \omega^{n-k} = 4^{n}k!(n-k)!H_{k}[u]d\lambda,$$

where $\omega = dd^c |z|^2$ is the fundamental Kähler form and $d\lambda$ is the volume form. When k = 1, $H_1[u] = \frac{1}{4}\Delta u$. When k = n, $H_n[u] = \det u_{i\bar{j}}$ is the complex Monge-Ampère operator.

1.1. Some known results and motivations. Let $S_k(D^2u)$ be the *k*-Hessian of a real C^2 function *u* in \mathbb{R}^n . When k > 1, the Hessian equations $S_k(D^2u) = f$ and $H_k[u] = f$ are both nonlinear. When f > 0, the Hessian equation is nondegenerate. When f vanishes somewhere, the Hessian equation is degenerate.

1.1.1. *Results on bounded domain*. For the Hessian equation on \mathbb{R}^n , its Dirichlet problem with positive *f*

$$\begin{cases} S_k(D^2 u) = f & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

was studied by Ivochkina [21] for k = 1, 2, 3, n on convex domain with further assumptions on f and by Caffarelli-Nirenberg-Spruck [10] for general f > 0 and $k = 1, 2, \dots, n$ by assuming Ω is (k - 1)-convex. B. Guan [16] showed the geometric condition on Ω could be removed by assumption of existence of a strict subsolution. In [35], Trudinger-Wang developed a Hessian measure theory for Hessian operator. One can see a survey in Wang [38] for more related topics. For the complex *k*-Hessian equation in \mathbb{C}^n , Li [30] solved its Dirichlet problem via the subsolution approach.

For Monge-Ampère equation in a bounded domain of \mathbb{R}^n , when f = 0, Caffarelli-Nirenberg-Spruck [11] proved the $C^{1,1}$ regularity in a bounded convex domain. For general $f \ge 0$, Guan-Trudinger-Wang [20] proved the $C^{1,1}$ regularity result when $f^{\frac{1}{n-1}} \in C^{1,1}$. Due to the counterexample by Wang [37], $C^{1,1}$ regularity is optimal. For *k*-Hessian equation in \mathbb{R}^n , the $C^{1,1}$ regularity is obtained by Krylov [24, 25] and Ivochkina-Trudinger-Wang [22].

For complex Monge-Ampère equation, Lempert [26, 27] proved the Dirichlet problem admits a smooth solution with a logarithm pole at the origin on a strictly convex punctured domain $\Omega \setminus \{0\}$ when f = 0. As for strongly pseudoconvex domain, Guan [17] and Błocki [5] proved the solution is $C^{1,1}$. In [19], Guan obtained the $C^{1,1}$ regularity for the solution on a ring domain. For general $f \ge 0$, the optimal $C^{1,1}$ regularity was in Caffarelli-Kohn-Nirenberg-Spruck [8],Krylov [24, 25] for strongly pseudoconvex domain.

1.1.2. *Results on unbounded domain.* The viscosity solution to nondegenerate *k*-Hessian equation on unbounded domain has been researched extensively. Caffarelli-Li [9] solved the viscosity solution to the Monge-Ampère equation det $D^2u = 1$ with prescribed asymptotic behavior at infinity. Bao-Li-Li [2] studied the *k*-Hessian equation case. For the related results on other type nondegenerate fully nonlinear equations, one can see [1,28,31].

In [29], Li-Wang consider the det $D^2 u = 0$ on a strip region $\Omega := \mathbb{R}^n \times [0, 1]$. By assuming two boundary functions are both strictly convex $C^{1,1}(\mathbb{R}^{n-1})$ functions, they obtained the solutions is $C^{1,1}(\overline{\Omega})$. If the boundary functions are locally uniformly convex $C^{k+2,\alpha}(\mathbb{R}^{n-1})$ function, then u is the unique $C^{k+2,\alpha}(\overline{\Omega})$ function.

Recently, Xiao [39] and Ma-Zhang [34] proved the $C^{1,1}$ regularity of Dirichlet fot the homogeneous k-Hessian equation out of $\Omega \subset \mathbb{R}^n$, by assuming Ω is starshaped, (k - 1)-convex and and $1 \leq k < \frac{n}{2}$ or Ω is (k - 1)-convex and $1 \leq k \leq n$ respectively. For homogeneous complex k-Hessian equation, Gao-Ma-Zhang [15] obtained the $C^{1,1}$ regularity.

1.1.3. *Motivations*. Our paper is motivated by the research on the regularity of extremal function or Green function. In [23], Klimek introduced the following extremal function

$$g_{\Omega}(z,\xi) = \sup\{v \in \mathcal{PSH}(\Omega) : v < 0, v(z) \le \log|z-\xi| + O(1)\}.$$

 $g_{\Omega}(z,\xi)$ is also call the pluricomplex Green function on Ω with a logarithminc pole at ξ . If Ω is hyperconvex, Demailly [13] showed that $u(z) = g_{\Omega}(z,\xi)$ is continuous and is a unique solution to the homogeneous complex Monge-Ampére equation,

(1.1)
$$\begin{cases} (dd^c u)^n = 0 & \text{in } \Omega \setminus \{\xi\}, \\ u = 0 & \text{on } \partial\Omega, \\ u(z) = \log|z - \xi| + O(1) & \text{as } z \to \xi. \end{cases}$$

If Ω is strictly convex domain in \mathbb{C}^n with smooth boundary, Lempert [26] proved (1.1) admits a unique plurisubharmonic solution which is smooth. In the strongly pseudonconvex case, B. Guan [17] proved $g_{\Omega}(z,\xi) \in C^{1,\alpha}(\overline{\Omega} \setminus \{\xi\})$ and later, Błocki improved it to $C^{1,1}(\overline{\Omega} \setminus \{\xi\})$ in [5] and generalized it to several poles in [6]. Due to the counterexamples found by Bedford-Demailly [3], $C^{1,1}$ regularity is optimal. P. Guan [19] established $C^{1,1}$ regularity of extremal function associated to intrinsic

P. Guan [19] established $C^{1,1}$ regularity of extremal function associated to intrinsic norms of Chen-Levine-Nirenberg [12] and Beford-Taylor [4] by considering

$$\begin{cases} (dd^{c}u)^{n} = 0 & \text{ in } \Omega_{0} \setminus (\cup_{i=1}^{m} \Omega_{i}), \\ u = 0 & \text{ on } \partial \Omega_{i}, \ i = 1, \cdots, n \\ u = 1 & \text{ on } \partial \Omega_{0}. \end{cases}$$

Applying the techniques from [19], B. Guan proved the $C^{1,1}$ regularity of pluricomplex Green function for the union of a finite collection of strongly pseudonconvex domains in \mathbb{C}^n .

In [14], we considered the following homogeneous (real) k-Hessian equation in a punctured domain

(1.2)
$$\begin{cases} S_k(D^2u) = 0 & \text{in } \Omega \setminus \{0\}, \\ u = c_k & \text{on } \partial \Omega, \\ u(x) = h_k(x) & \text{as } x \to 0 \end{cases}$$

where $c_k = 1$ and $h_k(x) = 0$ if $k > \frac{n}{2}$, $c_k = -1$ and $h_k(x) = -|x|^{2-\frac{n}{k}} + O(1)$ if $k < \frac{n}{2}$, $c_k = 0$ and $h_k(x) = \log |x| + O(1)$ if $k = \frac{n}{2}$. Assume that Ω is (k - 1)-convex, we proved the existence and uniqueness of $C^{1,1}$ solution to (1.2). Moreover the solution can be controlled pointwisely by fundamental solutions of homogenous k-Hessian equations up to the second order. If Ω is also starshaped with respect to the origin, we proved the positive lower bound of the gradient of the solution and then we show a nearly monotonicity formula along the level set of the approximating solution.

1.2. **Our result.** In this section, we consider the following problem for complex *k*-Hessian equation

(1.3)
$$\begin{cases} (dd^{c}u)^{k} \wedge \omega^{n-k} = 0 & \text{in } \Omega \setminus \{0\}, \\ u = -1 & \text{on } \partial\Omega, \\ u(z) = -|z|^{2-\frac{2n}{k}} + O(1) & \text{as } z \to 0. \end{cases}$$

Theorem 1.1. Assume $1 \le k < n$. Let Ω be a smooth (k - 1)-pseudoconvex domain containing the origin. Then there exists a unique k-subharmonic solution u of (1.3) in $C^{1,\alpha}(\overline{\Omega} \setminus \{0\})$. Moreover, u satisfies the estimate

(1.4)
$$-C \le u + |z|^{2-\frac{2n}{k}} \le 0,$$

(1.5)
$$|Du| + |z||\Delta u| \le C|z|^{1-\frac{2n}{k}}.$$

Here k-subharmonic function and (k-1)-pesudoconvex domain are introduced in Section 2. We suppose Ω contains the origin and we use the notation $\Omega_r = \Omega \setminus \overline{B}_r(0)$. We use B_r instead of $B_r(0)$ for short. To prove Theorem 1.1, we consider the approximating problem

(1.6)
$$\begin{cases} H_k[u^{\varepsilon,r}] = \varepsilon & \text{ in } \Omega_r, \\ u = \underline{u} & \text{ on } \partial B_r, \end{cases}$$

where <u>u</u> is a subsolution constructed in Section 3. The solution u to (1.3) with be obtained by approximating solution $u^{\varepsilon,r}$ to (1.6). The existence of $u^{\varepsilon,r}$ follows from subsolution method in [30].

The rest of the paper is organized as follows. In Section 2, we first give the definition and some notations. Then we recall some new gradient estimates and complex Hessian estimates in [15] motivated by B. Guan [18], which will be used in the proof of (1.5). In Section 3, we establish uniform gradient estimates and complex Hessian estimates. Theorem 1.1 will be proved in the last section.

2. Preliminaries

2.1. Elementary symmetric functions. For any $k = 1, \dots, n$ and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, the *k*-th elementary symmetric function on λ is defined by

$$S_k(\lambda) := \sum_{1 \le i_1 < \cdots < i_k \le n} \lambda_{i_1} \cdots \lambda_{i_k}$$

Let $S_k(\lambda|i)$ be the symmetric function with $\lambda_i = 0$. Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be an $n \times n$ matrix. Let $S_k(A)$ be the *k*-th elementary symmetric function on *A*, which is the sum of $k \times k$ principal minors of *A*. We use the convention that $S_0(A) = 1$. It is clear that $S_k(A) = S_k(\lambda(A))$, where $\lambda(A)$ are the eigenvalues of *A*.

The elementary symmetric functions have the following simple properties from [32].

(2.1)
$$S_k(\lambda) = S_k(\lambda|i) + \lambda_i S_{k-1}(\lambda|i),$$

and

(2.2)
$$\sum_{i=1}^{n} S_k(\lambda|i) = (n-k)S_k(\lambda).$$

Recall the Γ_k -cone is defined by

$$\Gamma_k := \{\lambda \in \mathbb{R}^n \mid S_i(\lambda) > 0, 1 \le i \le k\}$$

For $\lambda \in \Gamma_k$ and $1 \le l \le k$, the well-known MacLaurin inequality (see [32]) says

$$\left(\frac{S_k(\lambda)}{C_n^k}\right)^{\frac{1}{k}} \leq \left(\frac{S_l(\lambda)}{C_n^l}\right)^{\frac{1}{l}}.$$

One can find the concavity property of $S_k^{\frac{1}{k}}$ in [10].

Proposition 2.1. $S_k^{\frac{1}{k}}$ is a concave function in Γ_k .

2.2. *k*-subharmonic solutions. In this section we give the definition of *k*-subharmonic functions and definition of *k*-pseudoconvex domains. One can see the lecture notes by Wang [38] for more properties of the *k*-Hessian operator, and see Błocki [7] for those of the complex *k*-Hessian operator. We following the definition by Błocki [7] to give the definition of *k*-subharmonic functions.

Definition 2.2. Let α be a real (1, 1)-form in U, a domain of \mathbb{C}^n . We say that α is k-positive in U if the following inequalities hold

$$\alpha^{j} \wedge \omega^{n-j} \geq 0, \forall j = 1, \cdots, k.$$

Definition 2.3. *Let* U *be a domain in* \mathbb{C}^n *.*

(1). A function $u : U \to \mathbb{R} \cup \{-\infty\}$ is called k-subharmonic if it is subharmonic and for all k-positive real (1, 1)-form $\alpha_1, \dots, \alpha_{k-1}$ in U,

$$dd^{c}u \wedge \alpha_{1} \wedge \cdots \wedge \alpha_{k-1} \wedge \omega^{n-k} \geq 0.$$

The class of all k-subharmonic functions in U will be denoted by $SH_k(U)$.

(2). A function $u \in C^2(U)$ is called k-subharmonic (strictly k-subharmonic) if $\lambda(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j}) \in \overline{\Gamma}_k$ $(\lambda(\frac{\partial^2 u}{\partial z_i \partial \bar{z}_i}) \in \Gamma_k)$.

If $u \in SH_k(U) \cap C(U)$, $(dd^c u)^k \wedge \omega^{n-k}$ is well defined in pluripotential theory by Błocki [7]. We need the following comparison principle by Błocki [7] to prove the uniqueness of the continuous solution of the problem (1.3).

Lemma 2.4. Let U be a bounded domain in \mathbb{C}^n , $u, v \in SH_k(U) \cap C(\overline{U})$ satisfy

$$\begin{cases} (dd^{c}u)^{k} \wedge \omega^{n-k} \ge (dd^{c}v)^{k} \wedge \omega^{n-k} & \text{ in } U, \\ u \le v & \text{ on } \partial U \end{cases}$$

Then $u \leq v$ in U.

2.3. Gradient estimates and complex Hessian estimates. Motivated by [18], we proved the following new gradient estimates and complex Hessian estimates in [15].

Theorem 2.5. Let $u \in C^3(U) \cap C^1(\overline{U}) \cap S\mathcal{H}_k(U)$ be a negative solution to $H_k[u] = f$ in U, where $f \in C^1(\overline{U})$ is positive. Denote by

$$P = |Du|^2 (-u)^{-\frac{2n-k}{n-k}}.$$

Then

(2.3)
$$\max_{\overline{U}} P \le \max\left\{\max_{\partial U} P, \max_{\overline{U}} \left(\frac{2(n-k)}{k(2n-k)}\right)^2 (-u)^{-\frac{k}{n-k}} |D\log f|^2\right\}$$

Theorem 2.6. Let $u \in C^4(U) \cap C^2(\overline{U}) \cap S\mathcal{H}_k(U)$ be a negative solution to $H_k[u] = f$ in U, where $f \in C^2(\overline{U})$ is positive. Assume that $P = |Du|^2(-u)^{-\frac{2n-k}{n-k}}$, $(-u)^{-\frac{k}{n-k}}|D\log f|^2$ and $(-u)^{-\frac{k}{n-k}}|D^2\log f|$ are bounded. Denote by

$$H = u_{\xi\bar{\xi}}(-u)^{-\frac{n}{n-k}}(M-P)^{-\sigma},$$

where $M = 2 \max_{\overline{U}} P + 1$, $\sigma \leq \frac{n(n-k)}{8(2n-k)^2}$. Then we have

(2.4)
$$\max_{\overline{U}} H \le C + \max_{\partial U} H,$$

where C is a positive constant depending only on n, k, P, $(-u)^{-\frac{k}{n-k}}|D\log f|^2$ and $(-u)^{-\frac{k}{n-k}}|D^2\log f|$.

We need the following lemma by P. Guan [19] to construct the subsolution of the complex k-Hessian equation in a ring domain.

Lemma 2.7. Suppose that U is a bounded smooth domain in \mathbb{C}^n . For $h, g \in C^m(U)$, $m \ge 2$, for all $\delta > 0$, there is an $H \in C^m(U)$ such that (1) $H \ge \max\{h, g\}$ and

$$H(z) = \begin{cases} h(z), & \text{if } h(z) - g(z) > \delta, \\ g(z), & \text{if } g(z) - h(z) > \delta; \end{cases}$$

(2) There exists $|t(z)| \le 1$ such that

$$\left\{H_{i\bar{j}}(z)\right\} \ge \left\{\frac{1+t(z)}{2}g_{i\bar{j}} + \frac{1-t(z)}{2}h_{i\bar{j}}\right\}, \text{ for all } x \in \{|g-h| < \delta\}.$$

We can prove that *H* is *k*-subharmonic if *f* and *g* are both *k*-subharmonic by the concavity of $S^{\frac{1}{k}}$ in Proposition 2.1.

At the last of this subsection, we recall the definition of (k - 1)-pseudoconvex domain.

Definition 2.8. A C^2 domain U is called (k - 1)-pseudoconvex if there is $C_U > 0$, such that $\lambda(-d_{i\bar{j}} + C_U(d^2)_{i\bar{j}}) \in \Gamma_k$ on ∂U , where $d(z) = \text{dist}(z, \partial U)$ is the distance function from z to ∂U .

3. Solving the approximating problem in $\Omega \setminus B_r$

In this section, we will solve the approximating problem by a-priori estimates and the subsolution method. Before this, we make an assumption on Ω .

Assumption 3.1. Assume Ω contains the origin and $B_{r_0} \subset \Omega \subset B_{(1-\tau_0)R_0}$ for some $\tau_0 \in (0, \frac{1}{2})$.

Denote by $\Omega^{\mu} = \{z \in \Omega : d(z) < \mu\}$. In this section, we use *C* and *c* with subscript to denote some positve constant which are independent of ε and *r*.

The following lemma about (k - 1)-pesudoconvex domain in \mathbb{C}^n is a parallel version to (k - 1)-convex domain in \mathbb{R}^n with can be found in [10, Section 3]. It plays an important roles in constructing the subsolution.

Lemma 3.2. Let Ω be a smooth (k-1)-pseudoconvex bounded domain. There exists $\mu_0 \in (0, \frac{1}{2C_{\Omega}})$ small enough such that $B_{r_0} \subset \{z \in \Omega : d(z) > 2\mu_0\}$. Moreover $\rho := -d + C_{\Omega}d^2$ is smooth and strictly k-subharmonic and $H_k[\rho] \ge \epsilon_0$ in $\overline{\Omega^{2\mu_0}}$ for some $\epsilon_0 > 0$.

3.1. The approximating equation. We will approximate the solution to the homogeneous complex k-Hessian equation in $\Omega \setminus \{0\}$ by solutions to a sequence of nongenerate equation in Ω_r . The existance of approximating solution can be obtained if we can construct a smooth subsolution. In the following, we use the technique from P. Guan [19] to construct a subsolution.

Denote $w := -|z|^{2-\frac{2n}{k}} + R_0^{2-\frac{2n}{k}} - 1 + a_0 \frac{|z|^2}{R_0^2}$, where $a_0 = \frac{1}{2} ((1 - \tau_0)^{2-\frac{2n}{k}} - 1) R_0^{2-\frac{2n}{k}}$. Then by $\Omega \subset B_{(1-\tau_0)R_0}$, we have

$$w \leq -\frac{1}{2} ((1-\tau_0)^{2-\frac{2n}{k}} - 1) R_0^{2-\frac{2n}{k}} - 1 \quad \text{in } \overline{\Omega}.$$

By Proposition 2.1, we have

$$H_{k}^{\frac{1}{k}}[w] \ge H_{k}^{\frac{1}{k}}[-|z|^{2-\frac{2n}{k}}] + H_{k}^{\frac{1}{k}}[a_{0}\frac{|z|^{2}}{R_{0}^{2}}] = (C_{n}^{k})^{\frac{1}{k}}a_{0}R_{0}^{-2} \quad \text{in } \Omega.$$

Then by Lemma 2.7, we can construct a smooth and strictly k-subharmonic function \underline{u} from w and ρ .

Lemma 3.3. There is a strictly k-subharmonic function $u \in C^{\infty}(\overline{\Omega}_r)$ satisfying

$$\underline{u}(z) = \begin{cases} K_0 \rho(z) - 1 & \text{if } d(z) \le \frac{\mu_0}{M_0}, \\ w(z) & \text{if } d(z) > \mu_0, \end{cases}$$
$$\underline{u}(z) \ge \max\{K_0 \rho(z) - 1, w(z)\} & \text{if } \frac{\mu_0}{M_0} \le d(z) \le \mu_0 \\ H_k[\underline{u}] \ge \epsilon_1 := \min\{C_n^k a_0^k R_0^{-2k}, K_0^k \epsilon_0\} & \text{in } \Omega, \end{cases}$$

where K_0 and M_0 are uniform constants.

Proof. Since $B_{r_0} \subset \{z \in \Omega : d(z) > 2\mu_0\}$, by choosing $K_0 = \frac{r_0^{2-\frac{2n}{k}}}{C_{\Omega}\mu_0^2 - \mu_0}$, we find $\forall z \in$ $\overline{\Omega^{2\mu_0}} \setminus \Omega^{\mu_0}$, there holds

$$w - (K_0\rho - 1) = -|z|^{2-\frac{2n}{k}} + R_0^{2-\frac{2n}{k}} + a_0 \frac{|z|^2}{R_0^2} - K_0\rho$$

$$\geq -r_0^{2-\frac{2n}{k}} + R_0^{2-\frac{2n}{k}} - K_0(-\mu_0 + C_\Omega\mu_0^2)$$

$$\geq R_0^{2-\frac{2n}{k}},$$

For any $z \in \overline{\Omega_{\frac{\mu_0}{M_0}}} := \{z \in \overline{\Omega} : d(z) \le \frac{\mu_0}{M_0}\}$, there also holds

$$(K_0\rho - 1) - w \ge \frac{1}{2} \Big((1 - \tau_0)^{2 - \frac{2n}{k}} - 1) \Big) R_0^{2 - \frac{2n}{k}} + K_0 \Big(-\frac{\mu_0}{M_0} + C_\Omega (\frac{\mu_0}{M_0})^2 \Big) \ge \frac{1}{4} \Big((1 - \tau_0)^{2 - \frac{2n}{k}} - 1 \Big) R_0^{2 - \frac{2n}{k}} - 1 \Big)$$

provided that M_0 is a positive solution to

(3.1)
$$K_0(-\frac{\mu_0}{M_0} + C_\Omega(\frac{\mu_0}{M_0})^2) \ge -\frac{1}{4}((1-\tau_0)^{2-\frac{2n}{k}} - 1)R_0^{2-\frac{2n}{k}}$$

In fact, we can choose τ_0 small enough such that (3.1) holds if $M_0 > 1$. Take $\delta := \min\{\frac{1}{4}((1-\tau_0)^{2-\frac{2n}{k}}-1)R_0^{2-\frac{2n}{k}}, R_0^{2-\frac{2n}{k}}\}$ and we apply Lemma 2.7 with $g = K_0\rho - 1$, h = w and δ on $\Omega^{2\mu_0}$, we obtain a smooth and strictly k-subharmonic function \underline{u} in $\Omega^{2\mu_0}$. Moreover $\underline{u} = K_0 \rho - 1$ in $\Omega^{\frac{\mu_0}{M_0}}$, and $\underline{u} = w$ in $\overline{\Omega^{2\mu_0}} \setminus \Omega^{\mu_0}$. At last, we set $\underline{u} = w$ in $\Omega_r \setminus \Omega^{2\mu_0}$. By Lemma 2.7, we have

$$H_k[\underline{u}] \ge \min\{H_k[w], H_k[K_0\rho]\} \ge \min\{C_n^k a_0^k R_0^{-2k}, K_0^k \epsilon_0\}$$

We now consider the approximating equation

(3.2)
$$\begin{cases} H_k[u] = \varepsilon & \text{ in } \Omega_r, \\ u = \underline{u} & \text{ on } \partial \Omega_r \end{cases}$$

Then \underline{u} is a strictly subharmonic solution of above equation for any $\varepsilon < \epsilon_1$. By Li [30], (3.2) admits a strictly *k*-subharmonic solution $u^{\varepsilon,r} \in C^{\infty}(\overline{\Omega}_r)$. Let $r_1 = \min\{2\frac{2k}{2k-n}R_0, (\frac{2a_0}{R_0^2})^{-\frac{k}{2n}}\}$, $\forall r \leq r_1$, since $\underline{u} = -1$ on $\partial\Omega$ and $\underline{u} = -r^{2-\frac{2n}{k}} + R_0^{2-\frac{2n}{k}} - 1 + a_0\frac{r^2}{R_0^2}$ on ∂B_r , we have $\underline{u}|_{\partial\Omega_r} \leq -1$. By maximum principle, we have $u^{\varepsilon,r} \leq -1$ when $r \leq r_1, \varepsilon \leq \epsilon_1$.

In the following, we want to derive a (ε, r) -independent uniform C^2 estimate for $u^{\varepsilon, r}$. We prove the following

Theorem 3.4. Suppose Ω be a smooth (k - 1)-pseudoconvex bounded domain. Assume that Ω satisfies Assumption 3.1. For sufficient small r > 0 and $\varepsilon > 0$, (3.2) admits a k-subharmonic solution $u^{\varepsilon,r}$, where \underline{u} is constructed above. Moreover, $u^{\varepsilon,r}$ satisfies the following estimates,

$$(3.3) -|z|^{2-\frac{2n}{k}} + R_0^{2-\frac{2n}{k}} - 1 + a_0 \frac{|z|^2}{R_0^2} \le u^{\varepsilon,r} \le -|z|^{2-\frac{2n}{k}} + r_0^{2-\frac{2n}{k}} - 1,$$

$$|Du^{\varepsilon,r}| \le C|z|^{1-\frac{2n}{k}},$$

$$(3.5) |\partial \bar{\partial} u^{\varepsilon,r}| \le C|z|^{-\frac{2n}{k}},$$

where *C* is a uniform positive constant which is independent of ε and *r*.

In addition, if Ω is starshaped with respect to the origin, there is a uniform positive constant *c* independent of ε and *r* such that

$$(3.6) |Du^{\varepsilon,r}| \ge c_0 |z|^{1-\frac{2n}{k}}$$

3.2. C^0 estimate. Since <u>u</u> is a subsolution to (3.2), we obtain that

$$(3.7) u^{\varepsilon,r} \ge \underline{u} \quad \text{in } \Omega_r.$$

Let

$$\overline{u} = -|z|^{2-\frac{2n}{k}} + r_0^{2-\frac{2n}{k}} - 1.$$

By taking $r \le \min\{r_1, r_2\}$, where $r_2 = R_0 (r_0^{2 - \frac{2n}{k}} - R_0^{2 - \frac{2n}{k}})^{\frac{1}{2}} a_0^{-\frac{1}{2}}$, we have $u^{\varepsilon, r} \le \overline{u} \text{ on } \partial\Omega_r$.

Note that $H_k[u^{\varepsilon,r}] = \varepsilon > 0 = H_k[\overline{u}]$ in Ω_r , it follows that

$$(3.8) u^{\varepsilon,r} \le \overline{u} \quad \text{in } \Omega_r$$

By (3.7) and (3.8), we obtain

$$-|z|^{2-\frac{2n}{k}} + R_0^{2-\frac{2n}{k}} - 1 + a_0 \frac{|z|^2}{R_0^2} \le u^{\varepsilon,r} \le -|z|^{2-\frac{2n}{k}} + r_0^{2-\frac{2n}{k}} - 1.$$

This gives the C^0 estimate (3.3).

3.3. **Gradient estimates.** Base on the key estimate (2.3), we can prove the global gradient estimate in this subsection.

3.3.1. Reducing global gradient estimates to boundary gradient estimates. Since $u^{\varepsilon,r} < 0, f = \varepsilon$, by Theorem 2.5, we have

$$\max_{\overline{\Omega_r}} P = \max_{\partial \Omega_r} P.$$

3.3.2. Boundary gradient estimates. To prove boundary gradient estimates, we will construct barriers near $\partial \Omega$ and ∂B_r respectively.

Since $u^{\varepsilon,r} = \underline{u} = -1$ on $\partial \Omega$ and $u^{\varepsilon,r} \ge \underline{u}$ in Ω_r , we have

$$(3.9) |Du^{\varepsilon,r}| = u_v^{\varepsilon,r} \le \underline{u}_v \quad \text{on } \partial\Omega$$

where v is the unit outer normal to $\partial\Omega$. Let $r_3 \leq \min\{r_1, r_2, 1\}$ and h_1 be the harmonic function h_1 in $\Omega \setminus B_{r_3}$ with $h_1 = -1$ on $\partial\Omega$ and $h_1 = -r_3^{2-\frac{2n}{k}} + r_0^{2-\frac{2n}{k}} - 1$ on ∂B_{r_3} . Then we have $h_1 \geq u^{\varepsilon,r}$ on $\partial\Omega_{r_3}$. So $h_1 \geq u^{\varepsilon,r}$ in Ω_{r_3} , and it follows that

(3.10)
$$u_{\nu}^{\varepsilon,r} \ge h_{1,\nu} > 0 \quad \text{on } \partial\Omega,$$

That is there exist a positive constant C such that

$$(3.11) 0 < C^{-1} \le u_{\nu}^{\varepsilon,r} \le C \text{on } \partial\Omega.$$

Let h_2 be a harmonic function with $h_2 = \underline{u}$ on ∂B_r and $h_2 = \overline{u} = -\frac{1}{2}|z|^{2-\frac{2n}{k}}$ on ∂B_{2r} . Let

$$\tilde{h}_2(z) = r^{\frac{2n}{k}-2}(h_2(rz) + r^{2-\frac{2n}{k}}) = r^{\frac{2n}{k}-2}h_2(rz) + 1.$$

Then \tilde{h}_2 is a harmonic function in $B_2 \setminus B_1$ with $\tilde{h}_2 = a_0 r^{\frac{2n}{k}} R_0^{-2} + r^{\frac{2n}{k}-2} (R_0^{2-\frac{2n}{k}} - 1)$ on ∂B_1 and $\tilde{h}_2 = -2^{1-\frac{2n}{k}}$ on ∂B_2 . Let

(3.12)
$$\tilde{u} = r^{\frac{2n}{k}-2}u^{\varepsilon,r}(rz) + 1,$$

and

(3.13)
$$\underline{\tilde{u}} = r^{\frac{2n}{k}-2}\underline{u}(rz) + 1.$$

By maximum principle, we have

$$\underline{\tilde{u}} \leq \tilde{u} \leq \tilde{h}_2 \text{ in } B_2 \setminus B_1.$$

Note that

$$\underline{\tilde{u}} = \tilde{u} = \tilde{h}_2 = a_0 r^{\frac{2n}{k}} R_0^{-2} + r^{\frac{2n}{k}-2} (R_0^{2-\frac{2n}{k}} - 1) \text{ in } \partial B_1.$$

We obtain

$$D'\tilde{u} = D'\underline{\tilde{u}} = D'\tilde{h}_2 = 0 \quad \text{on } \partial B_1,$$

and

$$0 < c(n,k) \le \underline{\tilde{u}}_{\nu} \le \tilde{u}_{\nu} \le h_{2,\nu} \le C \quad \text{on } \partial B_{1,\nu}$$

where v is the unit outer normal to ∂B_1 , \tilde{C} is independent of r and ε . So we obtain

$$C^{-1} \leq |D\tilde{u}| \leq C$$
 on ∂B_1 .

By (3.3), we have

$$(3.14) |Du^{\varepsilon,r}| \le Cr^{1-\frac{2n}{k}} = C|z|^{1-\frac{2n}{k}} \le C(-u^{\varepsilon,r})^a \quad \text{on } \partial B_r,$$

and

$$(3.15) |Du^{\varepsilon,r}| \ge C^{-1}r^{1-\frac{2n}{k}} on \partial B_r.$$

By (2.3), (3.3), (3.9) and (3.14), we obtain

$$|Du^{\varepsilon,r}| \le C(-u^{\varepsilon,r})^a \le C(|z|^{2-\frac{2n}{k}} - R_0^{2-\frac{2n}{k}} + 1 - a_0 \frac{|z|^2}{R_0^2}) \le C|z|^{1-\frac{2n}{k}} \quad \text{in } \Omega_r.$$

3.3.3. **Positive lower bound of** $|Du^{\varepsilon,r}|$. Since $\partial\Omega$ is starshaped with respect to the origin, we have $t \cdot v > 0$ on $\partial\Omega$, where v is the unit outer normal to $\partial\Omega$, $t = (t_1, \dots, t_{2n}) = (y_1, \dots, y_n, x_1, \dots, x_n)$, $z_i = \frac{1}{\sqrt{2}}(x_i + \sqrt{-1}y_i)$. By (3.11), $|Du| \ge c$ for some uniform c on $\partial\Omega$. Then we have

$$\sum_{l=1}^{n} (z_l u_l^{\varepsilon,r} + \bar{z}_l u_{\bar{l}}^{\varepsilon,r}) = \sum_{l=1}^{2n} t_l u_{t_l}^{\varepsilon,r} = t \cdot v |Du^{\varepsilon,r}| \ge c \min_{\partial \Omega} t \cdot v := c_1 > 0.$$

Let $F^{i\bar{j}} = \frac{\partial}{\partial u_{i\bar{j}}^{\varepsilon,r}} (\log H_k[u^{\varepsilon,r}]), L = F^{i\bar{j}} \partial_{i\bar{j}}$. Consider the function

$$G := 2\operatorname{Re}\{z_l u_l^{\varepsilon,r}\} + A u^{\varepsilon,r} - B|z|^2,$$

where A, B are constants to be determined later. By calculation, we have

$$\mathcal{F} := \sum_{l=1}^{n} F^{l\bar{l}} = \sum_{l=1}^{n} \frac{S_{k}^{l\bar{l}}(\{u_{i\bar{j}}^{\varepsilon,r}\}_{1 \le i,j \le n})}{S_{k}(\{u_{i\bar{j}}^{\varepsilon,r}\}_{1 \le i,j \le n})} = (n-k+1) \frac{S_{k-1}(\{u_{i\bar{j}}^{\varepsilon,r}\}_{1 \le i,j \le n})}{S_{k}(\{u_{i\bar{j}}^{\varepsilon,r}\}_{1 \le i,j \le n})} \\ \ge k(C_{n}^{k})^{\frac{1}{k}} S_{k}^{-\frac{1}{k}}(\{u_{i\bar{j}}^{\varepsilon,r}\}_{1 \le i,j \le n}) = k(C_{n}^{k})^{\frac{1}{k}} \varepsilon^{-\frac{1}{k}}.$$

On ∂B_r , we have $Du^{\varepsilon,r} = |Du^{\varepsilon,r}|v = |Du^{\varepsilon,r}|\frac{t}{r}$. It follows by (3.15) that

$$t \cdot Du^{\varepsilon,r} = r|Du^{\varepsilon,r}| \ge c_2 r^{2-\frac{2n}{k}} \quad \text{on } \partial B_r.$$

By taking $r_4 = \min\{4^{\frac{2k}{2k-2n}}R_0, (\frac{4a_0}{R_0^2})^{-\frac{k}{2n}}\}$, we have

$$\underline{u} \leq -\frac{1}{2}r^{2-\frac{2n}{k}} \quad \text{on } \partial B_r.$$

It follows that if we take $A \leq \min\{\frac{c_1}{2}, c_2\}, B \leq \frac{c_1}{2R_0^2}, \varepsilon < \min\{\epsilon_1, \epsilon_2\}, r \leq \min\{r_3, r_4, r_5\},$ where $\epsilon_2 := \frac{C_n^k B^k}{(2+A)^k}, r_5 = (\frac{c_2}{2B})^{\frac{k}{2n}}$, then there holds

$$G \ge c_1 - A - BR_0^2 \ge 0$$
 on $\partial \Omega$,

$$G \ge (c_2 - \frac{A}{2})r^{2-\frac{2n}{k}} - Br^2 \ge 0$$
 on ∂B_r .

and

$$LG = (2+A)k - B\mathcal{F} = (2+A)k - Bk(C_n^k)^{\frac{1}{k}}\varepsilon^{-\frac{1}{k}} < 0 \quad \text{in } \Omega_r.$$

By maximum principle,

$$G \ge \min_{\partial \Omega_r} G > 0.$$

Thus we prove G > 0 in $\overline{\Omega}_r$ and (3.6) is obtained.

3.4. Second order estimates. Base on the key estimate (2.4), we can prove the global second order estimate in this subsection.

3.4.1. The global second order estimates can be reduced to the boundary second order estimates. By Theorem 2.6, we have

$$\max_{\overline{\Omega_r}} H = \max_{\partial \Omega_r} H + C.$$

So

(3.16)
$$u_{\xi\bar{\xi}}^{\varepsilon,r}(-u)^{-\frac{n}{n-k}} \le C(\max_{\partial\Omega_r}H+C) \le C(\max_{\partial\Omega_r}|\partial\bar{\partial}u^{\varepsilon,r}(-u^{\varepsilon,r})^{-\frac{n}{n-k}}|+1).$$

On the other hand, let $D_{\tau} = \sum_{i=1}^{2n} a_i \frac{\partial}{\partial t_i}$, with $\sum_{i=1}^{2n} a_i^2 = 1$, from $Lu_{\tau\tau}^{\varepsilon,r} \ge 0$, we obtain $u_{\tau\tau}^{\varepsilon,r} \le \max_{\partial \Omega_{\tau}} |D^2 u^{\varepsilon,r}|$ in Ω_r .

Since *u* is subharmonic, we have

$$-(2n-1)\max_{\partial\Omega_r}|D^2u^{\varepsilon,r}| \le u_{t_it_i} \le \max_{\partial\Omega_r}|D^2u^{\varepsilon,r}| \quad \text{in } \Omega_r.$$

Take $\tau = \frac{1}{\sqrt{2}} (\frac{\partial}{\partial t_i} \pm \frac{\partial}{\partial t_j})$, we get

$$|u_{t_i t_j}^{\varepsilon,r}| \le C \max_{\partial \Omega_r} |D^2 u^{\varepsilon,r}| \quad \text{in } \Omega_r.$$

Hence

$$|D^2 u^{\varepsilon,r}| \le C \max_{\partial \Omega_r} |D^2 u^{\varepsilon,r}| \quad \text{in } \Omega_r.$$

3.4.2. Second order estimates on the boundary $\partial \Omega_r$. The second order estimate on $\partial \Omega$ is almost the same as in [15]. So we only need to prove the second order estimate on ∂B_r .

Step 1. Pure tangential derivatives estimates

Near $p \in \partial B_r$, we may assume $p = (0, \dots, 0, r)$. Near $\tilde{p} = (0, \dots, 0, 1)$, ∂B_1 can be represented as a graph

$$x_n = \rho(t') = \left(1 - \sum_{i=1}^{2n-1} t_i^2\right)^{\frac{1}{2}},$$

where $t' = (1, \dots, t_{2n-1})$.

12

Let \tilde{u} and $\underline{\tilde{u}}$ be the functions defined in (3.12) and (3.13). Since \tilde{u} is equal to some constant on ∂B_1 , we have

$$\tilde{u}_{t_i t_j}(\tilde{p}) = \tilde{u}_{x_n}(\tilde{p})\delta_{ij}$$

It follows

$$|\tilde{u}_{t_i t_j}(\tilde{p})| \le C.$$

Hence

$$|u_{t_i t_j}^{\varepsilon, r}(p)| \le C r^{-\frac{2n}{k}}.$$

Furthermore, we have

(3.17) $\tilde{u}_{i\bar{j}}(\tilde{p}) = \tilde{u}_{x_n}(\tilde{p})\delta_{ij}.$

Step 2. Tangential-normal derivatives estimates

To estimate the tangential-normal second order derivatives on ∂B_r , we just estimate $\tilde{u}_{t_{\alpha}x_n}(\tilde{p})$ for $\alpha = 1, \dots, 2n-1$. Note that $F^{i\bar{j}}$ and $\tilde{u}_{i\bar{j}}$ are both Hermitian matrix, and can be diagonalized by a same unitrary matrix, $F^{i\bar{k}}u_{j\bar{k}}$ is also an Hermitian matrix. It follows that

$$F^{ij}(z_r\tilde{u}_s - \bar{z}_s\tilde{u}_{\bar{r}})_{i\bar{i}} = 0.$$

Now we estimate the mixed tangential-normal derivative $\tilde{u}_{t_i x_n}(\tilde{p})$ for $\tilde{p} \in \partial B_1$. Since $\tilde{u}(t', \rho(t'))$ is constant on $\partial B_1(0)$, we have

$$0=\tilde{u}_{t_{\alpha}}+\tilde{u}_{t_{2n}}\rho_{t_{\alpha}}=\tilde{u}_{t_{\alpha}}-\frac{t_{\alpha}}{\rho}\tilde{u}_{t_{2n}},\quad \alpha=1,\cdots,2n-1.$$

That is on $\partial B_1 \cap B_{\frac{1}{2}}(\tilde{p})$,

$$x_n \tilde{u}_{x_i} - x_i \tilde{u}_{x_n} = 0$$
 $i = 1, \dots, n-1$ and $x_n \tilde{u}_{y_i} - y_i \tilde{u}_{x_n} = 0$ $i = 1, \dots, n.$

It follows that

$$y_n \tilde{u}_{x_i} - x_i \tilde{u}_{y_n} = 0$$
 $i = 1, \dots, n-1$ and $y_n \tilde{u}_{y_i} - y_i \tilde{u}_{y_n} = 0$ $i = 1, \dots, n$.

To estimate $\tilde{u}_{x_i x_n}(\tilde{p})$ for $i = 1, \dots, n-1$, set

$$g^{1} = 2\operatorname{Re}(z_{i}\tilde{u}_{n} - \bar{z}_{n}\tilde{u}_{\bar{i}}) = x_{i}\tilde{u}_{x_{n}} - x_{n}\tilde{u}_{x_{i}} + y_{i}\tilde{u}_{y_{n}} - y_{n}\tilde{u}_{y_{i}}$$

Note that

$$F^{i\bar{j}}g_{i\bar{j}}=F^{i\bar{j}}(z_i\tilde{u}_n-\bar{z}_n\tilde{u}_{\bar{i}})_{i\bar{j}}+F^{i\bar{j}}(\bar{z}_i\tilde{u}_n-z_n\tilde{u}_i)_{i\bar{j}}=0.$$

On $\partial B_1(0) \cap B_{\frac{1}{2}}(\tilde{p})$, consider the barrier function

$$\Phi = A(1 - x_n) \pm g^1.$$

Since g_1 is bounded on $\partial B_1(0) \cap B_{\frac{1}{2}}(\tilde{p})$, $1 - x_n$ is bounded from below on $\partial B_{\frac{1}{2}}(\tilde{p}) \cap B_1(0)$, we can choose a postive A such that $\Phi \ge 0$ on $\partial(\partial B_1(0) \cap B_{\frac{1}{2}}(\tilde{p}))$. It follows

$$|g_{\chi_n}^1(\tilde{p})| \leq C.$$

However, at \tilde{p} , we have

$$g_{x_n}^1 = -\tilde{u}_{x_i} - \tilde{u}_{x_i x_n}.$$

Thus

$$\tilde{u}_{x_i x_n}(\tilde{p}) \leq C, \quad i = 1, \cdots, n-1.$$

To estimate $\tilde{u}_{y_i x_n}(\tilde{p})$ for $i = 1, \dots, n$, set

$$g^2 = 2\mathrm{Im}(z_i\tilde{u}_n - \bar{z}_n\tilde{u}_{\bar{i}}) = y_i\tilde{u}_{x_n} - x_n\tilde{u}_{y_i} + y_n\tilde{u}_{x_i} - x_i\tilde{u}_{y_n}.$$

Proceeding similarly, we obtain

$$|\tilde{u}_{v_i \chi_n}(\tilde{p})| \leq C, \quad i = 1, \cdots, n.$$

Step 3. Double normal derivative estimate

By pure tangential derivative estimate on ∂B_1 , we have

$$|\tilde{u}_{y_n y_n}(p)| \le C.$$

To estimate $\tilde{u}_{x_n x_n}(\tilde{p})$, it is suffices to estimate $\tilde{u}_{n\bar{n}}(\tilde{p})$. By rotating $\{z_1, \dots, z_{n-1}\}$, we may assume $\{\tilde{u}_{i\bar{j}}(\tilde{p})\}_{1 \le i,j \le n-1}$ is diagonal. Then

$$r^{2n}\varepsilon = H_k[\tilde{u}] = \tilde{u}_{n\bar{n}}S_{k-1}(\{\tilde{u}_{i\bar{j}}\}_{1\leq i,j\leq n-1}) - \sum_{\beta=1}^{n-1} |\tilde{u}_{\beta n}|^2 S_{k-2}(\{\tilde{u}_{i\bar{j}}\}_{1\leq i,j\leq n-1}).$$

By (3.17), we obtain

$$S_{k-1}(\{\tilde{u}_{i\bar{j}}\}_{1\leq i,j\leq n-1}) = S_{k-1}(\{\underline{\tilde{u}}_{i\bar{j}}\}_{1\leq i,j\leq n-1} + \frac{1}{2}(\tilde{u} - \underline{\tilde{u}})_{x_n}I_{n-1})$$

$$\geq S_{k-1}(\{\underline{\tilde{u}}_{i\bar{j}}\}_{1\leq i,j\leq n-1})$$

$$\geq C_n^{k-1}(C_n^k)^{\frac{1-k}{k}}\min_{\partial\Omega}H_k^{\frac{k-1}{k}}[\underline{\tilde{u}}] := c_1.$$

So

$$\tilde{u}_{n\bar{n}}(p) \leq C.$$

Combining these three cases together, and noting that \tilde{u} is sunharmonic, we obtain

$$|\partial \bar{\partial} \tilde{u}| \leq C \quad \text{on } \partial B_1.$$

Hence

$$|\partial \bar{\partial} u^{\varepsilon,r}| \leq Cr^{-\frac{2n}{k}} \quad \text{on } \partial B_r.$$

By (3.16) and C^0 estimate, we have

$$|\partial \bar{\partial} u^{\varepsilon,r}| \leq C |z|^{-\frac{2n}{k}} \quad \text{in } \Omega_r.$$

14

4. Proof of Theorem 1.1

4.1. Uniqueness. The uniqueness follows from comparison theorem 2.4.

Let u, v be two solutions to (1.3) in $\Omega \setminus \{0\}$. For any $z_0 \in \Omega \setminus \{0\}$, we first show $u(z_0) \leq v(z_0)$. In fact, for any $t \in (0, 1)$, since $u - tv = -(1 - t)|z|^{2-\frac{2n}{k}} + C$, there exists r sufficiently small such that $z_0 \in \Omega \setminus \overline{B}_r$ and u < tv on ∂B_r . Note that u = -1 < -t = tv on $\partial \Omega$. By comparison theorem 2.4, we get u < tv in $\Omega \setminus B_r$. Therefore $u(z_0) \leq tv(z_0)$. Let $t \to 1$, we obtain $u(z_0) \leq v(z_0)$. Hence $u \leq v$ in $\Omega \setminus \{0\}$. Similarly, we obtain $u \geq v$ in $\Omega \setminus \{0\}$.

4.2. **Existence.** The existence follows from the uniform C^2 -estimates for $u^{\varepsilon,r}$.

For $K = \Omega \setminus B_{r_0}$, for the solution to (3.2), by the estimate (3.3), we have

$$|u^{\varepsilon,r}|_{C^1(K)} + |\Delta u^{\varepsilon,r}| \le C(K).$$

By Evans-Krylov theory, we obtain for any $0 < \alpha < 1$,

$$|u^{\varepsilon,r}|_{C^{2,\alpha}(K)} \le C(K,\varepsilon).$$

By compactness, we can find a sequence $r_i \rightarrow 0$ such that

$$u^{\varepsilon,r_i} \to u^{\varepsilon}$$
 in $C^2(K)$.

where u^{ε} satisfies

$$\begin{cases} H_k[u^{\varepsilon}] = \varepsilon & \text{in } K, \\ u = -1 & \text{on } \partial\Omega, \end{cases}$$

and

(4.1)
$$\begin{aligned} -C - |z|^{2-\frac{2n}{k}} &\leq u^{\varepsilon}(z) \leq -|z|^{2-\frac{2n}{k}} \\ |Du^{\varepsilon}(z)| &\leq C|z|^{1-\frac{2n}{k}}, \end{aligned}$$

$$|\partial \bar{\partial} u^{\varepsilon}(z)| \le C |z|^{-\frac{2n}{k}}.$$

Moreover,

$$|u^{\varepsilon}|_{C^{2,\alpha}(K)} \leq C(K,\varepsilon).$$

By the classical Schauder theory, u^{ε} is smooth.

By above estimates (4.1) for u^{ε} , for any sequence $\varepsilon_j \to 0$, there is a subsequence of $\{u^{\varepsilon_j}\}$ converging to a function u in $C^{1,\alpha}$ norm on any compact subset of $\Omega \setminus \{0\}$. Thus $u \in C^{1,\alpha}(\Omega \setminus \{0\})$ and satisfies the estimates (1.4) and (1.5). By the convergence theorem of the complex *k*-Hessian operator proved by Trudinger-Zhang [36] (see also Lu [33]), u is a solution to (1.3).

ACKNOWLEDGEMENTS:

The second author was supported by National Natural Science Foundation of China (grants 11721101 and 12141105) and National Key Research and Development Project (grants SQ2020YFA070080). The third author was supported by NSFC grant No. 11901102.

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