LIOUVILLE THEOREM FOR ELLIPTIC EQUATIONS INVOLVING THE SUM OF THE FUNCTION AND ITS GRADIENT IN $\mathbb{R}^n$

XI-NAN MA, WANGZHE WU, AND QIQI ZHANG

ABSTRACT. We prove Liouville theorem for the equation $\Delta v + Nv^p + M|\nabla v|^q = 0$ in $\mathbb{R}^n$, with $M, N > 0, q = \frac{2p}{p+1}$ in the critical and subcritical case. The proof is based on an integral identity and Young inequality.

1. Introduction

In this paper, we are concerned with a Liouville type theorem for the global solutions in the critical and subcritical cases of the following equations in $\mathbb{R}^n$:

$$\Delta v + Nv^p + M|\nabla v|^q = 0, \tag{1}$$

where $q = \frac{2p}{p+1}$, p and q are exponents larger than 1 and $M \in \mathbb{R}$, $N > 0$. This kind of equations are widely concerned and studied depending on the different value of M, N.

For the case $M = 0, N = 1$, the work concerning the symmetry of solutions of second order elliptic equations on an unbounded domain was first done by Gidas, Ni and Nirenberg [14], and then generalized to infinite cylinders by Berestycki and Nirenberg [5]. In the elegant paper of Gidas, Ni and Nirenberg [14], one of the interesting results is on the symmetry of the solutions of the equation. They proved that for $p = \frac{n+2}{n-2}$, all the positive solutions with reasonable behavior at infinity, namely $v = O(|x|^{2-n})$ which are radially symmetric about some point, and hence assume the form

$$u = \Big[\frac{\lambda\sqrt{n(n-2)}}{\lambda^2 + |x - x_0|^2}\Big]^{\frac{n-2}{2}}.$$

This uniqueness result, as was pointed out by R. Schoen, is in fact equivalent to the geometric result due to Obata [20]. In the case that $1 < p < \frac{n+2}{n-2}$, Gidas and Spruck [15] showed that the only nonnegative solution is 0. The related equation

$$\Delta v + v^p = 0,$$

usually called the Lane-Emden equation, it is well-known that radial ground states exist if and only if the exponent p either is critical (i.e.$=(n + 2)/(n - 2)$) or is supercritical ($> (n+2)/(n-2)$), this result going back to the pioneering work of R.H. Fowler in [12] and [13]. A fairly straightforward modern proof of Fowler's result can be given using standard shooting methods together with generalized Pohožaev identities. Moreover, for the deformation Lane-Emden equation equation

$$-\Delta v = |v|^{p-1}v,$$

the first breakthrough in the study of it which plays an important role in modelling meteorological or astrophysical phenomena [9, 10] came in the consideration of the case

$1 < p < \frac{n+2}{n-2}$, it is exactly due to Gidas-Spruck [15], using Harnack inequality as in [22].

Due to Caffarelli, Gidas and Spruck [6], the treatment of the critical case $p = \frac{n+2}{n-2}$ was made possible thanks to a completely new approach based upon a combination of moving plane analysis and geometric measure theory. Later, Chen-Li [8] provided a new proof for [6]. As for the supercritical case, not much is known and the existence of radial ground states is a consequence of Pohožaev¡¯s identity [21], using a shooting method.

For the case $M = 1, N = 0$, it is evidently to be seen that equation (1) becomes the Hamilton-Jacobi equation

$$\Delta v + |\nabla v|^q = 0,$$

which is proved by P.L. Lions [18] that any C^2 solution in $\mathbb{R}^n$ has to be a constant for $q > 1$. In [7] and [23] most of the study deals with the case $q \neq \frac{2p}{p+1}$. In the critical case, then not only the sign of M but also its value plays a fundamental role, with a delicate interaction with the exponent p. Other work on this kind of equation can be seen in [4]. Peculiarly, it can be also seen as another special case of the equation which can be expressed as the product of v^p and $|\nabla v|^q$:

$$-\Delta v = v^p |\nabla v|^q, \text{ in } \quad \mathbb{R}^n. \tag{2}$$

Such equation was studied by Véron etc. in [1] and they obtained Liouville type theorem with the conditions on p, q. They studied local and global properties of positive solutions in the range $p + q > 1$, $p \geq 0$ and $0 \leq q < 2$. Their main results dealt with the subcritical range and proved a priori estimates of positive solutions of (2) in a punctured domain and existence of ground states in $\mathbb{R}^n$, based on two approaches for obtaining their results: the direct Bernstein method and the integral Bernstein method popularized by Lions [19] and Gidas and Spruck in [15] respectively. Both methods are based upon differentiating the equation.

Actually, Véron and his research partners have already studied the same type equation as (1) in [2] and [3] where it is concerned with local and global properties of positive solutions with $N = 1$ in a domain $\Omega \subset \mathbb{R}^n$, in the range $\min\{p, q\} > 1$, and $M \in \mathbb{R}$. Particularly, they pointed out that there exists no nontrivial nonnegative solution with $n \geq 1$, $p, q > 1$, $q \neq \frac{2p}{p+1}$ and some extra conditions on u. In fact, we observe that the equation (1) is invariant under the scaling transformation T_k with $k > 0$ if and only if q is critical with respect to p, i.e., $q = \frac{2p}{p+1}$, and

$$T_k[v](x) = k^{\frac{2}{p-1}} v(kx). \tag{3}$$

It follows that by rescaling, we can always assume $N = 1$ for our case. In the critical case, first studies in the case $M < 0$ are due to Chipot and Weissler [7] for $n = 1$. The case $n \geq 2$ was left open by Serrin and Zou [23] who performed a very detailed analysis and the first partial results are due to Fila and Quittner [11] and Voirol [24, 25]. Much less is known for $M > 0$. References [2] and [3] have done some work and also gave a Liouville type theorem:

Theorem 1.1 (Theorem C in [2]). *Let $n \geq 1$, $p > 1, q = \frac{2p}{p+1}$. For any*

$$M > \left(\frac{p-1}{p+1}\right)^{\frac{p-1}{p+1}} \left(\frac{n(p+1)^2}{4p}\right)^{\frac{p}{p+1}},$$

there exists no nontrivial nonnegative solution of (1) *in* $\mathbb{R}^n$.

Theorem 1.2 (Theorem D in [2]). *Let $n \geq 2$, $1 < p < \frac{n+3}{n-1}, q = \frac{2p}{p+1}$. For any $M > 0$, there exists no nontrivial nonnegative solution of* (1) *in* $\mathbb{R}^n$.

Theorem 1.3 (Theorem E in [2]). *Let $n \geq 3$, $1 < p < \frac{n+2}{n-2}, q = \frac{2p}{p+1}$. Then there exists $\epsilon_0 > 0$ depending on n and p such that for any $|M| \leq \epsilon_0$, there exists no nontrivial nonnegative solution of* (1) *in* $\mathbb{R}^n$.

By observation, we found that the result in [2] has not covered all the cases that $1 < p \leq \frac{n+2}{n-2}$ for $q = \frac{2p}{p+1}, M > 0$ and they did not give the expression of ϵ_0 for $M < 0$. Motivated by the results above, we aim to complete the corresponding results of the range of p with $1 < p \leq \frac{n+2}{n-2}, M > 0$, based on an integral identity and Young inequality. Besides we give a different proof for the case $M < 0$ and give the expression of the lower bound of M. Our main results in this framework are the following:

Theorem 1.4. *Let $n \geq 3$, $1 < p \leq \frac{n+2}{n-2}$, $q = \frac{2p}{p+1}$, then all the positive solutions of* (1) *are $v \equiv 0$ for any $M > 0$.*

In order to prove Theorem 1.4, we need to back up two other conclusions:

Theorem 1.5. *When $1 < p \leq \frac{n+2}{n-2}$, $q = \frac{2p}{p+1}$, $n \geq 3$, there exists $M_1 > 0$ such that if $0 < M < M_1$, then $v \equiv 0$. Especially, if $3 \leq n \leq 6$, then $M_1 = \infty$.*

Here, M_1 will be determined later in Section 4. For $3 \leq n \leq 6$, by simple computation, we can prove that for any M, Theorem 1.5 holds. But for $n \geq 7$, we need apply Young inequality.

Theorem 1.6. *For any $n \geq 3, 1 < p \leq \frac{n+2}{n-2}$, $q = \frac{2p}{p+1}$, there exists $M_2 > 0$, such that if $M > M_2$, then $v \equiv 0$.*

Besides, after computation, we have the following lemma:

Lemma 1.1. *For any $1 < p \leq \frac{n+2}{n-2}$, $q = \frac{2p}{p+1}$, we have $M_2 < M_1$.*

Therefore, we can think M can be taken from any positive constant that make sure the Liouville type theorem of equation (1) holds.

When $M < 0$, define

$$M_3(n,p) := \begin{cases} \max\limits_{T\geq 0} \frac{T^{\frac{2-q}{2}}}{T+1} \left(\frac{q}{2}\right)^{-\frac{q}{2}} \left(\frac{2}{2-q}\right)^{\frac{2-q}{2}} \Big[2.5 - p - T(p-1)\Big]^{\frac{q}{2}}, \text{ if } n = 3, 1 < p \leq 2; \\ \max\limits_{T\geq 0} \frac{T^{\frac{2-q}{2}}}{T+\frac{q+3}{2q-2}-1} \left(\frac{q}{2}\right)^{-\frac{q}{2}} \left(\frac{2}{2-q}\right)^{\frac{2-q}{2}} \left[5 - p - T\left(p-2\right)\right]^{\frac{q}{2}}, \text{ if } n = 3, 2 \leq p < 5; \\ \max\limits_{T\geq 0} \frac{T^{\frac{2-q}{2}}}{\frac{2(n+q)^2}{(n-1)^2q^2}+T-1} \left(\frac{q}{2}\right)^{-\frac{q}{2}} \left(\frac{2}{2-q}\right)^{\frac{2-q}{2}} \left[\frac{n+2}{n-2} - p - T\left(p - \frac{2}{n-2}\right)\right]^{\frac{q}{2}}, \text{ if } n \geq 4. \end{cases}$$

Then we get the following Liouville type theorem when $M < 0$:

Theorem 1.7. *When $n \geq 3, 1 < p < \frac{n+2}{n-2}, q = \frac{2p}{p+1}$, then for any $-M_3 < M < 0$, there exists no nontrivial nonnegative solution of* (1) *in* $\mathbb{R}^n$.

For convenience, we make a summary of ideas and article planning arrangement in this part. We expand the range of p of the result in [2] to $1 < p \leq \frac{n+2}{n-2}$ for Liouville type theorem of the equation (1), that is we not only prove the subcritical case but also the critical case which calls for much more complicated computations. Besides, contrast Theorem 1.3 and Theorem 1.4, we even remove the sufficient small condition for constants M in [2], leading us to find more suitable skills. The main method taken in our work is based on an integral identity and Young inequality. This paper is organized as follows: we use integration by parts to get the integral identity in Section 2. According to the integral identity, some conditions are given to make the nonnegative solution is zero in Section 3. On the basis of satisfying the above conditions, we introduce Young inequality and prove that there is only zero solution in Section 4. Finally, we're going to talk about the upper bound and lower bound of M separately in Section 5 and Section 6, and point out that the lower bound is smaller that the upper bound. Especially, for the upper bound M_1, we classify the cases where $2 \leq n \leq 6$ and $n \geq 7$. In conclusion, it follows that there is no requirement of positive constant M to make Liouville type theorem holds.

2. Integral identity

Consider the equation:

$$\Delta v + Nv^p + M|\nabla v|^q = 0. \tag{4}$$

Multiplying (4) by $v^\alpha|\nabla v|^\gamma \Delta v$,

$$\underset{①}{v^\alpha|\nabla v|^\gamma(\Delta v)^2} = \underset{I}{-Nv^{\alpha+p}|\nabla v|^\gamma \Delta v} \underset{II}{- Mv^\alpha|\nabla v|^{\gamma+q}\Delta v}. \tag{5}$$

We observed that the term ① is important. In the following contents, we always choose $\gamma = 0$ or $\gamma \geq 3$. And at one fixed point x_0, suppose that $v_1(x_0) = |\nabla v|(x_0)$ and define

$$G_{11} = v_{11} - S\Delta v,$$
$$G_{ij} = v_{ij} - Q\delta_{ij}\Delta v, \quad i + j > 2,$$

and

$$E_{ij} = v_{ij} - \frac{1}{n}\delta_{ij}\Delta v, \text{ for } i, j = 1, ..., n.$$

We remark that the definition of E_{ij} is global but G_{ij} is not.

2.1. **The term $v^{\alpha-1}|\nabla v|^{\gamma+2}\Delta v$.**

$$
\begin{aligned}
v^{\alpha-1}|\nabla v|^{\gamma+2}\Delta v &= (v^{\alpha-1}|\nabla v|^{\gamma+2}v_i)_i - (\alpha-1)v^{\alpha-2}|\nabla v|^{\gamma+4} - (\gamma+2)v^{\alpha-1}|\nabla v|^{\gamma}v_iv_jv_{ij} \\
&= (v^{\alpha-1}|\nabla v|^{\gamma+2}v_i)_i - (\alpha-1)v^{\alpha-2}|\nabla v|^{\gamma+4} \\
&- (\gamma+2)v^{\alpha-1}|\nabla v|^{\gamma+2}(G_{11}+S\Delta v), \\
\Rightarrow v^{\alpha-1}|\nabla v|^{\gamma+2}\Delta v &= \frac{1}{1+\gamma S+2S}(v^{\alpha-1}|\nabla v|^{\gamma+2}v_i)_i - \frac{\alpha-1}{1+\gamma S+2S}v^{\alpha-2}|\nabla v|^{\gamma+4} \\
&- \frac{\gamma+2}{1+\gamma S+2S}v^{\alpha-1}|\nabla v|^{\gamma+2}G_{11}.
\end{aligned} \tag{6}
$$

2.2. **The term ①.** In fact, we have

$$
\begin{aligned}
v^{\alpha}|\nabla v|^{\gamma}(\Delta v)^2 &= v^{\alpha}|\nabla v|^{\gamma}v_{ii}v_{jj} \\
&= (v^{\alpha}|\nabla v|^{\gamma}v_{ii}v_j)_j - \alpha v^{\alpha-1}|\nabla v|^{\gamma+2}v_{ii} - v^{\alpha}|\nabla v|^{\gamma}_j v_{ii}v_j \\
&- v^{\alpha}|\nabla v|^{\gamma}v_{iij}v_j,
\end{aligned}
$$

$$
\begin{aligned}
v^{\alpha}|\nabla v|^{\gamma}v_{iij}v_j &= (v^{\alpha}|\nabla v|^{\gamma}v_{ij}v_j)_i - (v^{\alpha}|\nabla v|^{\gamma}v_j)_iv_{ij} \\
&= (v^{\alpha}|\nabla v|^{\gamma}v_{ij}v_j)_i - \alpha v^{\alpha-1}|\nabla v|^{\gamma}v_iv_jv_{ij} - v^{\alpha}|\nabla v|^{\gamma}_i v_{ij}v_j \\
&- v^{\alpha}|\nabla v|^{\gamma}v_{ij}^2.
\end{aligned}
$$

As a result, we obtain

$$
\begin{aligned}
① &= v^{\alpha}|\nabla v|^{\gamma}(\Delta v)^2 \\
&= (v^{\alpha}|\nabla v|^{\gamma}v_{ii}v_j)_j - \alpha v^{\alpha-1}|\nabla v|^{\gamma+2}\Delta v - v^{\alpha}|\nabla v|^{\gamma}_j\Delta v\cdot v_j \\
&- (v^{\alpha}|\nabla v|^{\gamma}v_{ij}v_j)_i + \alpha v^{\alpha-1}v_iv_jv_{ij}|\nabla v|^{\gamma} + v^{\alpha}|\nabla v|^{\gamma}_i v_{ij}v_j \\
&+ v^{\alpha}|\nabla v|^{\gamma}v_{ij}^2 \\
&= (v^{\alpha}|\nabla v|^{\gamma}\Delta v v_i - v^{\alpha}|\nabla v|^{\gamma}v_{ij}v_j)_i + v^{\alpha}|\nabla v|^{\gamma}v_{ij}^2 - \alpha v^{\alpha-1}|\nabla v|^{\gamma+2}\Delta v \\
&+ \alpha v^{\alpha-1}|\nabla v|^{\gamma}v_iv_jv_{ij} - v^{\alpha}\Delta v\cdot v_j|\nabla v|^{\gamma}_j + v^{\alpha}v_jv_{ij}|\nabla v|^{\gamma}_i.
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\Rightarrow ① = & \left(1-\frac{1}{n}\right)\left(v^\alpha|\nabla v|^\gamma \Delta v v_i\right)_i - (v^\alpha|\nabla v|^\gamma E_{ij}v_j)_i + \sum_{i+j>2} v^\alpha|\nabla v|^\gamma v_{ij}^2 \\
& + v^\alpha|\nabla v|^\gamma G_{11}^2 + 2Sv^\alpha|\nabla v|^\gamma G_{11}\Delta v + S^2 v^\alpha|\nabla v|^\gamma(\Delta v)^2 - \alpha v^{\alpha-1}|\nabla v|^{\gamma+2}\Delta v \\
& + \alpha v^{\alpha-1}|\nabla v|^{\gamma+2}v_{11} - \gamma v^\alpha|\nabla v|^{\gamma-2}v_iv_jv_{ij}\Delta v + \gamma v^\alpha|\nabla v|^{\gamma-2}v_jv_{ij}v_lv_{il} \\
= & \left(1-\frac{1}{n}\right)\left(v^\alpha|\nabla v|^\gamma \Delta v v_i\right)_i - (v^\alpha|\nabla v|^\gamma E_{ij}v_j)_i + \sum_{i+j>2} v^\alpha|\nabla v|^\gamma v_{ij}^2 \\
& + v^\alpha|\nabla v|^\gamma G_{11}^2 + 2Sv^\alpha|\nabla v|^\gamma G_{11}\Delta v + S^2 v^\alpha|\nabla v|^\gamma(\Delta v)^2 - \alpha v^{\alpha-1}|\nabla v|^{\gamma+2}\Delta v \\
& + \alpha v^{\alpha-1}|\nabla v|^{\gamma+2}v_{11} - \gamma v^\alpha|\nabla v|^{\gamma}v_{11}\Delta v + \gamma v^\alpha|\nabla v|^{\gamma}v_{i1}^2 \\
= & \left(1-\frac{1}{n}\right)\left(v^\alpha|\nabla v|^\gamma \Delta v v_i\right)_i - (v^\alpha|\nabla v|^\gamma E_{ij}v_j)_i + \sum_{i+j>2} v^\alpha|\nabla v|^\gamma v_{ij}^2 \\
& + v^\alpha|\nabla v|^\gamma G_{11}^2 + 2Sv^\alpha|\nabla v|^\gamma G_{11}\Delta v + S^2 v^\alpha|\nabla v|^\gamma(\Delta v)^2 - \alpha v^{\alpha-1}|\nabla v|^{\gamma+2}\Delta v \\
& + \alpha v^{\alpha-1}|\nabla v|^{\gamma+2}\left(G_{11}+S\Delta v\right) - \gamma v^\alpha|\nabla v|^{\gamma}\left(G_{11}+S\Delta v\right)\Delta v + \gamma v^\alpha|\nabla v|^{\gamma}\left(G_{11}+S\Delta v\right)^2 \\
& + \sum_{i>1}\gamma v^\alpha|\nabla v|^\gamma G_{1i}^2.
\end{aligned}
$$

Thus we get that

$$
\begin{aligned}
\Rightarrow & \left(1-S^2+\gamma S-\gamma S^2\right)v^\alpha|\nabla v|^\gamma(\Delta v)^2 \\
& = \left(1-\frac{1}{n}\right)\left(v^\alpha|\nabla v|^\gamma \Delta v v_i\right)_i - \left(v^\alpha|\nabla v|^\gamma E_{ij}v_j\right)_i + \sum_{i+j>2} v^\alpha|\nabla v|^\gamma v_{ij}^2 \\
& + \left(\alpha S-\alpha\right)v^{\alpha-1}|\nabla v|^{\gamma+2}\Delta v + \alpha v^{\alpha-1}|\nabla v|^{\gamma+2}G_{11} \\
& + \left(2\gamma S+2S-\gamma\right)v^\alpha|\nabla v|^\gamma G_{11}\Delta v + \left(1+\gamma\right)v^\gamma|\nabla v|^\gamma G_{11}^2 + \sum_{i>1}\gamma v^\alpha|\nabla v|^\gamma G_{i1}^2 \\
& = \left(1-\frac{1}{n}\right)\left(v^\alpha|\nabla v|^\gamma \Delta v v_i\right)_i - \left(v^\alpha|\nabla v|^\gamma E_{ij}v_j\right)_i + \sum_{i+j>2} v^\alpha|\nabla v|^\gamma v_{ij}^2 \\
& + \left(\alpha S-\alpha\right)\cdot\left[\frac{1}{1+\gamma S+2S}(v^{\alpha-1}|\nabla v|^{\gamma+2}v_i)_i - \frac{\alpha-1}{1+\gamma S+2S}v^{\alpha-2}|\nabla v|^{\gamma+4}\right. \\
& \left. - \frac{\gamma+2}{1+\gamma S+2S}v^{\alpha-1}|\nabla v|^{\gamma+2}G_{11}\right] + \alpha v^{\alpha-1}|\nabla v|^{\gamma+2}G_{11} \\
& + \left(2\gamma S+2S-\gamma\right)v^\alpha|\nabla v|^\gamma G_{11}\Delta v + \left(1+\gamma\right)v^\gamma|\nabla v|^\gamma G_{11}^2 + \sum_{i>1}\gamma v^\alpha|\nabla v|^\gamma G_{i1}^2.
\end{aligned}
$$

$$\begin{aligned}
\Rightarrow&\Big(1-S^2+\gamma S-\gamma S^2\Big)v^\alpha|\nabla v|^\gamma(\Delta v)^2\\
&=\left(1-\frac{1}{n}\right)\Big(v^\alpha|\nabla v|^\gamma\Delta v v_i\Big)_i-\Big(v^\alpha|\nabla v|^\gamma E_{ij}v_j\Big)_i+\frac{\alpha S-\alpha}{1+\gamma S+2S}\Big(v^{\alpha-1}|\nabla v|^{\gamma+2}v_i\Big)_i\\
&+\sum_{i+j>2}v^\alpha|\nabla v|^\gamma v_{ij}^2+\frac{\alpha(\alpha-1)(1-S)}{1+\gamma S+2S}v^{\alpha-2}|\nabla v|^{\gamma+4}+\frac{\alpha(\gamma+3)}{1+\gamma S+2S}v^{\alpha-1}|\nabla v|^{\gamma+2}G_{11}\\
&+\Big(2\gamma S-\gamma+2S\Big)v^\alpha|\nabla v|^\gamma G_{11}\Delta v+(1+\gamma)v^\gamma|\nabla v|^\gamma G_{11}^2+\sum_{i>1}\gamma v^\alpha|\nabla v|^\gamma G_{i1}^2.
\end{aligned}\tag{8}$$

Recall that $G_{ij}:=v_{ij}-Q\delta_{ij}\Delta v, i+j>2$, then

$$\begin{aligned}
\sum_{i>1}v^\alpha|\nabla v|^\gamma v_{ii}^2&=\sum_{i>1}v^\alpha|\nabla v|^\gamma\Big(G_{ii}+Q\Delta v\Big)^2\\
&=\sum_{i>1}v^\alpha|\nabla v|^\gamma G_{ii}^2+2Q\sum_{i>1}v^\alpha|\nabla v|^\gamma G_{ii}\Delta v+(n-1)Q^2v^\alpha|\nabla v|^\gamma(\Delta v)^2.
\end{aligned}$$

If $(n-1)Q+S=1$, then

$$G_{11}+\sum_{i>1}G_{ii}=0.$$

and

$$\begin{aligned}
\sum_{i>1}v^\alpha|\nabla v|^\gamma v_{ii}^2&=\sum_{i>1}v^\alpha|\nabla v|^\gamma\Big(G_{ii}+Q\Delta v\Big)^2\\
&=\sum_{i>1}v^\alpha|\nabla v|^\gamma G_{ii}^2-2Qv^\alpha|\nabla v|^\gamma G_{11}\Delta v+(n-1)Q^2v^\alpha|\nabla v|^\gamma(\Delta v)^2
\end{aligned}$$

It follows that

$$\begin{aligned}
\Rightarrow&\Big[1-S^2+\gamma S-\gamma S^2-(n-1)Q^2\Big]v^\alpha|\nabla v|^\gamma(\Delta v)^2\\
&=\left(1-\frac{1}{n}\right)\Big(v^\alpha|\nabla v|^\gamma\Delta v v_i\Big)_i-(v^\alpha|\nabla v|^\gamma E_{ij}v_j)_i+\frac{\alpha S-\alpha}{1+\gamma S+2S}(v^{\alpha-1}|\nabla v|^{\gamma+2}v_i)_i\\
&+\sum_{i+j>2}v^\alpha|\nabla v|^\gamma G_{ij}^2+\frac{\alpha(\alpha-1)(1-S)}{1+\gamma S+2S}v^{\alpha-2}|\nabla v|^{\gamma+4}+\frac{\alpha(\gamma+3)}{1+\gamma S+2S}v^{\alpha-1}|\nabla v|^{\gamma+2}G_{11}\\
&+\Big(2\gamma S-\gamma+2S-2Q\Big)v^\alpha|\nabla v|^\gamma G_{11}\Delta v+(1+\gamma)v^\alpha|\nabla v|^\gamma G_{11}^2+\sum_{i>1}\gamma v^\alpha|\nabla v|^\gamma G_{i1}^2.
\end{aligned}\tag{9}$$

Remark 2.1. *The divergence term*

$$A:=(v^\alpha|\nabla v|^\gamma v_i\Delta v)_i$$

is important and will be explained later.

2.3. **The term A.**

$$\begin{aligned}
A &= (v^\alpha|\nabla v|^\gamma v_i\Delta v)_i \\
&= \alpha v^{\alpha-1}|\nabla v|^{\gamma+2}\Delta v+\gamma v^\alpha|\nabla v|^{\gamma-2}v_iv_jv_{ij}\Delta v+v^\alpha|\nabla v|^\gamma(\Delta v)^2 \\
&+\underbrace{v^\alpha|\nabla v|^\gamma v_i(\Delta v)_i}_{A_1} \\
&= \alpha v^{\alpha-1}|\nabla v|^{\gamma+2}\Delta v+\gamma v^\alpha|\nabla v|^{\gamma}G_{11}\Delta v+(1+\gamma S)v^\alpha|\nabla v|^\gamma(\Delta v)^2 \\
&+\underbrace{v^\alpha|\nabla v|^\gamma v_i(\Delta v)_i}_{A_1}.
\end{aligned}$$

Substitute (4) into A_1,

$$\begin{aligned}
A_1 &= v^\alpha|\nabla v|^\gamma v_i\Big(-Nv^p-M|\nabla v|^q\Big)_i \\
&= -pNv^{\alpha+p-1}|\nabla v|^{\gamma+2}-qMv^\alpha|\nabla v|^{\gamma+q-2}v_iv_jv_{ij}.
\end{aligned}$$

Multiply (4) by $v^{\alpha-1}|\nabla v|^{\gamma+2}$

$$-Nv^{\alpha+p-1}|\nabla v|^{\gamma+2}=v^{\alpha-1}|\nabla v|^{\gamma+2}\Delta v+Mv^{\alpha-1}|\nabla v|^{\gamma+q+2}. \tag{10}$$

Therefore we have

$$A_1=-qMv^\alpha|\nabla v|^{\gamma+q-2}v_iv_jv_{ij}+pv^{\alpha-1}|\nabla v|^{\gamma+2}\Delta v+pMv^{\alpha-1}|\nabla v|^{\gamma+q+2}. \tag{11}$$

On the other hand, we know

$$\begin{aligned}
&pMv^{\alpha-1}|\nabla v|^{\gamma+q+2}-qMv^\alpha|\nabla v|^{\gamma+q-2}v_iv_jv_{ij} \\
&=pMv^{\alpha-1}|\nabla v|^{\gamma+q+2}-\frac{qM}{\gamma+q}v^\alpha(|\nabla v|^{\gamma+q})_iv_i \\
&=\Big(p+\frac{q\alpha}{\gamma+q}\Big)Mv^{\alpha-1}|\nabla v|^{\gamma+q+2}-\frac{q}{\gamma+q}M(v^\alpha|\nabla v|^{\gamma+q}v_i)_i \\
&+\frac{qM}{\gamma+q}v^\alpha|\nabla v|^{\gamma+q}\Big(-Nv^p-M|\nabla v|^q\Big) \\
&=-\frac{qM}{\gamma+q}(v^\alpha|\nabla v|^{\gamma+q}v_i)_i+\Big(p+\frac{q\alpha}{\gamma+q}\Big)Mv^{\alpha-1}|\nabla v|^{\gamma+q+2} \\
&-\frac{qMN}{\gamma+q}v^{\alpha+p}|\nabla v|^{\gamma+q}-\frac{qM^2}{\gamma+q}v^\alpha|\nabla v|^{\gamma+2q}.
\end{aligned} \tag{12}$$

So we get that

$$\begin{aligned}
A &= (v^\alpha|\nabla v|^\gamma v_i\Delta v)_i \\
&= \alpha v^{\alpha-1}|\nabla v|^{\gamma+2}\Delta v+\gamma v^\alpha|\nabla v|^{\gamma}G_{11}\Delta v+(1+\gamma S)v^\alpha|\nabla v|^\gamma(\Delta v)^2 \\
&+\underbrace{v^\alpha|\nabla v|^\gamma v_i(\Delta v)_i}_{A_1} \\
&= \alpha v^{\alpha-1}|\nabla v|^{\gamma+2}\Delta v+\gamma v^\alpha|\nabla v|^{\gamma}G_{11}\Delta v+(1+\gamma S)v^\alpha|\nabla v|^\gamma(\Delta v)^2 \\
&+pv^{\alpha-1}|\nabla v|^{\gamma+2}\Delta v-\frac{qM}{\gamma+q}(v^\alpha|\nabla v|^{\gamma+q}v_i)_i-\frac{qM^2}{\gamma+q}v^\alpha|\nabla v|^{\gamma+2q} \\
&+\Big(p+\frac{q\alpha}{\gamma+q}\Big)Mv^{\alpha-1}|\nabla v|^{\gamma+q+2}-\frac{q}{\gamma+q}MNv^{\alpha+p}|\nabla v|^{\gamma+q},
\end{aligned}$$

Substitute (6) into it:

$$\begin{aligned}
&\Rightarrow -(1+\gamma S)v^{\alpha}|\nabla v|^{\gamma}(\Delta v)^2\\
&= -A + \gamma v^{\alpha}|\nabla v|^{\gamma}G_{11}\Delta v\\
&+ (\alpha+p)\cdot\Bigg[\frac{1}{1+\gamma S+2S}(v^{\alpha-1}|\nabla v|^{\gamma+2}v_i)_i - \frac{\alpha-1}{1+\gamma S+2S}v^{\alpha-2}|\nabla v|^{\gamma+4}\\
&- \frac{\gamma+2}{1+\gamma S+2S}v^{\alpha-1}|\nabla v|^{\gamma+2}G_{11}\Bigg]\\
&- \frac{qM}{\gamma+q}(v^{\alpha}|\nabla v|^{\gamma+q}v_i)_i - \frac{qM^2}{\gamma+q}v^{\alpha}|\nabla v|^{\gamma+2q} + \left(p+\frac{q\alpha}{\gamma+q}\right)Mv^{\alpha-1}|\nabla v|^{\gamma+q+2}\\
&- \frac{q}{\gamma+q}MNv^{\alpha+p}|\nabla v|^{\gamma+q},
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow -(1+\gamma S)v^{\alpha}|\nabla v|^{\gamma}(\Delta v)^2\\
&= -A + \frac{\alpha+p}{1+\gamma S+2S}(v^{\alpha-1}|\nabla v|^{\gamma+2}v_i)_i - \frac{qM}{\gamma+q}(v^{\alpha}|\nabla v|^{\gamma+q}v_i)_i + \gamma v^{\alpha}|\nabla v|^{\gamma}G_{11}\Delta v\\
&- (\alpha+p)\frac{\gamma+2}{1+\gamma S+2S}v^{\alpha-1}|\nabla v|^{\gamma+2}G_{11} - (\alpha+p)\frac{\alpha-1}{1+\gamma S+2S}v^{\alpha-2}|\nabla v|^{\gamma+4} - \frac{q}{\gamma+q}M^2v^{\alpha}|\nabla v|^{\gamma+2q}\\
&+ \left(p+\frac{q\alpha}{\gamma+q}\right)Mv^{\alpha-1}|\nabla v|^{\gamma+q+2} - \frac{q}{\gamma+q}MNv^{\alpha+p}|\nabla v|^{\gamma+q}.
\end{aligned} \tag{13}$$

Using the fact that

$$\begin{aligned}
E_{ij}^2 &= v_{ij}^2 - \frac{1}{n}(\Delta v)^2\\
&= G_{ij}^2 + 2(S-Q)G_{11}\Delta v + \left[S^2+(n-1)Q^2-\frac{1}{n}\right](\Delta v)^2,
\end{aligned} \tag{14}$$

and the equation (13), we know that for any $\varepsilon > 0$, it follows that

$$\begin{aligned}
&-(1+\gamma S-\varepsilon\tau)v^{\alpha}|\nabla v|^{\gamma}(\Delta v)^2\\
&= -A + \frac{\alpha+p}{1+\gamma S+2S}(v^{\alpha-1}|\nabla v|^{\gamma+2}v_i)_i - \frac{qM}{\gamma+q}(v^{\alpha}|\nabla v|^{\gamma+q}v_i)_i + \gamma v^{\alpha}|\nabla v|^{\gamma}G_{11}\Delta v\\
&- (\alpha+p)\frac{\gamma+2}{1+\gamma S+2S}v^{\alpha-1}|\nabla v|^{\gamma+2}G_{11} - (\alpha+p)\frac{\alpha-1}{1+\gamma S+2S}v^{\alpha-2}|\nabla v|^{\gamma+4} - \frac{q}{\gamma+q}M^2v^{\alpha}|\nabla v|^{\gamma+2q}\\
&+ \left(p+\frac{q\alpha}{\gamma+q}\right)Mv^{\alpha-1}|\nabla v|^{\gamma+q+2} - \frac{q}{\gamma+q}MNv^{\alpha+p}|\nabla v|^{\gamma+q}\\
&+ \varepsilon v^{\alpha}|\nabla v|^{\gamma}E_{ij}^2 - \varepsilon v^{\alpha}|\nabla v|^{\gamma}G_{ij}^2 - 2\varepsilon(S-Q)v^{\alpha}|\nabla v|^{\gamma}G_{11}\Delta v,
\end{aligned} \tag{15}$$

with $\tau := S^2 + (n-1)Q^2 - \frac{1}{n}$.

2.4. **The final equation.** Recalling in Section 2.2, we have already got that

$$\begin{aligned}
\Rightarrow&\Big[1-S^2+\gamma S-\gamma S^2-(n-1)Q^2\Big]v^\alpha|\nabla v|^\gamma(\Delta v)^2\\
&=\left(1-\frac{1}{n}\right)\Big(v^\alpha|\nabla v|^\gamma\Delta v v_i\Big)_i-(v^\alpha|\nabla v|^\gamma E_{ij}v_j)_i+\frac{\alpha S-\alpha}{1+\gamma S+2S}(v^{\alpha-1}|\nabla v|^{\gamma+2}v_i)_i\\
&+\sum_{i+j>2}v^\alpha|\nabla v|^\gamma G_{ij}^2+\frac{\alpha(\alpha-1)(1-S)}{1+\gamma S+2S}v^{\alpha-2}|\nabla v|^{\gamma+4}+\frac{\alpha(\gamma+3)}{1+\gamma S+2S}v^{\alpha-1}|\nabla v|^{\gamma+2}G_{11}\\
&+\Big(2\gamma S-\gamma+2S-2Q\Big)v^\alpha|\nabla v|^\gamma G_{11}\Delta v+(1+\gamma)v^\alpha|\nabla v|^\gamma G_{11}^2+\sum_{i>1}\gamma v^\alpha|\nabla v|^\gamma G_{i1}^2.
\end{aligned}\tag{16}$$

Thus we obtain

$$\begin{aligned}
0=&-A+\frac{\alpha+p}{1+\gamma S+2S}(v^{\alpha-1}|\nabla v|^{\gamma+2}v_i)_i-\frac{qM}{\gamma+q}(v^\alpha|\nabla v|^{\gamma+q}v_i)_i\\
&+\gamma v^\alpha|\nabla v|^\gamma G_{11}\Delta v-(\alpha+p)\frac{\gamma+2}{1+\gamma S+2S}v^{\alpha-1}|\nabla v|^{\gamma+2}G_{11}\\
&-(\alpha+p)\frac{\alpha-1}{1+\gamma S+2S}v^{\alpha-2}|\nabla v|^{\gamma+4}-\frac{q}{\gamma+q}M^2v^\alpha|\nabla v|^{\gamma+2q}\\
&+\left(p+\frac{q\alpha}{\gamma+q}\right)Mv^{\alpha-1}|\nabla v|^{\gamma+q+2}-\frac{q}{\gamma+q}MNv^{\alpha+p}|\nabla v|^{\gamma+q}\\
&+\varepsilon v^\alpha|\nabla v|^\gamma E_{ij}^2-\varepsilon v^\alpha|\nabla v|^\gamma G_{ij}^2-2\varepsilon(S-Q)v^\alpha|\nabla v|^\gamma G_{11}\Delta v\\
&+\frac{1+\gamma S-\varepsilon\tau}{1-S^2+\gamma S-\gamma S^2-(n-1)Q^2}\Bigg\{\left(1-\frac{1}{n}\right)\Big(v^\alpha|\nabla v|^\gamma\Delta v v_i\Big)_i-(v^\alpha|\nabla v|^\gamma E_{ij}v_j)_i\\
&+\frac{\alpha S-\alpha}{1+\gamma S+2S}(v^{\alpha-1}|\nabla v|^{\gamma+2}v_i)_i\\
&+\sum_{i+j>2}v^\alpha|\nabla v|^\gamma G_{ij}^2+\frac{\alpha(\alpha-1)(1-S)}{1+\gamma S+2S}v^{\alpha-2}|\nabla v|^{\gamma+4}+\frac{\alpha(\gamma+3)}{1+\gamma S+2S}v^{\alpha-1}|\nabla v|^{\gamma+2}G_{11}\\
&+\Big(2\gamma S-\gamma+2S-2Q\Big)v^\alpha|\nabla v|^\gamma G_{11}\Delta v+(1+\gamma)v^\gamma|\nabla v|^\gamma G_{11}^2+\sum_{i>1}\gamma v^\alpha|\nabla v|^\gamma G_{i1}^2\Bigg\},
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow 0 =& W + \varepsilon v^\alpha |\nabla v|^\gamma E_{ij}^2 \\
&+ \left[\gamma + \frac{1+\gamma S - \varepsilon\tau}{1 - S^2 + \gamma S - \gamma S^2 - (n-1)Q^2}\Big(2\gamma S - \gamma + 2S - 2Q\Big) - 2\varepsilon(S-Q)\right] v^\alpha |\nabla v|^\gamma G_{11}\Delta v \\
&+ \left[\frac{1+\gamma S - \varepsilon\tau}{1 - S^2 + \gamma S - \gamma S^2 - (n-1)Q^2} - \varepsilon\right] v^\alpha |\nabla v|^\gamma \sum_{i,j=2}^n G_{ij}^2 \\
&+ \left[\frac{1+\gamma S - \varepsilon\tau}{1 - S^2 + \gamma S - \gamma S^2 - (n-1)Q^2}(1+\gamma) - \varepsilon\right] v^\alpha |\nabla v|^\gamma G_{11}^2 \\
&+ \left[\frac{1+\gamma S - \varepsilon\tau}{1 - S^2 + \gamma S - \gamma S^2 - (n-1)Q^2}(2+\gamma) - 2\varepsilon\right] \sum_{i>1} v^\alpha |\nabla v|^\gamma G_{1i}^2 \\
&+ \left[-\Big(\alpha + p\Big)\frac{\alpha - 1}{1+\gamma S + 2S} + \frac{1+\gamma S - \varepsilon\tau}{1 - S^2 + \gamma S - \gamma S^2 - (n-1)Q^2}\cdot\frac{\alpha(\alpha-1)(1-S)}{1+\gamma S + 2S}\right] v^{\alpha-2}|\nabla v|^{\gamma+4} \\
&+ \left[\frac{1+\gamma S - \varepsilon\tau}{1 - S^2 + \gamma S - \gamma S^2 - (n-1)Q^2}\cdot\frac{\alpha(\gamma+3)}{1+\gamma S + 2S} - \Big(\alpha + p\Big)\frac{\gamma+2}{1+\gamma S + 2S}\right] v^{\alpha-1}|\nabla v|^{\gamma+2} G_{11} \\
&- \frac{q}{\gamma+q} M^2 v^\alpha |\nabla v|^{\gamma+2q} + \Big(p + \frac{q\alpha}{\gamma+q}\Big) M v^{\alpha-1}|\nabla v|^{\gamma+q+2} - \frac{q}{\gamma+q} MN v^{\alpha+p}|\nabla v|^{\gamma+q},
\end{aligned}
\tag{17}
$$

where W consists of all the divergence terms. If $\varepsilon > 0$ is small enough, $\gamma \geq 0$ and

$$
\frac{1+\gamma S}{1 - S^2 + \gamma S - \gamma S^2 - (n-1)Q^2} > 0,
$$

then we can rewrite (17) as

$$
\begin{aligned}
0 \geq& W + \varepsilon v^\alpha |\nabla v|^\gamma E_{ij}^2 + a_1 v^\alpha |\nabla v|^\gamma \sum_{i>1} G_{ii}^2 + a_2 v^\alpha |\nabla v|^\gamma G_{11}^2 + a_3 v^{\alpha-2}|\nabla v|^{\gamma+4} \\
&+ b_1 v^{\alpha-1}|\nabla v|^{\gamma+2} G_{11} + b_2 v^\alpha |\nabla v|^\gamma G_{11}\Delta v + c_2 M^2 v^\alpha |\nabla v|^{\gamma+2q} \\
&- b_3 N v^{\alpha+p-1}|\nabla v|^{\gamma+2} - b_4 M v^{\alpha-1}|\nabla v|^{\gamma+q+2} + b_5 MN v^{\alpha+p}|\nabla v|^{\gamma+q},
\end{aligned}
\tag{18}
$$

with

$$
\begin{aligned}
a_1 &= \frac{1+\gamma S-\varepsilon\tau}{1-S^2+\gamma S-\gamma S^2-(n-1)Q^2}-\varepsilon,\\
a_2 &= \frac{1+\gamma S-\varepsilon\tau}{1-S^2+\gamma S-\gamma S^2-(n-1)Q^2}(1+\gamma)-\varepsilon,\\
a_3 &= -\Big(\alpha+p\Big)\frac{\alpha-1}{1+\gamma S+2S}+\frac{1+\gamma S-\varepsilon\tau}{1-S^2+\gamma S-\gamma S^2-(n-1)Q^2}\cdot\frac{\alpha(\alpha-1)(1-S)}{1+\gamma S+2S},\\
b_1 &= \frac{1+\gamma S-\varepsilon\tau}{1-S^2+\gamma S-\gamma S^2-(n-1)Q^2}\cdot\frac{\alpha(\gamma+3)}{1+\gamma S+2S}-\Big(\alpha+p\Big)\frac{\gamma+2}{1+\gamma S+2S},\\
b_2 &= \gamma+\frac{1+\gamma S-\varepsilon\tau}{1-S^2+\gamma S-\gamma S^2-(n-1)Q^2}\Big(2\gamma S-\gamma+2S-2Q\Big)-2\varepsilon(S-Q),\\
c_2 &= -\frac{q}{\gamma+q},\\
b_3 &= 0,\\
b_4 &= -p-\frac{q\alpha}{\gamma+q},\\
b_5 &= -\frac{q}{\gamma+q}.
\end{aligned}
$$

3. Conditions

Multiply (4) by $Nv^{\alpha+p}|\nabla v|^{\gamma}$:

$$
\begin{aligned}
&N^2v^{\alpha+2p}|\nabla v|^{\gamma}+MNv^{\alpha+p}|\nabla v|^{\gamma+q}\\
&=-\frac{N}{1+\gamma S}(v^{\alpha+p}|\nabla v|^{\gamma}v_i)_i+\frac{\alpha+p}{1+\gamma S}Nv^{\alpha+p-1}|\nabla v|^{\gamma+2}\\
&+\frac{\gamma}{1+\gamma S}Nv^{\alpha+p}|\nabla v|^{\gamma}G_{11}.
\end{aligned}
\tag{19}
$$

Multiply (4) by $Mv^{\alpha}|\nabla v|^{\gamma+q}$:

$$
\begin{aligned}
&MNv^{\alpha+p}|\nabla v|^{\gamma+q}+M^2v^{\alpha}|\nabla v|^{\gamma+2q}\\
&=\frac{-M}{1+\gamma S+qS}(v^{\alpha}|\nabla v|^{\gamma+q}v_i)_i+\Big(\frac{\alpha}{1+\gamma S+qS}\Big)Mv^{\alpha-1}|\nabla v|^{\gamma+q+2}\\
&+\frac{\gamma+q}{1+\gamma S+qS}Mv^{\alpha}|\nabla v|^{\gamma+q}G_{11}.
\end{aligned}
\tag{20}
$$

Multiply (4) by $v^{\alpha-1}|\nabla v|^{\gamma+2}$:

$$
\begin{aligned}
-Mv^{\alpha-1}|\nabla v|^{\gamma+q+2}&=\frac{1}{1+\gamma S+2S}(v^{\alpha-1}|\nabla v|^{\gamma+2}v_i)_i-\frac{\alpha-1}{1+\gamma S+2S}v^{\alpha-2}|\nabla v|^{\gamma+4}\\
&-\frac{\gamma+2}{1+\gamma S+2S}v^{\alpha-1}|\nabla v|^{\gamma+2}G_{11}+Nv^{\alpha+p-1}|\nabla v|^{\gamma+2}.
\end{aligned}
$$

Besides, using the fact that

$$(n-1)\sum_{i>1} G_{ii}^2 \geq \left(\sum_{i>1} G_{ii}\right)^2 = G_{11}^2,$$

by (18) we get that if $a_1 > 0, b_2 = 0$, then

$$\begin{aligned}
0 \geq\ & W + \varepsilon v^{\alpha}|\nabla v|^{\gamma}E_{ij}^2 + \Big(a_2 + \frac{1}{n-1}a_1\Big)v^{\alpha}|\nabla v|^{\gamma}G_{11}^2 + a_3 v^{\alpha-2}|\nabla v|^{\gamma+4} + b_1 v^{\alpha-1}|\nabla v|^{\gamma+2}G_{11} \\
&+ c_2 M^2 v^{\alpha}|\nabla v|^{\gamma+2q} - b_3 N v^{\alpha+p-1}|\nabla v|^{\gamma+2} - b_4 M v^{\alpha-1}|\nabla v|^{\gamma+q+2} + b_5 M N v^{\alpha+p}|\nabla v|^{\gamma+q} \\
=\ & W + \varepsilon v^{\alpha}|\nabla v|^{\gamma}E_{ij}^2 + \Big(a_2 + \frac{1}{n-1}a_1\Big)v^{\alpha}|\nabla v|^{\gamma}G_{11}^2 + a_3 v^{\alpha-2}|\nabla v|^{\gamma+4} + b_1 v^{\alpha-1}|\nabla v|^{\gamma+2}G_{11} \\
&+ c_2 M^2 v^{\alpha}|\nabla v|^{\gamma+2q} - b_3 N v^{\alpha+p-1}|\nabla v|^{\gamma+2} - b_4 M v^{\alpha-1}|\nabla v|^{\gamma+q+2} + b_5 M N v^{\alpha+p}|\nabla v|^{\gamma+q} \\
&+ T\Bigg(N^2 v^{\alpha+2p}|\nabla v|^{\gamma} + MNv^{\alpha+p}|\nabla v|^{\gamma+q} - \frac{\alpha+p}{1+\gamma S}Nv^{\alpha+p-1}|\nabla v|^{\gamma+2} \\
&- \frac{\gamma}{1+\gamma S}Nv^{\alpha+p}|\nabla v|^{\gamma}G_{11}\Bigg) \\
&+ U\Bigg(MNv^{\alpha+p}|\nabla v|^{\gamma+q} + M^2 v^{\alpha}|\nabla v|^{\gamma+2q} \\
&- \frac{\alpha}{1+\gamma S+qS}Mv^{\alpha-1}|\nabla v|^{\gamma+q+2} - \frac{\gamma+q}{1+\gamma S+qS}Mv^{\alpha}|\nabla v|^{\gamma+q}G_{11}\Bigg) \\
&+ P\Bigg(-Nv^{\alpha+p-1}|\nabla v|^{\gamma+2} + \frac{\alpha-1}{1+\gamma S+2S}v^{\alpha-2}|\nabla v|^{\gamma+4} \\
&+ \frac{\gamma+2}{1+\gamma S+2S}v^{\alpha-1}|\nabla v|^{\gamma+2}G_{11} - Mv^{\alpha-1}|\nabla v|^{\gamma+q+2}\Bigg),
\end{aligned}$$

$$\begin{aligned}
0 \geq\ & W + \varepsilon v^{\alpha}|\nabla v|^{\gamma}E_{ij}^2 + \Big(a_2 + \frac{1}{n-1}a_1\Big)v^{\alpha}|\nabla v|^{\gamma}G_{11}^2 \\
&+ \left(a_3 + P\frac{\alpha-1}{1+\gamma S+2S}\right)v^{\alpha-2}|\nabla v|^{\gamma+4} + \Big(b_1 + P\frac{\gamma+2}{1+\gamma S+2S}\Big)v^{\alpha-1}|\nabla v|^{\gamma+2}G_{11} \\
&+ TN^2 v^{\alpha+2p}|\nabla v|^{\gamma} + \Big(c_2 + U\Big)M^2 v^{\alpha}|\nabla v|^{\gamma+2q} - \left[b_3 + P + \frac{T(\alpha+p)}{1+\gamma S}\right]Nv^{\alpha+p-1}|\nabla v|^{\gamma+2} \\
&- \frac{T\gamma}{1+\gamma S}Nv^{\alpha+p}|\nabla v|^{\gamma}G_{11} - U\frac{\gamma+q}{1+\gamma S+qS}Mv^{\alpha}|\nabla v|^{\gamma+q}G_{11} \\
&- \left(b_4 + P + U\frac{\alpha}{1+\gamma S+qS}\right)Mv^{\alpha-1}|\nabla v|^{\gamma+q+2} + \Big(b_5 + T + U\Big)MNv^{\alpha+p}|\nabla v|^{\gamma+q}.
\end{aligned}$$

Define $B_0 := a_2 + \frac{1}{n-1}a_1$ and if $B_0 > 0$, then

$$
\begin{aligned}
0 \geq\ & W + \varepsilon v^{\alpha}|\nabla v|^{\gamma}E_{ij}^2 \\
& - \frac{1}{4B_0}v^{\alpha}|\nabla v|^{\gamma}\left[\Big(b_1 + P\frac{\gamma+2}{1+\gamma S+2S}\Big)v^{-1}|\nabla v|^2 - \frac{T\gamma}{1+\gamma S}Nv^p - U\frac{\gamma+q}{1+\gamma S+qS}M|\nabla v|^q\right]^2 \\
& + \left(a_3 + P\frac{\alpha-1}{1+\gamma S+2S}\right)v^{\alpha-2}|\nabla v|^{\gamma+4} + TN^2v^{\alpha+2p}|\nabla v|^{\gamma} + \Big(c_2 + U\Big)M^2v^{\alpha}|\nabla v|^{\gamma+2q} \\
& - \left[b_3 + P + \frac{T(\alpha+p)}{1+\gamma S}\right]Nv^{\alpha+p-1}|\nabla v|^{\gamma+2} - \left(b_4 + P + U\frac{\alpha}{1+\gamma S+qS}\right)Mv^{\alpha-1}|\nabla v|^{\gamma+q+2} \\
& + \Big(b_5 + T + U\Big)MNv^{\alpha+p}|\nabla v|^{\gamma+q}.
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow 0 \geq\ & W + \varepsilon v^{\alpha}|\nabla v|^{\gamma}E_{ij}^2 + \left[a_3 + P\frac{\alpha-1}{1+\gamma S+2S} - \frac{1}{4B_0}\Big(b_1 + P\frac{\gamma+2}{1+\gamma S+2S}\Big)^2\right]v^{\alpha-2}|\nabla v|^{\gamma+4} \\
& + \left[T - \frac{1}{4B_0}\left(\frac{T\gamma}{1+\gamma S}\right)^2\right]N^2v^{\alpha+2p}|\nabla v|^{\gamma} \\
& + \left[c_2 + U - \frac{1}{4B_0}\Big(U\frac{\gamma+q}{1+\gamma S+qS}\Big)^2\right]M^2v^{\alpha}|\nabla v|^{\gamma+2q} \\
& - \left[\frac{T(\alpha+p)}{1+\gamma S} + P - \frac{1}{2B_0}\frac{T\gamma}{1+\gamma S}\Big(b_1 + P\frac{\gamma+2}{1+\gamma S+2S}\Big)\right]Nv^{\alpha+p-1}|\nabla v|^{\gamma+2} \\
& - \left[b_4 + P + U\frac{\alpha}{1+\gamma S+qS} - \frac{1}{2B_0}\Big(b_1 + P\frac{\gamma+2}{1+\gamma S+2S}\Big)U\frac{\gamma+q}{1+\gamma S+qS}\right]Mv^{\alpha-1}|\nabla v|^{\gamma+q+2} \\
& + \Big(b_5 + T + U - \frac{1}{2B_0}\frac{T\gamma}{1+\gamma S}\cdot U\frac{\gamma+q}{1+\gamma S+qS}\Big)MNv^{\alpha+p}|\nabla v|^{\gamma+q}.
\end{aligned}
$$

Without loss of generality, we can let $N=1$. In general, we hope that the following conditions hold for $\varepsilon>0$ small enough:

$$
\begin{cases}
a_3+P\Big(\dfrac{\alpha-1}{1+\gamma S+2S}\Big)-\dfrac{1}{4B_0}\Big(b_1+P\dfrac{\gamma+2}{1+\gamma S+2S}\Big)^2>0, & (21)\\
T-\dfrac{1}{4B_0}\left(\dfrac{T\gamma}{1+\gamma S}\right)^2>0, & (22)\\
c_2+U-\dfrac{1}{4B_0}\Big(U\dfrac{\gamma+q}{1+\gamma S+qS}\Big)^2>0, & (23)\\
\dfrac{T(\alpha+p)}{1+\gamma S}+P-\dfrac{1}{2B_0}\dfrac{T\gamma}{1+\gamma S}\Big(b_1+P\dfrac{\gamma+2}{1+\gamma S+2S}\Big)\le 0, & (24)\\
M\left[b_4+P+U\Big(\dfrac{\alpha}{1+\gamma S+qS}\Big)-\dfrac{1}{2B_0}\Big(b_1+P\dfrac{\gamma+2}{1+\gamma S+2S}\Big)U\dfrac{\gamma+q}{1+\gamma S+qS}\right]\le 0, & (25)\\
M\left(b_5+T+U-\dfrac{1}{2B_0}\dfrac{T\gamma}{1+\gamma S}\cdot U\dfrac{\gamma+q}{1+\gamma S+qS}\right)\ge 0. & (26)
\end{cases}
$$

We point that when $M>0$ the condition (63) can be deduced from (59) and (60) since $c_2=b_5$ and $B_0>0$. If these conditions are satisfied, then for any $\varepsilon>0$ small enough, we have

$$
\begin{aligned}
&\varepsilon v^{\alpha}|\nabla v|^{\gamma}|E_{ij}|^2+\varepsilon v^{\alpha-2}|\nabla v|^{\gamma+4}+\varepsilon v^{\alpha+2p}|\nabla v|^{\gamma}+\varepsilon v^{\alpha}|\nabla v|^{\gamma+2q}\\
&\le B_1(v^{\alpha}|\nabla v|^{\gamma}v_jE_{ij})_i+B_2(v^{\alpha+p}|\nabla v|^{\gamma}v_i)_i+B_3(v^{\alpha-1}|\nabla v|^{\gamma+2}v_i)_i\\
&+B_4(v^{\alpha}|\nabla v|^{\gamma+q}v_i)_i.
\end{aligned}
\tag{27}
$$

4. Young inequality and cut-off functions

In this part, we assume (27) holds and prove that $|\nabla v|\equiv 0$. Define η is a smooth cut-off function, satisfying that

$$
\begin{aligned}
&\eta\equiv 1 \quad \text{in} \quad B_R,\\
&\eta\equiv 0 \quad \text{in} \quad \mathbb{R}^n\backslash B_{2R}.
\end{aligned}
$$

Multiply (27) by η^δ and integrate over $\mathbb{R}^n$,

$$\begin{aligned}
\mathrm{RHS} = & B_1\delta\int v^\alpha|\nabla v|^\gamma v_j E_{ij}\eta^{\delta-1}\eta_i + B_2\delta\int v^{\alpha+p}|\nabla v|^\gamma v_i\eta^{\delta-1}\eta_i \\
& + B_3\delta\int v^{\alpha-1}|\nabla v|^{\gamma+2}v_i\eta^{\delta-1}\eta_i + B_4\delta\int v^\alpha|\nabla v|^{\gamma+q}v_i\eta^{\delta-1}\eta_i \\
& \le \frac{\varepsilon}{2}\int v^\alpha|\nabla v|^\gamma|E_{ij}|^2\eta^\delta + C\int v^\alpha|\nabla v|^{\gamma+2}\eta^{\delta-2}|\nabla\eta|^2 \\
& + \frac{\varepsilon}{2}\int v^{\alpha+2p}|\nabla v|^\gamma\eta^\delta + C\int v^\alpha|\nabla v|^{\gamma+2}\eta^{\delta-2}|\nabla\eta|^2 \\
& + \frac{\varepsilon}{2}\int v^{\alpha-2}|\nabla v|^{\gamma+4} + C\int v^\alpha|\nabla v|^{\gamma+2}\eta^{\delta-2}|\nabla\eta|^2 \\
& + \frac{\varepsilon}{2}\int v^\alpha|\nabla v|^{\gamma+2q} + C\int v^\alpha|\nabla v|^{\gamma+2}\eta^{\delta-2}|\nabla\eta|^2.
\end{aligned}$$

Therefore we get that

$$\begin{aligned}
& \Rightarrow \int v^{\alpha+2p}|\nabla v|^\gamma\eta^\delta + \int v^{\alpha-2}|\nabla v|^{\gamma+4}\eta^\delta \\
& \le C\int v^\alpha|\nabla v|^{\gamma+2}\eta^{\delta-2}|\nabla\eta|^2.
\end{aligned} \tag{28}$$

Define $p_1, q_1, \sigma_1 \ge 0$, such that

$$\frac{1}{p_1} + \frac{1}{q_1} + \frac{1}{\sigma_1} = 1, \text{ and } p_1, q_1, \sigma_1 \ge 0. \tag{29}$$

So by Young inequality, we know

$$\begin{aligned}
& \int v^\alpha|\nabla v|^{\gamma+2}\eta^{\delta-2}|\nabla\eta|^2 \\
& = \int v^{\alpha-A}|\nabla v|^{\gamma+2-B}v^A|\nabla v|^B\eta^{\delta-2}|\nabla\eta|^2 \\
& \le \varepsilon_0\int v^{\alpha-2}|\nabla v|^{\gamma+4}\eta^\delta + \varepsilon_0\int v^{\alpha+2p}|\nabla v|^\gamma\eta^\delta \\
& + C\int \eta^{\delta-2\sigma_1}|\nabla\eta|^{2\sigma_1} \\
\Rightarrow & \int v^{\alpha+2p}|\nabla v|^\gamma\eta^\delta + \int v^{\alpha-2}|\nabla v|^{\gamma+4}\eta^\delta \\
& \le C\int \eta^{\delta-2\sigma_1}|\nabla\eta|^{2\sigma_1} \\
& \le CR^{n-2\sigma_1} \to 0, \text{as R tends to infinity.}
\end{aligned}$$

The last two steps use the condition that

$$\begin{aligned}
\delta - 2\sigma_1 > 0, \\
n - 2\sigma_1 < 0.
\end{aligned} \tag{30}$$

So we always take δ large enough. In conclusion, we need

$$\begin{aligned} &\frac{\alpha+2p}{A}=\frac{\gamma}{B}=p_1,\\ &\frac{\alpha-2}{\alpha-A}=\frac{\gamma+4}{\gamma+2-B}=q_1,\\ &1-\frac{2}{n}<\frac{1}{p_1}+\frac{1}{q_1}<1. \end{aligned} \tag{31}$$

Therefore, as long as the conditions (58) - (63) are satisfied, and there exists such p_1, q_1, σ_1, then $u \equiv 0$.

5. Proof of Theorem 1.5

In this section, we choose $\gamma = 0, S = Q = \frac{1}{n}$, then

$$\begin{aligned} G_{ij} &= E_{ij},\\ \tau &= 0,\\ a_1 &= \frac{n}{n-1}-\varepsilon,\\ a_3 &= -\frac{n}{n+2}p(\alpha-1),\\ b_1 &= \frac{n}{n-1}\alpha-\frac{2np}{n+2},\\ b_2 &= 0,\\ c_2 &= -1,\\ b_3 &= 0,\\ b_4 &= -p-\alpha,\\ b_5 &= -1,\\ B_0 &= \frac{n}{n-1}\left(\frac{n}{n-1}-\varepsilon\right). \end{aligned}$$

And the conditions (58) - (63) become

$$\begin{cases} a_3+P\cdot\dfrac{\alpha-1}{1+2S}-\dfrac{1}{4B_0}\Big(b_1+\dfrac{2P}{1+2S}\Big)^2>0, & (32)\\ T>0, & (33)\\ c_2+U-\dfrac{1}{4B_0}\Big(U\dfrac{q}{1+qS}\Big)^2>0, & (34)\\ T(\alpha+p)+P\le 0, & (35)\\ M\left[b_4+P+U\Big(\dfrac{\alpha}{1+qS}\Big)-\dfrac{1}{2B_0}\Big(b_1+P\dfrac{2}{1+2S}\Big)U\dfrac{q}{1+qS}\right]\le 0, & (36)\\ M\Big(b_5+T+U\Big)\ge 0. & (37) \end{cases}$$

5.1. **Proof of Theorem 1.5.** In this subsection, we consider the case $M > 0$ and choose $T = \frac{\varepsilon}{\alpha+p} > 0$ small enough. By direct computation, we deduce that if $n \geq 3$, $\varepsilon > 0$ small is enough and

$$\alpha = -\frac{2(p-P)}{n+2},$$
$$0 < p - P < \frac{n+2}{n-2},$$

then (32) holds. Besides we know that if $|P|$ is small enough, then

$$\alpha + p = \frac{np}{n+2} + \frac{2P}{n+2} > 0.$$

At this time, we can solve the equations in (31):

$$\begin{cases} \frac{\alpha+2p}{A} = p_1, \\ \frac{\alpha-2}{\alpha-A} = 2 = q_1. \end{cases}$$

We obtain that

$$A = \frac{\alpha+2}{2},$$

and

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2} + \frac{1}{2} \cdot \frac{\alpha+2}{\alpha+2p}.$$

If $p > 1$ and $p - P < \frac{n+2}{n-2}$, then $\alpha + 2p > \alpha + 2 \geq 0$. So we have

$$\frac{1}{p_1} + \frac{1}{q_1} < 1,$$

and

$$p_1, q_1 \geq 0.$$

Besides since $\alpha + 2p > 0$, then

$$\begin{aligned} \frac{1}{p_1} + \frac{1}{q_1} > 1 - \frac{2}{n} \Leftrightarrow & \frac{\alpha+2}{2(\alpha+2p)} > \frac{1}{2} - \frac{2}{n}, \\ \Leftrightarrow & -\frac{p-P}{n+2} + 1 > \left[-\frac{2(p-P)}{n+2} + 2p\right]\left(\frac{1}{2} - \frac{2}{n}\right), \\ \Leftrightarrow & 1 > p \cdot \frac{n^2-2n-4}{n(n+2)} - \frac{4P}{n(n+2)}. \end{aligned}$$

In fact, we have

$$\begin{aligned} p \cdot \frac{n^2-2n-4}{n(n+2)} &\leq \frac{n+2}{n-2} \cdot \frac{n^2-2n-4}{n(n+2)} \\ &= \frac{n^2-2n-4}{n^2-2n} < 1. \end{aligned}$$

Thus when $|P|$ is small enough, we have

$$\frac{1}{p_1} + \frac{1}{q_1} > 1 - \frac{2}{n}.$$

Then we can prove Theorem 1.5 :

Proof of Theorem 1.5 . In this case, we choose $\gamma = 0, S = \frac{1}{n}$. If $1 < p \le \frac{n+2}{n-2}$, we can choose $\alpha = -\frac{2(p-P)}{n+2}$ and $P = \varepsilon$, such that (32) hold. Then we only need to consider conditions (34), (35), (36) and (37).

Firstly we need to check the condition (35). If $\varepsilon > 0$ is much smaller than M and

$$\frac{\varepsilon q}{M^{\frac{2}{q}}}\Big(\frac{2}{2-q}\Big)^{-\frac{2-q}{q}} < \varepsilon^{\frac{1}{2}},$$

then using the following Young inequality:

$$v^{\alpha+p-1}|\nabla v|^{\gamma+2} \le \frac{q}{2}\Big(M\cdot\frac{2}{2-q}\Big)^{-\frac{2-q}{q}} v^{\alpha+p}|\nabla v|^{\gamma+q} + Mv^{\alpha-1}|\nabla v|^{\gamma+q+2},$$

with $(\frac{2}{q}, \frac{2}{2-q})$, we get that

$$\begin{aligned}\Big[T(\alpha+p)+P\Big]v^{\alpha+p-1}|\nabla v|^{\gamma+2} &\le \frac{\varepsilon q}{M^{\frac{2}{q}}}\Big(\frac{2}{2-q}\Big)^{-\frac{2-q}{q}} Mv^{\alpha+p}|\nabla v|^{\gamma+q} + 2\varepsilon Mv^{\alpha-1}|\nabla v|^{\gamma+q+2}\\ &\le \varepsilon^{\frac{1}{2}} Mv^{\alpha+p}|\nabla v|^{\gamma+q} + 2\varepsilon Mv^{\alpha-1}|\nabla v|^{\gamma+q+2}.\end{aligned}$$

Next let us consider the condition (36):

$$\begin{aligned}LHS &= -\frac{n}{n+2}p + U\frac{\alpha}{1+qS} - \frac{1}{2B_0}b_1U\frac{q}{1+qS}\\ &= \Big[-\frac{2p}{n+2} + \frac{(n-1)^2}{2n^2}q\cdot\frac{2n^2p}{(n-1)(n+2)}\Big]\frac{U}{1+qS} - \frac{n}{n+2}p + O(\varepsilon)\\ &= \Big[-2+(n-1)q\Big]\frac{p}{n+2}\cdot\frac{U}{1+qS} - \frac{n}{n+2}p + O(\varepsilon).\end{aligned}$$

Note that there always exists $U > 0$ such that (34) holds for $U > 0$. For $n \ge 3, q > 1$, we know

$$-2+(n-1)q > 0.$$

For the easier case, we hope that the following hold

$$\begin{cases} c_2 + U - \dfrac{1}{4B_0}\Big(U\dfrac{q}{1+qS}\Big)^2 > 0, & (38)\\ b_4 + U\Big(\dfrac{\alpha}{1+qS}\Big) - \dfrac{1}{2B_0}b_1U\dfrac{q}{1+qS} + 2\varepsilon \le 0. & (39)\end{cases}$$

If so, then the condition (37) also holds and the conclusion holds for any $M > 0$. We find that if $3 \le n < 5$, we can choose $U = 1 + \frac{q}{n}$; if $n = 5, 6$, we can choose $U = 1 + \frac{2q}{n}$. But for $n \ge 7$, the above conditions will not hold at the same time. As a result, we need to consider the case for $n \ge 7$

$$\begin{cases} c_2 + U - \dfrac{1}{4B_0}\Big(U\dfrac{q}{1+qS}\Big)^2 < 0, & (40)\\ b_4 + U\Big(\dfrac{\alpha}{1+qS}\Big) - \dfrac{1}{2B_0}b_1U\dfrac{q}{1+qS} + 2\varepsilon \le 0. & (41)\end{cases}$$

Note that by Young inequality, we have

$$M^2 v^\alpha |\nabla v|^{\gamma+2q} \leq KMv^{\alpha-1}|\nabla v|^{\gamma+q+2} + \frac{\left(K \cdot \frac{p+1}{p}\right)^{-p}}{p+1} M^{p+2} v^{\alpha+p} |\nabla v|^{\gamma+q},$$

with $(\frac{p+1}{p}, p+1)$. We only need that

$$\begin{cases} b_4 + U\Big(\dfrac{\alpha}{1+qS}\Big) - \dfrac{b_1 U}{2B_0}\dfrac{q}{1+qS} + 2\varepsilon - K\Big[c_2 + U - \dfrac{1}{4B_0}\Big(U\dfrac{q}{1+qS}\Big)^2\Big] = 0, & (42) \\ b_5 + T + U - \varepsilon^{\frac{1}{2}} + \dfrac{\left(K \cdot \frac{p+1}{p}\right)^{-p}}{p+1} M^{p+1}\Big[c_2 + U - \dfrac{1}{4B_0}\Big(U\dfrac{q}{1+qS}\Big)^2\Big] > 0. & (43) \end{cases}$$

$$\Rightarrow b_5 + T + U - \varepsilon^{\frac{1}{2}} - \frac{p^p}{(p+1)^{p+1}} M^{p+1}\Big[-c_2 - U + \frac{1}{4B_0}\Big(U\frac{q}{1+qS}\Big)^2\Big]^{p+1}\Big[-b_4 - U\Big(\frac{\alpha}{1+qS}\Big) + \frac{b_1 U}{2B_0}\frac{q}{1+qS} - 2\varepsilon\Big]^{-p} > 0. \tag{44}$$

We still choose $U = 1 + \frac{q}{n}$, then

$$b_5 + T + U = \frac{q}{n} + \varepsilon > 0,$$

so the condition (37) holds and the condition (44) becomes:

$$\begin{aligned} &\Rightarrow \frac{q}{n} > \frac{p^p}{(p+1)^{p+1}} M^{p+1}\left[\frac{(n-1)^2 q^2}{4n^2} - \frac{q}{n}\right]^{p+1}\left[p - \frac{(n-1)pq}{n+2}\right]^{-p}, \\ &\Rightarrow M < \Big(\frac{q}{n}\Big)^{\frac{1}{p+1}} \frac{p+1}{p^{\frac{p}{p+1}}}\left[\frac{(n-1)^2 q^2}{4n^2} - \frac{q}{n}\right]^{-1}\left[p - \frac{(n-1)pq}{n+2}\right]^{\frac{p}{p+1}} \\ &\quad = (p+1)\left[\frac{(n-1)^2 q}{4n} - 1\right]^{-1}\left[\frac{n}{q} - \frac{n(n-1)}{n+2}\right]^{\frac{q}{2}} \\ &\quad =: M_1. \end{aligned}$$

□

5.2. **Proof of Theorem 1.7.** In this case, we choose $M < 0$ and the conditions become:

$$\begin{cases} a_3 + P\cdot\dfrac{\alpha-1}{1+2S} - \dfrac{1}{4B_0}\Big(b_1 + \dfrac{2P}{1+2S}\Big)^2 > 0, & (45)\\ T \geq 0, & (46)\\ c_2 + U - \dfrac{1}{4B_0}\Big(U\dfrac{q}{1+qS}\Big)^2 \geq 0, & (47)\\ T(\alpha+p) + P \leq 0, & (48)\\ b_4 + P + U\Big(\dfrac{\alpha}{1+qS}\Big) - \dfrac{1}{2B_0}\Big(b_1 + P\dfrac{2}{1+2S}\Big)U\dfrac{q}{1+qS} \geq 0, & (49)\\ b_5 + T + U \leq 0. & (50) \end{cases}$$

There are three cases:

- If $n = 3$, we choose
$$\alpha = -0.4(p-P).$$
- If $n \geq 4$, we choose
$$p - P = \frac{n+2}{n-2} - \varepsilon,$$
$$\alpha = -\frac{2(p-P)}{n+2} = -\frac{2}{n-2} + \frac{2\varepsilon}{n+2},$$

  with $\varepsilon > 0$ is small enough.

By direct computation, we know all of the cases satisfy that
$$1 - \frac{2}{n} < \frac{1}{p_1} + \frac{1}{q_1} < 1.$$

Now we can prove the Theorem 1.7:

Proof of Theorem 1.7.

- If $n \geq 4$, the condition (49) becomes that
$$b_4 + P + U\Big(\frac{\alpha}{1+qS}\Big) - \frac{1}{2B_0}\Big(b_1 + P\frac{2}{1+2S}\Big)U\frac{q}{1+qS}$$
$$= -\frac{n}{n-2} + \frac{U}{1+qS}\frac{(n-1)q-2}{n-2} + O(\varepsilon).$$

  We choose
$$U = \frac{2B_0(1+qS)^2}{q^2} = \frac{2(n+q)^2}{(n-1)^2q^2}.$$

  claim 1. *When $n \geq 4$, we have*
$$\frac{2(n+q)^2}{(n-1)^2q^2} > \frac{n+q}{(n-1)q-2}.$$

Then conditions (47) and (49) hold at the same time. Using the Young inequality

$$|M|v^{\alpha+p}|\nabla v|^{\gamma+q} \le Kv^{\alpha+2p}|\nabla v|^{\gamma} + \left(K\cdot\frac{2}{2-q}\right)^{-\frac{2-q}{q}}\frac{q}{2}|M|^{\frac{2}{q}}v^{\alpha+p-1}|\nabla v|^{\gamma+2}, \tag{51}$$

with $\left(\frac{2}{2-q},\frac{2}{q}\right)$, we only need that

$$\begin{cases} K^{-\frac{2}{q}}\frac{q}{2}\left(\frac{2}{2-q}\right)^{-\frac{2-q}{q}}|M|^{\frac{2}{q}} = -\dfrac{T(p-\frac{2}{n-2}+\frac{2\varepsilon}{n+2})+P}{T}, & (52)\\ \dfrac{T}{K}+1-T-U>0. & (53)\end{cases}$$

We deduce that

$$M > -\frac{T^{\frac{2-q}{2}}}{\frac{2(n+q)^2}{(n-1)^2q^2}+T-1}\left(\frac{q}{2}\right)^{-\frac{q}{2}}\left(\frac{2}{2-q}\right)^{\frac{2-q}{2}}\left[\frac{n+2}{n-2}-p-T\left(p-\frac{2}{n-2}\right)\right]^{\frac{q}{2}}.$$

where $T\ge 0$ is arbitrary.

- If $n=3$ and $1<p\le 2$, choose $p-P=2.5$, then $\alpha=-1$ and

$$a_3+P\cdot\frac{\alpha-1}{1+2S}-\frac{1}{4B_0}\Big(b_1+\frac{2P}{1+2S}\Big)^2=0.75.$$

Choose $U=2$, then for any $1<q<1.5$, we know

$$\begin{aligned}&4\left[a_3+P\cdot\frac{\alpha-1}{1+2S}-\frac{1}{4B_0}\Big(b_1+\frac{2P}{1+2S}\Big)^2\right]\cdot\left[c_2+U-\frac{1}{4B_0}\Big(U\frac{q}{1+qS}\Big)^2\right]\\ &-\left[b_4+P+U\Big(\frac{\alpha}{1+qS}\Big)-\frac{1}{2B_0}\Big(b_1+P\frac{2}{1+2S}\Big)U\frac{q}{1+qS}\right]^2\\ &=4\cdot 0.75\cdot\frac{-3q^2+6q+9}{(q+3)^2}-\left(\frac{4.5q-10.5}{q+3}\right)^2\\ &=\frac{-29.25q^2+112.5q-83.25}{(q+3)^2}>\frac{-29.25+112.5-83.25}{(q+3)^2}=0.\end{aligned}$$

Then by (51) we only need that

$$\begin{cases} K^{-\frac{2}{q}}\frac{q}{2}\left(\frac{2}{2-q}\right)^{-\frac{2-q}{q}}|M|^{\frac{2}{q}} = -\dfrac{T(\alpha+p)+P}{T}, & (54)\\ \dfrac{T}{K}>T+1. & (55)\end{cases}$$

We deduce that

$$M > -\frac{T^{\frac{2-q}{2}}}{T+1}\left(\frac{q}{2}\right)^{-\frac{q}{2}}\left(\frac{2}{2-q}\right)^{\frac{2-q}{2}}\Big[2.5-p-T(p-1)\Big]^{\frac{q}{2}},$$

where $T\ge 0$ is arbitrary.

- If $n = 3$ and $2 < p < 5$, choose $p - P = 5 - \varepsilon$, where $\varepsilon > 0$ is small enough. Then $\alpha = -2 + 0.4\varepsilon$ and the condition (45) holds. Let

$$b_4 + P + U\Big(\frac{\alpha}{1+qS}\Big) - \frac{1}{2B_0}\Big(b_1 + P\frac{2}{1+2S}\Big)U\frac{q}{1+qS} = 0,$$

then

$$U = \frac{q+3}{2q-2} + O(\varepsilon).$$

Since $\varepsilon > 0$ is small and $\frac{4}{3} < q < \frac{5}{3}$, we know that

$$c_2 + U - \frac{1}{4B_0}\Big(U\frac{q}{1+qS}\Big)^2 = \frac{-3q^2+12q-10}{(2q-2)^2} > 0.$$

Then by (51) we only need that

$$\begin{cases} K^{-\frac{2}{q}}\frac{q}{2}\left(\frac{2}{2-q}\right)^{-\frac{2-q}{q}}|M|^{\frac{2}{q}} = -\frac{T(\alpha+p)+P}{T}, & (56)\\ \frac{T}{K} > T + \frac{q+3}{2q-2} - 1. & (57)\end{cases}$$

We deduce that

$$M > -\frac{T^{\frac{2-q}{2}}}{T+\frac{q+3}{2q-2}-1}\left(\frac{q}{2}\right)^{-\frac{q}{2}}\left(\frac{2}{2-q}\right)^{\frac{2-q}{2}}\Big[5-p-T(p-2)\Big]^{\frac{q}{2}},$$

where $T \geq 0$ is arbitrary.

□

Finally we give the proof of claim 1:

Proof of claim 1. When $n \geq 4$, $1 < q < \frac{n+2}{n}$,

$$\begin{aligned} &\frac{2(n+q)^2}{(n-1)^2q^2} > \frac{n+q}{(n-1)q-2},\\ \Leftrightarrow &\frac{2(n+q)}{(n-1)^2q^2} > \frac{1}{(n-1)q-2},\\ \Leftrightarrow &2(n+q)\Big[(n-1)q-2\Big] - (n-1)^2q^2 > 0,\\ \Leftrightarrow &f_1(q) := -(n-1)(n-3)q^2 + 2(n+1)(n-2)q - 4n > 0.\end{aligned}$$

We only need to check that

$$f_1(1), f_1\left(\frac{n+2}{n}\right) > 0.$$

In fact, we have

$$f_1(1) = n^2 - 2n - 7 \geq 16 - 8 - 7 > 0.$$

and

$$f_1\left(\frac{n+2}{n}\right) = \frac{n^4-2n^3+n^2-4n-12}{n^2} > 0.$$

□

6. Proof of Theorem 1.6

In this section we consider the case that $n \geq 7$ and M is large enough. At this case, we can not directly choose $\gamma = 0$ any more. Recall that in (31):

$$B = \gamma \cdot \frac{\alpha+\gamma+2}{(\gamma+4)p+2\alpha+\gamma},$$
$$G(p) := \frac{1}{p_1} + \frac{1}{q_1} = \frac{(\gamma+2)p+2\alpha+\gamma+2}{(\gamma+4)p+2\alpha+\gamma}.$$

We hope that

$$\alpha+\gamma+2 > 0,$$

then if $\gamma > 0$ we have

$$\begin{aligned}
&(\gamma+4)p+2\alpha+\gamma > \gamma+4+2\alpha+\gamma > 0,\\
&B > 0 \Rightarrow p_1 > 0,\\
&B < \frac{\gamma}{2} < \gamma+2 \Rightarrow q_1 > 0.
\end{aligned}$$

Besides we have $G(p)$ is decreasing and

$$G(p) < G(1) = 1.$$

Thus if we choose

$$\gamma = n-4,$$
$$\alpha+\gamma+2 > 0,$$

then

$$1-\frac{2}{n} < G(p) < 1.$$

If $\gamma \neq 0$, solving the equation

$$b_2 = 0,$$

we get that

$$b_2 = \gamma + \frac{1+\gamma S-\varepsilon\tau}{1-S^2+\gamma S-\gamma S^2-(n-1)Q^2}\Big(2\gamma S-\gamma+2S-2Q\Big) - 2\varepsilon(S-Q),$$

$$\begin{aligned}
\Rightarrow 0 &= \gamma-\gamma S^2+\gamma^2 S-\gamma^2 S^2-\frac{\gamma}{n-1}(1-S)^2+(1+\gamma S-\varepsilon\tau)\Big(2\gamma S-\gamma+\frac{2nS}{n-1}-\frac{2}{n-1}\Big)\\
&\quad -2\varepsilon\cdot\frac{nS-1}{n-1}\Big[1-S^2+\gamma S-\gamma S^2-\frac{(1-S)^2}{n-1}\Big]\\
&= 2\varepsilon\cdot\frac{n}{n-1}\Big(\gamma+\frac{n}{n-1}\Big)S^3+...
\end{aligned}$$

Since $2\varepsilon \cdot \frac{n}{n-1}\Big(\gamma + \frac{n}{n-1}\Big) \neq 0$, the cubic equation with respect to S at least has one solution $S = S(n, \gamma, \varepsilon)$. Next, we need to estimate the solution when $\varepsilon > 0$ is small enough. If $\varepsilon = 0$, then

$$\begin{aligned}
0 &= \gamma - \gamma S^2 + \gamma^2 S - \gamma^2 S^2 - \frac{\gamma}{n-1}(1-S)^2 + (1+\gamma S)\Big(2\gamma S - \gamma + \frac{2nS}{n-1} - \frac{2}{n-1}\Big) \\
&= \Big(-\gamma - \gamma^2 - \frac{\gamma}{n-1} + 2\gamma^2 + \frac{2n\gamma}{n-1}\Big)S^2 \\
&+ \Big(\gamma^2 + \frac{2\gamma}{n-1} - \gamma^2 - \frac{2\gamma}{n-1} + 2\gamma + \frac{2n}{n-1}\Big)S \\
&+ \gamma - \frac{\gamma}{n-1} - \gamma - \frac{2}{n-1} \\
&= \Big(\gamma^2 + \frac{n\gamma}{n-1}\Big)S^2 + 2\Big(\gamma + \frac{n}{n-1}\Big)S - \frac{\gamma+2}{n-1}.
\end{aligned}$$

Choose

$$\begin{aligned}
S &= \frac{-\gamma - \frac{n}{n-1} + \sqrt{\Big(\gamma + \frac{n}{n-1}\Big)^2 + \Big(\gamma^2 + \frac{n\gamma}{n-1}\Big)\frac{\gamma+2}{n-1}}}{\gamma^2 + \frac{n\gamma}{n-1}} \\
&= \frac{\gamma+2}{(n-1)\gamma + n + \sqrt{\Big[(n-1)\gamma + n\Big]^2 + \Big[(n-1)\gamma^2 + n\gamma\Big](\gamma+2)}}.
\end{aligned}$$

If $\gamma = n-4$, then

$$\begin{aligned}
S &= \frac{n-2}{(n-1)(n-4) + n + \sqrt{\Big[(n-1)(n-4)+n\Big]^2 + \Big[(n-1)(n-4)+n\Big](n-2)(n-4)}} \\
&= \frac{n-2}{(n-2)^2 + \sqrt{(n-2)^4 + (n-2)^3(n-4)}} \\
&= \frac{1}{n-2 + \sqrt{(n-2)(2n-6)}} \\
&= \frac{\sqrt{(n-2)(2n-6)} - n + 2}{(n-2)(n-4)}.
\end{aligned}$$

So when $\varepsilon > 0$ is small enough, we have

$$S = \frac{\sqrt{(n-2)(2n-6)} - n + 2}{(n-2)(n-4)} + E(\varepsilon),$$

where $E(\varepsilon)$ tends to 0 when ε tends to 0. We hope the following conditions hold:

$$\begin{cases} a_3 + P\Big(\dfrac{\alpha-1}{1+\gamma S+2S}\Big) - \dfrac{1}{4B_0}\Big(b_1 + P\dfrac{\gamma+2}{1+\gamma S+2S}\Big)^2 > 0, & (58)\\ T - \dfrac{1}{4B_0}\left(\dfrac{T\gamma}{1+\gamma S}\right)^2 > 0, & (59)\\ c_2 + U - \dfrac{1}{4B_0}\Big(U\dfrac{\gamma+q}{1+\gamma S+qS}\Big)^2 > 0, & (60)\\ \dfrac{T(\alpha+p)}{1+\gamma S} + P - \dfrac{1}{2B_0}\dfrac{T\gamma}{1+\gamma S}\Big(b_1 + P\dfrac{\gamma+2}{1+\gamma S+2S}\Big) < 0, & (61)\\ M\left[b_4 + P + U\Big(\dfrac{\alpha}{1+\gamma S+qS}\Big) - \dfrac{1}{2B_0}\Big(b_1 + P\dfrac{\gamma+2}{1+\gamma S+2S}\Big)U\dfrac{\gamma+q}{1+\gamma S+qS}\right] < 0, & (62)\\ M\left(b_5 + T + U - \dfrac{1}{2B_0}\dfrac{T\gamma}{1+\gamma S}\cdot U\dfrac{\gamma+q}{1+\gamma S+qS}\right) > 0. & (63) \end{cases}$$

Next, we are devoted to check conditions (58) and (60). In the following computation, by the continuity of ε, we only need to check the conditions for $\varepsilon = 0$.

Lemma 6.1. *If $n \geq 7, \gamma = n-4, \varepsilon = 0, \alpha = -\gamma - \frac{4}{n}, p - P = \frac{n+2}{n-2} - \frac{1}{n^2}$, then*

$$a_3 + P\Big(\frac{\alpha-1}{1+\gamma S+2S}\Big) - \frac{1}{4B_0}\Big(b_1 + P\frac{\gamma+2}{1+\gamma S+2S}\Big)^2 > 0.$$

Proof of Lemma 6.1. Recall that

$$\begin{aligned} a_1 &= \frac{1+\gamma S}{1-S^2+\gamma S-\gamma S^2-(n-1)Q^2},\\ a_2 &= \frac{1+\gamma S}{1-S^2+\gamma S-\gamma S^2-(n-1)Q^2}(1+\gamma),\\ a_3 &= -\Big(\alpha+p\Big)\frac{\alpha-1}{1+\gamma S+2S} + \frac{1+\gamma S}{1-S^2+\gamma S-\gamma S^2-(n-1)Q^2}\cdot\frac{\alpha(\alpha-1)(1-S)}{1+\gamma S+2S},\\ b_1 &= \frac{1+\gamma S}{1-S^2+\gamma S-\gamma S^2-(n-1)Q^2}\cdot\frac{\alpha(\gamma+3)}{1+\gamma S+2S} - \Big(\alpha+p\Big)\frac{\gamma+2}{1+\gamma S+2S},\\ b_2 &= \gamma + \frac{1+\gamma S}{1-S^2+\gamma S-\gamma S^2-(n-1)Q^2}\Big(2\gamma S-\gamma+2S-2Q\Big),\\ c_2 &= -\frac{q}{\gamma+q},\\ b_3 &= 0,\\ b_4 &= -p-\frac{q\alpha}{\gamma+q},\\ b_5 &= -\frac{q}{\gamma+q},\\ B_0 &= a_1\Big(\frac{n}{n-1}+\gamma\Big). \end{aligned}$$

Besides using $b_2 \equiv 0$ we have

$$\begin{aligned}
a_1 &= -\frac{\gamma}{2\gamma S - \gamma + 2S - 2Q} \\
&= -\frac{\gamma}{2\gamma S - \gamma + 2S - \frac{2}{n-1}(1-S)} \\
&= -\frac{\gamma}{2\gamma S - \gamma + \frac{2n}{n-1}S - \frac{2}{n-1}} \\
&= -\frac{(n-4)(n-1)}{2S(n-2)^2 - (n-2)(n-3)} \\
&= -\frac{(n-4)(n-1)}{2\frac{\sqrt{(n-2)(2n-6)}-n+2}{n-4}(n-2) - (n-2)(n-3)} \\
&= \frac{2\sqrt{(n-2)(2n-6)} + n^2 - 5n + 8}{(n-1)(n-2)}.
\end{aligned}$$

In order to prove inequality (58) , we can simplify the computation and it is equivalent to prove that $4a_1\left(\frac{n}{n-1} + \gamma\right) a_3 - b_1^2 > 0$ with $p = \frac{n+2}{n-2} - \frac{1}{n^2}$.

$$\begin{aligned}
&4a_1\left(\frac{n}{n-1} + \gamma\right) a_3 - b_1^2 \\
&= 4a_1\left(\frac{n}{n-1} + \gamma\right)\left\{ - \frac{p(\alpha-1)}{1+\gamma S + 2S} + \Big[a_1(1-S) - 1\Big]\frac{\alpha(\alpha-1)}{1+\gamma S + 2S}\right\} \\
&- \left\{ - \frac{(\gamma+2)p}{1+\gamma S + 2S} + \frac{\alpha}{1+\gamma S + 2S}\Big[(\gamma+3)a_1 - (\gamma+2)\Big]\right\}^2 \\
&= 4a_1\left(\frac{n}{n-1} + n - 4\right)\left\{ - \frac{p(\alpha-1)}{1+(n-2)S} + \Big[a_1(1-S) - 1\Big]\frac{\alpha(\alpha-1)}{1+(n-2)S}\right\} \\
&- \left\{ - \frac{(n-2)p}{1+(n-2)S} + \frac{\alpha}{1+(n-2)S}\Big[(n-1)a_1 - n + 2\Big]\right\}^2.
\end{aligned}$$

As a result, we know

$$\begin{aligned}
&4a_1\left(\frac{n}{n-1} + \gamma\right) a_3 - b_1^2 > 0, \\
\Leftrightarrow H :=&4\Big[1 + (n-2)S\Big]a_1\left(\frac{n}{n-1} + n - 4\right)\left\{ - p(\alpha-1) + \Big[a_1(1-S) - 1\Big]\alpha(\alpha-1)\right\} \\
&- \left\{ - (n-2)p + \alpha\Big[(n-1)a_1 - n + 2\Big]\right\}^2 > 0.
\end{aligned}$$

Using the facts that

- $$1+(\gamma+2)S=\frac{\sqrt{(n-2)(2n-6)}-2}{n-4}.$$

- $$\begin{aligned}&(n-1)a_1-n+2\\&=\frac{2\sqrt{(n-2)(2n-6)}+n^2-5n+8}{n-2}-n+2\\&=\frac{2\sqrt{(n-2)(2n-6)}-n+4}{n-2}.\end{aligned}$$

- $$\begin{aligned}&a_1(1-S)-1\\&=a_1-1-a_1S\\&=\frac{2\sqrt{(n-2)(2n-6)}+n^2-5n+8}{(n-1)(n-2)}-1\\&-\frac{2\sqrt{(n-2)(2n-6)}+n^2-5n+8}{(n-1)(n-2)}\cdot\frac{\sqrt{(n-2)(2n-6)}-n+2}{(n-2)(n-4)}\\&=\frac{2\sqrt{(n-2)(2n-6)}-2n+6}{(n-1)(n-2)}\\&-\frac{2\sqrt{(n-2)(2n-6)}+n^2-5n+8}{(n-1)(n-2)}\cdot\frac{\sqrt{(n-2)(2n-6)}-n+2}{(n-2)(n-4)},\end{aligned}$$

$$\begin{aligned}&\Rightarrow a_1(1-S)-1\\&=\frac{1}{(n-1)(n-2)^2(n-4)}\Bigg\{\Big[2\sqrt{(n-2)(2n-6)}-2n+6\Big](n^2-6n+8)\\&-\Big[2\sqrt{(n-2)(2n-6)}+n^2-5n+8\Big]\cdot\Big[\sqrt{(n-2)(2n-6)}-n+2\Big]\Bigg\}\\&=\frac{1}{(n-1)(n-2)^2(n-4)}\Bigg\{(n^2-5n+8)(n-2)-2(n-2)(2n-6)\\&-(2n-6)(n^2-6n+8)+\Big[2n^2-12n+16+2n-4-n^2+5n-8\Big]\sqrt{(n-2)(2n-6)}\Bigg\}\\&=\frac{1}{(n-1)(n-2)^2(n-4)}\Bigg\{-(n-1)(n-2)(n-4)+(n^2-5n+4)\sqrt{(n-2)(2n-6)}\Bigg\}\\&=\frac{1}{(n-2)^2}\Big[-(n-2)+\sqrt{(n-2)(2n-6)}\Big].\end{aligned}$$

Therefore, we obtain:

$$H = 4\frac{\sqrt{(n-2)(2n-6)}-2}{n-4}\cdot\frac{2\sqrt{(n-2)(2n-6)}+n^2-5n+8}{(n-1)^2}\cdot(n-2)\Big\{-p(\alpha-1)$$
$$+\frac{1}{(n-2)^2}\Big[-(n-2)+\sqrt{(n-2)(2n-6)}\Big]\alpha(\alpha-1)\Big\}$$
$$-\Big\{-(n-2)p+\alpha\frac{2\sqrt{(n-2)(2n-6)}-n+4}{n-2}\Big\}^2.$$

If $\alpha=-n+4-\frac{4}{n}=-\frac{(n-2)^2}{n}$, then we find that

$$H(p) = 4\frac{\sqrt{(n-2)(2n-6)}-2}{n-4}\cdot\frac{2\sqrt{(n-2)(2n-6)}+n^2-5n+8}{(n-1)^2}\cdot(n-2)\Big\{-p$$
$$+\frac{1}{(n-2)^2}\Big[-(n-2)+\sqrt{(n-2)(2n-6)}\Big]\left(-n+4-\frac{4}{n}\right)\Big\}\left(-n+3-\frac{4}{n}\right)$$
$$-\Big\{-(n-2)p+\left(-n+4-\frac{4}{n}\right)\frac{2\sqrt{(n-2)(2n-6)}-n+4}{n-2}\Big\}^2.$$

claim 2. *When $n\geq 7$, we have*

$$H\left(\frac{n+2}{n-2}-\frac{1}{n^2}\right)>0.$$

□

Lemma 6.2. *If $n\geq 7, 1<p\leq\frac{n+2}{n-2}$ and $U_2>U_1>0$ are two solutions of*

$$c_2+U-\frac{1}{4B_0}\Big(U\frac{\gamma+q}{1+\gamma S+qS}\Big)^2=0. \tag{64}$$

Define

$$U_0:=\frac{2B_0(1+\gamma S+qS)^2}{(\gamma+q)^2}\left[1-\frac{1}{\sqrt{2}}\left(1-\frac{2}{n}\right)\right],$$

then for any $p-P=\frac{n+2}{n-2}-\frac{1}{n^2}$, we have

$$U_1<U_0<U_2, \tag{65}$$

and

$$b_4+P+\frac{U_0\alpha}{1+\gamma S+qS}-\frac{1}{2B_0}\Big(b_1+P\frac{\gamma+2}{1+\gamma S+2S}\Big)U_0\frac{\gamma+q}{1+\gamma S+qS}<0.$$

If Lemma 6.2 holds, then for $1<p<\frac{n+2}{n-2}-\frac{1}{n^2}$, we get that

$$P=p-\frac{n+2}{n-2}+\frac{1}{n^2}<0.$$

And for the choice of T, we let $0 < T << |P|$ small enough. Then as a result, the conditions (58) ...(63) all hold. So we deduce that $|\nabla v| \equiv 0$. Therefore we only need to consider the case that $\frac{n+2}{n-2} - \frac{1}{n^2} \le p \le \frac{n+2}{n-2}$. At this time, $P \ge 0$ and we can give the lower bound of M. By Young inequality, we have

$$v^{\alpha+p-1}|\nabla v|^{\gamma+2} \le \frac{q}{2}\Big(JM \cdot \frac{2}{2-q}\Big)^{-\frac{2-q}{q}} v^{\alpha+p}|\nabla v|^{\gamma+q} + JMv^{\alpha-1}|\nabla v|^{\gamma+q+2},$$

with $(\frac{2}{q}, \frac{2}{2-q})$. We only need that

$$b_4 + P + U_0\Big(\frac{\alpha}{1+\gamma S+qS}\Big) - \frac{1}{2B_0}\Big(b_1 + P\frac{\gamma+2}{1+\gamma S+2S}\Big)U_0\frac{\gamma+q}{1+\gamma S+qS} + PJ < 0, \tag{66}$$

and

$$b_5 + T + U_0 - P \cdot \frac{q}{2}\Big(J \cdot \frac{2}{2-q}\Big)^{-\frac{2-q}{q}} M^{-\frac{2}{q}} > 0. \tag{67}$$

Proof of Lemma 6.2 . Firstly, we can rewrite (64) as:

$$-\frac{q}{\gamma+q} + U - \frac{1}{4a_1\Big(\frac{n}{n-1}+\gamma\Big)}\Big(U\frac{\gamma+q}{1+\gamma S+qS}\Big)^2 = 0.$$

Since we know that

$$\begin{aligned}
\Delta &= 1 - \frac{q(\gamma+q)}{a_1(\frac{n}{n-1}+\gamma)(1+\gamma S+qS)^2} \\
&> 1 - \frac{q(\gamma+q)}{a_1(\frac{n}{n-1}+\gamma)(1+\gamma S)^2} \\
&\ge 1 - \frac{(1+\frac{2}{n})(\gamma+1+\frac{2}{n})}{a_1(\frac{n}{n-1}+\gamma)(1+\gamma S)^2} \\
&= 1 - \frac{(1+\frac{2}{n})(n-3+\frac{2}{n})(n-2)^2(n-1)}{(\frac{n}{n-1}+n-4)(2n-6)\Big[2\sqrt{(n-2)(2n-6)}+n^2-5n+8\Big]}.
\end{aligned}$$

When $n = 7$, we have $\Delta > 0.284 > 0.5 - \frac{2}{n} + \frac{2}{n^2}$; when $n = 8$, we have $\Delta > 0.322 > 0.5 - \frac{2}{n} + \frac{2}{n^2}$.

claim 3. *When $n \ge 9$, then*

$$\Delta > 0.5 - \frac{2}{n} + \frac{2}{n^2} > 0.$$

Thus there always exists two solutions $U_2 > U_1 > 0$. Define

$$\begin{aligned}
U_0 &:= \frac{2B_0(1+\gamma S+qS)^2}{(\gamma+q)^2}\left[1 - \frac{1}{\sqrt{2}}\left(1-\frac{2}{n}\right)\right] \\
&= \frac{2B_0(1+\gamma S+qS)^2}{(\gamma+q)^2}\left(\frac{2-\sqrt{2}}{2} + \frac{\sqrt{2}}{n}\right),
\end{aligned}$$

then $U_1 < U_0 < U_2$. Define

$$\begin{aligned}
I(U,p-P) &:= b_4 + P + \frac{U\alpha}{1+\gamma S+qS} - \frac{1}{2B_0}\Big(b_1 + P\frac{\gamma+2}{1+\gamma S+2S}\Big)U\frac{\gamma+q}{1+\gamma S+qS}\\
&= -(p-P) - \frac{q\alpha}{\gamma+q} + \frac{U\alpha}{1+\gamma S+qS}\\
&- \frac{1}{2B_0}\Big[a_1\cdot\frac{\alpha(\gamma+3)}{1+\gamma S+2S} - \Big(\alpha+p-P\Big)\frac{\gamma+2}{1+\gamma S+2S}\Big]U\frac{\gamma+q}{1+\gamma S+qS}\\
&= \left[\frac{U(\gamma+2)(\gamma+q)}{2B_0(1+\gamma S+2S)(1+\gamma S+qS)} - 1\right](p-P) - \frac{q\alpha}{\gamma+q} + \frac{U\alpha}{1+\gamma S+qS}\\
&- \frac{1}{2B_0}\Big[a_1\cdot\frac{\alpha(\gamma+3)}{1+\gamma S+2S} - \frac{\alpha(\gamma+2)}{1+\gamma S+2S}\Big]U\frac{\gamma+q}{1+\gamma S+qS}.
\end{aligned}$$

If $p-P = \frac{n+2}{n-2} - \frac{1}{n^2}$,

$$\begin{aligned}
I\left(U_0, \frac{n+2}{n-2} - \frac{1}{n^2}\right) &= \left[\left(\frac{2-\sqrt{2}}{2} + \frac{\sqrt{2}}{n}\right)\frac{(\gamma+2)(1+\gamma S+qS)}{(\gamma+q)(1+\gamma S+2S)} - 1\right]\left(\frac{n+2}{n-2} - \frac{1}{n^2}\right) - \frac{q\alpha}{\gamma+q}\\
&+ \left(\frac{2-\sqrt{2}}{2} + \frac{\sqrt{2}}{n}\right)\frac{2B_0(1+\gamma S+qS)\alpha}{(\gamma+q)^2}\\
&- \Big[a_1\cdot\frac{\alpha(\gamma+3)}{1+\gamma S+2S} - \frac{\alpha(\gamma+2)}{1+\gamma S+2S}\Big]\frac{1+\gamma S+qS}{\gamma+q}\left(\frac{2-\sqrt{2}}{2} + \frac{\sqrt{2}}{n}\right)\\
&= \Bigg[\underbrace{\left(\frac{2-\sqrt{2}}{2} + \frac{\sqrt{2}}{n}\right)\frac{(\gamma+2)(1+\gamma S+qS)}{(\gamma+q)(1+\gamma S+2S)}}_{①} - 1\Bigg]\left(\frac{n+2}{n-2} - \frac{1}{n^2}\right)\\
&+ \frac{\alpha}{\gamma+q}\Bigg[\underbrace{-q + \left(\frac{2-\sqrt{2}}{2} + \frac{\sqrt{2}}{n}\right)\frac{2B_0(1+\gamma S+qS)}{\gamma+q}}_{②}\\
&- \underbrace{\frac{2\sqrt{(n-2)(2n-6)} - n + 4}{n-2}\cdot\frac{1+\gamma S+qS}{1+\gamma S+2S}\left(\frac{2-\sqrt{2}}{2} + \frac{\sqrt{2}}{n}\right)}_{③}\Bigg].
\end{aligned}$$

Then we need to estimate ①, ② and ③. Using the fact that if $x = \frac{1}{n-2} \le \frac{1}{5}$,

$$\begin{aligned}
&1 - \frac{1}{2}x - \frac{1}{6}x^2 < \sqrt{1-x} < 1 - \frac{1}{2}x - \frac{1}{8}x^2,\\
\Rightarrow &\sqrt{2}\Big[n - \frac{5}{2} - \frac{1}{6(n-2)}\Big] < \sqrt{(n-2)(2n-6)} = \sqrt{2}(n-2)\sqrt{1-\frac{1}{n-2}} < \sqrt{2}\Big[n - \frac{5}{2} - \frac{1}{8(n-2)}\Big],
\end{aligned}$$

we get the following estimates:

- ①: Firstly, we know that

$$
\begin{aligned}
\frac{(\gamma+2)(1+\gamma S+qS)}{(\gamma+q)(1+\gamma S+2S)}-1 &= \frac{2-q}{(\gamma+q)(1+\gamma S+2S)} \\
&= \frac{2-q}{n-4+q}\cdot\frac{n-4}{\sqrt{(n-2)(2n-6)}-2} \\
&< \frac{2-q}{n-4+q}\cdot\frac{n-4}{\sqrt{2}(n-2)\Big[1-\frac{1}{2(n-2)}-\frac{1}{6(n-2)^2}\Big]-2} \\
&= \frac{2-q}{n-4+q}\cdot\frac{n-4}{\sqrt{2}\Big[n-\frac{5}{2}-\frac{1}{6(n-2)}-\sqrt{2}\Big]},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(-\frac{1}{2}+\frac{1}{n}\right)\left(\frac{n+2}{n-2}-\frac{1}{n^2}\right) \\
&= -\frac{1}{2}-\frac{1}{n}+\frac{n-2}{2n^3} < -\frac{1}{2}-\frac{1}{n}+\frac{1}{2n^2}.
\end{aligned}
$$

Besides, we have the following estimates:

claim 4. *When $n\geq 7$, we have*

$$
\left(\frac{2-\sqrt{2}}{2}+\frac{\sqrt{2}}{n}\right)\cdot\frac{n-4}{\sqrt{2}\Big(n-\frac{5}{2}-\frac{1}{6(n-2)}-\sqrt{2}\Big)}\left(\frac{n+2}{n-2}-\frac{1}{n^2}\right) < \frac{\sqrt{2}-1}{2}+\frac{3}{n}.
$$

Then we get that

$$
\begin{aligned}
&\left[\left(\frac{2-\sqrt{2}}{2}+\frac{\sqrt{2}}{n}\right)\underbrace{\frac{(\gamma+2)(1+\gamma S+qS)}{(\gamma+q)(1+\gamma S+2S)}}_{①}-1\right]\left(\frac{n+2}{n-2}-\frac{1}{n^2}\right) \\
&< \left\{\left(\frac{2-\sqrt{2}}{2}+\frac{\sqrt{2}}{n}\right)\left[1+\frac{2-q}{n-4+q}\cdot\frac{n-4}{\sqrt{2}\Big(n-\frac{5}{2}-\frac{1}{6(n-2)}-\sqrt{2}\Big)}\right]-1\right\}\left(\frac{n+2}{n-2}-\frac{1}{n^2}\right) \\
&= \left(\frac{2-\sqrt{2}}{2}+\frac{\sqrt{2}}{n}\right)\frac{2-q}{n-4+q}\cdot\frac{n-4}{\sqrt{2}\Big(n-\frac{5}{2}-\frac{1}{6(n-2)}-\sqrt{2}\Big)}\left(\frac{n+2}{n-2}-\frac{1}{n^2}\right) \\
&+ \left(-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{n}\right)\left(\frac{n+2}{n-2}-\frac{1}{n^2}\right) \\
&< -\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{n}+\frac{\sqrt{2}}{2n^2}+\frac{2-q}{n-4+q}\left(\frac{\sqrt{2}-1}{2}+\frac{3}{n}\right).
\end{aligned}
$$

- ②:

$$
\begin{aligned}
&\frac{2B_0(1+\gamma S+qS)}{\gamma+q}\\
&=2\left(\gamma+\frac{n}{n-1}\right)a_1\left(\frac{1}{\gamma+q}+S\right)\\
&=2\left(n-3+\frac{1}{n-1}\right)\frac{2\sqrt{(n-2)(2n-6)}+n^2-5n+8}{(n-1)(n-2)}\left[\frac{1}{n-4+q}+\frac{\sqrt{(n-2)(2n-6)}-n+2}{(n-2)(n-4)}\right]\\
&>2\left(n-3+\frac{1}{n-1}\right)\frac{2\sqrt{2}\left[n-\frac{5}{2}-\frac{1}{6(n-2)}\right]+n^2-5n+8}{(n-1)(n-2)}\left[\frac{1}{n-4+q}+\frac{\sqrt{2}\left[n-\frac{5}{2}-\frac{1}{6(n-2)}\right]-n+2}{(n-2)(n-4)}\right]\\
&\geq 2\left(n-3+\frac{1}{n-1}\right)\frac{2\sqrt{2}\left(n-\frac{38}{15}\right)+n^2-5n+8}{(n-1)(n-2)}\left[\frac{1}{n-4+q}+\frac{\sqrt{2}\left(n-\frac{38}{15}\right)-n+2}{(n-2)(n-4)}\right].
\end{aligned}
$$

Using the fact that when $n=7$,

$$\frac{1}{n}+\frac{1}{n(n-1)}=\frac{0.5}{n-4},$$

and when $n\geq 8$

$$\frac{1}{n}\geq\frac{0.5}{n-4},$$

we get that

$$
\begin{aligned}
\frac{1}{n-4+q}&\geq\frac{1}{n-3+\frac{2}{n}}=\frac{1}{n-2}\cdot\frac{1}{1-\frac{1}{n}}\\
&=\frac{1}{n-2}\cdot\left[1+\frac{1}{n}+\frac{1}{n(n-1)}\right]\\
&\geq\frac{1}{n-2}\cdot\left(1+\frac{0.5}{n-4}\right)\\
&=\frac{1}{n-2}+\frac{0.5}{(n-2)(n-4)},
\end{aligned}
$$

we get that:

$$
\begin{aligned}
&\frac{2B_0(1+\gamma S+qS)}{\gamma+q} \\
&> 2\left(n-3+\frac{1}{n-1}\right)\frac{2\sqrt{2}\left(n-\frac{38}{15}\right)+n^2-5n+8}{(n-1)(n-2)}\left[\frac{1}{n-2}+\frac{0.5}{(n-2)(n-4)}+\frac{\sqrt{2}\left(n-\frac{38}{15}\right)-n+2}{(n-2)(n-4)}\right] \\
&\geq 2\left(n-3+\frac{1}{n-1}\right)\frac{2\sqrt{2}\left(n-\frac{38}{15}\right)+n^2-5n+8}{(n-1)(n-2)^2(n-4)}\left[n-3.5+\sqrt{2}\left(n-\frac{38}{15}\right)-n+2\right] \\
&= 2\left(n-3+\frac{1}{n-1}\right)\frac{2\sqrt{2}\left(n-\frac{38}{15}\right)+n^2-5n+8}{(n-1)(n-2)^2(n-4)}\left[\sqrt{2}\left(n-\frac{53}{21}\right)-1.5\right].
\end{aligned}
$$

claim 5. *When $n \geq 7$, we have*

$$
\begin{aligned}
&2\left(n-3+\frac{1}{n-1}\right)\frac{2\sqrt{2}\left(n-\frac{38}{15}\right)+n^2-5n+8}{(n-1)(n-2)^2(n-4)}\left[\sqrt{2}\left(n-\frac{38}{15}\right)-1.5\right] \\
&> 2\sqrt{2}+\frac{0.4\sqrt{2}}{n}.
\end{aligned}
$$

Thus we obtain that

$$
\frac{2B_0(1+\gamma S+qS)}{\gamma+q} > 2\sqrt{2}+\frac{0.4\sqrt{2}}{n}.
$$

- ③:

$$
\begin{aligned}
\frac{1+\gamma S+qS}{1+\gamma S+2S}-1 &= -\frac{(2-q)S}{1+\gamma S+2S} \\
&= -\frac{2-q}{n-2}\cdot\frac{\sqrt{(n-2)(2n-6)}-n+2}{\sqrt{(n-2)(2n-6)}-2} \\
&< -\frac{2-q}{n-2}\cdot\frac{\sqrt{2}\left[n-\frac{5}{2}-\frac{1}{6(n-2)}\right]-n+2}{\sqrt{2}\left[n-\frac{5}{2}-\frac{1}{8(n-2)}\right]-2} \\
&< -\frac{2-q}{n-2}\cdot\frac{\frac{2-\sqrt{2}}{2}n-\frac{5}{2}-\frac{1}{6(n-2)}+\sqrt{2}}{n-\frac{5}{2}-\sqrt{2}}.
\end{aligned}
$$

$$\frac{2\sqrt{(n-2)(2n-6)}-n+4}{n-2}\cdot\frac{1+\gamma S+qS}{1+\gamma S+2S}\left(\frac{2-\sqrt{2}}{2}+\frac{\sqrt{2}}{n}\right)$$
$$<\frac{2\sqrt{(n-2)(2n-6)}-n+4}{n-2}\cdot\left(1-\frac{2-q}{n-2}\cdot\frac{\frac{2-\sqrt{2}}{2}n-\frac{5}{2}-\frac{1}{6(n-2)}+\sqrt{2}}{n-\frac{5}{2}-\sqrt{2}}\right)\left(\frac{2-\sqrt{2}}{2}+\frac{\sqrt{2}}{n}\right)$$
$$=\frac{2\sqrt{(n-2)(2n-6)}-n+4}{n-2}\cdot\left(\frac{2-\sqrt{2}}{2}+\frac{\sqrt{2}}{n}\right)$$
$$-\frac{2-q}{n-2}\cdot\frac{\frac{2-\sqrt{2}}{2}n-\frac{5}{2}-\frac{1}{6(n-2)}+\sqrt{2}}{n-\frac{5}{2}-\sqrt{2}}\cdot\frac{2\sqrt{(n-2)(2n-6)}-n+4}{n-2}\left(\frac{2-\sqrt{2}}{2}+\frac{\sqrt{2}}{n}\right).$$

claim 6. *When $n \geq 7$, we have*

$$\frac{2\sqrt{(n-2)(2n-6)}-n+4}{n-2}\cdot\left(\frac{2-\sqrt{2}}{2}+\frac{\sqrt{2}}{n}\right)<(2\sqrt{2}-1)\cdot\frac{2-\sqrt{2}}{2}+\frac{2.76}{n}+\frac{1.7}{n^2}.$$

claim 7. *When $n \geq 7$, we have*

$$\frac{1}{n-2}\cdot\frac{\frac{2-\sqrt{2}}{2}n-\frac{5}{2}-\frac{1}{6(n-2)}+\sqrt{2}}{n-\frac{5}{2}-\sqrt{2}}\cdot\frac{2\sqrt{(n-2)(2n-6)}-n+4}{n-2}\left(\frac{2-\sqrt{2}}{2}+\frac{\sqrt{2}}{n}\right)$$
$$>\frac{8\sqrt{2}-11}{2}\left(\frac{1}{n}+\frac{7}{n^2}\right).$$

Hence we get that

$$\frac{2\sqrt{(n-2)(2n-6)}-n+4}{n-2}\cdot\frac{1+\gamma S+qS}{1+\gamma S+2S}\left(\frac{2-\sqrt{2}}{2}+\frac{\sqrt{2}}{n}\right)$$
$$<(2\sqrt{2}-1)\cdot\frac{2-\sqrt{2}}{2}+\frac{2.76}{n}+\frac{1.7}{n^2}$$
$$-(2-q)\frac{8\sqrt{2}-11}{2}\left(\frac{1}{n}+\frac{7}{n^2}\right).$$

By direct computation, we get that

$$\lim_{n\to\infty} I\left(U_0, \frac{n+2}{n-2}-\frac{1}{n^2}\right)=0.$$

So we should pay more attention to the terms $O(\frac{1}{n})$.

$$
\begin{aligned}
\Rightarrow & I\left(U_0, \frac{n+2}{n-2}-\frac{1}{n^2}\right) \\
\leq & -\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{n}+\frac{\sqrt{2}}{2n^2}+\frac{2-q}{n-4+q}\left(\frac{\sqrt{2}-1}{2}+\frac{3}{n}\right) \\
& +\frac{-n+4-\frac{4}{n}}{n-4+q}\Bigg\{-q+\left(\frac{2-\sqrt{2}}{2}+\frac{\sqrt{2}}{n}\right)\cdot\left(2\sqrt{2}+\frac{0.4\sqrt{2}}{n}\right) \\
& -(2\sqrt{2}-1)\cdot\frac{2-\sqrt{2}}{2}-\frac{2.76}{n}-\frac{1.7}{n^2}+(2-q)\frac{8\sqrt{2}-11}{2}\left(\frac{1}{n}+\frac{7}{n^2}\right)\Bigg\} \\
\leq & -\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{n}+\frac{\sqrt{2}}{2n^2}+\frac{2-q}{n-3}\left(\frac{\sqrt{2}-1}{2}+\frac{3}{n}\right) \\
& -\frac{n-4+\frac{4}{n}}{n-4+q}\Bigg\{-q+\left(\frac{2-\sqrt{2}}{2}+\frac{\sqrt{2}}{n}\right)\cdot\left(2\sqrt{2}+\frac{0.4\sqrt{2}}{n}\right) \\
& +3-2.5\sqrt{2}-\frac{2.76}{n}-\frac{1.7}{n^2}+(2-q)\frac{8\sqrt{2}-11}{2}\left(\frac{1}{n}+\frac{7}{n^2}\right)\Bigg\},
\end{aligned}
$$

Thus for $n \geq 7$, we have:

$$
\begin{aligned}
\Rightarrow & I\left(U_0, \frac{n+2}{n-2}-\frac{1}{n^2}\right) \\
< & -\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{n}+\frac{\sqrt{2}}{2n^2}+\frac{2-q}{n-3}\left(\frac{\sqrt{2}-1}{2}+\frac{3}{n}\right) \\
& -\frac{n-4+\frac{4}{n}}{n-4+q}\left[1-q-0.5\sqrt{2}+\frac{0.84+0.4\sqrt{2}}{n}-\frac{0.9}{n^2}+(2-q)\frac{8\sqrt{2}-11}{2}\left(\frac{1}{n}+\frac{7}{n^2}\right)\right].
\end{aligned} \tag{68}
$$

- When $n \geq 9$, we have

$$
\begin{aligned}
& 1-q-0.5\sqrt{2}+\frac{0.84+0.4\sqrt{2}}{n}-\frac{0.9}{n^2}+(2-q)\frac{8\sqrt{2}-11}{2}\left(\frac{1}{n}+\frac{7}{n^2}\right) \\
< & -0.5\sqrt{2}+\frac{0.84+0.4\sqrt{2}}{n}-\frac{0.9}{n^2}+\frac{8\sqrt{2}-11}{2}\left(\frac{1}{n}+\frac{7}{n^2}\right) \\
= & -0.5\sqrt{2}+\frac{4.4\sqrt{2}-4.66}{n}+\frac{28\sqrt{2}-39.4}{n^2} \\
\leq & -0.5\sqrt{2}+\frac{4.4\sqrt{2}-4.66}{7}+\frac{28\sqrt{2}-39.4}{49}<0.
\end{aligned}
$$

- When $n \geq 9$

$$\frac{1}{n-3} = \frac{1}{n}\left(1 + \frac{3}{n} + \frac{9}{n^2}\cdot\frac{1}{1-\frac{3}{n}}\right) \leq \frac{1}{n}\left(1 + \frac{3}{n} + \frac{1}{n}\cdot\frac{9}{6}\right) = \frac{1}{n} + \frac{4.5}{n^2}.$$

claim 8. *When $n \geq 9$, we have*

$$\frac{n-4+\frac{4}{n}}{n-4+q} < 1 - \frac{q}{n} + \frac{1.5q^2 - 8q + 12}{n^2}.$$

Noting that when $n \geq 9$ and $1 < q \leq 1 + \frac{2}{n} \leq \frac{11}{9}$, we have that

$$\begin{aligned}
\Rightarrow & I\left(U_0, \frac{n+2}{n-2} - \frac{1}{n^2}\right) \\
& \leq -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{n} + \frac{\sqrt{2}}{2n^2} + (2-q)\left(\frac{1}{n} + \frac{4.5}{n^2}\right)\left(\frac{\sqrt{2}-1}{2} + \frac{3}{n}\right) \\
& - \left(1 - \frac{q}{n} + \frac{1.5q^2 - 8q + 12}{n^2}\right)\left[1 - q - 0.5\sqrt{2} + \frac{0.84 + 0.4\sqrt{2}}{n} - \frac{0.9}{n^2}\right. \\
& \left. + (2-q)\frac{8\sqrt{2}-11}{2}\left(\frac{1}{n} + \frac{7}{n^2}\right)\right] \\
& = \left[-\sqrt{2} + \frac{\sqrt{2}-1}{2}(2-q) + n(q-1) - 0.84 - 0.4\sqrt{2} - (2-q)\frac{8\sqrt{2}-11}{2} - 0.5\sqrt{2}q\right]\frac{1}{n} \\
& + \left[\frac{\sqrt{2}}{2} + 3(2-q) + \frac{9\sqrt{2}-9}{4}(2-q) + 0.9 - (2-q)\frac{56\sqrt{2}-77}{2} + nq(1-q)\right. \\
& \left. + q(0.84 + 0.4\sqrt{2}) + q(2-q)\frac{8\sqrt{2}-11}{2} + 0.5\sqrt{2}(1.5q^2 - 8q + 12)\right]\frac{1}{n^2} \\
& + \left[13.5(2-q) - 0.9q + q(2-q)\frac{56\sqrt{2}-77}{2}\right]\frac{1}{n^3} \\
& - \frac{1.5q^2 - 8q + 12}{n^2}\left[1 - q + \frac{0.84 + 0.4\sqrt{2}}{n} - \frac{0.9}{n^2} + (2-q)\frac{8\sqrt{2}-11}{2}\left(\frac{1}{n} + \frac{7}{n^2}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow & I\left(U_0, \frac{n+2}{n-2}-\frac{1}{n^2}\right)\\
< & \left[-2.8+0.051(2-q)+n(q-1)-0.5\sqrt{2}q\right]\frac{1}{n}\\
+ & \left[1.61+2.84(2-q)+nq(1-q)+1.41q+q(2-q)\frac{8\sqrt{2}-11}{2}+0.5\sqrt{2}(1.5-8+12)\right]\frac{1}{n^2}\\
+ & \left[13.5(2-q)-0.9q+q(2-q)\frac{56\sqrt{2}-77}{2}\right]\frac{1}{n^3}\\
- & \frac{1.5q^2-8q+12}{n^2}\left[-\frac{2}{n}+\frac{0.84+0.4\sqrt{2}}{n}-\frac{0.9}{n^2}+\left(1-\frac{2}{n}\right)\frac{8\sqrt{2}-11}{2}\left(\frac{1}{n}+\frac{7}{n^2}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow & I\left(U_0, \frac{n+2}{n-2}-\frac{1}{n^2}\right)\\
< & \left[-2.8+0.051(2-q)+n(q-1)-0.5\sqrt{2}q\right]\frac{1}{n}\\
+ & \left[5.5+2.84(2-q)+nq(1-q)+1.41q+q(2-q)\frac{8\sqrt{2}-11}{2}\right]\frac{1}{n^2}\\
+ & \left[13.5(2-q)-0.9q+q(2-q)\frac{56\sqrt{2}-77}{2}\right]\frac{1}{n^3}\\
- & \frac{1.5q^2-8q+12}{n^2}\left(-\frac{0.44}{n}-\frac{0.12}{n^2}-\frac{2.2}{n^3}\right)\\
< & \left[-2.8+0.051(2-q)+n(q-1)-0.5\sqrt{2}q\right]\frac{1}{n}\\
+ & \left[5.5+2.84(2-q)+nq(1-q)+1.41q+q(2-q)\frac{8\sqrt{2}-11}{2}\right]\frac{1}{n^2}\\
+ & \left[13.5(2-q)-0.9+\frac{56\sqrt{2}-77}{2}\right]\frac{1}{n^3}+\frac{5.5}{n^2}\left(\frac{0.44}{n}+\frac{0.12}{9n}+\frac{2.2}{81n}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow I\left(U_0, \frac{n+2}{n-2}-\frac{1}{n^2}\right)\\
&\quad< \left[-2.8+0.051(2-q)+n(q-1)-0.5\sqrt{2}q\right]\frac{1}{n}\\
&\quad+\left[5.5+2.84(2-q)+nq(1-q)+1.41q+q(2-q)\frac{8\sqrt{2}-11}{2}\right]\frac{1}{n^2}\\
&\quad+\frac{13.5(2-q)+2.85}{n^3}\\
&\quad< \left[-2.8+0.051(2-q)+n(q-1)-0.5\sqrt{2}q\right]\frac{1}{n}\\
&\quad+\left[5.5+2.84(2-q)+nq(1-q)+1.41q+\frac{8\sqrt{2}-11}{2}\right]\frac{1}{n^2}\\
&\quad+\frac{13.5(2-q)+2.85}{9n^2}.
\end{aligned}
$$

$$
\begin{aligned}
&\Rightarrow I\left(U_0, \frac{n+2}{n-2}-\frac{1}{n^2}\right)\\
&\quad< \left[-2.8+0.051(2-q)+n(q-1)-0.5\sqrt{2}q\right]\frac{1}{n}\\
&\quad+\left[6+4.34(2-q)+nq(1-q)+1.41q\right]\frac{1}{n^2}.
\end{aligned}
\tag{69}
$$

claim 9. *When $n = 7, 8$, the inequality (69) still holds.*

Since $n \geq 7$ and $1 < q \leq 1 + \frac{2}{n}$, we know

$$
\begin{aligned}
& I\left(U_0, \frac{n+2}{n-2} - \frac{1}{n^2}\right) \\
< & \left[- 2.8 + 0.051(2-q) + n(q-1) - 0.5\sqrt{2}q \right] \frac{1}{n} \\
+ & \left[6 + 4.34(2-q) + 1.41q \right] \frac{1}{n^2} \\
< & \left(n - 0.758 - \frac{2.93}{n} \right) \frac{q}{n} + \frac{-n - 2.698}{n} + \frac{14.68}{n^2} \\
\leq & \left(n - 0.758 - \frac{2.93}{n} \right) \frac{n+2}{n^2} + \frac{-n - 2.698}{n} + \frac{14.68}{n^2} \\
= & \frac{-0.758n^2 - 2.93n + 2n^2 - 1.516n - 5.86 - 2.698n^2 + 14.68n}{n^3} \\
= & \frac{-1.456n^2 + 10.234n - 5.86}{n^3} < 0.
\end{aligned}
$$

□

Lemma 6.3. *If* $1 < p \leq \frac{n+2}{n-2}, q = \frac{2p}{p+1}$*, then there exists* $0 \leq M_2 < M_1$*, where* M_1 *comes from Theorem 1.5, such that for any* $M > M_2$*, the conditions* (66) *and* (67) *are satisfied.*

Proof. By Lemma 6.2, we know when $1 < p < \frac{n+2}{n-2} - \frac{1}{n^2}$, we can choose $P < 0$, such that

$$p - P = \frac{n+2}{n-2} - \frac{1}{n^2},$$

and the conditions (66) and (67) are satisfied. At this time, we can choose $M_2 = 0 < M_1$.

claim 10. *When* $\frac{n+2}{n-2} - \frac{1}{n^2} \leq p \leq \frac{n+2}{n-2}$ *and* $n \geq 7$*, we have*

$$1 + \frac{1.9}{n} < q < 1 + \frac{2}{n}.$$

Thus by (69) we get that

$$\begin{aligned}\Rightarrow & I\left(U_0, \frac{n+2}{n-2}-\frac{1}{n^2}\right)\\ & < \left[-2.8+0.051(2-q)+n(q-1)-0.5\sqrt{2}q\right]\frac{1}{n}\\ & + \left[6+4.34(2-q)+nq(1-q)+1.41q\right]\frac{1}{n^2}\\ & < \left[-2.8+0.051(2-q)+n(q-1)-0.5\sqrt{2}q\right]\frac{1}{n}\\ & + \left[6+4.32(2-q)-1.9\left(1+\frac{1.9}{n}\right)+1.41q\right]\frac{1}{n^2}.\end{aligned}$$

$$\begin{aligned}\Rightarrow & I\left(U_0, \frac{n+2}{n-2}-\frac{1}{n^2}\right)\\ & < \left(n-0.758-\frac{2.93}{n}\right)\frac{q}{n}+\frac{-n-2.698}{n}+\frac{12.78}{n^2},\end{aligned}$$

Combine (66), we can see

$$\begin{aligned}PJ = & -I\left(U_0, \frac{n+2}{n-2}-\frac{1}{n^2}\right)\\ & > -\left(n-0.758-\frac{2.93}{n}\right)\frac{q}{n}+\frac{n+2.698}{n}-\frac{12.78}{n^2}.\end{aligned} \tag{70}$$

By (67), we get

$$\begin{aligned}& P\cdot\frac{q}{2}\Big(J\cdot\frac{2}{2-q}\Big)^{-\frac{2-q}{q}}M^{-\frac{2}{q}} < -\frac{q}{\gamma+q}+U_0,\\ \Leftrightarrow & P\cdot\frac{q}{2}\Big(J\cdot\frac{2}{2-q}\Big)^{-\frac{2-q}{q}}\left(-\frac{q}{\gamma+q}+U_0\right)^{-1} < M^{\frac{2}{q}},\\ \Leftrightarrow & P^{\frac{q}{2}}\cdot\left(\frac{q}{2}\right)^{\frac{q}{2}}\left(J\cdot\frac{2}{2-q}\right)^{-\frac{2-q}{2}}\left(-\frac{q}{\gamma+q}+U_0\right)^{-\frac{q}{2}} < M.\end{aligned}$$

We only need to show that

$$\begin{aligned}& P^{\frac{q}{2}}\cdot\left(\frac{q}{2}\right)^{\frac{q}{2}}\left(J\cdot\frac{2}{2-q}\right)^{-\frac{2-q}{2}}\left(-\frac{q}{n-4+q}+U_0\right)^{-\frac{q}{2}}\\ & < (p+1)\left[\frac{(n-1)^2q}{4n}-1\right]^{-1}\left[\frac{n}{q}-\frac{n(n-1)}{n+2}\right]^{\frac{q}{2}} = M_1.\end{aligned}$$

$$
\begin{aligned}
RHS &= (p+1)\left[\frac{(n-1)^2q}{4n}-1\right]^{-1}\left[\frac{n}{q}-\frac{n(n-1)}{n+2}\right]^{\frac{q}{2}} \\
&> (p+1)\left[\frac{(n-1)^2q}{4n}-1\right]^{-1}\left[\frac{n^2}{n+2}-\frac{n(n-1)}{n+2}\right]^{\frac{q}{2}} \\
&> \left(\frac{n+2}{n-2}-\frac{1}{n^2}+1\right)\left[\frac{(n-1)^2(n+2)}{4n^2}-1\right]^{-1}\left(\frac{n}{n+2}\right)^{\frac{n+2}{2n}} \\
&> \left(\frac{n+1}{n-2}+1\right)\left[\frac{(n-1)^2(n+2)}{4n^2}-1\right]^{-1}\left(\frac{7}{9}\right)^{\frac{n+2}{2n}} \\
&> 2\left(\frac{7}{9}\right)^{\frac{9}{14}}\left[\frac{(n-1)^2(n+2)}{4n^2}-1\right]^{-1} \\
&> 1.7\left[\frac{(n-1)^2(n+2)}{4n^2}-1\right]^{-1}.
\end{aligned}
$$

claim 11. *When $n \geq 7$, we have*

$$U_0 > (4-2\sqrt{2})\left(\frac{1}{n}+\frac{7}{n^2}\right).$$

Besides we know that the function $y = x^{-\frac{1}{x}}$ is decreasing when $x > 1$. So we get that

$$\left(\frac{2}{2-q}\right)^{-\frac{2-q}{2}} < \left(\frac{2}{2-1}\right)^{-\frac{2-1}{2}} = \frac{\sqrt{2}}{2}.$$

Hence we obtain that:

$$
\begin{aligned}
LHS &= P^{\frac{q}{2}} \cdot \left(\frac{q}{2}\right)^{\frac{q}{2}} \left(J \cdot \frac{2}{2-q}\right)^{-\frac{2-q}{2}} \left(-\frac{q}{n-4+q} + U_0\right)^{-\frac{q}{2}} \\
&= P \cdot \left(\frac{q}{2}\right)^{\frac{q}{2}} \left(PJ \cdot \frac{2}{2-q}\right)^{-\frac{2-q}{2}} \left(-\frac{q}{n-4+q} + U_0\right)^{-\frac{q}{2}} \\
&< \frac{1}{n^2}\left[-\left(n - 0.758 - \frac{2.93}{n}\right)\frac{q}{n} + \frac{n+2.698}{n} - \frac{12.78}{n^2}\right]^{-\frac{2-q}{2}} \\
&\cdot \left(\frac{2}{2-q}\right)^{-\frac{2-q}{2}} \cdot \left(-\frac{1+\frac{2}{n}}{n-3+\frac{2}{n}} + \frac{4-2\sqrt{2}}{n} + \frac{28-14\sqrt{2}}{n^2}\right)^{-\frac{q}{2}} \\
&< \frac{1}{n^2}\left[-\left(n - 0.758 - \frac{2.93}{n}\right)\frac{n+2}{n^2} + \frac{n+2.698}{n} - \frac{12.78}{n^2}\right]^{-\frac{2-q}{2}} \\
&\cdot \frac{\sqrt{2}}{2}\left(-\frac{1+\frac{2}{n}}{n-3} + \frac{4-2\sqrt{2}}{n} + \frac{28-14\sqrt{2}}{n^2}\right)^{-\frac{q}{2}} \\
&< \frac{1}{n^q}\left(1.456n - 8.334\right)^{-\frac{2-q}{2}} \cdot \frac{\sqrt{2}}{2}\left(-\frac{1+\frac{2}{n}}{n-3} + \frac{4-2\sqrt{2}}{n} + \frac{28-14\sqrt{2}}{n^2}\right)^{-\frac{q}{2}} \\
&< n^{-1-\frac{1.9}{n}}\left(1.456n - 8.334\right)^{-\frac{n-2}{2n}} \cdot \frac{\sqrt{2}}{2}\left(-\frac{1+\frac{2}{n}}{n-3} + \frac{4-2\sqrt{2}}{n} + \frac{28-14\sqrt{2}}{n^2}\right)^{-\frac{q}{2}}
\end{aligned}
$$

When $n = 7$, we have

$$RHS > 2.6,$$

and

$$
\begin{aligned}
LHS &< n^{-1-\frac{1.9}{n}}\left(1.456n - 8.334\right)^{-\frac{n-2}{2n}} \cdot \frac{\sqrt{2}}{2}\left(-\frac{1+\frac{2}{n}}{n-3} + \frac{4-2\sqrt{2}}{n} + \frac{28-14\sqrt{2}}{n^2}\right)^{-\frac{n+2}{2n}} \\
&< 0.8.
\end{aligned}
$$

When $n \geq 8$, we have

$$\begin{aligned}
&-\frac{1+\frac{2}{n}}{n-3}+\frac{1}{n}+\frac{28-14\sqrt{2}}{n^2}\\
&>-\frac{1+\frac{2}{n}}{n-3}+\frac{1}{n}+\frac{8.2}{n^2}\\
&=\frac{-n^2-2n+n^2-3n+8.2n-24.6}{n^2(n-3)}\\
&=\frac{3.2n-24.6}{n^2(n-3)}>0.
\end{aligned}$$

Thus we deduce that

$$\begin{aligned}
LHS &< n^{-1-\frac{1.9}{n}}\left(1.456n-8.334\right)^{-\frac{n-2}{2n}}\cdot\frac{\sqrt{2}}{2}\cdot\left(\frac{3-2\sqrt{2}}{n}\right)^{-\frac{q}{2}}\\
&< n^{-1-\frac{1.9}{n}}\left(1.456n-8.334\right)^{-\frac{n-2}{2n}}\cdot\frac{\sqrt{2}}{2}\cdot\left(\frac{3-2\sqrt{2}}{n}\right)^{-\frac{n+2}{2n}}\\
&= n^{-1}\left(1.456n-8.334\right)^{-\frac{1}{2}}\cdot\frac{\sqrt{2}}{2}\cdot\left(\frac{3-2\sqrt{2}}{n}\right)^{-\frac{1}{2}}\left[\frac{1.456n-8.334}{(3-2\sqrt{2})n}\right]^{\frac{1}{n}}n^{\frac{0.1}{n}}.
\end{aligned}$$

Since the function $y=n^{\frac{1}{n}}$ is decreasing when $n \geq e$, we get that

$$\begin{aligned}
LHS &< n^{-1}\left(1.456n-8.334\right)^{-\frac{1}{2}}\cdot\frac{\sqrt{2}}{2}\cdot\left(\frac{3-2\sqrt{2}}{n}\right)^{-\frac{1}{2}}\left[\frac{1.456}{3-2\sqrt{2}}\right]^{\frac{1}{8}}8^{\frac{0.1}{8}}\\
&< n^{-1}\left(1.456n-8.334\right)^{-\frac{1}{2}}\left(\frac{3-2\sqrt{2}}{n}\right)^{-\frac{1}{2}}\\
&< n^{-1}\left(0.24-\frac{1.43}{n}\right)^{-\frac{1}{2}}.
\end{aligned}$$

Therefore, if $n \geq 8$, we need to show that

$$
\begin{aligned}
& n^{-1}\left(0.24-\frac{1.43}{n}\right)^{-\frac{1}{2}}<1.7\left[\frac{(n-1)^2(n+2)}{4n^2}-1\right]^{-1}, \\
\Leftrightarrow & \left[\frac{(n-1)^2(n+2)}{4n^2}-1\right]<1.7n\left(0.24-\frac{1.43}{n}\right)^{\frac{1}{2}}, \\
\Leftarrow & \left[\frac{(n-1)^2(n+2)}{4n^2}-1\right]<1.7n\left(0.24-\frac{1.43}{8}\right)^{\frac{1}{2}}, \\
\Leftarrow & \frac{(n-1)^2(n+2)}{4n^2}-1<0.4n, \\
\Leftarrow & 0.6n^3+4n^2+3n-2>0.
\end{aligned}
$$

□

A. Proof of claims in Section 6

Proof of claim 2. Denote

$$
\begin{aligned}
D_1 &= \frac{\sqrt{(n-2)(2n-6)}-2}{n-4} \\
D_2 &= \frac{2\sqrt{(n-2)(2n-6)}+n^2-5n+8}{(n-1)^2}
\end{aligned}
$$

then

$$
\begin{aligned}
H = {} & 4(n-2)D_1D_2\left(-n+3-\frac{4}{n}\right)\times\left\{\frac{1}{n^2}-\frac{n+2}{n-2}+\frac{n-2-\sqrt{(n-2)(2n-6)}}{n}\right\} \\
& -\left\{-(n+2)+\frac{n-2}{n^2}-\frac{n-2}{n}\left(2\sqrt{(n-2)(2n-6)}-n+4\right)\right\}^2 \\
= {} & 4(n-2)D_1D_2\left(n-3+\frac{4}{n}\right)\frac{4n^2-n+2+n(n-2)\left(2+\sqrt{(n-2)(2n-6)}\right)}{n^2(n-2)} \\
& -\left\{\frac{1}{n^2}\left[8n^2-9n+2+2n(n-2)\sqrt{(n-2)(2n-6)}\right]\right\}^2.
\end{aligned}
$$

$\Rightarrow$

$$\begin{aligned} n^4H =&4n^2D_1D_2\left(n-3+\frac{4}{n}\right)\left[6n^2-5n+2+n(n-2)\sqrt{(n-2)(2n-6)}\right]\\ &-\left\{8n^2-9n+2+2n(n-2)\sqrt{(n-2)(2n-6)}\right\}^2\\ D_1\times D_2 =&\frac{\left(\sqrt{(n-2)(2n-6)}-2\right)\left(2\sqrt{(n-2)(2n-6)}+n^2-5n+8\right)}{(n-1)^2(n-4)}\\ =&\frac{2(n^2-5n+4)+(n-1)(n-4)\sqrt{(n-2)(2n-6)}}{(n-1)^2(n-4)}\\ =&\frac{2+\sqrt{(n-2)(2n-6)}}{n-1}. \end{aligned}$$

Let $\Delta' = (n-2)(2n-6)$

$$\begin{aligned} \Rightarrow n^4H =&4n^2\frac{2+\sqrt{\Delta'}}{n-1}\frac{n^2-3n+4}{n}\left[6n^2-5n+2+n(n-2)\sqrt{\Delta'}\right]\\ &-\left[8n^2-9n+2+2n(n-2)\sqrt{\Delta'}\right]^2, \end{aligned}$$

so

$$\begin{aligned} (n-1)n^4H =&4n(n^2-3n+4)(2+\sqrt{\Delta'})\left[6n^2-5n+2+n(n-2)\sqrt{\Delta'}\right]\\ &-(n-1)\left[8n^2-9n+2+2n(n-2)\sqrt{\Delta'}\right]^2\\ =&8n(n^2-3n+4)(6n^2-5n+2)\\ &+4n^2(n-2)^2(n^2-3n+4)(2n-6)-4n^2(n-2)^3(2n-6)(n-1)\\ &-(8n^2-9n+2)^2(n-1)+\sqrt{\Delta'}\cdot 8n(8n^2-9n+2)\\ =&8n(n^2-3n+4)(6n^2-5n+2)+8n^2(n-2)^2(2n-6)\\ &-(8n^2-9n+2)^2(n-1)+\sqrt{\Delta'}\cdot 8n(8n^2-9n+2)\\ =&-88n^4+327n^3-251n^2+24n+4+\sqrt{\Delta'}\cdot 8n(8n^2-9n+2). \end{aligned}$$

Since we know when $n \geq 7$,

$$\begin{aligned} \sqrt{\Delta'} &> \sqrt{2}\left[n-2.5-\frac{1}{6(n-2)}\right]\\ &> \sqrt{2}\left(n-2.5-\frac{1}{30}\right)\\ &= \sqrt{2}\left(n-\frac{38}{15}\right), \end{aligned}$$

then we get that

$$(n-1)n^4H > -88n^4 + 327n^3 - 251n^2 + 24n + 4 + 8\sqrt{2}n\left(n - \frac{38}{15}\right) \cdot (8n^2 - 9n + 2)$$
$$> 2.5n^4 - 4.2n^3 + 29n^2 - 34n + 4 > 0.$$

□

Proof of claim 3 . In fact, we know

$$\begin{aligned}
&\left(1+\frac{2}{n}\right)\left(n-3+\frac{2}{n}\right)(n-2)^2(n-1)\\
&=\left(n-1-\frac{4}{n}+\frac{4}{n^2}\right)(n-2)^2(n-1)\\
&<\left(n-1-\frac{3}{n}\right)(n-2)^2(n-1)\\
&=\left(n^2-3n\right)(n-2)(n-1),
\end{aligned}$$

and

$$\begin{aligned}
&2\sqrt{(n-2)(2n-6)}+n^2-5n+8\\
&>2\sqrt{2}(n-3)+n^2-5n+8\\
&=n^2-2.3n+(2\sqrt{2}-2.7)n+8-6\sqrt{2}\\
&>n^2-2.3n.
\end{aligned}$$

Then we get that

$$\begin{aligned}
\Delta &> 1-\frac{(1+\frac{2}{n})(n-3+\frac{2}{n})(n-2)^2(n-1)}{(\frac{n}{n-1}+n-4)(2n-6)\left[2\sqrt{(n-2)(2n-6)}+n^2-5n+8\right]}\\
&> 1-\frac{\left(n^2-3n\right)(n-2)(n-1)}{(n-3+\frac{1}{n-1})(2n-6)(n^2-2.3n)}\\
&= 1-\frac{(n-2)(n-1)}{2(n-3+\frac{1}{n-1})(n-2.3)}\\
&= 1-\frac{(n-2)(n-1)}{2(n^2-5.3n+7.9-\frac{1.3}{n-1})}\\
&> 1-\frac{(n-2)(n-1)}{2(n^2-5.3n+7.6)}.
\end{aligned}$$

Then

$$\begin{aligned}
\Delta > 0.5 - \frac{2}{n} + \frac{2}{n^2}&,\\
\Leftrightarrow 1 - \frac{(n-2)(n-1)}{2(n^2-5.3n+7.6)} > 0.5 - \frac{2}{n} + \frac{2}{n^2}&,\\
\Leftrightarrow \frac{(n-2)(n-1)}{(n^2-5.3n+7.6)} < \frac{n^2+4n-4}{n^2}&,\\
\Leftrightarrow h_1(n) := -1.7n^3 + 19.6n^2 - 51.6n + 30.4 < 0&
\end{aligned}$$

Since $n \geq 9$, we know

$$h'(n) = -5.1n^2 + 39.2n - 51.6 \leq h'(9) < 0.$$

So we deduce that

$$h(n) < h(9) < 0.$$

□

Proof of claim 4 .

$$\begin{aligned}
LHS &= \left(\frac{\sqrt{2}-1}{2} + \frac{1}{n}\right) \cdot \frac{n-4}{n - \frac{5}{2} - \frac{1}{6(n-2)} - \sqrt{2}} \left(1 + \frac{4}{n-2} - \frac{1}{n^2}\right)\\
&= \left(\frac{\sqrt{2}-1}{2} + \frac{1}{n} + \frac{2\sqrt{2}-2}{n-2} + \frac{4}{n(n-2)} - \frac{\sqrt{2}-1}{2n^2} - \frac{1}{n^3}\right) \cdot \frac{n-4}{n - \frac{5}{2} - \frac{1}{6(n-2)} - \sqrt{2}}\\
&< \left(\frac{\sqrt{2}-1}{2} + \frac{1}{n} + \frac{2\sqrt{2}-2}{n-2} + \frac{4}{n(n-2)} - \frac{\sqrt{2}-1}{2n^2} - \frac{1}{n^3}\right) \cdot \frac{n-4}{n - \frac{5}{2} - \frac{1}{30} - \sqrt{2}}\\
&< \frac{\sqrt{2}-1}{2} + \frac{1}{n} + \frac{2\sqrt{2}-2}{n} + \frac{2\sqrt{2}-2}{n-2} - \frac{2\sqrt{2}-2}{n} + \frac{4}{n(n-2)} - \frac{\sqrt{2}-1}{2n^2} - \frac{1}{n^3}
\end{aligned}$$

Thus we get that

$$\begin{aligned}
LHS &< \frac{\sqrt{2}-1}{2} + \frac{2\sqrt{2}-1}{n} + \frac{4\sqrt{2}}{n(n-2)}\\
&\leq \frac{\sqrt{2}-1}{2} + \frac{2\sqrt{2}-1}{n} + \frac{4\sqrt{2}}{5n}\\
&< \frac{\sqrt{2}-1}{2} + \frac{3}{n}.
\end{aligned}$$

□

Proof of claim 5 .

$$
\begin{aligned}
LHS &= 2\sqrt{2}\left(n-3+\frac{1}{n-1}\right)\frac{2\sqrt{2}\left(n-\frac{38}{15}\right)+n^2-5n+8}{(n-1)(n-2)^2(n-4)}\left(n-\frac{38}{15}-\frac{1.5}{\sqrt{2}}\right)\\
&> 2\sqrt{2}\left(n-3+\frac{1}{n-1}\right)\frac{n^2-2.2n+0.83}{(n-1)(n-2)^2(n-4)}(n-3.6)\\
&> 2\sqrt{2}\left(n-3+\frac{1}{n}\right)\frac{n^2-2.2n+0.83}{(n-1)(n-2)^2(n-4)}(n-3.6)\\
&> 2\sqrt{2}\left(n^3-5.2n^2+8.43n-4.69\right)\frac{n-3.6}{(n-1)(n-2)^2(n-4)}\\
&= 2\sqrt{2}\cdot\frac{n^3-5.2n^2+8.43n-4.69}{n^3-8n^2+20n-16}\cdot\frac{n-3.6}{n-1}.
\end{aligned}
$$

So we only need to show that

$$
\begin{aligned}
&\frac{n^3-5.2n^2+8.43n-4.69}{n^3-8n^2+20n-16}\cdot\frac{n-3.6}{n-1} > \frac{n+0.2}{n},\\
\Leftrightarrow\, & n(n-3.6)(n^3-5.2n^2+8.43n-4.69)-(n-1)(n+0.2)(n^3-8n^2+20n-16) > 0.
\end{aligned}
$$

Actually, we know

$$
\begin{aligned}
LHS &= n^5-8.8n^4+27.15n^3-35.038n^2+16.884n-(n^5-8.8n^4+26.2n^3-30.4n^2+8.8n+3.2)\\
&= 0.95n^3-4.638n^2+8.084n-3.2\\
&> 0.95\cdot 7^3-4.638\cdot 7^2+8.084\cdot 7-3.2 > 0.
\end{aligned}
$$

□

Proof of claim 6 . When $n=7$, we have

$$LHS < 0.96 < RHS.$$

When $n=8$, we have

$$LHS < 0.9 < RHS.$$

When $n \geq 9$, we have

$$\begin{aligned}
LHS &< \frac{2\sqrt{2}\left[n - \frac{5}{2} - \frac{1}{8(n-2)}\right] - n + 4}{n-2} \cdot \left(\frac{2-\sqrt{2}}{2} + \frac{\sqrt{2}}{n}\right) \\
&< \frac{2\sqrt{2}\left[n - \frac{5}{2} - \frac{1}{8n}\right] - n + 4}{n-2} \cdot \left(\frac{2-\sqrt{2}}{2} + \frac{\sqrt{2}}{n}\right) \\
&= \left[(2\sqrt{2}-1)n + 4 - 5\sqrt{2} - \frac{\sqrt{2}}{4n}\right] \cdot \frac{1}{n}\left(1 + \frac{2}{n} + \frac{4}{n^2} \cdot \frac{n}{n-2}\right) \cdot \left(\frac{2-\sqrt{2}}{2} + \frac{\sqrt{2}}{n}\right) \\
&\leq \left(2\sqrt{2} - 1 + \frac{4-5\sqrt{2}}{n} - \frac{\sqrt{2}}{4n^2}\right) \cdot \left(1 + \frac{2}{n} + \frac{4}{n^2} \cdot \frac{9}{7}\right) \cdot \left(\frac{2-\sqrt{2}}{2} + \frac{\sqrt{2}}{n}\right)
\end{aligned}$$

$$\begin{aligned}
LHS &= \left[2\sqrt{2} - 1 + \frac{2-\sqrt{2}}{n} + \frac{72\sqrt{2}-36}{7n^2} + \frac{8-10\sqrt{2}}{n^2} - \frac{\sqrt{2}}{4n^2} + \frac{144-180\sqrt{2}}{7n^3} - \frac{\sqrt{2}}{2n^3}\right. \\
&\quad \left. - \frac{9\sqrt{2}}{7n^4}\right] \cdot \left(\frac{2-\sqrt{2}}{2} + \frac{\sqrt{2}}{n}\right) \\
&< \left(2\sqrt{2} - 1 + \frac{2-\sqrt{2}}{n} + \frac{2.91}{n^2} - \frac{16.5}{n^3}\right) \cdot \left(\frac{2-\sqrt{2}}{2} + \frac{\sqrt{2}}{n}\right) \\
&= (2\sqrt{2}-1)\frac{2-\sqrt{2}}{2} + \frac{4-\sqrt{2}}{n} + \frac{3-2\sqrt{2}}{n} + \frac{2\sqrt{2}-2}{n^2} + \frac{2.91(2-\sqrt{2})}{2n^2} + \frac{2.91\sqrt{2}}{n^3} \\
&\quad - \frac{16.5(2-\sqrt{2})}{2n^3} - \frac{16.5\sqrt{2}}{n^4} \\
&= (2\sqrt{2}-1)\frac{2-\sqrt{2}}{2} + \frac{7-3\sqrt{2}}{n} + \frac{0.91+0.545\sqrt{2}}{n^2} + \frac{11.16\sqrt{2}-16.5}{n^3} - \frac{16.5\sqrt{2}}{n^4} \\
&< (2\sqrt{2}-1)\frac{2-\sqrt{2}}{2} + \frac{7-3\sqrt{2}}{n} + \frac{0.91+0.545\sqrt{2}}{n^2} \\
&< (2\sqrt{2}-1)\frac{2-\sqrt{2}}{2} + \frac{2.76}{n} + \frac{1.7}{n^2}.
\end{aligned}$$

□

Proof of claim 7 . When $n = 7$,

$$LHS > 0.05 > RHS.$$

When $n = 8$,

$$LHS > 0.04 > RHS.$$

When $n \geq 9$

$$\begin{aligned}
LHS &> \frac{1}{n-2} \cdot \frac{\frac{2-\sqrt{2}}{2}n - \frac{53}{21} + \sqrt{2}}{n - \frac{5}{2} - \sqrt{2}} \cdot \frac{2\sqrt{2}\left[n - \frac{5}{2} - \frac{1}{6(n-2)}\right] - n + 4}{n-2} \left(\frac{2-\sqrt{2}}{2} + \frac{\sqrt{2}}{n}\right) \\
&\geq \frac{1}{n-2} \cdot \frac{\frac{2-\sqrt{2}}{2}n - \frac{53}{21} + \sqrt{2}}{n - \frac{5}{2} - \sqrt{2}} \cdot \frac{2\sqrt{2}\left[n - \frac{5}{2} - \frac{1}{42}\right] - n + 4}{n-2} \left(\frac{2-\sqrt{2}}{2} + \frac{\sqrt{2}}{n}\right) \\
&> \frac{2-\sqrt{2}}{2}(2\sqrt{2}-1) \cdot \frac{2-\sqrt{2}}{2} \cdot \frac{(n-3.8)(n-1.72)}{(n-2)^2(n-3.9)} \cdot \frac{n+4.8}{n} \\
&= \frac{8\sqrt{2}-11}{2} \cdot \frac{(n-3.8)(n-1.72)}{(n-2)^2(n-3.9)} \cdot \frac{n+4.8}{n}
\end{aligned}$$

So we only need to check that

$$\begin{aligned}
&\frac{(n-3.8)(n-1.72)(n+4.8)}{(n-2)^2(n-3.9)} > \frac{n+7}{n}, \\
\Leftrightarrow & n(n-3.8)(n-1.72)(n+4.8) - (n+7)(n-2)^2(n-3.9) > 0, \\
\Leftarrow & n(n-1.72)(n+4.8) - (n+7)(n-2)^2 = 0.08n^2 + 15.744n - 28 > 0.
\end{aligned}$$

□

Proof of claim 8 .

$$\begin{aligned}
\frac{n-4+\frac{4}{n}}{n-4+q} &= \left(1 - \frac{4}{n} + \frac{4}{n^2}\right) \sum_{k=0}^{\infty} \left(\frac{4-q}{n}\right)^k \\
&< \left(1 - \frac{4}{n} + \frac{4}{n^2}\right) \left(1 + \frac{4-q}{n} + \frac{(4-q)^2}{n^2} \cdot \frac{1}{1 - \frac{4-q}{n}}\right) \\
&\leq \left(1 - \frac{4}{n} + \frac{4}{n^2}\right) \left(1 + \frac{4-q}{n} + \frac{(4-q)^2}{n^2} \cdot \frac{1}{1 - \frac{4-1}{9}}\right) \\
&= \left(1 - \frac{4}{n} + \frac{4}{n^2}\right) \left(1 + \frac{4-q}{n} + \frac{1.5(4-q)^2}{n^2}\right) \\
&= 1 - \frac{q}{n} + \frac{1.5(4-q)^2 - 16 + 4q + 4}{n^2} + \frac{4-q}{n^3}\left[-6(4-q) + 4 + \frac{6(4-q)}{n}\right] \\
&< 1 - \frac{q}{n} + \frac{1.5q^2 - 8q + 12}{n^2}
\end{aligned}$$

□

Proof of claim 9. Recall that we have already known (68) :

$$
\begin{aligned}
&\Rightarrow I\left(U_0, \frac{n+2}{n-2}-\frac{1}{n^2}\right)\\
&\quad\le -\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{n}+\frac{\sqrt{2}}{2n^2}+\frac{2-q}{n-3}\left(\frac{\sqrt{2}-1}{2}+\frac{3}{n}\right)\\
&\quad-\frac{n-4+\frac{4}{n}}{n-4+q}\left[1-q-0.5\sqrt{2}+\frac{0.84+0.4\sqrt{2}}{n}-\frac{0.9}{n^2}+(2-q)\frac{8\sqrt{2}-11}{2}\left(\frac{1}{n}+\frac{7}{n^2}\right)\right].
\end{aligned}
$$

When $n=7$ and $1<q\le\frac{9}{7}$,

$$
\begin{aligned}
&I\left(U_0, \frac{n+2}{n-2}-\frac{1}{n^2}\right)\\
&<-0.89+0.16(2-q)-\frac{25}{7(q+3)}\Big[-q+0.47+0.044(2-q)\Big]\\
&<-0.89+0.16(2-q)-\frac{25}{28}\Big[-q+0.47+0.044(2-q)\Big]\\
&<0.773q-1.068.
\end{aligned}
$$

and

$$
\begin{aligned}
&\Big[-2.8+0.051(2-q)+n(q-1)-0.5\sqrt{2}q\Big]\frac{1}{n}\\
&+\Big[6+4.34(2-q)+nq(1-q)+1.41q\Big]\frac{1}{n^2}\\
&>-\frac{1}{7}q^2+0.974q-1.086>0.773q-1.068.
\end{aligned}
$$

When $n=8$ and $1<q\le\frac{5}{4}$,

$$
\begin{aligned}
&I\left(U_0, \frac{n+2}{n-2}-\frac{1}{n^2}\right)\\
&<-0.87+0.12(2-q)-\frac{4.5}{4+q}\Big[-q+0.45+0.036(2-q)\Big]\\
&<-0.87+0.12(2-q)-\frac{4.5}{5}\Big[-q+0.45+0.036(2-q)\Big]\\
&<0.8124q-1.0998,
\end{aligned}
$$

and

$$\begin{aligned}
&\left[-2.8+0.051(2-q)+n(q-1)-0.5\sqrt{2}q\right]\frac{1}{n}\\
&+\left[6+4.34(2-q)+nq(1-q)+1.41q\right]\frac{1}{n^2}\\
&>-\frac{1}{8}q^2+0.98q-1.108>0.8124q-1.0998.
\end{aligned}$$

□

Proof of claim 10. If $q>1+\frac{1.9}{n}$, then

$$p=\frac{q}{2-q}>\frac{n+1.9}{n-1.9}.$$

So we only need to check that

$$\begin{aligned}
&\frac{n+2}{n-2}-\frac{1}{(n-2)^2}>\frac{n+1.9}{n-1.9},\\
\Leftrightarrow&\frac{n^2-5}{(n-2)^2}-\frac{n+1.9}{n-1.9}>0,\\
\Leftrightarrow&(n^2-5)(n-1.9)-(n+1.9)(n-2)^2>0,\\
\Leftrightarrow&0.2n^2-1.4n+1.9>0.
\end{aligned}$$

The last inequality follows from $n\geq 7$.

□

Proof of claim 11 . By claim 5 we know

$$\begin{aligned}
U_0&=\frac{2B_0(1+\gamma S+qS)^2}{(\gamma+q)^2}\left(\frac{2-\sqrt{2}}{2}+\frac{\sqrt{2}}{n}\right)\\
&=\frac{2B_0(1+\gamma S+qS)}{\gamma+q}\left(\frac{1}{\gamma+q}+S\right)\left(\frac{2-\sqrt{2}}{2}+\frac{\sqrt{2}}{n}\right)\\
&>\left(2\sqrt{2}+\frac{0.4\sqrt{2}}{n}\right)\frac{\sqrt{2}(n-\frac{38}{15})-1.5}{(n-2)(n-4)}\left(\frac{2-\sqrt{2}}{2}+\frac{\sqrt{2}}{n}\right)\\
&>\frac{2-\sqrt{2}}{2}\left(4+\frac{0.8}{n}\right)\frac{n-3.6}{(n-2)(n-4)}\left(1+\frac{4.8}{n}\right)\\
&=(4-2\sqrt{2})\left(1+\frac{0.2}{n}\right)\frac{n-3.6}{(n-2)(n-4)}\left(1+\frac{4.8}{n}\right).
\end{aligned}$$

We only need to show that

$$\frac{(n+0.2)(n-3.6)(n+4.8)}{n^2(n-2)(n-4)} > \frac{n+7}{n^2},$$
$$\Leftrightarrow (n+0.2)(n-3.6)(n+4.8)-(n+7)(n-2)(n-4)>0.$$

$$LHS = 0.4n^2 + 16.96n - 59.456 > 0.$$

□

References

[1] M. F. Bidaut-Véron, M. Garcia-Huidobro and L. Véron, Estimates of solutions of elliptic equations with a source reaction term involving the product of the function and its gradient. Duke Math. J. 168 (2019), 1487-1537.

[2] M. F. Bidaut-Véron, M. Garcia-Huidobro and L. Véron, A priori estimates for elliptic equations with reaction terms involving the function and its gradient. Math. Ann. 378(1-2) (2020), 13-56.

[3] M. F. Bidaut-Véron, M. Garcia-Huidobro and L. Véron, Radial solutions of scaling invariant nonlinear elliptic equations with mixed reaction terms. (2019), arXiv:1812.08418.

[4] M. F. Bidaut-Véron, M. Garcia-Huidobro and L. Véron, Local and global properties of solutions of quasilinear Hamilton-Jacobi equations. J. Funct. Anal. 267(9) (2014), 3294-3331.

[5] H. Berestycki, L. Nirenberg, Some qualitative properties of solutions of semilinear elliptic equations in cylindrical domains [M]. Analysis, et cetera. Academic Press, (1990), 115-164.

[6] L. Caffarelli, B. Gidas and J. Spruck, Asymptotic symmetry and local behaviour of semilinear elliptic equations with critical Sobolev growth. Comm. Pure Appl. Math. 42 (1989), 271-297.

[7] M. Chipot, F. Weissler, Some blow-up results for a nonlinear parabolic equation with a gradient term. S.I.A.M. J. of Num. Anal. 20 (1989), 886-907.

[8] W. X. Chen, C. M. Li, Classification of solutions of some nonlinear elliptic equations. Duke Math. 63(3) (1991), 615-622.

[9] S. Chandrasekar, An introduction to the Study of Steller Structure [M]. Dover Publ. Inc. (1967).

[10] V. R. Emden, Gaskugeln, B.G. Teubner, Leipzig (1897).

[11] M. Fila, Quittner, Radial positive solutions for a semilinear elliptic equation with a gradient term. Adv. Math. Sci. Appl. 2 (1993), 39-45.

[12] R. H. Fowler, The form near infinity of real continuous solutions of a certain differential equation of the second order. Quart. J. Pure Appl. Math. 45 (1914), 289-350.

[13] R. H. Fowler, Further studies of Emden's equation and similar differential equations. Quart. J. Math. Oxford series, 2 (1930), 259-288.

[14] B. Gidas, W. M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$ in Mathematical analysis and applications. Part A ed. L. Nachbin. Adv. in Math. Suppl. Stud. Academic Press, New York, 7 (1981), 369-402.

[15] B. Gidas, J. Spruck, Global and local behaviour of positive solutions of nonlinear elliptic equations. Comm. Pure Appl. Math. 34 (1981), 525-598.

[16] D. W. Lin, X. N. Ma, Liouville type theorem for a class quasilinear p-Laplace type equation on the sphere. (2022), arXiv:2211.03532.

[17] D. W. Lin, Q. Z. Ou, Liouville type theorems for positive harmonic functions on the unit ball with a nonlinear boundary condition. Calc. Var. Partial Differential Equations. 62(1) (2023), Paper No. 34, 13 pp.
[18] P. L. Lions, Quelques remarques sur les problèmes elliptiques quasilinéaires du second ordre. (French) [Some remarks on second-order elliptic quasilinear problems] J. Analyse Math. 45(1985), 234-254.
[19] P. L. Lions, Résolution de problèmes elliptiques quasilinéaires. Arch. Rational Mech. Anal. 74 (1980), 335-353.
[20] M. Obata, The conjectures on conformal transformations of Riemannian manifolds. Differential Geometry 6 (1971), 247-258.
[21] S. I. Pohožaev, On the eigenfunctions of the equation $\Delta u = \lambda f(u)$. Dokl. Akad. Nauk. SSSR, 165 (1965), 33-39.
[22] J. Serrin, Local behavior of solutions of quasi-linear equations. Acta Math. 111 (1964), 247-302.
[23] J. Serrin, H. Zou, Existence and non-existence results for ground states of quasilinear elliptic equations. Arch. Rat. Mech. Anal. 121 (1992), 101-130.
[24] F. X. Voirol, Thése de Doctorat, Université de Metz, (1994).
[25] F. X. Voirol, Coexistence of singular and regular solutions for the equation of Chipot and Weissler. Acta Math. Univ-Comenianae 65 (1996), 53-64.

Department of Mathematics, University of Science and Technology of China, Hefei, China
Email address: xinan@ustc.edu.cn

Department of Mathematics, University of Science and Technology of China, Hefei, China
Email address: wuwz18@mail.ustc.edu.cn

College of Mathematics and Information Science, Nanchang Hangkong University, Nanchang, China
Email address: 71131@nchu.edu.cn