JERISON-LEE IDENTITIES AND SEMI-LINEAR SUBELLIPTIC EQUATIONS ON CR MANIFOLDS

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ABSTRACT. In the study of the extremal for Sobolev inequality on the Heisenberg group and the Cauchy-Riemann(CR) Yamabe problem, Jerison-Lee found a three-dimensional family of differential identities for critical exponent subelliptic equation on Heisenberg group \mathbb{H}^n by using the computer in [11]. They wanted to know whether there is a theoretical framework that would predict the existence and the structure of such formulae. With the help of dimensional conservation and invariant tensors, we can answer the above question. For a class of subcritical exponent subelliptic equations on the CR manifold, several new types of differential identities are found. Then we use those identities to get the rigidity result, where rigidity means that subelliptic equations have no other solution than some constant at least when parameters are in a certain range. The rigidity result also deduces the sharp Folland-Stein inequality on closed CR manifolds.

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1. INTRODUCTION

Let M be a real manifold. A distinguished complex subbundle $T^{(1,0)}M$ of $\mathbb{C}TM$ is a CR structure, if $T^{(1,0)} \cap T^{(0,1)} = 0$, where $T^{(0,1)} := \overline{T^{(1,0)}}$, and M is called a CR manifold. An CR manifold M is hypersurface type, if $\dim_{\mathbb{R}} M = 2n + 1$ and $\dim_{\mathbb{C}} T^{(1,0)}M = n$.

If M is oriented, a globally defined real one-form θ that annihilates $T^{(1,0)}M$ and $T^{(0,1)}M$ exists. The Levi form $\langle V, W \rangle_{L_{\theta}} = L_{\theta}(V, \overline{W}) := -2\sqrt{-1}d\theta(V \wedge \overline{W})$ is a hermitian form on $T^{(1,0)}M$. We say the CR structure is strictly pseudoconvex if L_{θ} is positive definite on

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 $T^{(1,0)}M$ for some choice of θ , in which case θ defines a contact structure on M and we call θ a contact form associated with the CR structure.

The Reeb vector field T is defined by $\theta(T) = 1$, and $d\theta(T, X) = 0$, $\forall X \in TM$, then $TM = T^{(1,0)}M \oplus T^{(0,1)}M \oplus \text{span}\{T\}$. Let $\{Z_i\}_{i=1}^n$ be an orthogonal basis w.r.t. the Levi form, and $\{\theta^i\}_{i=1}^n$ be the dual base of $\{Z_i\}_{i=1}^n$, then $\theta^i(T) = 0$. Set $d\theta = 2\sqrt{-1}h_{i\bar{j}}\theta^i \wedge \theta^{\bar{j}}$, and we'll use the hermitian matrix $h_{i\bar{j}}$ and its inverse $h^{i\bar{j}}$ to raise and lower indices. Let $R_{i\bar{j}k\bar{l}}$ be the Webster curvature tensor, $\text{Tor}(Z_i, Z_j) = A_{ij}$ the Webster torsion tensor, $\text{Ric}(Z_i, Z_j) = R_{i\bar{j}} = R_i^{\ k}_{\ k\bar{j}}$ the pseudohermitian Ricci tensor, and $R = R_i^{\ i}$ the pseudohermitian scalar curvature.

In this article, all small English letters in lower or upper place will be considered as summation indices taking part in the process of summing from 1 to n. Besides, all Greek letters and Arabic numbers in the lower or upper place won't participate in the summation process. Denote $Z_i f$ as $f_{,i}$, $Z_{\bar{i}} f$ as $f_{,\bar{i}}$, Tf as $f_{,0}$. Commutation formulae are presented as follows:

$$f_{,ij} = f_{,ji}, \ f_{,i\bar{j}} - f_{,\bar{j}i} = 2\sqrt{-1}h_{i\bar{j}}f_{,0}, \ f_{,0i} - f_{,i0} = A_{ij}f_{,j}, \ f_{,ij\bar{k}} - f_{,i\bar{k}j} = 2\sqrt{-1}h_{j\bar{k}}f_{,i0} + R_{i\,j\bar{k}}^{\ l}f_{,l}.$$

Define $\Delta f := \frac{1}{2}(f_{,i}^{i} + f_{,i}^{i})$ as the sub-Laplacian operator on M, then $\Delta f = \operatorname{Re} f_{,i}^{i}$, $f_{,i}^{i} = \Delta f + n\sqrt{-1}f_{,0}$. Denote $f_{,i}f_{,i}^{i}$ as $|\nabla f|^{2}$. In this article, M is a closed, oriented, strictly pseudoconvex CR manifold of hypersurface

In this article, M is a closed, oriented, strictly pseudoconvex CR manifold of hypersurface type, and curvature and torsion satisfy the following pointwise condition:

$$\operatorname{Ric}(Z,Z) \ge (n+1)\langle Z,Z\rangle_{L_{\theta}}, \quad \operatorname{Tor}(Z,Z) = 0, \quad \forall Z \in T^{(1,0)}M.$$
(1.1)

Let $\alpha > 1, \lambda > 0, u \in C^{\infty}(M)$ is positive, we study the following equation:

$$\Delta u - \lambda u + u^{\alpha} = 0. \tag{1.2}$$

',' would be omitted while writing derivatives of solution u in this article.

Positive solutions don't exist if $\lambda \leq 0$ by directly integrated by part, hence assume that $\lambda > 0$. The existence and regularity of solutions are discussed carefully in [10], then only the classification of smooth solutions is considered in this article.

In celebrated paper [11], Jerison-Lee introduced remarkable identities to deduce the following theorem.

Theorem 1.1 ([11] Theorem A). Assume that
$$u > 0$$
 satisfying (1.2) with $\alpha = \frac{n+2}{n}$,
 $\lambda = \frac{n^2}{4}$, and $(M^{2n+1}, \theta) = (\mathbb{S}^{2n+1}, \theta_c)$, then there exists $s \ge 0$, $\xi \in \mathbb{S}^{2n+1}$ such that
 $u(z) = c_{n,s} |\cosh s + (\sinh s) \langle z, \xi \rangle |^{-n}$, $z \in \mathbb{S}^{2n+1}$.

For the flat CR manifold Heisenberg group \mathbb{H}^n case, Jerison-Lee studied critical exponent $\alpha = \frac{n+2}{n}$ in [11], where they found three-dimensional family identities. The positive

solution of the CR-Yamabe equation can be classified with the assumption of finite energy by using those identities:

Theorem 1.2 ([11] Corollary C). Assume that $u \in L^{\frac{2n+2}{n}}(\mathbb{H}^n)$ is the positive solution of $\Delta u + u^{\frac{n+2}{n}} = 0$ in \mathbb{H}^n , then there exists $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{C}^n$ satisfying $\operatorname{Im} \lambda > \frac{|\mu|^2}{4}$, such that

$$u(z,t) = c_{n,\lambda,\mu} \left| t + \sqrt{-1} |z|^2 + z \cdot \mu + \lambda \right|^{-n}.$$

Theorem 1.1 is covered by Theorem 1.2 by Cayley transformation. The proofs of the above theorems were based on the same idea as Obata's [13] proof of the analogous result in Riemannian geometry: the only Riemannian metrics on the sphere that are conformal to the standard one and have constant scalar curvature are obtained from the standard metric by a conformal diffeomorphism of the sphere. Since the pseudohermitian Bianchi identities involve extra torsion terms and on the Heisenberg group, this reflects the nontrivial commutation relations. Using computer algebra, Jerison-Lee found a three-dimensional family $((4.2)\sim(4.4)$ in [11]) of solutions with divergence terms on the left-hand side and positive terms on the right-hand side. Then, the divergence theorem would prove that the right-hand side vanishes identically and gets above classification results. In page 4 of [11], Jerison-Lee raised the following problem:

An interesting (but vaguely defined) problem raised by this work is to find an "explanation" for the existence of divergence formulas such as (4.2) and (3.1). Is there a theoretical framework that would predict the existence and the structure of such formulas, so that they could be discovered more systematically?

With the help of dimensional conservation and invariant tensors, we state the following theorem, which answers the problem above.

Theorem 1.3. Assume that u is the positive solution of $\Delta u + u^{\frac{n+2}{n}} = 0$ in \mathbb{H}^n , then all the useful identities of $\{(0,0), 2, 6, +\}$ type must lie in the three-dimensional family.

The meaning of dimensional symbol $\{(0,0), 2, 6, +\}$ is given in Section 2. In [14], Wang extended Theorem 1.1 from $(\mathbb{S}^{2n+1}, \theta_c)$ to closed Einstein pseudohermitian manifold, and to closed pseudohermitian manifold under condition (1.1) in [15] where he raised the following conjecture.

Conjecture 1.4 ([15] Conjecture 1). If $1 < \alpha \leq \frac{n+2}{n}$ and $0 < \lambda \leq \frac{n}{2(\alpha-1)}$, the only positive solution of (1.2) under the condition (1.1) must be $u \equiv \lambda^{\frac{1}{\alpha-1}}$, otherwise $\alpha = \frac{n+2}{n}$, $\lambda = \frac{n^2}{4}$, $(M^{2n+1}, \theta) = (\mathbb{S}^{2n+1}, \theta_c)$ is standard CR sphere, with some $s \geq 0$, $\xi \in \mathbb{S}^{2n+1}$ such that

$$u(z) = c_{n,s} |\cosh s + (\sinh s) \langle z, \xi \rangle|^{-n}, \ z \in \mathbb{S}^{2n+1}.$$

Remark 1.5. In [15], Wang proved the above conjecture when $\alpha = \frac{n+2}{n}$ by using the Jerison-Lee identity (3.1) in [11].

By dimensional conservation and invariant tensors, several new differential identities are obtained for the subcritical exponent case, which leads to our following main theorem. This theorem gives the positive answer to the subcritical exponent case of Conjecture 1.4.

Theorem 1.6. If $1 < \alpha < \frac{n+2}{n}$ and $0 < \lambda \leq \frac{n}{2(\alpha-1)}$, the only positive solution of (1.2) under the condition (1.1) must be $u \equiv \lambda^{\frac{1}{\alpha-1}}$.

We let $u \in HW^{1,2}(M)$, which is the usual Sobolev space on M (see [7] or [5] for details). In [15], the following Corollary 1.7 was raised as the consequence of Theorem 1.6.

Corollary 1.7. Let (M^{2n+1}, θ) be a closed, oriented, strictly-pseudoconvex CR manifold of hypersurface type satisfying (1.1), then for $2 < q < \frac{2Q}{Q-2} = \frac{2n+2}{n}$, $u \in HW^{1,2}(M)$,

$$\frac{4(q-2)}{Q-2} \int_{M} |\nabla u|^2 + \int_{M} |u|^2 \ge \operatorname{vol}(M)^{1-\frac{2}{q}} \left(\int_{M} |u|^q \right)^{\frac{2}{q}}$$

Equality holds if and only if u is constant. Notice that Q = 2n + 2, and q is equivalent to $\alpha + 1$ in Theorem 1.6.

The proof of Corollary 1.7 is similar to the Riemannian case as corollary 6.2 in [1]. It's noteworthy that Frank-Lieb got this inequality on standard CR sphere $(\mathbb{S}^{2n+1}, \theta_c)$ by harmonic polynomials extension in [8] (see corollary 2.3), and in fact, Frank-Lieb derived the sharp constants for Hardy-Littlewood-Sobolev inequalities on Heisenberg group. Besides, Moser-Trudinger and Beckner-Onofri's inequalities on the CR sphere are obtained by Branson-Fontana-Morpurgo in [2].

Remark 1.8. Motivated by the Jerison-Lee identity (4.2) in [11], Ma-Ou [12] proved that there is no positive solution of $\Delta u + u^{\alpha} = 0$ in \mathbb{H}^n while $1 < \alpha < \frac{n+2}{n}$. Recently, Catino-Li-Monticelli-Roncoron [3] and Flynn-Vétois in [6] got more generalizations for the critical exponent case.

The result in Ma-Ou [12] can be proved by the identity (3.2) raised in this article with curvature terms discarded.

For semilinear elliptic equations on compact Riemannian manifolds, M-F. B. Véron and L. Véron [[1], Theorem 6.1] introduced the Bochner-Lichnerowicz-Weitzenbeck formula in such a way that they could extend and simplify Gidas-Spruck's [9] results:

Theorem 1.9 ([1] Theorem 6.1). Assume that (M^n, g) is a closed Riemannian manifold of dimension $n \ge 2$, $\alpha > 1$, $\lambda > 0$, and u is a positive solution of

$$\Delta u - \lambda u + u^{\alpha} = 0.$$

Assume also that the spectrum $\sigma(R(x))$ of the Ricci tensor R of the metric g satisfies

$$\inf_{x \in M} \min \sigma(R(x)) \ge \frac{n-1}{n} (\alpha - 1)\lambda, \quad \alpha \le \frac{n+2}{n-2}.$$

Moreover, assume that one of the two inequalities is strict if $(M^n, g) = (\mathbb{S}^n, g_c)$ is the standard sphere, then $u \equiv \lambda^{\frac{1}{\alpha-1}}$.

The direct extension of Jerison-Lee identity fails in the process of solving the subcritical exponent case in CR geometry, because of the concurrence of curvature, torsion, and the second layer of the CR manifold as a Carnot group. Hence, dimensional conservation and invariant tensors are introduced in Section 2. Then, we get some new differential identities in Section 3 to prove Theorem 1.6. Besides, the new method can explain the existence of such a three-dimensional family in the critical exponent case for the Heisenberg group. We discuss it in detail in Section 4 and prove Theorem 1.3, which answers the question of the theoretical framework for finding differential identities raised by Jerison-Lee [11].

2. Preparations: dimensional conservation and invariant tensors

In this section, **dimensional conservation** and **invariant tensors** are introduced for preparing useful differential identities. Target identities are composed of divergence of some vector fields and summation of positive terms which contain the complete square of some tensors, then all tensors in complete square terms are zero by divergence theorem. Thus, how to find those tensors priorly is essential.

We say a tensor S(u) is of $\{(r, s), x, y, +/-\}$ type, if it's linearly composed of some $T^{(r,s)}$ tensors with x-degree u, y-order derivatives, and the number of $\sqrt{-1}$ plus the number of vector field T is even/odd for every tensors. For example:

$$\{(2,0), 1, 2, +\}: D_{ij} = u_{ij} + c_1 \frac{u_i u_j}{u};$$

$$\{(1,1),1,2,+\}: E_{i\bar{j}} = u_{i\bar{j}} + c_2 \frac{u_i u_{\bar{j}}}{u} + c_3 \Delta u h_{i\bar{j}} + c_4 n \sqrt{-1} u_0 h_{i\bar{j}} + c_5 \frac{|\nabla u|^2}{u} h_{i\bar{j}} + c_6 \lambda u h_{i\bar{j}}, h_{i\bar{j}} + c_6 \lambda u h_{i\bar{j}} + c_6 \lambda u h_{i\bar{j}}, h_{$$

where $\{c_l\}_{l=1}^6$ are constants. It's noteworthy that u_0 is of $\{(0,0), 1, 2, -\}$ type, and λ is of $\{(0,0), 0, 2, +\}$ type. The type of tensors is additive when several types of tensors are multiplied together. The type of tensors must be conserved in differential identities. We call this phenomenon as **dimensional conservation**.

Recall the Riemannian case. From Obata [13], Gidas-Spurck [13], M-F. B. Véron and L. Véron [1], and especially Dolbeault-Esteban-Loss [4], we know that differential identities are found by multiplying Δu in both sides of the equation, and using divergence theorem. Namely,

$$(\Delta u)^{2} = (\Delta u u_{i})_{,}^{i} - (\Delta u)_{,}^{i} u_{i} = -(u^{ji} u_{i})_{,j} + \sum_{i,j=1}^{n} |u_{ij}|^{2} + (\Delta u u_{i})_{,}^{i},$$

then u_{ij} becomes the main term of some target tensor hoped to be zero. Similar as Riemannian case, by multiplying equation (1.2) with Δu and divergence theorem,

$$(\Delta u)^{2} = (\Delta u u_{i})_{,}^{i} - (\Delta u)_{,}^{i} u_{i} - n\sqrt{-1}u_{0}\Delta u$$

= $-(u^{j}_{j} + n\sqrt{-1}u_{0})_{,}^{i} u_{i} + (\Delta u u_{i})_{,}^{i} - n\sqrt{-1}u_{0}\Delta u$
= $-(u^{ij}u_{i})_{,j} + \sum_{i,j=1}^{n} |u_{ij}|^{2} - (n+2)\sqrt{-1}u_{0}^{i}u_{i} + (\Delta u u_{i})_{,}^{i} - n\sqrt{-1}u_{0}\Delta u$,

then we can yield $\sum_{i,j=1}^{n} |u_{ij}|^2$ term, hence consider u_{ij} as the main term of one of the target

tensors. By dimensional conservation, we need a $\{(2,0), 1, 2, +\}$ type tensor, hence consider D_{ij} defined as above.

Similarly, use the divergence theorem in another way:

$$(\Delta u)^{2} = (\Delta u u_{i})^{i}_{,i} - (\Delta u)^{i}_{,i} u_{i} - n\sqrt{-1}u_{0}\Delta u$$

$$= - (u_{j}^{j} - n\sqrt{-1}u_{0})^{i}_{,i} u_{i} + (\Delta u u_{i})^{i}_{,i} - n\sqrt{-1}u_{0}\Delta u$$

$$= - (u_{j}^{i}u_{i})^{j}_{,i} + u_{j}^{i}u_{i}^{j}_{,i} + n\sqrt{-1}u_{0}^{i}u_{i} + (\Delta u u_{i})^{i}_{,i} - n\sqrt{-1}u_{0}\Delta u$$

$$= - (u_{j}^{i}u_{i})^{j}_{,i} + \sum_{i,j=1}^{n} |u_{i\bar{j}}|^{2} + 2\sqrt{-1}u_{0}u_{i}^{i} + n\sqrt{-1}u_{0}^{i}u_{i} + (\Delta u u_{i})^{i}_{,i} - n\sqrt{-1}u_{0}\Delta u,$$

then $|u_{i\bar{j}}|^2$ term can be attained, hence consider a $\{(1,1), 1, 2, +\}$ type tensor $E_{i\bar{j}}$ defined as above. For $\sum_{i,j=1}^{n} |D_{ij}|^2$ and $\sum_{i,j=1}^{n} |E_{i\bar{j}}|^2$, $\{(0,0), 2, 4, +\}$ type identity is enough, such as Riemannian case. However, a $\{(1,0), 1, 3, -\}$ type tensor G_i occurs by the following invariant tensors argument because of non-commutativity of Z_i and $Z_{\bar{i}}$ caused by the second layer of CR manifold. At last, we need a $\{(0,0), 2, 6, +\}$ type identity to deal with $\sum_{i=1}^{n} |G_i|^2$ term, which is just the key identity (2.10) in this paper. The identity (2.10) has the same dimensional as Jerison-Lee's identities in [11].

Now, we hope that D_{ij} and $E_{i\bar{j}}$ are zero for some α and λ . By $E_i^{\ i} = 0$, we yield

$$c_3 = -\frac{1}{n}, \quad c_4 = -\frac{1}{n}, \quad c_5 = -\frac{1}{n}c_2, \quad c_6 = 0,$$

then
$$E_{i\overline{j}} = u_{i\overline{j}} + c_2 \frac{u_i u_{\overline{j}}}{u} - \frac{1}{n} \left(\Delta u + n\sqrt{-1}u_0 + c_2 \frac{|\nabla u|^2}{u} \right) h_{i\overline{j}}.$$

Set
$$D_{i} = \frac{D_{ij}u^{j}}{u}, E_{i} = \frac{E_{i\bar{j}}u^{\bar{j}}}{u}$$
. By direct computation and using equation (1.2):
 $D_{ij,}{}^{i} = u_{ij}{}^{i} + c_{1}\frac{u_{j}{}^{i}u_{i}}{u} + c_{1}\frac{u_{j}(\Delta u + n\sqrt{-1}u_{0})}{u} - c_{1}\frac{|\nabla u|^{2}}{u^{2}}u_{j}$

$$= (\Delta u + n\sqrt{-1}u_{0})_{j} + 2\sqrt{-1}u_{0j} + R_{j\bar{i}}u^{\bar{i}} + c_{1}\left[E_{j\bar{i}} - c_{2}\frac{u_{j}u_{\bar{i}}}{u} + \frac{1}{n}\left(\Delta u + n\sqrt{-1}u_{0} + c_{2}\frac{|\nabla u|^{2}}{u}\right)h_{j\bar{i}}\right]\frac{u^{\bar{i}}}{u} + c_{1}\frac{u_{j}(\Delta u + n\sqrt{-1}u_{0})}{u} - c_{1}\frac{|\nabla u|^{2}}{u^{2}}u_{j}$$

$$= c_{1}E_{j} + (n+2)\sqrt{-1}u_{0j} + (n+1)c_{1}\frac{\sqrt{-1}u_{0}u_{j}}{u} + (\frac{n+1}{n}c_{1}+\alpha)\frac{\Delta u}{u}u_{j}$$

$$- (\frac{n-1}{n}c_{2} + 1)c_{1}\frac{|\nabla u|^{2}}{u^{2}}u_{j} + R_{j\bar{i}}u^{\bar{i}} + (1-\alpha)\lambda u_{j},$$
(2.1)

and

$$\begin{split} E_{i\overline{j},}{}^{i} = & u_{i\overline{j}}{}^{i} + c_{2}\frac{u_{\overline{j}}{}^{i}u_{i}}{u} + c_{2}\frac{u_{\overline{j}}(\Delta u + n\sqrt{-1}u_{0})}{u} - c_{2}\frac{|\nabla u|^{2}}{u^{2}}u_{\overline{j}} \\ & - \frac{(\Delta u + n\sqrt{-1}u_{0})_{\overline{j}}}{n} - \frac{c_{2}}{n}\frac{u_{\overline{i}\overline{j}}u^{\overline{i}}}{u} - \frac{c_{2}}{n}\frac{u_{i\overline{j}}u^{\overline{i}}}{u} + \frac{c_{2}}{n}\frac{|\nabla u|^{2}}{u^{2}}u_{\overline{j}} \\ = & (\Delta u + n\sqrt{-1}u_{0})_{\overline{j}} + \frac{n-1}{n}c_{2}\frac{u_{\overline{i}\overline{j}}u^{\overline{i}}}{u} + c_{2}\frac{u_{\overline{j}}(\Delta u + n\sqrt{-1}u_{0})}{u} \\ & - \frac{c_{2}}{n}\frac{u_{i\overline{j}}u^{\overline{i}}}{u} - \frac{(\Delta u + n\sqrt{-1}u_{0})_{\overline{j}}}{n} - \frac{n-1}{n}c_{2}\frac{|\nabla u|^{2}}{u^{2}}u_{\overline{j}} \\ = & - \frac{c_{2}}{n}\left[E_{i\overline{j}} - c_{2}\frac{u_{i}u_{\overline{j}}}{u} + \frac{1}{n}\left(\Delta u + n\sqrt{-1}u_{0} + c_{2}\frac{|\nabla u|^{2}}{u}\right)h_{i\overline{j}}\right]\frac{u^{\overline{i}}}{u} \\ & + \frac{n-1}{n}c_{2}\left(D_{\overline{i}\overline{j}} - c_{1}\frac{u_{\overline{i}}u_{\overline{j}}}{u}\right)\frac{u^{\overline{i}}}{u} + (n-1)\sqrt{-1}u_{0\overline{j}} + nc_{2}\frac{\sqrt{-1}u_{0}u_{\overline{j}}}{u} \\ & + (c_{2} + \frac{n-1}{n}\alpha)\frac{\Delta u}{u}u_{\overline{j}} - \frac{n-1}{n}c_{2}\frac{|\nabla u|^{2}}{u^{2}}u_{\overline{j}} + \frac{n-1}{n}(1-\alpha)\lambda u_{\overline{j}} \\ = & \frac{n-1}{n}c_{2}D_{\overline{j}} - \frac{c_{2}}{n}E_{\overline{j}} + (n-1)\sqrt{-1}u_{0\overline{j}} + \frac{n^{2}-1}{n}c_{2}\frac{\sqrt{-1}u_{0}u_{\overline{j}}}{u} + \frac{n-1}{n}\times \\ & (\frac{n+1}{n}c_{2} + \alpha)\frac{\Delta u}{u}u_{\overline{j}} - \frac{n-1}{n}(c_{1} - \frac{c_{2}}{n} + 1)c_{2}\frac{|\nabla u|^{2}}{u^{2}}u_{\overline{j}} + \frac{n-1}{n}(1-\alpha)\lambda u_{\overline{j}}. \end{split}$$

If D_{ij} and $E_{i\overline{j}}$ are 0, then D_{ij} , i and $E_{i\overline{j}}$, i are also 0, hence

$$0 = (n+2)\sqrt{-1}u_{0j} + (n+1)c_1\frac{\sqrt{-1}u_0u_j}{u} + (\frac{n+1}{n}c_1 + \alpha)\frac{\Delta u}{u}u_j - (\frac{n-1}{n}c_2 + 1)c_1\frac{|\nabla u|^2}{u^2}u_j + R_{j\bar{i}}u^{\bar{i}} + (1-\alpha)\lambda u_j,$$
(2.3)

and

$$0 = (n-1)\sqrt{-1}u_{0\overline{j}} + \frac{n^2 - 1}{n}c_2\frac{\sqrt{-1}u_0u_{\overline{j}}}{u} + \frac{n-1}{n}(\frac{n+1}{n}c_2 + \alpha)\frac{\Delta u}{u}u_{\overline{j}} - \frac{n-1}{n}(c_1 - \frac{c_2}{n} + 1)c_2\frac{|\nabla u|^2}{u^2}u_{\overline{j}} + \frac{n-1}{n}(1-\alpha)\lambda u_{\overline{j}}.$$
(2.4)

Let the coefficients of $\sqrt{-1}u_{0j}$, $\frac{\sqrt{-1}u_0u_j}{u}$, $\frac{\Delta u}{u}u_j$ and $\frac{|\nabla u|^2}{u^2}u_j$ in (2.3) and (2.4) are proportional:

$$\frac{n+2}{-(n-1)} = \frac{(n+1)c_1}{-\frac{n^2-1}{n}c_2} = \frac{\frac{n+1}{n}c_1 + \alpha}{\frac{n-1}{n}(\frac{n+1}{n}c_2 + \alpha)} = \frac{-(\frac{n-1}{n}c_2 + 1)c_1}{-\frac{n-1}{n}(c_1 - \frac{c_2}{n} + 1)c_2},$$

then $c_1 = c_2 = \alpha = 0$ or $c_1 = -\frac{n+2}{n}$, $c_2 = -1$, $\alpha = \frac{n+2}{n}$. Hence the critical exponent $\alpha = \frac{n+2}{n}$ can be determined with this method. $c_1 = -\frac{n+2}{n}$ and $c_2 = -1$ in the critical exponent case are essential when we answer the question raised by Jerison-Lee and prove the Theorem 1.3 in Section 4.

In the following, we concentrate on the subcritical exponent case $1 < \alpha < \frac{n+2}{n}$. By the rigidity theorem in Riemannian case, such as [4], the terms with $|\nabla u|^4$ are needed in the proof of Theorem 1.6.

Hence, only let the coefficients of $\sqrt{-1}u_{0i}$, $\frac{\sqrt{-1}u_0u_i}{u}$ and $\frac{\Delta u}{u}u_i$ in (2.3) and (2.4) are proportional:

$$\frac{n+2}{-(n-1)} = \frac{(n+1)c_1}{-\frac{n^2-1}{n}c_2} = \frac{\frac{n+1}{n}c_1 + \alpha}{\frac{n-1}{n}(\frac{n+1}{n}c_2 + \alpha)},$$

then $c_1 = -\alpha$, $c_2 = -\frac{n\alpha}{n+2}$. Rewrite D_{ij} , $E_{i\overline{j}}$, and define a $\{(1,0), 1, 3, +\}$ type tensor G_i :

$$D_{ij} = u_{ij} - \alpha \frac{u_i u_j}{u},$$

$$E_{i\overline{j}} = u_{i\overline{j}} - \frac{n\alpha}{n+2} \frac{u_i u_{\overline{j}}}{u} - \frac{1}{n} \left(\Delta u + n\sqrt{-1}u_0 - \frac{n\alpha}{n+2} \frac{|\nabla u|^2}{u} \right) h_{i\overline{j}}.$$

$$G_{i} = n\sqrt{-1}u_{0i} - \frac{n(n+1)}{n+2}\alpha \frac{\sqrt{-1}u_{0}u_{i}}{u} - \frac{\alpha}{n+2}\frac{\Delta u}{u}u_{i} + \frac{n\alpha}{n+2}\left(\frac{n+1}{n+2}\alpha - 1\right)\frac{|\nabla u|^{2}}{u^{2}}u_{i} + (\alpha - 1)\lambda u_{i}.$$

Rewrite D_{ij} , ^{*i*} and $E_{i\overline{j}}$, ^{*i*} in (2.1) and (2.2):

$$D_{ij,\,\,^{i}} = -\alpha E_j + \frac{n+2}{n}G_j + 2\alpha(1-\frac{n\alpha}{n+2})\frac{|\nabla u|^2}{u^2}u_j + R_{j\bar{i}}u^{\bar{i}} - \frac{2(n+1)}{n}(\alpha-1)\lambda u_j, \quad (2.5)$$

$$E_{i\bar{j},\ }^{\ i} = -\frac{n-1}{n+2}\alpha D_{\bar{j}} + \frac{\alpha}{n+2}E_{\bar{j}} - \frac{n-1}{n}G_{\bar{j}}.$$
(2.6)

Now, the covariant derivatives of D_{ij} and $E_{i\bar{j}}$ and G_i are also composed of D_{ij} , $E_{i\bar{j}}$, G_i and $\frac{|\nabla u|^2}{u^2}u_j$ terms in some suitable curvature condition. The $\frac{|\nabla u|^2}{u^2}u_j$ term is vanishing in the critical exponent case. The invariance of D_{ij} , $E_{i\bar{j}}$, and G_i in differentiating process are reasonable since those tensors are hoped to be zero. Hence, we call D_{ij} , $E_{i\bar{j}}$ and G_i as **invariant tensors**. With the invariance arguments above, invariant tensors can be deduced without any geometric background.

Some notations are needed:

$$E_{\bar{i}j} = \overline{E_{i\bar{j}}}, \quad L_{i\bar{j}} = \frac{u_i u_{\bar{j}}}{u} - \frac{1}{n} \frac{|\nabla u|^2}{u} h_{i\bar{j}}, \quad \mathscr{R} = R_{i\bar{j}} u^i u^{\bar{j}} - \frac{2(n+1)}{n} (\alpha - 1) \lambda |\nabla u|^2.$$

For convenience, the following four lemmas are needed.

Lemma 2.1.

$$\begin{array}{l} (1) \ E_{i}{}^{i} := E_{i\bar{j}}h^{i\bar{j}} = 0, \ E_{i\bar{j}} = E_{\bar{j}i}, \ E_{i}u^{i} \in \mathbb{R}; \\ (2) \ L_{i}{}^{i} = 0, \ E_{i}u^{i} = E_{i\bar{j}}L^{i\bar{j}}, \ \sum_{i,j=1}^{n} |L_{i\bar{j}}|^{2} = \frac{n-1}{n} \frac{|\nabla u|^{4}}{u^{2}}; \\ (3) \ Assume \ that \ (1.1) \ and \ \lambda \leqslant \frac{n}{2(\alpha - 1)} \ hold, \ then \ \mathscr{R} \geqslant 0. \\ Proof. \ E_{i}{}^{i} = u_{i}{}^{i} - \frac{n\alpha}{n+2} \frac{|\nabla u|^{2}}{u} - \left(\Delta u + n\sqrt{-1}u_{0} - \frac{n\alpha}{n+2} \frac{|\nabla u|^{2}}{u}\right) = 0, \ then \ L_{i}{}^{i} = 0 \ is \\ proved \ similarly. \ Hence, \ E_{i\bar{j}}L^{i\bar{j}} = E_{i\bar{j}} \cdot \frac{u^{i}u^{\bar{j}}}{u} = E_{i}u^{i}, \ \sum_{i,j=1}^{n} |L_{i\bar{j}}|^{2} = L_{i\bar{j}} \cdot \frac{u^{i}u^{\bar{j}}}{u} = \frac{n-1}{n} \frac{|\nabla u|^{4}}{u^{2}}. \end{array}$$

By $u_{i\overline{j}} - u_{\overline{j}i} = 2\sqrt{-1}h_{i\overline{j}}u_0$, we yield that

$$E_{i\overline{j}} = u_{i\overline{j}} - \frac{n\alpha}{n+2} \frac{u_i u_{\overline{j}}}{u} - \frac{1}{n} \left(\Delta u + n\sqrt{-1}u_0 - \frac{n\alpha}{n+2} \frac{|\nabla u|^2}{u} \right) h_{i\overline{j}}$$
$$= u_{\overline{j}i} - \frac{n\alpha}{n+2} \frac{u_{\overline{j}}u_i}{u} - \frac{1}{n} \left(\Delta u - n\sqrt{-1}u_0 - \frac{n\alpha}{n+2} \frac{|\nabla u|^2}{u} \right) h_{\overline{j}i} = E_{\overline{j}i}$$

then $E_i u^i = \frac{E_{i\overline{j}} u^i u^{\overline{j}}}{u} = \frac{E_{\overline{j}i} u^i u^{\overline{j}}}{u} = E_{\overline{j}} u^{\overline{j}}$, i.e. $E_i u^i \in \mathbb{R}$. If $\operatorname{Ric}(Z, Z) \ge (n+1)\langle Z, Z \rangle_{L_0}$ and $\lambda \le \frac{n}{2}$.

$$Z, Z \rangle \ge (n+1)\langle Z, Z \rangle_{L_{\theta}} \text{ and } \lambda \leqslant \frac{n}{2(\alpha-1)},$$

 $\mathscr{R} \ge (n+1)|\nabla u|^2 - \frac{2(n+1)}{n}(\alpha-1)\lambda|\nabla u|^2 \ge 0.$

Lemma 2.2. $\frac{|\nabla u|^2}{u^2} \sum_{i,j} |D_{ij}|^2 \ge \sum_i |D_i|^2, \ \frac{|\nabla u|^2}{u^2} \sum_{i,j} |E_{i\overline{j}}|^2 \ge \frac{n}{n-1} \sum_i |E_i|^2 \ if \ n \ge 2.$

Proof. Assume that $A \in \mathbb{C}^{n \times n}$ is Hermitian, $\mu \in \mathbb{C}^{n \times 1}$. By Cauchy inequality,

$$\sum_{j=1}^{n} |A_{ij}\mu_j|^2 \leqslant \sum_{j=1}^{n} |A_{ij}|^2 ||\mu||^2.$$

Sum *i* from 1 to *n*: $\sum_{i,j=1}^{n} |A_{ij}\mu_j|^2 \leq \sum_{i,j=1}^{n} |A_{ij}|^2 ||\mu||^2$. Then $u^2 \sum_i |D_i|^2 \leq |\nabla u|^2 \sum_{i,j} |D_{ij}|^2$. For $n \geq 2$, assume that tr A = 0 additionally. Without loss of generality, assume that $A_{ij} = 0$ if $i \neq j$ and $i, j \geq 2$, $\mu = (1, 0, \dots, 0)^T$, then

$$\sum_{i,j=1}^{n} |A_{ij}|^2 \|\mu\|^2 - \frac{n}{n-1} \sum_{i,j=1}^{n} |A_{ij}\mu_j|^2$$

$$= \sum_{i=1}^{n} |A_{ii}|^2 + 2 \sum_{i=2}^{n} |A_{i1}|^2 - \frac{n}{n-1} |A_{11}|^2 - \frac{n}{n-1} \sum_{i=2}^{n} |A_{i1}|^2$$

$$\geqslant \sum_{i=2}^{n} |A_{ii}|^2 - \frac{1}{n-1} |A_{11}|^2$$

$$\xrightarrow{\operatorname{tr} A=0} \frac{1}{n-1} \sum_{2 \leq i < j \leq n} |A_{ii} - A_{jj}|^2 \ge 0.$$

Hence $|\nabla u|^2 \sum_{i,j} |E_{i\overline{j}}|^2 \ge \frac{n}{n-1} u^2 \sum_i |E_i|^2$ for $n \ge 2$.

Lemma 2.3.

$$(\Delta u)_{,i} = \lambda u_i - \alpha u^{\alpha - 1} u_i = \alpha \frac{\Delta u}{u} u_i + (1 - \alpha) \lambda u_i,$$

$$(|\nabla u|^2)_{,\overline{i}} = u D_{\overline{i}} + u E_{\overline{i}} + \frac{2n + 1}{n + 2} \alpha \frac{|\nabla u|^2}{u} u_{\overline{i}} + \frac{1}{n} \Delta u u_{\overline{i}} + \sqrt{-1} u_0 u_{\overline{i}},$$

$$n \sqrt{-1} u_{0\overline{i}} = -G_{\overline{i}} + \frac{n(n+1)}{n+2} \alpha \frac{\sqrt{-1} u_0 u_{\overline{i}}}{u} - \frac{\alpha}{n+2} \frac{\Delta u}{u} u_{\overline{i}}$$

$$+ \frac{n\alpha}{n+2} \left(\frac{n+1}{n+2}\alpha - 1\right) \frac{|\nabla u|^2}{u^2} u_{\overline{i}} + (\alpha - 1) \lambda u_{\overline{i}}.$$

Proof. They can be checked by equation (1.2) and definitions of D_{ij} , $E_{i\bar{j}}$ and G_i easily. Lemma 2.4.

$$D_{i,i}^{i} = u^{-1} \sum_{i,j=1}^{n} |D_{ij}|^{2} + (\alpha - 1) \frac{D_{i}u^{i}}{u} - \alpha \frac{E_{i}u^{i}}{u} + \frac{n+2}{n} \frac{G_{i}u^{i}}{u} + 2\alpha(1 - \frac{n\alpha}{n+2}) \frac{|\nabla u|^{4}}{u^{3}} + u^{-1}\mathscr{R},$$
(2.7)

$$E_{i,\ i} = u^{-1} \sum_{i,j=1}^{n} |E_{i\overline{j}}|^2 - \frac{n-1}{n+2} \alpha \frac{D_{\overline{i}} u^{\overline{i}}}{u} + \left(\frac{n+1}{n+2} \alpha - 1\right) \frac{E_{\overline{i}} u^{\overline{i}}}{u} - \frac{n-1}{n} \frac{G_{\overline{i}} u^{\overline{i}}}{u}, \tag{2.8}$$

$$\operatorname{Im} G_{i,\,i} = \operatorname{Im} \left[\frac{n\alpha}{n+2} \left(\frac{n+1}{n+2} \alpha - 1 \right) \frac{D_{\overline{i}} u^{\overline{i}}}{u} + \frac{n+1}{n+2} \alpha \frac{G_{\overline{i}} u^{\overline{i}}}{u} \right].$$
(2.9)

Proof. (2.7) and (2.8) can be checked directly by (2.5) and (2.6). By Lemma 2.3,

$$\begin{split} &\operatorname{Im} G_{i,}{}^{i} \\ =& n\operatorname{Im} \sqrt{-1} (\Delta u)_{,0} - \frac{n(n+1)}{n+2} \alpha \left(\operatorname{Im} \frac{\sqrt{-1}u_{0\bar{i}}u^{\bar{i}}}{u} + \frac{u_{0}\Delta u}{u} - \frac{u_{0}|\nabla u|^{2}}{u^{2}} \right) \\ &- \frac{n\alpha}{n+2} \frac{u_{0}\Delta u}{u} + \frac{n\alpha}{n+2} \left(\frac{n+1}{n+2}\alpha - 1 \right) \left(\operatorname{Im} \frac{D_{\bar{i}}u^{\bar{i}}}{u} + (n+1)\frac{u_{0}|\nabla u|^{2}}{u^{2}} \right) + n(\alpha-1)\lambda u_{0} \\ &= \frac{n\alpha}{n+2} \left(\frac{n+1}{n+2}\alpha - 1 \right) \operatorname{Im} \frac{D_{\bar{i}}u^{\bar{i}}}{u} - \frac{n(n+1)}{n+2}\alpha \operatorname{Im} \frac{\sqrt{-1}u_{0\bar{i}}u^{\bar{i}}}{u} + \frac{n(n+1)^{2}}{(n+2)^{2}}\alpha^{2}\frac{u_{0}|\nabla u|^{2}}{u^{2}} \\ &= \operatorname{Im} \left[\frac{n\alpha}{n+2} \left(\frac{n+1}{n+2}\alpha - 1 \right) \frac{D_{\bar{i}}u^{\bar{i}}}{u} + \frac{n+1}{n+2}\alpha \frac{G_{\bar{i}}u^{\bar{i}}}{u} \right]. \end{split}$$

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By invariance argument above, we need an identity including $\sum_{i,j} |D_{ij}|^2$, $\sum_{i,j} |E_{i\bar{j}}|^2$ and $\sum_i |G_i|^2$. Because of $\sum_i |G_i|^2$, $\{(0,0), 2, 4, +\}$ type identity is not enough. Hence consider the following $\{(0,0), 2, 6, +\}$ type identity, which is crucial for the proof of Theorem 1.6. **Proposition 2.5.** Let $\{d_l\}_{l=1}^4$, $\{e_l\}_{l=1}^4$, μ and β be undetermined constants, then

$$u^{-\beta} \operatorname{Re} \left\{ u^{\beta} \left[\left(d_{1} \frac{|\nabla u|^{2}}{u} + d_{2}u^{\alpha} + d_{3}\lambda u + d_{4}n\sqrt{-1}u_{0} \right) D_{i} + \left(e_{1} \frac{|\nabla u|^{2}}{u} + e_{2}u^{\alpha} + e_{3}\lambda u + e_{4}n\sqrt{-1}u_{0} \right) E_{i} - \mu n\sqrt{-1}u_{0}G_{i} \right] \right\}^{i}_{,i} \\ = \left[d_{1} \frac{|\nabla u|^{2}}{u^{2}} + d_{2}u^{\alpha-1} + d_{3}\lambda \right] \left[\sum_{i,j} |D_{ij}|^{2} + 2\alpha(1 - \frac{n\alpha}{n+2}) \frac{|\nabla u|^{4}}{u^{2}} + \mathscr{R} \right] \\ + \left[e_{1} \frac{|\nabla u|^{2}}{u^{2}} + e_{2}u^{\alpha-1} + e_{3}\lambda \right] \sum_{i,j} |E_{i\bar{j}}|^{2} + d_{1}\sum_{i} |D_{i}|^{2} + e_{1}\sum_{i} |E_{i}|^{2} \\ + \mu \sum_{i} |G_{i}|^{2} + (d_{1} + e_{1}) \operatorname{Re} D_{i}E^{i} - d_{4} \operatorname{Re} D_{i}G^{i} - e_{4} \operatorname{Re} E_{i}G^{i} \\ + \operatorname{Re} \left[\Delta_{1} \frac{|\nabla u|^{2}}{u^{2}} + \Delta_{2}u^{\alpha-1} + \Delta_{3}\lambda + \Delta_{4} \frac{n\sqrt{-1}u_{0}}{u} \right] D_{i}u^{i} \\ + \left[\Theta_{1} \frac{|\nabla u|^{2}}{u^{2}} + \Theta_{2}u^{\alpha-1} + \Theta_{3}\lambda \right] E_{i}u^{i} \\ + \operatorname{Re} \left[\Xi_{1} \frac{|\nabla u|^{2}}{u^{2}} + \Xi_{2}u^{\alpha-1} + \Xi_{3}\lambda + \Xi_{4} \frac{n\sqrt{-1}u_{0}}{u} \right] G_{i}u^{i}. \end{cases}$$

$$(2.10)$$

The coefficients are:

$$\begin{split} \Delta_1 &= \left(\beta + \frac{3(n+1)}{n+2}\alpha - 2\right) d_1 - \frac{n-1}{n+2}\alpha e_1 + \frac{n\alpha}{n+2} (\frac{n+1}{n+2}\alpha - 1) d_4, \\ \Delta_2 &= -\frac{1}{n} d_1 + (\beta + 2\alpha - 1) d_2 - \frac{n-1}{n+2}\alpha e_2 + \frac{\alpha}{n+2} d_4, \\ \Delta_3 &= \frac{1}{n} d_1 + (\beta + \alpha) d_3 - \frac{n-1}{n+2}\alpha e_3 + (\frac{n+1}{n+2}\alpha - 1) d_4, \\ \Delta_4 &= \frac{1}{n} d_1 + \left(\beta + \frac{2n+3}{n+2}\alpha - 1\right) d_4 + \frac{n-1}{n+2}\alpha e_4 + \frac{n\alpha}{n+2} (\frac{n+1}{n+2}\alpha - 1) \mu, \\ \Theta_1 &= -\alpha d_1 + \left(\beta + \frac{3n+2}{n+2}\alpha - 2\right) e_1 + \frac{n\alpha}{n+2} (\frac{n+1}{n+2}\alpha - 1) e_4, \\ \Theta_2 &= -\frac{1}{n} e_1 - \alpha d_2 + \left(\beta + \frac{2n+3}{n+2}\alpha - 1\right) e_2 + \frac{\alpha}{n+2} e_4, \\ \Theta_3 &= \frac{1}{n} e_1 - \alpha d_3 + \left(\beta + \frac{n+1}{n+2}\alpha\right) e_3 + (\frac{n+1}{n+2}\alpha - 1) e_4, \\ \Xi_1 &= \frac{n+2}{n} d_1 - \frac{n-1}{n} e_1 - \frac{n\alpha}{n+2} (\frac{n+1}{n+2}\alpha - 1) \mu, \\ \Xi_2 &= \frac{n+2}{n} d_2 - \frac{n-1}{n} e_2 - \frac{\alpha}{n+2} \mu, \end{split}$$

$$\Xi_{3} = \frac{n+2}{n}d_{3} - \frac{n-1}{n}e_{3} - (\frac{n+1}{n+2}\alpha - 1)\mu,$$

$$\Xi_{4} = \frac{n+2}{n}d_{4} + \frac{n-1}{n}e_{4} - \beta\mu.$$

$$\begin{split} Proof. \mbox{ By Lemma 2.3 and Lemma 2.4, we yield the following } \{(0,0),2,6,+\} \mbox{ type identity:} \\ & u^{-\beta} \operatorname{Re}(u^{\beta-1}|\nabla u|^2 D_i),^i \\ & = \frac{|\nabla u|^2}{u^2} \sum_{i,j} \left[|D_{ij}|^2 + 2\alpha(1 - \frac{n\alpha}{n+2}) \frac{|\nabla u|^4}{u^2} + \mathscr{R} \right] + \sum_i |D_i|^2 + \operatorname{Re} D_i E^i \\ & + \operatorname{Re} \left[\left(\beta + \frac{3(n+1)}{n+2} \alpha - 2 \right) \frac{|\nabla u|^2}{u^2} - \frac{1}{n} u^{\alpha-1} + \frac{1}{n} \lambda + \frac{1}{n} \frac{n\sqrt{-1}u_0}{u} \right] D_i u^i \\ & - \alpha \frac{|\nabla u|^2}{u^2} E_i u^i + \frac{n+2}{n} \frac{|\nabla u|^2}{u^2} \operatorname{Re} G_i u^i, \\ & u^{-\beta} \operatorname{Re}(u^{\beta+\alpha} D_i),^i \\ & = u^{\alpha-1} \sum_{i,j} \left[|D_{ij}|^2 + 2\alpha(1 - \frac{n\alpha}{n+2}) \frac{|\nabla u|^4}{u^2} + \mathscr{R} \right] \\ & + (\beta + 2\alpha - 1)u^{\alpha-1} \operatorname{Re} D_i u^i - \alpha u^{\alpha-1} E_i u^i + \frac{n+2}{n} u^{\alpha-1} \operatorname{Re} G_i u^i, \\ & u^{-\beta} \operatorname{Re}(u^{\beta+1} \cdot \lambda D_i),^i \\ & = \lambda \sum_{i,j} \left[|D_{ij}|^2 + 2\alpha(1 - \frac{n\alpha}{n+2}) \frac{|\nabla u|^4}{u^2} + \mathscr{R} \right] \\ & + (\beta + \alpha)\lambda \operatorname{Re} D_i u^i - \alpha \lambda E_i u^i + \frac{n+2}{n} \lambda \operatorname{Re} G_i u^i, \end{split}$$

$$\begin{split} u^{-\beta} \operatorname{Re}(u^{\beta} \cdot n\sqrt{-1}u_{0}D_{i})_{,}^{i} \\ &= -\operatorname{Re} D_{i}G^{i} + \left(\beta + \frac{2n+3}{n+2}\alpha - 1\right)\operatorname{Re} \frac{n\sqrt{-1}u_{0}}{u}D_{i}u^{i} + \frac{n+2}{n}\operatorname{Re} \frac{n\sqrt{-1}u_{0}}{u}G_{i}u^{i} \\ &+ \left[\frac{n\alpha}{n+2}\left(\frac{n+1}{n+2}\alpha - 1\right)\frac{|\nabla u|^{2}}{u^{2}} + \frac{\alpha}{n+2}u^{\alpha-1} + \left(\frac{n+1}{n+2}\alpha - 1\right)\lambda\right]\operatorname{Re} D_{i}u^{i}, \\ u^{-\beta}\operatorname{Re}(u^{\beta-1}|\nabla u|^{2}E_{i})_{,}^{i} \\ &= \frac{|\nabla u|^{2}}{u^{2}}\sum_{i,j}|E_{i\overline{j}}|^{2} + \sum_{i}|E_{i}|^{2} + \operatorname{Re} D_{i}E^{i} - \frac{n-1}{n+2}\alpha\frac{|\nabla u|^{2}}{u^{2}}\operatorname{Re} D_{i}u^{i} \\ &+ \left[\left(\beta + \frac{3n+2}{n+2}\alpha - 2\right)\frac{|\nabla u|^{2}}{u^{2}} - \frac{1}{n}u^{\alpha-1} + \frac{1}{n}\lambda\right]E_{i}u^{i} - \frac{n-1}{n}\frac{|\nabla u|^{2}}{u^{2}}\operatorname{Re} G_{i}u^{i}, \\ &u^{-\beta}\operatorname{Re}(u^{\beta+\alpha}E_{i})_{,}^{i} \end{split}$$

$$= u^{\alpha - 1} \sum_{i,j} |E_{i\bar{j}}|^2 - \frac{n - 1}{n + 2} \alpha u^{\alpha - 1} \operatorname{Re} D_i u^i + \left(\beta + \frac{2n + 3}{n + 2} \alpha - 1\right) u^{\alpha - 1} E_i u^i - \frac{n - 1}{n} u^{\alpha - 1} \operatorname{Re} G_i u^i,$$

$$u^{-\beta} \operatorname{Re}(u^{\beta+1} \cdot \lambda E_i)_{,i}^{i}$$
$$= \lambda \sum_{i,j} |E_{i\overline{j}}|^2 - \frac{n-1}{n+2} \alpha \lambda \operatorname{Re} D_i u^i + \left(\beta + \frac{n+1}{n+2}\alpha\right) \lambda E_i u^i - \frac{n-1}{n} \lambda \operatorname{Re} G_i u^i.$$

$$u^{-\beta} \operatorname{Re}(u^{\beta} \cdot n\sqrt{-1}u_{0}E_{i}), i$$

$$= -\operatorname{Re}E_{i}G^{i} + \frac{n-1}{n+2}\alpha \operatorname{Re}\frac{n\sqrt{-1}u_{0}}{u}D_{i}u^{i} + \frac{n-1}{n}\operatorname{Re}\frac{n\sqrt{-1}u_{0}}{u}G_{i}u^{i}$$

$$+ \left[\frac{n\alpha}{n+2}\left(\frac{n+1}{n+2}\alpha - 1\right)\frac{|\nabla u|^{2}}{u^{2}} + \frac{\alpha}{n+2}u^{\alpha-1} + \left(\frac{n+1}{n+2}\alpha - 1\right)\lambda\right]E_{i}u^{i},$$

$$u^{-\beta} \operatorname{Re}(-n\sqrt{-1}u^{\beta}u_{0}G_{i})_{,}^{i}$$

$$= \sum_{i} |G_{i}|^{2} + \frac{n\alpha}{n+2} \left(\frac{n+1}{n+2}\alpha - 1\right) \operatorname{Re} \frac{n\sqrt{-1}u_{0}}{u} D_{i}u^{i} - \beta \operatorname{Re} \frac{n\sqrt{-1}u_{0}}{u} G_{i}u^{i}$$

$$- \left[\frac{n\alpha}{n+2} \left(\frac{n+1}{n+2}\alpha - 1\right) \frac{|\nabla u|^{2}}{u^{2}} + \frac{\alpha}{n+2}u^{\alpha-1} + \left(\frac{n+1}{n+2}\alpha - 1\right)\lambda\right] \operatorname{Re} G_{i}u^{i}.$$

Then identity (2.10) can be proved by linearly combining them.

When n = 1, because $E_{1\overline{1}} = 0$, we need some other vector fields, whose divergence is composed of invariant tensors as well. The following identity satisfies our demand.

Proposition 2.6. If n = 1, let β be an undetermined constant, then

$$u^{-\beta} \operatorname{Re} \left\{ u^{\beta} \left[\left(\frac{\alpha}{3} (\frac{1}{2} - \frac{\alpha}{3}) \frac{|\nabla u|^{2}}{u^{2}} - \frac{\alpha}{6} u^{\alpha - 1} + (\frac{1}{2} - \frac{\alpha}{3}) \lambda \right) \frac{|\nabla u|^{2}}{u} + \left(\frac{1}{2} (\beta + \frac{4}{3} \alpha - 1) \frac{|\nabla u|^{2}}{u^{2}} - u^{\alpha - 1} + \lambda - \frac{\sqrt{-1}u_{0}}{u} \right) \sqrt{-1} u_{0} \right] u_{1} \right\}^{-1}_{,}$$

$$= \operatorname{Re} \left[\frac{\alpha}{3} (1 - \frac{2}{3} \alpha) \frac{|\nabla u|^{2}}{u^{2}} - \frac{\alpha}{6} u^{\alpha - 1} + (\frac{1}{2} - \frac{\alpha}{3}) \lambda - \frac{1}{2} (\beta + \frac{4}{3} \alpha - 1) \frac{\sqrt{-1}u_{0}}{u} \right] D_{1} u^{1} \qquad (2.11)$$

$$+ \operatorname{Re} \left[-\frac{1}{2} (\beta + \frac{4}{3} \alpha - 1) \frac{|\nabla u|^{2}}{u^{2}} + u^{\alpha - 1} - \lambda - 2 \frac{\sqrt{-1}u_{0}}{u} \right] G_{1} u^{1}$$

$$- \frac{\alpha}{3} (1 - \frac{\alpha}{3}) (1 - \frac{2}{3} \alpha) \frac{|\nabla u|^{6}}{u^{4}}.$$

Proof. Notice that $E_{1\overline{1}} = 0$ in the case n = 1. By Lemma 2.3, we yield the following $\{(0,0), 2, 6, +\}$ type identities:

$$\begin{split} u^{-\beta} \operatorname{Re}(u^{\beta-3} | \nabla u|^4 u_1), \, ^1 &= 2 \frac{|\nabla u|^2}{u^2} \operatorname{Re} D_1 u^1 + (\beta + 2\alpha - 3) \frac{|\nabla u|^6}{u^4} + 3\Delta u \frac{|\nabla u|^4}{u^3}, \quad (2.12) \\ u^{-\beta} \operatorname{Re}(u^{\beta+\alpha-2} | \nabla u|^2 u_1), \, ^1 &= u^{\alpha-1} \operatorname{Re} D_1 u^1 + (\beta + 2\alpha - 2) u^{\alpha-3} | \nabla u|^4 + 2u^{\alpha-2} \Delta u | \nabla u|^2, \\ u^{-\beta} \operatorname{Re}(u^{\beta-1} \lambda | \nabla u|^2 u_1), \, ^1 &= \lambda \operatorname{Re} D_1 u^1 + (\beta + \alpha - 1) \lambda \frac{|\nabla u|^4}{u^2} + 2\lambda \Delta u \frac{|\nabla u|^2}{u}, \\ u^{-\beta} \operatorname{Re}(u^{\beta-2} | \nabla u|^2 \sqrt{-1} u_0 u_1), \, ^1 \\ &= -\operatorname{Re} \frac{\sqrt{-1} u_0}{u} D_1 u^1 - \frac{|\nabla u|^2}{u^2} \operatorname{Re} G_1 u^1 - \frac{\alpha}{3} (1 - \frac{2}{3} \alpha) \frac{|\nabla u|^6}{u^4} \quad (2.13) \\ &- \frac{\alpha}{3} \Delta u \frac{|\nabla u|^4}{u^3} + (\alpha - 1) \lambda \frac{|\nabla u|^4}{u^2} - 2 \frac{|\nabla u|^2}{u^2} u_0^2, \\ u^{-\beta} \operatorname{Re}(u^{\beta+\alpha-1} \sqrt{-1} u_0 u_1), \, ^1 \\ &= -u^{\alpha-1} \operatorname{Re} G_1 u^1 - \frac{\alpha}{3} (1 - \frac{2}{3} \alpha) u^{\alpha-3} | \nabla u|^4 \\ &- \frac{\alpha}{3} u^{\alpha-2} \Delta u | \nabla u|^2 + (\alpha - 1) \lambda u^{\alpha-1} | \nabla u|^2 - u^{\alpha-1} u_0^2, \\ u^{-\beta} \operatorname{Re}(u^{\beta} \lambda \sqrt{-1} u_0 u_1), \, ^1 \\ &= -\lambda \operatorname{Re} G_1 u^1 - \frac{\alpha}{3} (1 - \frac{2}{3} \alpha) \lambda \frac{|\nabla u|^4}{u^2} - \frac{\alpha}{3} \lambda \Delta u \frac{|\nabla u|^2}{u} + (\alpha - 1) \lambda^2 |\nabla u|^2 - \lambda u_0^2, \end{split}$$

$$u + \operatorname{Re}(u - u_0 u_1),$$

= $-2\operatorname{Re}\frac{\sqrt{-1}u_0}{u}G_1 u^1 + (\beta + \frac{4}{3}\alpha - 1)\frac{|\nabla u|^2}{u^2}u_0^2 + \frac{\Delta u}{u}u_0^2$

Use equation (1.2), and linearly combine seven identities together with coefficients

$$\left\{\frac{\alpha}{3}(\frac{1}{2}-\frac{\alpha}{3}), -\frac{\alpha}{6}, \frac{1}{2}-\frac{\alpha}{3}, \frac{1}{2}(\beta+\frac{4}{3}\alpha-1), -1, 1, 1\right\},\$$

then identity (2.11) can be proved.

The following identity is also useful, which provides a positive u_0^2 term. **Proposition 2.7.** If n = 1, let β be an undetermined constant, then

$$u^{-\beta} \operatorname{Re} \left\{ u^{\beta} \left[\frac{1}{3} (\frac{2}{3}\alpha - 1) \frac{|\nabla u|^{4}}{u^{3}} - \frac{\sqrt{-1}u_{0}}{u^{2}} |\nabla u|^{2} \right] u_{1} \right\}_{,}^{1}$$

$$= \frac{2}{3} (\frac{2}{3}\alpha - 1) \frac{|\nabla u|^{2}}{u^{2}} \operatorname{Re} D_{1}u^{1} + \operatorname{Re} \frac{\sqrt{-1}u_{0}}{u} D_{1}u^{1} + \frac{|\nabla u|^{2}}{u^{2}} \operatorname{Re} G_{1}u^{1}$$

$$+ (\frac{\beta + \alpha}{3} - 1) (\frac{2}{3}\alpha - 1) \frac{|\nabla u|^{6}}{u^{4}} - (\alpha - 1)u^{\alpha - 3} |\nabla u|^{4} + 2\frac{|\nabla u|^{2}}{u^{2}} u_{0}^{2}.$$
(2.14)

Proof. It's just
$$\frac{1}{3}(\frac{2}{3}\alpha - 1) \times (2.12) - (2.13).$$

In conclusion, inspired by the Riemannian case, the $\{(0,0,), 2, 6, +\}$ type identity (2.10) is found by the method of dimensional conservation and invariant tensors. As a supplement, (2.11) and (2.14) are proposed to deal with special case n = 1.

3. Proof of Theorem 1.6

In this section, we'll prove Theorem 1.6 by discussing four different cases. $\{(0,0), 2, 4, +\}$ type identity should be considered first, which is Case 1. The identity $\{(0,0), 2, 6, +\}$ type (2.10) deals with the "near-critical" exponent in Case 2. For n = 1, Case 3 and Case 4 are finished with the help of (2.11) and (2.14) separately.

Case 1.
$$1 < \alpha < \frac{n+2}{n+\frac{1}{2n}}$$
 for $n \ge 2$.
By (2.7), (2.8), and use Lemma 2.1 to write as square terms:

$$\operatorname{Re}[(n-1)D_{ij}u^{j} + (n+2)E_{i\overline{j}}u^{j}]^{i} = (n+2)\sum_{i,j} |E_{i\overline{j}}|^{2} + (n-1)\sum_{i,j} |D_{ij}|^{2} + 2\alpha E_{i}u^{i} + 2(n-1)\alpha(1 - \frac{n\alpha}{n+2})\frac{|\nabla u|^{4}}{u^{2}} + (n-1)\mathscr{R} = (n+2)\sum_{i,j} \left|E_{i\overline{j}} + \frac{\alpha}{n+2}L_{i\overline{j}}\right|^{2} + (n-1)\left[\sum_{i,j} |D_{ij}|^{2} + \alpha\left(2 - \frac{2n^{2}+1}{n(n+2)}\alpha\right)\frac{|\nabla u|^{4}}{u^{2}} + \mathscr{R}\right].$$

$$(3.1)$$

$$f(1 < \alpha < \frac{n+2}{1}, \text{ the RHS of identity (3.1) is non-negative with positive } \frac{|\nabla u|^{4}}{u^{2}} \operatorname{term},$$

If $1 < \alpha < \frac{n+2}{n+\frac{1}{2n}}$, the RHS of identity (3.1) is non-negative with positive $\frac{|\nabla u|}{u^2}$ term,

and the LHS of identity (3.1) is composed of divergence of vector fields, which are vanished while integrating over M. Hence $|\nabla u|^4 = 0$, i.e., u is constant by integrating over M on both sides of (3.1).

Remark 3.1. The method discussed in Case 1 is similar to the one used by Xu in [16]. Though the $\{(0,0), 2, 4, +\}$ type identity (3.1) is simple, it's trivial when n = 1 because of $E_{1\overline{1}} = 0$. Hence the case n = 1 can't be taken into consideration in Case 1.

Case 2.
$$\frac{n+2}{n+\frac{1}{2n}} \leq \alpha < \frac{n+2}{n}$$
 for $n \in \mathbb{N}^*$.

Consider the subcritical exponent case $\frac{n+2}{n+\frac{1}{2n}} \leq \alpha < \frac{n+2}{n}$ first. In identity (2.10),

take

$$d_1 = e_1 = \frac{n^2 \alpha [3n + 6 - (n - 1)\alpha]}{(2n + 1)(n + 2)^2}, \quad d_2 = e_2 = \frac{n\alpha}{n + 2}, \quad d_3 = e_3 = n(\frac{n + 1}{n + 2}\alpha - 1),$$

$$d_4 = \frac{n}{2n+1} \left(3 - \frac{7n+2}{n+2}\alpha\right), \quad e_4 = \frac{n(3+\alpha)}{2n+1}, \quad \mu = 3, \quad \beta = 1 - \alpha.$$

Rewrite all coefficients with the parameters above:

$$\Delta_{1} = \frac{2n^{2}\alpha[(4n+5)\alpha - 3n - 6]}{(2n+1)(n+2)^{2}} (1 - \frac{n\alpha}{n+2}), \quad \Theta_{1} = -\frac{6n^{2}\alpha(\alpha + n + 2)}{(2n+1)(n+2)^{2}} (1 - \frac{n\alpha}{n+2}),$$
$$\Xi_{1} = \frac{6n\alpha}{2n+1} (1 - \frac{n\alpha}{n+2}), \quad \Delta_{3} = \Theta_{3} = \frac{2n(\alpha - 1)(2 + n - n\alpha)}{2n+1},$$
$$\Delta_{2} = \Theta_{2} = \Xi_{2} = \Xi_{3} = \Delta_{4} = \Xi_{4} = 0.$$
Notice that $\sum_{i \neq k} |D_{ij}u_{\overline{k}} + E_{i\overline{k}}u_{j}|^{2} = |\nabla u|^{2} \sum_{i \neq i} (|D_{ij}|^{2} + |E_{i\overline{j}}|^{2}) + 2u^{2} \operatorname{Re} D_{i}E^{i}.$ Use Lemma

2.1 to rewrite identity (2.10) as square terms:

$$u^{-\beta} \operatorname{Re} \left\{ u^{\beta} \left[\left(d_{1} \frac{|\nabla u|^{2}}{u} + d_{2} u^{\alpha} + d_{3} \lambda u \right) (D_{i} + E_{i}) + n \sqrt{-1} u_{0} (d_{4} D_{i} + e_{4} E_{i} - 3G_{i}) \right] \right\}_{,}^{i}$$

$$= d_{1} u^{-2} \sum_{i,j,k} |D_{ij} u_{\overline{k}} + E_{i\overline{k}} u_{j}|^{2} + d_{2} u^{\alpha - 1} \left[\sum_{i,j} (|D_{ij}|^{2} + |E_{i\overline{j}}|^{2}) + 2\alpha (1 - \frac{n\alpha}{n+2}) \frac{|\nabla u|^{4}}{u^{2}} \right] \quad (3.2)$$

$$+ d_{3} \lambda \left[\sum_{i,j} \left(\left| D_{ij} + \frac{\Delta_{3}}{2d_{3}} \frac{u_{i} u_{j}}{u} \right|^{2} + \left| E_{i\overline{j}} + \frac{\Delta_{3}}{2d_{3}} L_{i\overline{j}} \right|^{2} \right) + \left(2\alpha (1 - \frac{n\alpha}{n+2}) - \frac{2n-1}{n} \frac{\Delta_{3}^{2}}{4d_{3}^{2}} \right) \frac{|\nabla u|^{4}}{u^{2}} \right] + \left[d_{1} \frac{|\nabla u|^{2}}{u^{2}} + d_{2} u^{\alpha - 1} + d_{3} \lambda \right] \mathscr{R} + \mathbf{Q}_{1},$$

where \mathbf{Q}_1 can be written as a quadratic form:

$$\begin{aligned} \mathbf{Q}_{1} = & d_{1} \sum_{i} |D_{i}|^{2} + d_{1} \sum_{i} |E_{i}|^{2} + 3 \sum_{i} |G_{i}|^{2} - d_{4} \operatorname{Re} D_{i} G^{i} - e_{4} \operatorname{Re} E_{i} G^{i} \\ & + \Delta_{1} \frac{|\nabla u|^{2}}{u^{2}} \operatorname{Re} D_{i} u^{i} + \Theta_{1} \frac{|\nabla u|^{2}}{u^{2}} E_{i} u^{i} + \Xi_{1} \frac{|\nabla u|^{2}}{u^{2}} \operatorname{Re} G_{i} u^{i} + 2d_{1} \alpha (1 - \frac{n\alpha}{n+2}) \frac{|\nabla u|^{6}}{u^{4}}. \end{aligned}$$

If
$$\alpha \in (\frac{n+2}{n+1}, \frac{n+2}{n}), d_1 = \frac{n\alpha}{n+2} \cdot \frac{n[3n+6-(n-1)\alpha]}{(2n+1)(n+2)} \ge \frac{n\alpha}{n+2} > 0, d_2 = \frac{n\alpha}{n+2} > 0$$

 $d_3 = n(\frac{n+1}{n+2}\alpha - 1) > 0$, which are correct for $\alpha \in [\frac{n+2}{n+\frac{1}{2n}}, \frac{n+2}{n})$ of course.

Check the positivity of $\lambda \frac{|\nabla u|^4}{u^2}$ term:

$$d_3\left(2\alpha(1-\frac{n\alpha}{n+2})-\frac{2n-1}{n}\frac{\Delta_3^2}{4d_3^2}\right) = \frac{n}{(2n+1)^2d_3}(1-\frac{n\alpha}{n+2})f_1(\alpha),$$

where $f_1(\alpha)$ is a polynomial of α :

$$f_1(\alpha) = \frac{n(10n^4 + 35n^3 + 44n^2 + 16n - 6)}{(n+2)^2} \alpha^3 - \frac{22n^4 + 57n^3 + 46n^2 - 8}{n+2} \alpha^2 + (14n^3 + 25n^2 + 8n - 8)\alpha - (n+2)^2(2n-1).$$

Study the monotonicity of f_1 :

$$f_1'(\alpha) = 3n(10n^4 + 35n^3 + 44n^2 + 16n - 6)\left(\frac{\alpha}{n+2}\right)^2 - 2(22n^4 + 57n^3 + 46n^2 - 8)\left(\frac{\alpha}{n+2}\right) + (14n^3 + 25n^2 + 8n - 8).$$

Compare the symmetry axis of $f'_1((n+2)x)$ with $\left(n+\frac{1}{2n}\right)^{-1}$, the minimum of $\frac{\alpha}{n+2}$:

$$\frac{22n^4 + 57n^3 + 46n^2 - 8}{3n(10n^4 + 35n^3 + 44n^2 + 16n - 6)} \Big/ \left(n + \frac{1}{2n}\right)^{-1} - 1$$
$$= -\frac{16n^6 + 96n^5 + 150n^4 + 39n^3 - 66n^2 + 8}{6n^2(10n^4 + 35n^3 + 44n^2 + 16n - 6)} < 0,$$

then $f_1'(\alpha) \ge f_1'(\frac{n+2}{n+\frac{1}{2n}}) = \frac{(2n-1)(32n^5 + 96n^4 + 64n^3 - 41n^2 - 24n + 8)}{(2n^2+1)^2} > 0$, hence

$$f_1(\alpha) \ge f_1(\frac{n+2}{n+\frac{1}{2n}}) = \frac{(n+2)(2n-1)(32n^5 + 16n^4 - 24n^3 - 28n^2 + 15n - 2)}{(2n^2+1)^3} > 0,$$

from which we have $2\alpha(1-\frac{n\alpha}{n+2}) > \frac{2n-1}{n}\frac{\Delta_3^2}{4d_3^2} \ge 0, \forall \alpha \in [\frac{n+2}{n+\frac{1}{2n}}, \frac{n+2}{n}].$

Check the positivity of the quadratic form \mathbf{Q}_1 , which corresponds to a matrix as

$$\begin{pmatrix} d_1 & 0 & -\frac{d_4}{2} & \frac{\Delta_1}{2} \\ 0 & d_1 & -\frac{e_4}{2} & \frac{\Theta_1}{2} \\ -\frac{d_4}{2} & -\frac{e_4}{2} & 3 & \frac{\Xi_1}{2} \\ \frac{\Delta_1}{2} & \frac{\Theta_1}{2} & \frac{\Xi_1}{2} & 2d_1\alpha(1-\frac{n\alpha}{n+2}) \end{pmatrix}.$$
(3.3)

Compute principal minor sequence of matrix (3.3):

$$\begin{vmatrix} d_1 & 0 & -\frac{d_4}{2} \\ 0 & d_1 & -\frac{e_4}{2} \\ -\frac{d_4}{2} & -\frac{e_4}{2} & 3 \end{vmatrix} = \frac{n^4 \alpha (3 - \frac{n-1}{n+2}\alpha)}{2(n+2)(2n+1)^3} f_2(\alpha),$$

where $3 - \frac{n-1}{n+2}\alpha > 3 - \frac{n-1}{n+2} \cdot \frac{n+2}{n} = \frac{2n+1}{n} > 0,$ $f_2(\alpha) = -(37n^2 + 10n - 2)\left(\frac{\alpha}{n+2}\right)^2 + 18(3n+1)\left(\frac{\alpha}{n+2}\right) - 9$ $\ge \min\left\{f_2(\frac{n+2}{n+\frac{1}{2n}}), f_2(\frac{n+2}{n})\right\}$

$$= \min\left\{\frac{32n^4 + 32n^3 + 80n^2 + 36n - 9}{(2n^2 + 1)^2}, \frac{2(2n+1)^2}{n^2}\right\} > 0.$$

The only left term we need to check is the determinant of \mathbf{Q}_1 :

$$\det (3.3) = \frac{n^6 \alpha^3 (3 - \frac{n-1}{n+2}\alpha)^2}{(n+2)^3 (2n+1)^4} (1 - \frac{n\alpha}{n+2}) f_3(\alpha),$$

hence we only need to prove that $f_3(\alpha)$ is positive:

$$f_{3}(\alpha) = -2(2n-1)(11n^{2}+14n+2)\left(\frac{\alpha}{n+2}\right)^{2} + (79n^{2}+58n-2)\left(\frac{\alpha}{n+2}\right) - 27n$$
$$\geqslant \min\left\{f_{3}(\frac{n+2}{n+\frac{1}{2n}}), f_{3}(\frac{n+2}{n})\right\}$$
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$$=\min\left\{\frac{n(32n^4+96n^3+122n^2+132n-31)}{(2n^2+1)^2},\frac{2(n+2)(2n+1)^2}{n^2}\right\}>0.$$

In conclusion, the matrix (3.3) is strictly positive definite. Then the RHS of identity (3.2) are non-negative with some positive $\frac{|\nabla u|^6}{u^4}$ terms left while $\alpha \in [\frac{n+2}{n+\frac{1}{2m}}, \frac{n+2}{n}]$, hence $|\nabla u|^6 = 0$ by multiplying u^β on both sides of (3.2) and integrating over M, then u

is constant.

Remark 3.2. For the critical exponent case $\alpha = \frac{n+2}{n}$, the parameters discussed above are

$$d_k = e_k = 1, \ k = 1, 2, 3, \ d_4 = -2, \ e_4 = 2, \ \beta = -\frac{2}{n},$$

$$\Delta_l = \Theta_l = \Xi_l = 0, \quad l = 1, 2, 3, 4.$$

Then the identity (3.2) becomes Jerison-Lee type identity:

$$u^{\frac{2}{n}} \operatorname{Re} \left\{ u^{-\frac{2}{n}} \left[\left(\frac{|\nabla u|^{2}}{u} + u^{\frac{n+2}{n}} + \lambda u \right) (D_{i} + E_{i}) - n\sqrt{-1}u_{0}(2D_{i} - 2E_{i} + 3G_{i}) \right] \right\}_{,}^{i}$$

= $u^{-2} \sum_{i,j,k} |D_{ij}u_{\overline{k}} + E_{i\overline{k}}u_{j}|^{2} + \frac{|\nabla u|^{2}}{u^{2}} \mathscr{R} + (u^{\frac{2}{n}} + \lambda) \sum_{i,j} (|D_{ij}|^{2} + |E_{i\overline{j}}|^{2} + \mathscr{R})$
+ $\sum_{i} (|G_{i} + D_{i}|^{2} + |G_{i} - E_{i}|^{2} + |G_{i}|^{2}).$ (3.4)

The rigidity and the existence of non-trivial solutions can be deduced by discussing the positivity of \mathscr{R} . We recommend readers to see [11] and [14] for more details.

Case 3. 1.06 $\leq \alpha < 3$ for n = 1.

It's a pity that the method in Case 1 failed when n = 1. Besides, Case 2 can cover $2 \leq \alpha < 3$ with n = 1 only. In this section, the identity (2.11) will be used to cover those difficulties.

Notice that $E_{1\overline{1}} = 0$, $|\nabla u|^2 |D_{11}|^2 = u^2 |D_1|^2$. Rewrite (2.10) while n = 1:

$$u^{-\beta} \operatorname{Re} \left\{ u^{\beta} \left[\left(d_{1} \frac{|\nabla u|^{2}}{u} + d_{2}u^{\alpha} + d_{3}\lambda u \right) D_{1} + \sqrt{-1}u_{0}(d_{4}D_{1} - \mu G_{1}) \right] \right\}_{,}^{1}$$

$$= \left(2d_{1} \frac{|\nabla u|^{2}}{u^{2}} + d_{2}u^{\alpha-1} + d_{3}\lambda \right) |D_{11}|^{2} + \left(d_{1} \frac{|\nabla u|^{2}}{u^{2}} + d_{2}u^{\alpha-1} + d_{3}\lambda \right)$$

$$\times \left[2\alpha(1 - \frac{\alpha}{3}) \frac{|\nabla u|^{4}}{u^{2}} + \mathscr{R} \right] + \mu |G_{1}|^{2} - d_{4} \operatorname{Re} D_{1}G^{1} \qquad (3.5)$$

$$+ \operatorname{Re} \left[\Delta_{1} \frac{|\nabla u|^{2}}{u^{2}} + \Delta_{2}u^{\alpha-1} + \Delta_{3}\lambda + \Delta_{4} \frac{\sqrt{-1}u_{0}}{u} \right] D_{1}u^{1}$$

$$+ \operatorname{Re} \left[\Xi_{1} \frac{|\nabla u|^{2}}{u^{2}} + \Xi_{2}u^{\alpha-1} + \Xi_{3}\lambda + \Xi_{4} \frac{\sqrt{-1}u_{0}}{u} \right] G_{1}u^{1}.$$

The coefficients are:

$$\Delta_1 = (\beta + 2\alpha - 2)d_1 + \frac{\alpha}{3} \left(\frac{2}{3}\alpha - 1\right) d_4, \quad \Delta_2 = -d_1 + (\beta + 2\alpha - 1) d_2 + \frac{\alpha}{3} d_4,$$

$$\Delta_3 = d_1 + (\beta + \alpha) \, d_3 + \left(\frac{2}{3}\alpha - 1\right) d_4, \quad \Delta_4 = d_1 + \left(\beta + \frac{5}{3}\alpha - 1\right) d_4 + \frac{\alpha}{3} \left(\frac{2}{3}\alpha - 1\right) \mu,$$

$$\Xi_1 = 3d_1 - \frac{\alpha}{3} \left(\frac{2}{3}\alpha - 1\right)\mu, \quad \Xi_2 = 3d_2 - \frac{\alpha}{3}\mu, \quad \Xi_3 = 3d_3 - \left(\frac{2}{3}\alpha - 1\right)\mu, \quad \Xi_4 = 3d_4 - \beta\mu.$$

Take $d_1 = \frac{\alpha}{36}(5\alpha - 3), d_2 = d_3 = \frac{\alpha - 1}{2}, d_4 = 2 - \frac{4}{3}\alpha, \mu = 3, \beta = 1 - \alpha$. Rewrite all coefficients with the parameters above:

$$\Delta_1 = \frac{\alpha}{108}(3-\alpha)(17\alpha - 21), \quad \Delta_2 = \frac{\alpha}{12}(3-\alpha), \quad \Delta_3 = \frac{1}{12}(3-\alpha)(9\alpha - 10),$$

$$\Delta_4 = \frac{\alpha}{12}(3-\alpha), \quad \Xi_1 = \frac{\alpha}{4}(3-\alpha), \quad \Xi_2 = \frac{\alpha-3}{2}, \quad \Xi_3 = \frac{3-\alpha}{2}, \quad \Xi_4 = 3-\alpha.$$

$$\begin{aligned} \text{Consider } (3.5) &+ \frac{3-\alpha}{2} \times (2.11): \\ u^{\alpha-1} \operatorname{Re} \left\{ u^{1-\alpha} \Big[\left(\frac{\alpha}{36} (5\alpha - 3) \frac{|\nabla u|^2}{u} + \frac{\alpha - 1}{2} (u^{\alpha} + \lambda u) \right) D_1 \\ &+ \sqrt{-1} u_0 \left((2 - \frac{4}{3} \alpha) D_1 - 3G_1 \right) \\ &+ \frac{3-\alpha}{2} \left(\frac{\alpha}{3} (\frac{1}{2} - \frac{\alpha}{3}) \frac{|\nabla u|^2}{u^2} - \frac{\alpha}{6} u^{\alpha-1} + (\frac{1}{2} - \frac{\alpha}{3}) \lambda \right) \frac{|\nabla u|^2}{u} u_1 \\ &+ \frac{3-\alpha}{2} \left(\frac{\alpha}{6} \frac{|\nabla u|^2}{u^2} - u^{\alpha-1} + \lambda - \frac{\sqrt{-1} u_0}{u} \right) \sqrt{-1} u_0 u_1 \Big] \Big\},^1 \\ &= \left(\frac{\alpha}{18} (5\alpha - 3) \frac{|\nabla u|^2}{u^2} + \frac{\alpha - 1}{2} (u^{\alpha-1} + \lambda) \right) |D_{11}|^2 + 3|G_1|^2 + (\frac{4}{3}\alpha - 2) \operatorname{Re} D_1 G^1 \\ &+ \frac{\alpha}{6} (\alpha - 1) (\alpha + 3) (1 - \frac{\alpha}{3}) \frac{|\nabla u|^6}{u^4} + \frac{\alpha}{36} (5\alpha - 3) \frac{|\nabla u|^2}{u^2} \mathscr{R} \\ &+ \frac{\alpha - 1}{2} (u^{\alpha-1} + \lambda) \left[2\alpha (1 - \frac{\alpha}{3}) \frac{|\nabla u|^4}{u^2} + \mathscr{R} \right] \\ &+ (3 - \alpha) \operatorname{Re} \left[\frac{\alpha}{108} (5\alpha - 3) \frac{|\nabla u|^2}{u^2} D_1 u^1 + \frac{7}{12} (\alpha - 1) \lambda D_1 u^1 + \frac{\alpha}{6} \frac{|\nabla u|^2}{u^2} G_1 u^1 \right] \\ &= \frac{\alpha - 1}{2} \lambda \left[\left| D_{11} + \frac{7}{12} (3 - \alpha) \frac{u_1 u_1}{u} \right|^2 + \frac{1}{144} (3 - \alpha) (145\alpha - 147) \frac{|\nabla u|^4}{u^2} + \mathscr{R} \right] \\ &+ \frac{\alpha - 1}{2} u^{\alpha - 1} \left[|D_{11}|^2 + 2\alpha (1 - \frac{\alpha}{3}) \frac{|\nabla u|^4}{u^2} + \mathscr{R} \right] + \frac{\alpha}{36} (5\alpha - 3) \frac{|\nabla u|^2}{u^2} \mathscr{R} + \mathbf{Q}_2, \end{aligned}$$

where \mathbf{Q}_2 can be written as a quadratic form:

$$\mathbf{Q}_{2} = \frac{\alpha}{18} (5\alpha - 3) |D_{1}|^{2} + 3|G_{1}|^{2} + (\frac{4}{3}\alpha - 2) \operatorname{Re} D_{1}G^{1} + (3 - \alpha) \frac{|\nabla u|^{2}}{u^{2}} \\ \times \left[\frac{\alpha}{108} (5\alpha - 3) \operatorname{Re} D_{1}u^{1} + \frac{\alpha}{6} \operatorname{Re} G_{1}u^{1} + \frac{\alpha}{18} (\alpha - 1)(\alpha + 3) \frac{|\nabla u|^{4}}{u^{2}} \right].$$

While $\alpha \in [1.06,3)$, $\alpha - 1 > 0$, $(3 - \alpha)(145\alpha - 147) > 0$, Hence we only need to check the positivity of quadratic form \mathbf{Q}_2 , which corresponds to a matrix as

$$\begin{pmatrix} \frac{\alpha}{18}(5\alpha-3) & \frac{2}{3}\alpha-1 & \frac{\alpha}{216}(3-\alpha)(5\alpha-3) \\ \frac{2}{3}\alpha-1 & 3 & \frac{\alpha}{12}(3-\alpha) \\ \frac{\alpha}{216}(3-\alpha)(5\alpha-3) & \frac{\alpha}{12}(3-\alpha) & \frac{\alpha}{18}(3-\alpha)(\alpha-1)(\alpha+3) \end{pmatrix}.$$
 (3.7)

Compute principal minor sequence of matrix (3.7):

$$\frac{\alpha}{18}(5\alpha - 3) > \frac{\alpha}{9} > 0,$$

$$\begin{vmatrix} \frac{\alpha}{18}(5\alpha - 3) & \frac{2}{3}\alpha - 1 \\ \frac{2}{3}\alpha - 1 & 3 \end{vmatrix} = \frac{1}{18}(\alpha + 3)(7\alpha - 6) > 0,$$

$$\det (3.7) = \frac{\alpha}{5184} (3-\alpha)(3+\alpha)f_4(\alpha),$$

where $f_4(\alpha) = 117\alpha^3 + 110\alpha^2 - 519\alpha + 288$. Study the monotonicity of f_4 :

$$f_4'(\alpha) = 351\alpha^2 + 220\alpha - 519 > 351 + 220 - 519 = 52 > 0,$$

then $f_4(\alpha) \ge f_4(1.06) = 0.804872 > 0$, hence the matrix (3.7) is strictly positive definite.

In conclusion, the RHS of identity (3.6) are non-negative with some positive $u^{\alpha-3}|\nabla u|^4$ terms left while $\alpha \in [1.06, 2)$. Then $|\nabla u|^4 = 0$, i.e. u is a constant, by multiplying u^β on both sides of (3.6) and integrating over M.

Remark 3.3. In the critical exponent case $\alpha = 3$, the identity (3.6) becomes (3.4) in the case n = 1 again. Besides, the lower bound of 1.06 can be decreased to 1.052327, the approximation of

$$\frac{2}{351} \left\{ \sqrt{194269} \cos \left[\frac{1}{3} \left(\arccos \frac{84611717}{194269^{\frac{3}{2}}} - \pi \right) \right] - 55 \right\},\$$

which is the biggest root of f_4 . However, we'll use a new identity in Case 4, hence it's meaningless to compute such finely.

Case 4. $1 < \alpha \leq 1.06$ for n = 1. Take $d_1 = \frac{1}{18}, d_2 = d_3 = \frac{\alpha - 1}{2}, d_4 = \frac{2}{3}, \mu = 3, \beta = \frac{1}{2}$ in identity (3.5). Rewrite all coefficients in (3.5) with parameters above:

$$\Delta_1 = \frac{1}{108}(16\alpha^2 - 12\alpha - 9), \quad \Delta_2 = \frac{1}{36}(4\alpha - 1)(9\alpha - 7), \quad \Delta_3 = \frac{1}{36}(18\alpha^2 + 7\alpha - 31),$$

$$\Delta_4 = \frac{1}{18}(12\alpha^2 + 2\alpha - 5), \quad \Xi_1 = -\frac{1}{6}(4\alpha^2 - 6\alpha - 1), \quad \Xi_2 = \frac{\alpha - 3}{2}, \quad \Xi_3 = \frac{3 - \alpha}{2}, \quad \Xi_4 = \frac{1}{2}$$

$$\begin{split} & \text{Consider } (3.5) + \frac{3-\alpha}{2} \times (2.11) + \frac{9}{40} \times (2.14) \text{ with } \beta = \frac{1}{2}; \\ & u^{-\frac{1}{2}} \operatorname{Re} \left\{ u^{\frac{1}{2}} \left[\left(\frac{1}{18} \frac{|\nabla u|^2}{u} + \frac{\alpha - 1}{2} (u^{\alpha} + \lambda u) \right) D_1 + \sqrt{-1} u_0 \left(\frac{2}{3} D_1 - 3G_1 \right) \right. \\ & + \frac{3-\alpha}{2} \left[\left(-\frac{\alpha}{6} u^{\alpha-1} + (\frac{1}{2} - \frac{\alpha}{3}) \lambda \right) \frac{|\nabla u|^2}{u} + \left(-u^{\alpha-1} + \lambda - \frac{\sqrt{-1} u_0}{u} \right) \sqrt{-1} u_0 \right] u_1 \\ & + \left((2\alpha - 3)(\frac{1}{36} \alpha^2 - \frac{1}{12} \alpha + \frac{1}{40}) \frac{|\nabla u|^2}{u} - (\frac{1}{3} \alpha^2 - \frac{9}{8} \alpha + \frac{3}{5}) \sqrt{-1} u_0 \right) \frac{|\nabla u|^2}{u^2} u_1 \right] \right\}_{,}^{-1} \\ & = \left[\frac{1}{9} \frac{|\nabla u|^2}{u^2} + \frac{\alpha - 1}{2} (u^{\alpha-1} + \lambda) \right] |D_{11}|^2 + 3|G_1|^2 - \frac{2}{3} \operatorname{Re} D_1 G^1 \\ & + \frac{1}{2160} (80\alpha^4 - 600\alpha^3 + 1468\alpha^2 - 1272\alpha + 405) \frac{|\nabla u|^6}{u^4} + \frac{1}{18} \frac{|\nabla u|^2}{u^2} \mathscr{R} \\ & + \frac{\alpha - 1}{2} u^{\alpha-1} \left[-\frac{1}{60} (40\alpha^2 - 120\alpha + 27) \frac{|\nabla u|^4}{u^2} + \mathscr{R} \right] \\ & + \frac{\alpha - 1}{2} \lambda \left[2\alpha(1 - \frac{\alpha}{3}) \frac{|\nabla u|^4}{u^2} + \mathscr{R} \right] + \frac{9}{20} \frac{|\nabla u|^2}{u^2} u_0^2 \\ & + \frac{1}{270} (30\alpha^3 - 95\alpha^2 + 132\alpha - 63) \frac{|\nabla u|^2}{u^2} \operatorname{Re} D_1 u^1 \\ & + (\alpha - 1) \left[\frac{1}{36} (39 - 7\alpha) u^{\alpha-1} + \frac{1}{9} (6\alpha + 1)\lambda \right] \operatorname{Re} D_1 u^1 \\ & + \frac{1}{360} (360\alpha^2 - 365\alpha + 116) \operatorname{Re} \frac{\sqrt{-1} u_0}{u} D_1 u^1 \\ & - \frac{1}{120} (40\alpha^2 + 15\alpha - 92) \frac{|\nabla u|^2}{u^2} \operatorname{Re} G_1 u^1 + (\alpha - \frac{5}{2}) \operatorname{Re} \frac{\sqrt{-1} u_0}{u} G_1 u^1 \\ & - \frac{1}{120} (40\alpha^2 + 15\alpha - 92) \frac{|\nabla u|^2}{u^2} \operatorname{Re} G_1 u^1 + (\alpha - \frac{5}{2}) \operatorname{Re} \frac{\sqrt{-1} u_0}{u} G_1 u^1 \\ & - \frac{\alpha - 1}{2} \lambda \left[\left| D_{11} + \frac{1}{36} (39 - 7\alpha) \frac{u_1 u_1}{u} \right|^2 - \frac{4565\alpha^2 - 15690\alpha + 10521}{6480} \frac{|\nabla u|^4}{u^2} \right] \\ & + \left(\frac{1}{18} \frac{|\nabla u|^2}{u^2} + \frac{\alpha - 1}{2} (u^{\alpha-1} + \lambda) \right] \mathscr{R} + \mathbf{Q}_3, \end{split}$$

where \mathbf{Q}_3 can be written as a quadratic form:

$$\begin{aligned} \mathbf{Q}_{3} = &\frac{1}{9} |D_{1}|^{2} + 3|G_{1}|^{2} - \frac{2}{3} \operatorname{Re} D_{1}G^{1} + \frac{9}{20} \frac{|\nabla u|^{2}}{u^{2}} u_{0}^{2} + \Delta_{4}' \operatorname{Re} \frac{\sqrt{-1}u_{0}}{u} D_{1}u^{1} \\ &+ \Xi_{4}' \operatorname{Re} \frac{\sqrt{-1}u_{0}}{u} G_{1}u^{1} + \Delta_{1}' \frac{|\nabla u|^{2}}{u^{2}} \operatorname{Re} D_{1}u^{1} + \Xi_{1}' \frac{|\nabla u|^{2}}{u^{2}} \operatorname{Re} G_{1}u^{1} + A \frac{|\nabla u|^{6}}{u^{4}} \end{aligned}$$

with the coefficients as

$$\begin{split} \Delta_1' &= \frac{1}{270} (30\alpha^3 - 95\alpha^2 + 132\alpha - 63), \quad \Delta_4' &= \frac{1}{360} (360\alpha^2 - 365\alpha + 116), \\ &= \frac{1}{120} (40\alpha^2 + 15\alpha - 92), \quad \Xi_4' = \alpha - \frac{5}{2}, \\ A &= \frac{1}{2160} (80\alpha^4 - 600\alpha^3 + 1468\alpha^2 - 1272\alpha + 405). \end{split}$$
 While $\alpha \in (1, 1.06]$, check the positivity of $u^{\alpha - 1} \frac{|\nabla u|^4}{u^2}$ term and $\lambda \frac{|\nabla u|^4}{u^2}$ term:
 $-(4565\alpha^2 - 15690\alpha + 10521) \ge -(4565 \times 1 - 15690 \times 1 + 10521) = 604 > 0, \end{split}$

$$-(90\alpha^2 - 150\alpha + 1) \ge -(90 \times 1.06^2 - 150 + 1) = 47.876 > 0,$$

Hence we only need to check the positivity of quadratic form \mathbf{Q}_2 , which corresponds to a matrix as

$$\begin{pmatrix} \frac{1}{9} & -\frac{1}{3} & \frac{\Delta'_4}{2} & \frac{\Delta'_1}{2} \\ -\frac{1}{3} & 3 & \frac{\Xi'_4}{2} & \frac{\Xi'_1}{2} \\ \frac{\Delta'_4}{2} & \frac{\Xi'_4}{2} & \frac{9}{20} & 0 \\ \frac{\Delta'_1}{2} & \frac{\Xi'_1}{2} & 0 & A \end{pmatrix}.$$
(3.9)

Compute principal minor sequence of matrix (3.9):

$$\begin{vmatrix} \frac{1}{9} & -\frac{1}{3} \\ -\frac{1}{3} & 3 \end{vmatrix} = \frac{2}{9} > 0, \quad \begin{vmatrix} \frac{1}{9} & -\frac{1}{3} & \frac{\Delta'_4}{2} \\ -\frac{1}{3} & 3 & \frac{\Xi'_4}{2} \\ \frac{\Delta'_4}{2} & \frac{\Xi'_4}{2} & \frac{9}{20} \end{vmatrix} = \frac{f_5(\alpha)}{57600}, \quad \det(3.9) = \frac{f_6(\alpha)}{29859840000},$$

where $f_5(\alpha) = -43200\alpha^4 + 78000\alpha^3 - 40115\alpha^2 + 8800\alpha - 992$, $f_6(\alpha) = -460800000\alpha^8 + 6320640000\alpha^7 - 25055552000\alpha^6 + 44595172000\alpha^5 - 42848423575\alpha^4 + 24660626800\alpha^3 - 8756098960\alpha^2 + 1823449600\alpha - 252801536$.

Study the concavity of f_5 :

$$\begin{aligned} f_5''(\alpha) &= 10(-51840\alpha^2 + 46800\alpha - 8023) < 10(-51840 + 46800 \times 1.06 - 8023) = -102550 < 0, \\ \text{then } f_5(\alpha) &\ge \min\{f_5(1), f_5(1.06)\} = \min\{2493, 1623.03\} > 0. \\ \text{Similarly, check the positivity of } f_6: \end{aligned}$$

$$f_6''(\alpha) = -20(1290240000\alpha^6 - 13273344000\alpha^5 + 37583328000\alpha^4 - 44595172000\alpha^3 + 25709054145\alpha^2 - 7398188040\alpha + 875609896),$$

$$\begin{split} f_6^{(3)}(\alpha) &= -600(258048000\alpha^5 - 2212224000\alpha^4 + 5011110400\alpha^3 \\ &\quad -4459517200\alpha^2 + 1713936943\alpha - 246606268), \\ f_6^{(4)}(\alpha) &= -600(1290240000\alpha^4 - 8848896000\alpha^3 \\ &\quad +15033331200\alpha^2 - 8919034400\alpha + 1713936943), \\ f_6^{(5)}(\alpha) &= -480000(6451200\alpha^3 - 33183360\alpha^2 + 37583328\alpha - 11148793), \\ f_6^{(6)}(\alpha) &= 46080000(-201600\alpha^2 + 691320\alpha - 391493) \geqslant f_6^{(6)}(1) = 46080000 \times 98227 > 0, \\ f_6^{(6)}(\alpha) &= 46080000(-201600\alpha^2 + 691320\alpha - 391493) \geqslant f_6^{(6)}(1) = 46080000 \times 98227 > 0, \\ f_6^{(5)}(\alpha) &\geq f_6^{(5)}(1) = 480000 \times 297625 > 0, \\ f_6^{(4)}(\alpha) &\leq f_6^{(4)}(1.06) = -600 \times 240932969.8544 < 0, \\ f_6^{(3)}(\alpha) &\leq f_6^{(3)}(1) = -600 \times 64747875 < 0, \\ f_6^{(3)}(\alpha) &\leq f_6^{(3)}(1) = -20 \times 191528001 < 0, \\ \end{split}$$

then f_6 is concave while $\alpha \in (1, 1.06]$, hence

$$f_6(\alpha) \ge \min\{f_6(1), f_6(1.06)\} = \min\{26212329, 2.38 \times 10^7\} > 0.$$

In conclusion, the matrix (3.9) is strictly positive definite, hence the RHS of identity (3.8) are non-negative with some positive $u^{\alpha-3}|\nabla u|^4$ terms left while $\alpha \in (1, 1.06]$. Then $|\nabla u|^4 = 0$, i.e. u is a constant, by multiplying u^β on both sides of (3.8) and integrating over M.

Combine Case $1 \sim \text{Case } 4$, then Theorem 1.6 is proved.

4. Theorem 1.3: Answer to the problem raised by Jerison-Lee

In [11], Jerison-Lee found a three-dimensional family of differential identities with positive RHS for the Yamabe equation on Heisenberg group \mathbb{H}^n by using the computer. However, they care about whether there exists a theoretical framework that would predict the existence and the structure of such formulae.

In this section, we unravel the mystery of the existence of these identities and prove that all useful identities of $\{(0,0), 2, 6, +\}$ type must be the three-dimensional family in [11]. From now on, $(M^{2n+1}, \theta) = \mathbb{H}^n$ is Heisenberg group, then $h_{i\bar{j}} = \delta_{i\bar{j}}, R_{i\bar{j}} = 0$. All the couple indices, such as i and \overline{i} , will be considered as summation indices taking part in the process of summing from 1 to n. We study the Yamabe equation:

$$\Delta u + u^{\frac{n+2}{n}} = 0 \quad \text{on} \quad \mathbb{H}^n, \tag{4.1}$$

which is identical with (1.2) in the case $\lambda = 0$ and $\alpha = \frac{n+2}{n}$. Then, Lemma 2.4 becomes

$$D_{i,\bar{i}} = u^{-1} \sum_{i,j=1}^{n} |D_{ij}|^2 + \frac{2}{n} \frac{D_i u_{\bar{i}}}{u} - \frac{n+2}{n} \frac{E_i u_{\bar{i}}}{u} + \frac{n+2}{n} \frac{G_i u_{\bar{i}}}{u},$$

$$\begin{split} E_{i,\overline{i}} &= u^{-1}\sum_{i,j=1}^{n}|E_{i\overline{j}}|^{2} - \frac{n-1}{n}\frac{D_{\overline{i}}u_{i}}{u} + \frac{1}{n}\frac{E_{\overline{i}}u_{i}}{u} - \frac{n-1}{n}\frac{G_{\overline{i}}u_{i}}{u},\\ \operatorname{Im} G_{i,\overline{i}} &= \operatorname{Im}\left[\frac{1}{n}\frac{D_{\overline{i}}u_{i}}{u} + \frac{n+1}{n}\frac{G_{\overline{i}}u_{i}}{u}\right]. \end{split}$$

In the critical exponent case, target identities are composed of divergence of some vector fields and positive quadratic form of invariant tensors only. Similar with discussion about $\sum_{i} |G_i|^2$ in Section 2, we need $\{(0,0), 2, 6, +\}$ type identity. Rewrite (2.10) first:

$$u^{-\beta} \operatorname{Re} \left\{ u^{\beta} \left[\left(d_{1} \frac{|\nabla u|^{2}}{u} + d_{2} u^{\frac{n+2}{n}} + d_{4} n \sqrt{-1} u_{0} \right) D_{i} + \left(e_{1} \frac{|\nabla u|^{2}}{u} + e_{2} u^{\frac{n+2}{n}} + e_{4} n \sqrt{-1} u_{0} \right) E_{i} - \mu n \sqrt{-1} u_{0} G_{i} \right] \right\}_{,\bar{i}} \\ = \left[d_{1} \frac{|\nabla u|^{2}}{u^{2}} + d_{2} u^{\frac{2}{n}} \right] \sum_{i,j} |D_{ij}|^{2} + d_{1} \sum_{i} |D_{i}|^{2} + \left[e_{1} \frac{|\nabla u|^{2}}{u^{2}} + e_{2} u^{\frac{2}{n}} \right] \sum_{i,j} |E_{i\bar{j}}|^{2} + e_{1} \sum_{i} |E_{i}|^{2} + \mu \sum_{i} |G_{i}|^{2} + (d_{1} + e_{1}) \operatorname{Re} D_{i} E_{\bar{i}} - d_{4} \operatorname{Re} D_{i} G_{\bar{i}} - e_{4} \operatorname{Re} E_{i} G_{\bar{i}} + \operatorname{Re} \left[\Delta_{1} \frac{|\nabla u|^{2}}{u^{2}} + \Delta_{2} u^{\frac{2}{n}} + \Delta_{4} \frac{n \sqrt{-1} u_{0}}{u} \right] D_{i} u_{\bar{i}} + \left[\Theta_{1} \frac{|\nabla u|^{2}}{u^{2}} + \Theta_{2} u^{\frac{2}{n}} \right] E_{i} u_{\bar{i}} + \operatorname{Re} \left[\Xi_{1} \frac{|\nabla u|^{2}}{u^{2}} + \Xi_{2} u^{\frac{2}{n}} + \Xi_{4} \frac{n \sqrt{-1} u_{0}}{u} \right] G_{i} u_{\bar{i}}.$$

$$(4.2)$$

The coefficients are:

$$\begin{split} \Delta_1 &= (\beta + \frac{n+3}{n})d_1 - \frac{n-1}{n}e_1 + \frac{1}{n}d_4, \\ \Delta_2 &= -\frac{1}{n}d_1 + (\beta + \frac{n+4}{n})d_2 - \frac{n-1}{n}e_2 + \frac{1}{n}d_4, \\ \Delta_4 &= \frac{1}{n}d_1 + (\beta + \frac{n+3}{n})d_4 + \frac{n-1}{n}e_4 + \frac{1}{n}\mu, \\ \Theta_1 &= -\frac{n+2}{n}d_1 + (\beta + \frac{n+2}{n})e_1 + \frac{1}{n}e_4, \\ \Theta_2 &= -\frac{1}{n}e_1 - \frac{n+2}{n}d_2 + (\beta + \frac{n+3}{n})e_2 + \frac{1}{n}e_4, \\ \Xi_1 &= \frac{n+2}{n}d_1 - \frac{n-1}{n}e_1 - \frac{1}{n}\mu, \\ \Xi_2 &= \frac{n+2}{n}d_2 - \frac{n-1}{n}e_2 - \frac{1}{n}\mu, \\ \Xi_4 &= \frac{n+2}{n}d_4 + \frac{n-1}{n}e_4 - \beta\mu. \end{split}$$

Besides, notice that

$$\operatorname{Re}[u_{jk,\overline{i}}u_{\overline{j}}u_{\overline{k}}u_i - u_{j\overline{k},\overline{i}}u_{\overline{j}}u_ku_i]_{27}$$

$$= \operatorname{Re}[u_{j\overline{i},k}u_{\overline{j}}u_{\overline{k}}u_i + 2\sqrt{-1}u_{j0}u_{\overline{j}}|\nabla u|^2 - u_{j\overline{k},\overline{i}}u_{\overline{j}}u_ku_i]$$

$$= \operatorname{Re}[u_{\overline{i}j,k}u_{\overline{j}}u_{\overline{k}}u_i + 4\sqrt{-1}u_{0i}u_{\overline{i}}|\nabla u|^2 - u_{j\overline{k},\overline{i}}u_{\overline{j}}u_ku_i]$$

$$= 4|\nabla u|^2 \operatorname{Re}\sqrt{-1}u_{0i}u_{\overline{i}}$$

then

$$\begin{aligned} \operatorname{Re}[u_{jk}u_{\overline{j}}u_{\overline{k}}u_{i} - u_{j\overline{k}}\overline{u}_{\overline{j}}u_{k}u_{i}]_{,\overline{i}} \\ &= \operatorname{Re}[u_{jk,\overline{i}}u_{\overline{j}}u_{\overline{k}}u_{i} - u_{j\overline{k},\overline{i}}u_{\overline{j}}u_{k}u_{i}] + \operatorname{Re} u_{jk}u_{\overline{j}i}u_{\overline{k}}u_{i} + \operatorname{Re} u_{jk}u_{\overline{j}}u_{\overline{k}\overline{i}}u_{i} \\ &- \operatorname{Re} u_{j\overline{k}}u_{\overline{j}i}u_{k}u_{i} - \operatorname{Re} u_{j\overline{k}}u_{\overline{j}}u_{k}u_{i} + u_{jk}u_{\overline{j}}u_{\overline{k}}u_{i\overline{i}} - \operatorname{Re} u_{j\overline{k}}u_{\overline{j}}u_{k}u_{i\overline{i}} \\ &= 4|\nabla u|^{2}\operatorname{Re}\sqrt{-1}u_{0i}u_{\overline{i}} + 2\sum_{j}|u_{jk}u_{\overline{k}}|^{2} - \operatorname{Re} u_{j\overline{k}}u_{k}u_{\overline{j}\overline{i}}u_{i} - \sum_{k}|u_{j\overline{k}}u_{\overline{j}}|^{2} \\ &- 2\operatorname{Re}\sqrt{-1}u_{0}u_{i\overline{j}}u_{\overline{i}}u_{j} + \Delta u\operatorname{Re} u_{ij}u_{\overline{i}}u_{\overline{j}} + \operatorname{Re} n\sqrt{-1}u_{0}u_{ij}u_{\overline{i}}u_{\overline{j}} \\ &- \Delta u\operatorname{Re} u_{i\overline{j}}u_{\overline{i}}u_{j} - n\operatorname{Re}\sqrt{-1}u_{0}u_{i\overline{j}}u_{\overline{i}}u_{j} \\ &- \Delta u\operatorname{Re} u_{i\overline{j}}u_{\overline{i}}u_{j} - n\operatorname{Re}\sqrt{-1}u_{0}u_{i\overline{j}}u_{\overline{i}}u_{j} \\ &= 2u^{2}\sum_{i}|D_{i}|^{2} - u^{2}\sum_{i}|E_{i}|^{2} - u^{2}\operatorname{Re} D_{i}E_{\overline{i}} + \frac{4}{n}|\nabla u|^{2}\operatorname{Re} G_{i}u_{\overline{i}} \\ &+ \operatorname{Re}\left[\frac{3(n+3)}{n}|\nabla u|^{2} + \frac{n-1}{n}u\Delta u + (n+1)\sqrt{-1}uu_{0}\right]D_{i}u_{\overline{i}} \\ &- \left[3|\nabla u|^{2} + \frac{n+2}{n}u\Delta u\right]E_{i}u_{\overline{i}} + \frac{9n+5}{n^{2}}\frac{|\nabla u|^{6}}{u^{2}} + \frac{4}{n^{2}}\frac{|\nabla u|^{4}\Delta u}{u} \\ &- \frac{n+1}{n^{2}}|\nabla u|^{2}(\Delta u)^{2} + (n+1)|\nabla u|^{2}u_{0}^{2}. \end{aligned}$$

Hence another $\{(0,0), 2, 6, +\}$ type identity is found:

Proposition 4.1. Let β be an undetermined constant, then

$$u^{-\beta} \operatorname{Re}[u^{\beta-1}(D_{j}u_{\overline{j}} - E_{j}u_{\overline{j}})u_{i}]_{,\overline{i}}$$

$$= 2\sum_{i} |D_{i}|^{2} - \sum_{i} |E_{i}|^{2} - \operatorname{Re} D_{i}E_{\overline{i}} + \frac{3}{n} \frac{|\nabla u|^{2}}{u^{2}} \operatorname{Re} G_{i}u_{\overline{i}}$$

$$+ \operatorname{Re}\left[(\beta + \frac{n+3}{n})\frac{|\nabla u|^{2}}{u^{2}} - u^{\frac{2}{n}} + \frac{n\sqrt{-1}u_{0}}{u}\right] D_{i}u_{\overline{i}}$$

$$- \left[(\beta + \frac{n+6}{n})\frac{|\nabla u|^{2}}{u^{2}} - \frac{n+1}{n}u^{\frac{2}{n}}\right] \operatorname{Re} E_{i}u_{\overline{i}},$$
(4.4)

Proof. By Lemma 2.3 and (4.3), we yield that $\operatorname{Re}[D_{ik}u_{\overline{i}}u_{\overline{k}}u_i - E_{i\overline{k}}u_{\overline{i}}u_ku_i]_{\overline{i}}$

$$\operatorname{Re}[D_{jk}u_{\overline{j}}u_{\overline{k}}u_i - E_{j\overline{k}}u_{\overline{j}}u_ku_i]_{,\overline{i}} = \operatorname{Re}[u_{jk}u_{\overline{j}}u_{\overline{k}}u_i - u_{j\overline{k}}u_{\overline{j}}u_ku_i]_{,\overline{i}} + \operatorname{Re}\left[-\frac{3}{n}\frac{|\nabla u|^4}{u}u_i + \frac{1}{n}\Delta u|\nabla u|^2u_i + \sqrt{-1}u_0|\nabla u|^2u_i\right]_{,\overline{i}} = 2u^2\sum_i |D_i|^2 - u^2\sum_i |E_i|^2 - u^2\operatorname{Re}D_iE_{\overline{i}} + \frac{3}{n}|\nabla u|^2\operatorname{Re}G_iu_{\overline{i}}$$

$$= 2u^2\sum_i |D_i|^2 - u^2\sum_i |E_i|^2 - u^2\operatorname{Re}D_iE_{\overline{i}} + \frac{3}{n}|\nabla u|^2\operatorname{Re}G_iu_{\overline{i}}$$

$$+\operatorname{Re}\left[\frac{3(n+1)}{n}|\nabla u|^{2}+u\Delta u+n\sqrt{-1}uu_{0}\right]D_{i}u_{\overline{i}}-\left[\frac{3(n+2)}{n}|\nabla u|^{2}+\frac{n+1}{n}u\Delta u\right]E_{i}u_{\overline{i}},$$

then (4.4) is proved by inserting $u^{\beta-2}$ into vector field.

Remark 4.2. It's noteworthy that, the (4.4) type identities in general CR manifolds are omitted, because some Webster curvature terms occur. Without other assumptions of Webster curvature, those terms are tricky.

To seek for all $\{(0,0), 2, 6, +\}$ type identities with invariant tensors as RHS, the vector fields composed of non-invariant things are also needed. Let β be an undetermined constant, and consider

$$u^{-\beta} \operatorname{Re}[u^{\beta-3} |\nabla u|^4 u_i]_{,\bar{i}}$$

=2 $\frac{|\nabla u|^2}{u^2} (\operatorname{Re} D_i u_{\bar{i}} + E_i u_{\bar{i}}) + (\beta + \frac{n+2}{n}) \frac{|\nabla u|^6}{u^4} - \frac{n+2}{n} u^{\frac{2}{n}-2} |\nabla u|^4,$ (4.5)

$$u^{-\beta} \operatorname{Re}[u^{\beta+\frac{2}{n}-1}|\nabla u|^{2}u_{i}]_{,\bar{i}}$$

= $u^{\frac{2}{n}} (\operatorname{Re} D_{i}u_{\bar{i}} + E_{i}u_{\bar{i}}) + (\beta + \frac{n+3}{n})u^{\frac{2}{n}-2}|\nabla u|^{4} - \frac{n+1}{n}u^{\frac{4}{n}}|\nabla u|^{2},$ (4.6)

$$u^{-\beta} \operatorname{Re}[u^{\beta-2}|\nabla u|^{2} \cdot n\sqrt{-1}u_{0}u_{i}]_{,\overline{i}}$$

$$= -\operatorname{Re}\frac{n\sqrt{-1}u_{0}}{u}D_{i}u_{\overline{i}} - \frac{|\nabla u|^{2}}{u^{2}}\operatorname{Re}G_{i}u_{\overline{i}} + \frac{1}{n}\frac{|\nabla u|^{6}}{u^{4}}$$

$$+ \frac{1}{n}u^{\frac{2}{n}-2}|\nabla u|^{4} - n(n+1)\frac{|\nabla u|^{2}u_{0}^{2}}{u^{2}},$$
(4.7)

$$u^{-\beta} \operatorname{Re}[u^{\beta + \frac{4}{n} + 1} u_i]_{,\overline{i}} = (\beta + \frac{n+4}{n})u^{\frac{4}{n}} |\nabla u|^2 - u^{\frac{2n+6}{n}},$$

$$u^{-\beta} \operatorname{Re}[u^{\beta+\frac{2}{n}} \cdot n\sqrt{-1}u_0 u_i]_{,\bar{i}}$$

= $-u^{\frac{2}{n}} \operatorname{Re} G_i u_{\bar{i}} + \frac{1}{n} u^{\frac{2}{n}-2} |\nabla u|^4 + \frac{1}{n} u^{\frac{4}{n}} |\nabla u|^2 - n^2 u^{\frac{2}{n}} u_0^2,$ (4.8)

$$u^{-\beta} \operatorname{Re}[u^{\beta-1}n^{2}u_{0}^{2}u_{i}]_{,\bar{i}}$$

= $-2 \operatorname{Re} \frac{n\sqrt{-1}u_{0}}{u}G_{i}u_{\bar{i}} + n(n\beta + n + 2)\frac{|\nabla u|^{2}u_{0}^{2}}{u^{2}} - n^{2}u^{\frac{2}{n}}u_{0}^{2}.$ (4.9)

Eliminate all terms except for invariant tensor terms, we yield the following identity:

Proposition 4.3. Let β be an undetermined constant, then

$$\begin{split} u^{-\beta} \operatorname{Re} & \left\{ u^{\beta-1} \Big[\frac{|\nabla u|^4}{u^2} + u^{\frac{2}{n}} |\nabla u|^2 - (n\beta + n + 2) \frac{|\nabla u|^2 \cdot n\sqrt{-1}u_0}{u} \\ & + (n+1)u^{\frac{n+2}{n}} \cdot n\sqrt{-1}u_0 - (n+1)n^2 u_0^2 \Big] u_i \right\}_{,\overline{i}} \\ &= \operatorname{Re} \left[2 \frac{|\nabla u|^2}{u^2} + u^{\frac{2}{n}} + (n\beta + n + 2) \frac{n\sqrt{-1}u_0}{u} \right] D_i u_{\overline{i}} + \left[2 \frac{|\nabla u|^2}{u^2} + u^{\frac{2}{n}} \right] E_i u_{\overline{i}} \\ & + \operatorname{Re} \left[(n\beta + n + 2) \frac{|\nabla u|^2}{u^2} - (n+1)u^{\frac{2}{n}} + 2(n+1) \frac{n\sqrt{-1}u_0}{u} \right] G_i u_{\overline{i}}. \end{split}$$

$$\begin{aligned} & \text{Proof.} \ (4.5) + (4.6) - (n\beta + n + 2) \times (4.7) + (n+1) \times (4.8) - (n+1) \times (4.9). \end{aligned}$$

Remark 4.4. Because of $u^{\frac{2n+6}{n}}$ term, vector field $u^{-\beta} \operatorname{Re}[u^{\beta+\frac{4}{n}+1}u_i]_{,i}$ is useless. Besides, identity (4.10) with n = 1 is identical to (2.11) with $\alpha = 3$ and $\lambda = 0$.

Similar as Riemannian $\{(0,0), 2, 4, +\}$ case in [1] and [4], all reasonable $\{(0,0), 2, 6, +\}$ type identities with invariant tensors as RHS are found. Linearly combine (4.2), (4.4) and (4.10) together. Since we hope that a non-trivial solution exists in the critical exponent case, the cross terms must vanish. Here the cross terms are:

$$\frac{|\nabla u|^2}{u^2} \operatorname{Re} D_i u_{\overline{i}}, \quad u^{\frac{2}{n}} \operatorname{Re} D_i u_{\overline{i}}, \quad \operatorname{Re} \frac{n\sqrt{-1}u_0}{u} D_i u_{\overline{i}}, \quad \frac{|\nabla u|^2}{u^2} E_i u_{\overline{i}},$$
$$u^{\frac{2}{n}} E_i u_{\overline{i}}, \quad \frac{|\nabla u|^2}{u^2} \operatorname{Re} G_i u_{\overline{i}}, \quad u^{\frac{2}{n}} \operatorname{Re} G_i u_{\overline{i}}, \quad \operatorname{Re} \frac{n\sqrt{-1}u_0}{u} G_i u_{\overline{i}}.$$

If not, take some D_i term for example, then we'll yield that $D_{ij} + c \frac{u_i u_j}{u} = 0$ for some $c \neq 0$ by writing into a complete square form. However, $D_{ij} + c \frac{u_i u_j}{u}$ is not an invariant tensor, i.e. $0 = \left(D_{ij} + c \frac{u_i u_j}{u}\right)_{\bar{i}}$ is composed by some non-invariant things, which will cause that u can only be a constant.

With the idea above, linearly combine RHS of (4.2), (4.4) and (4.10):

$$\begin{aligned} \text{RHS of } \left[(4.2) + a \times (4.4) + b \times (4.10) \right] \\ &= \left[d_1 \frac{|\nabla u|^2}{u^2} + d_2 u^{\frac{2}{n}} \right] \sum_{i,j} |D_{ij}|^2 + (d_1 + 2a) \sum_i |D_i|^2 + \left[e_1 \frac{|\nabla u|^2}{u^2} + e_2 u^{\frac{2}{n}} \right] \sum_{i,j} |E_{i\overline{j}}|^2 \\ &+ (e_1 - a) \sum_i |E_i|^2 + \mu \sum_i |G_i|^2 + (d_1 + e_1 - a) \operatorname{Re} D_i E_{\overline{i}} - d_4 \operatorname{Re} D_i G_{\overline{i}} \\ &- e_4 \operatorname{Re} E_i G_{\overline{i}} + \operatorname{Re} \left[\widetilde{\Delta}_1 \frac{|\nabla u|^2}{u^2} + \widetilde{\Delta}_2 u^{\frac{2}{n}} + \widetilde{\Delta}_4 \frac{n\sqrt{-1}u_0}{u} \right] \operatorname{Re} D_i u_{\overline{i}} \\ &+ \left[\widetilde{\Theta}_1 \frac{|\nabla u|^2}{u^2} + \widetilde{\Theta}_2 u^{\frac{2}{n}} \right] E_i u_{\overline{i}} + \operatorname{Re} \left[\widetilde{\Xi}_1 \frac{|\nabla u|^2}{u^2} + \widetilde{\Xi}_2 u^{\frac{2}{n}} + \widetilde{\Xi}_4 \frac{n\sqrt{-1}u_0}{u} \right] \operatorname{Re} G_i u_{\overline{i}}, \end{aligned}$$

where a and b are undertermined constants. The coefficients are:

$$\begin{split} \widetilde{\Delta}_1 &= (\beta + \frac{n+3}{n})d_1 - \frac{n-1}{n}e_1 + \frac{1}{n}d_4 + (\beta + \frac{n+3}{n})a + 2b, \\ \widetilde{\Delta}_2 &= -\frac{1}{n}d_1 + (\beta + \frac{n+4}{n})d_2 - \frac{n-1}{n}e_2 + \frac{1}{n}d_4 - a + b, \\ \widetilde{\Delta}_4 &= \frac{1}{n}d_1 + (\beta + \frac{n+3}{n})d_4 + \frac{n-1}{n}e_4 + \frac{1}{n}\mu + a + (n\beta + n + 2)b, \\ \widetilde{\Theta}_1 &= -\frac{n+2}{n}d_1 + (\beta + \frac{n+2}{n})e_1 + \frac{1}{n}e_4 - (\beta + \frac{n+6}{n})a + 2b, \\ \widetilde{\Theta}_2 &= -\frac{1}{n}e_1 - \frac{n+2}{n}d_2 + (\beta + \frac{n+3}{n})e_2 + \frac{1}{n}e_4 + \frac{n+1}{n}a + b, \\ \widetilde{\Xi}_1 &= \frac{n+2}{n}d_1 - \frac{n-1}{n}e_1 - \frac{1}{n}\mu + \frac{3}{n}a + (n\beta + n + 2)b, \\ \widetilde{\Xi}_2 &= \frac{n+2}{n}d_2 - \frac{n-1}{n}e_2 - \frac{1}{n}\mu - (n+1)b, \\ \widetilde{\Xi}_4 &= \frac{n+2}{n}d_4 + \frac{n-1}{n}e_4 - \beta\mu + 2(n+1)b. \end{split}$$

Case $n \ge 2$: Let $\widetilde{\Delta}_l$, $\widetilde{\Theta}_l$ and $\widetilde{\Xi}_l$ be 0. Fix d_l , μ , a, b, and solve e_l from $\widetilde{\Xi}_1 = \widetilde{\Xi}_2 = \widetilde{\Xi}_4 = 0$: $e_1 = \frac{(n+2)d_1 - \mu + 3a + n(n\beta + n + 2)b}{n-1},$ $e_2 = \frac{(n+2)d_2 - \mu - n(n+1)b}{n-1},$ $e_4 = \frac{-(n+2)d_4 + n\beta\mu - 2n(n+1)b}{n-1}.$

Insert e_l into $\widetilde{\Delta}_1 - \widetilde{\Delta}_4$, then $\widetilde{\Delta}_1 - \widetilde{\Delta}_4 = \beta(d_1 - d_4 + a - 2nb - \mu)$. If $\beta = 0$, we have $\widetilde{\Delta}_1 = \widetilde{\Delta}_4$. Fix d_1 , μ , a, b, and solve d_2 , d_4 from $\widetilde{\Delta}_1 = \widetilde{\Delta}_2 = 0$:

$$d_2 = d_1 + na - n(n+1)b, \quad d_4 = -d_1 - na + n^2b - \mu.$$

Insert d_2 , d_4 , e_l and β into $\widetilde{\Theta}_1$ and $\widetilde{\Theta}_2$, then

$$\widetilde{\Theta}_1 = \frac{2[2(n+2)d_1 + 6a + n^2b]}{n(n-1)}, \quad \widetilde{\Theta}_2 = \frac{2(n+2)[2d_1 + (3n-1)a - n(3n+4)b]}{n(n-1)}.$$

Fix *b*, and solve d_1 , *a* from $\widetilde{\Theta}_1 = \widetilde{\Theta}_2 = 0$: $d_1 = -\frac{n(n+3)b}{2(n-1)}$, $a = \frac{n(n+1)b}{n-1}$. To ensure the positivity or negativity of the RHS of (4.11), the coefficients of $\frac{|\nabla u|^2}{u^2} \sum_{i,j} |D_{ij}|^2$ and $u^{\frac{2}{n}} \sum_{i,j} |D_{ij}|^2$ must have the same sign, i.e. $d_1 d_2 \ge 0$. Insert d_1 and *a* into d_2 : $d_2 = \frac{n}{2}b$, hence b = 0. Similarly, the coefficients of $u^{\frac{2}{n}} \sum_{i,j} |E_{i\bar{j}}|^2$ and $\sum_i |G_i|^2$ must have the same sign, i.e. $e_2\mu \ge 0$. Insert $d_2 = b = 0$ into e_2 : $e_2 = -\frac{\mu}{n-1}$, hence $\mu = 0$. Now, all parameters are 0, and the identity (4.11) is trivial.

If $\beta \neq 0$, then $d_4 = d_1 + a - 2nb - \mu$. Insert d_4 and e_l into $\widetilde{\Delta}_1$, $\widetilde{\Delta}_2$, and $\widetilde{\Delta}_4$:

$$\widetilde{\Delta}_{1} = \widetilde{\Delta}_{4} = \frac{1}{n} [(n\beta + 2)d_{1} + (n\beta + n + 1)a - (n\beta + n + 2)nb]$$
$$\widetilde{\Delta}_{2} = \frac{1}{n} [(n\beta + 2)d_{2} - (n - 1)a + n^{2}b].$$

If $\beta \neq 0$ and $\beta \neq -\frac{2}{n}$, d_1 and d_2 can be solved from $\widetilde{\Delta}_1 = \widetilde{\Delta}_2 = \widetilde{\Delta}_4 = 0$: $d_1 = \frac{-(n\beta + n + 1)a + (n\beta + n + 2)nb}{n\beta + 2}, \quad d_2 = \frac{(n-1)a - n^2b}{n\beta + 2}.$

Insert d_l and e_l into $\widetilde{\Theta}_1$ and $\widetilde{\Theta}_2$:

$$\widetilde{\Theta}_1 = \frac{(n\beta + n + 4)[-2(n-1)a + (n\beta + 2n + 2)nb]}{n(n-1)},$$

$$\widetilde{\Theta}_2 = -\frac{(n+2)(n\beta + 4)[-2(n-1)a + (n\beta + 2n + 2)nb]}{n(n-1)(n\beta + 2)}$$

then $a = \frac{(n\beta + 2n + 2)nb}{2(n-1)}$. To ensure the positivity or negativity of the RHS of (4.11), the coefficients of $u^{\frac{2}{n}} \sum_{i,j} |D_{ij}|^2$, $u^{\frac{2}{n}} \sum_{i,j} |E_{i\bar{j}}|^2$ and $\sum_i |G_i|^2$ must have the same sign, i.e.

 d_2 , e_2 and μ have the same sign. Insert *a* into d_2 and e_2 : $d_2 = \frac{n}{2}b$, $e_2 = -\frac{2\mu + n^2b}{2(n-1)}$, hence $b = \mu = 0$. Then, all parameters are 0, which means that the identity (4.11) is trivial again.

From discussions above, $\beta = -\frac{2}{n}$ is the only possible case when $n \ge 2$. When $\beta = -\frac{2}{n}$, rewrite d_4 and e_l : $(n \pm 2)d = n + 2a \pm n^2h$

$$d_4 = d_1 - \mu + a - 2nb, \quad e_1 = \frac{(n+2)d_1 - \mu + 3a + n^2b}{n-1},$$
$$e_2 = \frac{(n+2)d_2 - \mu - n(n+1)b}{n-1}, \quad e_4 = \frac{-(n+2)d_1 + n\mu - (n+2)a + 2nb}{n-1}$$

Insert them and $\beta = -\frac{2}{n}$ into $\widetilde{\Delta}_l$ and $\widetilde{\Theta}_l$:

$$\widetilde{\Delta}_{1} = -\widetilde{\Delta}_{2} = \widetilde{\Delta}_{4} = -\frac{n-1}{2(n+2)}\widetilde{\Theta}_{1} = \frac{n-1}{n}a - nb,$$

$$\widetilde{\Theta}_{2} = \frac{(n+2)}{n(n-1)}[-2d_{1} + 2d_{2} + (n-3)a - n^{2}b].$$

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Fix d_1 and a, and solve d_2 and b from $\widetilde{\Delta}_1 = \widetilde{\Theta}_2 = 0$: $d_2 = d_1 + a$, $b = \frac{n-1}{n^2}a$. Then

$$\beta = -\frac{2}{n}, \quad b = \frac{n-1}{n^2}a, \quad d_2 = d_1 + a, \quad d_4 = d_1 - \frac{n-2}{n}a - \mu, \quad e_1 = \frac{(n+2)(d_1 + a) - \mu}{n-1},$$
$$e_2 = \frac{(n+2)d_1 + (2+\frac{1}{n})a - \mu}{n-1}, \quad e_4 = \frac{-(n+2)d_1 - (n+\frac{2}{n})a + n\mu}{n-1}.$$

Rewrite the identity (4.11) with the parameters above as the following proposition. This identity has three undetermined parameters $\{d_1, a, \mu\}$.

Proposition 4.5. For $n \ge 2$. The only positive $\{(0,0), 2, 6, +\}$ type identity is

$$\begin{split} u^{\frac{2}{n}} \operatorname{Re} \left\{ u^{-\frac{2}{n}} \left\{ \left(d_{1} \frac{|\nabla u|^{2}}{u} + (d_{1} + a)u^{\frac{n+2}{n}} + (d_{1} - \frac{n-2}{n}a - \mu)n\sqrt{-1}u_{0} \right) D_{i} \right. \\ &+ \left(\frac{(n+2)(d_{1} + a) - \mu}{n-1} \frac{|\nabla u|^{2}}{u} + \frac{(n+2)d_{1} + (2 + \frac{1}{n})a - \mu}{n-1} u^{\frac{n+2}{n}} \right. \\ &+ \frac{-(n+2)d_{1} - (n + \frac{2}{n})a + n\mu}{n-1} \cdot n\sqrt{-1}u_{0} \right) E_{i} - \mu n\sqrt{-1}u_{0}G_{i} \\ &+ a \left[D_{j}u_{\overline{j}} - E_{j}u_{\overline{j}} + \frac{n-1}{n^{2}} \left(\frac{|\nabla u|^{4}}{u^{2}} + u^{\frac{2}{n}} |\nabla u|^{2} - n \frac{|\nabla u|^{2} \cdot n\sqrt{-1}u_{0}}{u} \right. \\ &+ \left(n + 1)u^{\frac{n+2}{n}} \cdot n\sqrt{-1}u_{0} - (n+1)n^{2}u_{0}^{2} \right) \right] \frac{u_{i}}{u} \right\} \right\}_{,\overline{i}} \\ &+ \left[d_{1} \frac{|\nabla u|^{2}}{u^{2}} + (d_{1} + a)u^{\frac{2}{n}} \right] \sum_{i,j} |D_{ij}|^{2} + (d_{1} + 2a) \sum_{i} |D_{i}|^{2} \\ &+ \left[\frac{(n+2)(d_{1} + a) - \mu}{n-1} \frac{|\nabla u|^{2}}{u^{2}} + \frac{(n+2)d_{1} + (2 + \frac{1}{n})a - \mu}{n-1} u^{\frac{2}{n}} \right] \sum_{i,j} |E_{i\overline{j}}|^{2} \\ &+ \left(\frac{(n+2)d_{1} + 3a - \mu}{n-1} \sum_{i} |E_{i}|^{2} + \mu \sum_{i} |G_{i}|^{2} + \frac{(2n+1)d_{1} + 3a - \mu}{n-1} \operatorname{Re} D_{i}E_{\overline{i}} \\ &+ \left(-d_{1} + \frac{n-2}{n}a + \mu \right) \operatorname{Re} D_{i}G_{\overline{i}} + \frac{(n+2)d_{1} + (n+\frac{2}{n})a - n\mu}{n-1} \operatorname{Re} E_{i}G_{\overline{i}}. \end{split}$$

The parameters d_1 , a, and μ satisfy

$$d_1 \ge \max\{0, -a\}, \quad (n+2)d_1 - \mu \ge \max\{-(n+2)a, -(2+\frac{1}{n})a\}, \tag{4.13}$$

and the following matrix is semi-positive:

$$\begin{pmatrix} \mu & \frac{1}{2}(-d_1 + \frac{n-2}{n}a + \mu) & \frac{(n+2)d_1 + (n+\frac{2}{n})a - n\mu}{2(n-1)} \\ \frac{1}{2}(-d_1 + \frac{n-2}{n}a + \mu) & 2(d_1 + a) & \frac{(2n+1)d_1 + 3a - \mu}{2(n-1)} \\ \frac{(n+2)d_1 + (n+\frac{2}{n})a - n\mu}{2(n-1)} & \frac{(2n+1)d_1 + 3a - \mu}{2(n-1)} & \frac{2n-1}{(n-1)^2}[(n+2)d_1 + \frac{n^2 + 5n - 3}{2n-1}a - \mu] \end{pmatrix}$$

$$(4.14)$$

Proof. The coefficients of $\frac{|\nabla u|^2}{u^2} \sum_{i,j} |D_{ij}|^2$, $u^{\frac{2}{n}} \sum_{i,j} |D_{ij}|^2$, $\frac{|\nabla u|^2}{u^2} \sum_{i,j} |E_{i\overline{j}}|^2$ and $u^{\frac{2}{n}} \sum_{i,j} |E_{i\overline{j}}|^2$ are non-negative, i.e. (4.13). By Lemma 2.2, the RHS of identity is greater than or equal to a quadratic form with (4.14) as matrix.

Now we prove Theorem 1.3.

Proof of Theorem 1.3. From Prop.4.5. we know that three constants d_1 , a, and μ determine a three-dimensional family of differential identities as Jerison-Lee stated.

Three constants d_1 , a, and μ determine a three-dimensional family of differential identities as Jerison-Lee stated. If $d_1 = 1$, a = 0 and $\mu = 3$, we yield the classical Jerison-Lee identity (4.2) in [11]:

$$\begin{split} u^{\frac{2}{n}} \operatorname{Re} \left\{ u^{-\frac{2}{n}} \left[\left(\frac{|\nabla u|^2}{u} + u^{\frac{n+2}{n}} \right) (D_i + E_i) - n\sqrt{-1}u_0 (2D_i - 2E_i + 3G_i) \right] \right\}_{,\overline{i}} \\ &= \left[\frac{|\nabla u|^2}{u^2} + u^{\frac{2}{n}} \right] \sum_{i,j} (|D_{ij}|^2 + |E_{i\overline{j}}|^2) + \sum_i (|D_i|^2 + |E_i|^2 + 3|G_i|^2) \\ &+ 2\operatorname{Re} D_i E_{\overline{i}} + 2\operatorname{Re} D_i G_{\overline{i}} - 2\operatorname{Re} E_i G_{\overline{i}} \\ &= u^{\frac{2}{n}} \sum_{i,j} (|D_{ij}|^2 + |E_{i\overline{j}}|^2) + \sum_i (|G_i|^2 + |G_i + D_i|^2 + |G_i - E_i|^2) + u^{-2} \sum_{i,j,k} |D_{ij}u_{\overline{k}} + E_{i\overline{k}}u_j|^2. \end{split}$$

If $d_1 = 0$, a = n and $\mu = n + 2$, we yield the identity (4.3) in [11], which is also positive:

$$u^{\frac{2}{n}} \operatorname{Re} \left\{ u^{-\frac{2}{n}} \left\{ \left(nu^{\frac{n+2}{n}} - 2n^2 \sqrt{-1} u_0 \right) D_i + \left((n+2) \frac{|\nabla u|^2}{u} + u^{\frac{n+2}{n}} + 2n \sqrt{-1} u_0 \right) E_i - (n+2)n \sqrt{-1} u_0 G_i + n \left[D_j u_{\overline{j}} - E_j u_{\overline{j}} + \frac{n-1}{n^2} \left(\frac{|\nabla u|^4}{u^2} + u^{\frac{2}{n}} |\nabla u|^2 - n \frac{|\nabla u|^2 \cdot n \sqrt{-1} u_0}{u} + (n+1)u^{\frac{n+2}{n}} \cdot n \sqrt{-1} u_0 - (n+1)n^2 u_0^2 \right) \right] \frac{u_i}{u} \right\} \right\}_{,\overline{i}}$$
$$= nu^{\frac{2}{n}} \sum_{i,j} |D_{ij}|^2 + 2n \sum_i |D_i|^2 + \left[(n+2) \frac{|\nabla u|^2}{u^2} + u^{\frac{2}{n}} \right] \sum_{i,j} |E_{i\overline{j}}|^2$$

$$+ 2\sum_{i} |E_{i}|^{2} + (n+2)\sum_{i} |G_{i}|^{2} + 2 \operatorname{Re} D_{i}E_{\overline{i}} + 2n \operatorname{Re} D_{i}G_{\overline{i}} - 2 \operatorname{Re} E_{i}G_{\overline{i}}$$
$$= (n+2)\frac{|\nabla u|^{2}}{u^{2}}\sum_{i,j} |E_{i\overline{j}}|^{2} + \sum_{i} |E_{i}|^{2} + (n-2)\sum_{i} |D_{i}|^{2} + (n+1)\sum_{i} |G_{i} + D_{i}|^{2}$$
$$+ \sum_{i} |G_{i} - D_{i} - E_{i}|^{2} + u^{\frac{2}{n}}\sum_{i,j} (|E_{i\overline{j}}|^{2} + n|D_{ij}|^{2}).$$

if $d_1 = 1$, a = 0 and $\mu = 3n$, we yield the identity (4.4) in [11], which is not positive:

$$u^{\frac{2}{n}} \operatorname{Re} \left\{ u^{-\frac{2}{n}} \left\{ \left(\frac{|\nabla u|^2}{u} + u^{\frac{n+2}{n}} \right) (D_i - 2E_i) - n\sqrt{-1}u_0 [(3n-1)D_i - (3n+2)E_i + 3nG_i] \right\} \right\}_{,\bar{i}}$$

$$= \left[\frac{|\nabla u|^2}{u^2} + u^{\frac{2}{n}} \right] \sum_{i,j} (|D_{ij}|^2 - 2|E_{i\bar{j}}|^2) + \sum_i (|D_i|^2 - 2|E_i|^2 + 3n|G_i|^2) - \operatorname{Re} D_i E_{\bar{i}} + (3n-1)\operatorname{Re} D_i G_{\bar{i}} - (3n+2)\operatorname{Re} E_i G_{\bar{i}}.$$

Notice that the matrix (4.14) can't be semi-positive if $\mu = 0$. W.L.O.G., assume that $\mu = 3$, then the positivity condition (4.13) and matrix (4.14) ≥ 0 determine the range for d_1 and a, which can be described by the following figure:



FIGURE 4.1. The range for d_1 and a when identity (4.12) is positive.

The coordinates of key points are:

P1 = (1,0), **P2** =
$$(0, \frac{3n}{n+2})$$
, **P3** = $(\frac{1}{n}, 0)$, **A1** = $(0, \frac{3n}{2n+1})$,
₃₅

$$\begin{split} \mathrm{A2} = & \left(0, \frac{2\sqrt{n(73n^7 + 538n^6 + 1435n^5 + 134n^4 - 1439n^3 - 120n^2 + 292n + 48)}}{3n^4 - 2n^3 - 5n^2 + 26n + 8} \\ & \times \cos\left\{\frac{1}{3}\arccos\left[\sqrt{n}(595n^{10} + 7017n^9 + 30666n^8 + 55019n^7 - 7692n^6 - 82095n^5 - 12345n^4 + 38598n^3 + 2556n^2 - 6920n - 1440}\right] \\ & \times (73n^7 + 538n^6 + 1435n^5 + 134n^4 - 1439n^3 - 120n^2 + 292n + 48)^{-3/2}\right] \right\} \\ & + \frac{n(10n^3 + 35n^2 + 4)}{(n+2)(3n^3 - 8n^2 + 11n + 4)} \Big), \\ \mathrm{B1} = \left(\frac{\sqrt{468n^4 + 1380n^3 + n^2 - 1500n + 612}}{3n^2 + 8n + 4}\cos\frac{\theta - 2\pi}{3} + \frac{24n^2 + 43n - 18}{2(n+2)(3n+2)}, 0\right), \\ \mathrm{B2} = \left(\frac{\sqrt{468n^4 + 1380n^3 + n^2 - 1500n + 612}}{3n^2 + 8n + 4}\cos\frac{\theta}{3} + \frac{24n^2 + 43n - 18}{2(n+2)(3n+2)}, 0\right), \\ \theta = \arccos\frac{9936n^6 + 44172n^5 + 32202n^4 - 66149n^3 - 35622n^2 + 54756n - 15336}{(468n^4 + 1380n^3 + n^2 - 1500n + 612)^{\frac{3}{2}}}, \end{split}$$

where P1, P2, P3 correspond with identity (4.2), (4.3), (4.4) in [11], and A1, A2, B1, B2 are intersections of coordinate axes and boundary of the range. From the figure, it's obvious that (4.2) and (4.3) are positive, and (4.4) is not positive.

Case n = 1: It's easy to check that the identity (4.12) degenerates to classical Jerison-Lee identity (4.2) in [11] when n = 1, hence identity (4.2), (4.3) and (4.4) in [11] are identical. In fact, it's the only possible $\{(0,0), 2, 6, +\}$ type identity when n = 1.

Proposition 4.6. For n = 1, the only positive $\{(0,0), 2, 6, +\}$ type identity is the classical Jerison-Lee identity (4.2) in [11].

Proof. Notice that (4.4) degenerates to $u^{-\beta}[u^{\beta-1}|\nabla u|^2 D_1]_{,\overline{1}}$, which is the d_1 term in the vector field of (4.2). Hence we assume that a = 0. Rewrite (4.11) as

ī.

$$\begin{split} & [(4.2) + b \times (4.10)] \Big|_{n=1} \\ &= \left[d_1 \frac{|\nabla u|^2}{u^2} + d_2 u^{\frac{2}{n}} \right] \sum_{i,j} |D_{ij}|^2 + d_1 \sum_i |D_i|^2 + \mu \sum_i |G_i|^2 - d_4 \operatorname{Re} D_i G_{\overline{i}} \\ &+ \operatorname{Re} \left[\widetilde{\Delta}_1 \frac{|\nabla u|^2}{u^2} + \widetilde{\Delta}_2 u^{\frac{2}{n}} + \widetilde{\Delta}_4 \frac{n\sqrt{-1}u_0}{u} \right] \operatorname{Re} D_i u_{\overline{i}} \\ &+ \operatorname{Re} \left[\widetilde{\Xi}_1 \frac{|\nabla u|^2}{u^2} + \widetilde{\Xi}_2 u^{\frac{2}{n}} + \widetilde{\Xi}_4 \frac{n\sqrt{-1}u_0}{u} \right] \operatorname{Re} G_i u_{\overline{i}}, \end{split}$$
(4.15)

The coefficients are:

$$\widetilde{\Delta}_1 = (\beta + 4)d_1 + d_4 + 2b, \quad \widetilde{\Delta}_2 = -d_1 + (\beta + 5)d_2 + d_4 + b, \\ \widetilde{\Delta}_4 = d_1 + (\beta + 4)d_4 + \mu + (\beta + 3)b, \quad \widetilde{\Xi}_1 = 3d_1 - \mu + (\beta + 3)b, \\ 36$$

 $\widetilde{\Xi}_2 = 3d_2 - \mu - 2b, \quad \widetilde{\Xi}_4 = 3d_4 - \beta\mu + 4b.$ Fix β , μ , b, and solve d_1 , d_2 , d_4 from $\widetilde{\Xi}_1 = \widetilde{\Xi}_2 = \widetilde{\Xi}_4 = 0$:

$$d_1 = \frac{\mu + (\beta + 3)b}{3}, \quad d_2 = \frac{\mu + 2b}{3}, \quad d_4 = \frac{\beta\mu + 4b}{3}.$$

Insert them into Δ_l :

$$\widetilde{\Delta}_{1} = \frac{2(\beta+2)\mu + (\beta^{2}+7\beta+22)b}{3},$$
$$\widetilde{\Delta}_{2} = \frac{2(\beta+2)\mu + (\beta+14)b}{3},$$
$$\widetilde{\Delta}_{4} = \frac{(\beta+2)^{2}\mu + 4(2\beta+7)}{3}.$$

Consider $\widetilde{\Delta}_1 - \widetilde{\Delta}_2 = 0$, i.e. $(\beta + 2)(\beta + 4)b = 0$. If $\beta = -4$, then $\widetilde{\Delta}_1 = \frac{-4\mu + 10b}{3} = 0$, $\widetilde{\Delta}_4 = \frac{4\mu - 4b}{3} = 0$, hence $\mu = b = 0$, then $d_1 = d_2 = d_4 = 0$, which means that the identity is trivial.

If $\beta \neq -2$ and $\beta \neq -4$, b = 0, $\mu = 0$, then identity is trivial as well. From discussions above, $\beta = -2$ is the only possible case when n = 1. H

f
$$\beta = -2$$
, then $\Delta_2 = 0$ yields that $b = 0$. All parameters are:

$$d_1 = d_2 = \frac{\mu}{3}, \quad d_4 = -\frac{2}{3}\mu, \quad \beta = -2, \quad b = 0,$$

which is just identical to the classical Jerison-Lee identity case.

All possible identities can be found by dimensional conservation and invariant tensors, and then a linear combination of them with some appropriate parameters will always work. For the 'near-critical' subcritical exponent case in CR geometry, the best choice is to use the same structure of identity as the critical exponent case. For example, the vector fields in Case 2 in Section 3 are composed of invariant tensors. However, we need a little change with (2.11) for Case 3 in Section 3, but we don't break the continuity when $\alpha \to 3$. For the case α far away from the critical exponent, without caring about the critical exponent case, our choices of parameters are flexible, such as Case 1 and Case 4 in Section 3.

References

- [1] Marie-Françoise Bidaut-Véron and Laurent Véron. Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations. Invent. Math., 106(3):489–539, 1991.
- [2] Thomas P. Branson, Luigi Fontana, and Carlo Morpurgo. Moser-Trudinger and Beckner-Onofri's inequalities on the CR sphere. Ann. of Math. (2), 177(1):1-52, 2013.
- [3] Giovanni Catino, Yanyan Li, Dario D. Monticelli, and Alberto Roncoron. A Liouville theorem in the Heisenberg group. Preprint, arXiv:2310.10469.
- [4] Jean Dolbeault, Maria J. Esteban, and Michael Loss. Nonlinear flows and rigidity results on compact manifolds. J. Funct. Anal., 267(5):1338–1363, 2014.
- Sorin Dragomir and Giuseppe Tomassini. Differential geometry and analysis on CR manifolds, volume $\left|5\right|$ 246 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 2006.

- [6] Joshua Flynn and Jérôme Vétois. Liouville-type results for the CR Yamabe equation in the Heisenberg group. *Preprint*, arXiv:2310.14048.
- [7] G. B. Folland and E. M. Stein. Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group. Comm. Pure Appl. Math., 27:429–522, 1974.
- [8] Rupert L. Frank and Elliott H. Lieb. Sharp constants in several inequalities on the Heisenberg group. Ann. of Math. (2), 176(1):349–381, 2012.
- B. Gidas and J. Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations. Comm. Pure Appl. Math., 34(4):525–598, 1981.
- [10] David Jerison and John M. Lee. The Yamabe problem on CR manifolds. J. Differential Geom., 25(2):167–197, 1987.
- [11] David Jerison and John M. Lee. Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem. J. Amer. Math. Soc., 1(1):1–13, 1988.
- [12] Xi-Nan Ma and Qianzhong Ou. A Liouville theorem for a class semilinear elliptic equations on the Heisenberg group. Adv. Math., 413:Paper No. 108851, 20, 2023.
- [13] Morio Obata. The conjectures on conformal transformations of Riemannian manifolds. J. Differential Geometry, 6:247-258, 1971/72.
- [14] Xiaodong Wang. On a remarkable formula of Jerison and Lee in CR geometry. Math. Res. Lett., 22(1):279–299, 2015.
- [15] Xiaodong Wang. Uniqueness results on a geometric PDE in Riemannian and CR geometry revisited. Math. Z., 301(2):1299–1314, 2022.
- [16] Lu Xu. Semi-linear Liouville theorems in the Heisenberg group via vector field methods. J. Differential Equations, 247(10):2799–2820, 2009.

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