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Research article

Gradient estimates for the solutions of higher order curvature equations with prescribed contact angle†

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Abstract: In this paper, we use the maximum principle and moving frame technique to prove the global gradient estimates for the higher-order curvature equations with prescribed contact angle problems.

Keywords: fully nonlinear elliptic; higher-order curvature equations; gradient estimate; prescribed contact angle

Dedicated to the 80th birthday of Professor Neil Trudinger.

1. Introduction

Let Ω be a bounded C^3 domain in $\mathbb{R}^n$ and $u \in C^3(\overline{\Omega})$. In this paper, we will establish a priori gradient estimates for solutions of the prescribed k-curvature equation with the prescribed contact angle boundary value

$$\begin{cases} \sigma_k(\kappa) = f(x,u), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \phi(x)\sqrt{1+|Du|^2}, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where $\kappa = (\kappa_1, \cdots, \kappa_n)$ are the principal curvatures of the graph $\mathcal{M} = \{(x, u(x)) \in \mathbb{R}^{n+1} | x \in \Omega\}$, $n \geq 2$, f is a smooth, positive function in Ω and ϕ is a smooth function on $\overline{\Omega}$ such that $-1 < \phi < 1$. And for any

$k = 1, 2, \cdots, n$,

$$\sigma_k(\lambda) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}, \tag{1.2}$$

the k-order fundamental symmetric function of $\lambda \in \mathbb{R}^n$. For $k = 1, n$, the (1.1) is the mean curvature and Gaussian curvature equation respectively.

The gradient estimate for the prescribed mean curvature equation has been extensively studied. The interior gradient estimate, for the minimal surface equation, was obtained in the case of two variables by Finn [2]. Bombieri-De Giorgi-Miranda [1] obtained the estimate for high dimensional cases. For the general mean curvature equation, such an estimate had also been obtained by Ladyzhenskaya and Ural'tseva [10], Trudinger [17] and Simon [13]. All their methods were used by test function argument and a resulting Sobolev inequality. A more detailed history could be found in Gilbarg and Trudinger [3]. In 1983, Korevaar [5] introduced the normal variation technique and got the maximum principle proof for the interior gradient estimate on the minimal surface equation, then in 1987 Korevaar [6] got the interior gradient estimates for the higher order curvature equations. Trudinger [18] also studied the curvature equations and got the interior gradient estimates for a class curvature equation. In 1998, Wang [19] gave new proof for the interior gradient estimates on the general k-curvature equation via the standard Bernstein technique. In 2012, Sheng-Trudinger-Wang [16] also gave a new proof for the general Weingarten curvatures equation by the moving frame on the hypersurface.

For the mean curvature equation with the Neumann boundary value problem, Ma and Xu [11] used the technique developed by Spruck [14], Lieberman [7], Wang [19] and Jin-Li-Li [4] to get the global gradient estimates. As a consequence, they obtained an existence theorem for a class of mean curvature equations with the Neumann boundary value. For a fully nonlinear elliptic equation with Neumann boundary value or oblique derivative problem, we recommend Lieberman [8] to readers. Recently, Ma and Wang [12] used the technique developed by Sheng-Trudinger-Wang [16] to give a simpler new proof of the gradient estimates for the mean curvature equation with Neumann boundary value and prescribed contact angle boundary value. In this paper, we use the same technique to get the global gradient estimates for the k-curvature equation with the prescribed contact angle boundary value. Precisely, we have the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a C^3 boundary, $n \ge 2$, ν is the unit inner normal to $\partial\Omega$. Suppose $f \in C^1(\overline{\Omega} \times [-M_0, M_0])$ satisfies that $f_z \ge 0$ with $\inf_\Omega f \ge f_0 > 0$, and $\phi \in C^3(\overline{\Omega})$ and $-1 < \phi < 1$. If $u \in C^2(\overline{\Omega}) \cap C^3(\Omega)$ is a bounded k-admissible solution of the k-curvature equation (1.1), then we have*

$$\sup_{\overline{\Omega}} |Du| \le C, \tag{1.3}$$

where C is a positive constant depending on n, k, f_0, Ω, $|f|_{C^1(\overline{\Omega}\times[-M_0,M_0])}$, $|\phi|_{C^3(\overline{\Omega})}$, and $M_0 = |u|_{C^0(\overline{\Omega})}$.

Remark 1.2. We define the Garding's cone as $\Gamma_k = \{\lambda \in \mathbb{R}^n | \sigma_i(\lambda) > 0,\ 1 \le i \le k\}$. Then we say a function u is k-admissible if $\lambda(D^2u) \in \Gamma_k$, where $\lambda(D^2u) = (\lambda_1, \cdots, \lambda_n)$ are eigenvalues of the Hessian matrix D^2u.

Remark 1.3. In order to prove the existence theorem for the k-curvature equations with the prescribed contact angle boundary value problem, we still need global estimates for second-order derivatives. In another paper, we had gotten the global gradient estimates for the k-curvature equation with the Neumann boundary value problem.

The rest of the paper is organized as follows. In Section 2, we first give the definitions and some notations. We also give some basic properties of the fundamental symmetric functions. In Section 3, we prove the main Theorem 1.1 by the moving frame on the hypersurface.

2. Preliminary

$A, B, \cdots$ will be from 1 to $n+1$ and $i, j, \alpha, \cdots$ from 1 to n, the repeated indices denote summation over the indices.

Let Ω be a bounded domain in $\mathbb{R}^n$ and $u \in C^\infty(\overline{\Omega})$. Then the graph of u is a hypersurface in $\mathbb{R}^{n+1}$, denoted by $\mathcal{M}$, given by the smooth embedding $X : \Omega \to \mathbb{R}^{n+1}$,

$$X(x_1, \cdots, x_n) = (x_1, \cdots, x_n, u(x_1, \cdots, x_n)). \tag{2.1}$$

Denote $u_i = u_{x_i}$, $u_{ij} = u_{x_i x_j}$, and $Du = (u_1, \cdots, u_n)$. Then the downward unit normal of $\mathcal{M}$ is

$$N = \frac{(Du, -1)}{\sqrt{1 + |Du|^2}}. \tag{2.2}$$

Let $\{\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n+1}\}$ be the standard orthonormal basis in $\mathbb{R}^{n+1}$. We choose an orthonormal frame in $\mathbb{R}^{n+1}$ such that $\{e_1, e_2, ..., e_n\}$ are tangent to $\mathcal{M}$ and $e_{n+1} = N$ is the downward unit normal. Let the corresponding coframe be denoted by $\{\omega_A\}$ and the connection forms by $\{\omega_{A,B}\}$. The pullback of them through the embedding are still denoted by $\{\omega_A\}$,$\{\omega_{A,B}\}$ in the abuse of notation. Therefore on $\mathcal{M}$

$$\omega_{n+1} = 0.$$

The second fundamental form is defined by the symmetry matrix $\{h_{ij}\}$ with

$$\omega_{i,n+1} = h_{ij}\omega_j. \tag{2.3}$$

The principal curvatures $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_n)$ are the eigenvalues of the second fundamental form (h_{ij}).

The first and second-order covariant derivatives will be denoted by ∇_i, $\nabla_i\nabla_j$ respectively. We recall the following fundamental formulas of a hypersurface in $\mathbb{R}^{n+1}$.

$$\nabla_j\nabla_i X = \nabla_j e_i = -h_{ij}e_{n+1}, \quad \text{(Gauss formula)} \tag{2.4}$$

$$\nabla_i e_{n+1} = h_{ij}e_j. \quad \text{(Weingarten equation)} \tag{2.5}$$

We denote

$$d(x) = dist(x, \partial\Omega),$$
$$\Omega_\mu = \{x \in \Omega | \; d(x) < \mu\}.$$

It is well known that there exists a small positive universal constant μ_0 such that $d(x) \in C^3(\overline{\Omega_\mu})$, $\forall 0 < \mu \le \mu_0$, provided $\partial\Omega \in C^3$. As in Simon-Spruck [15] or Lieberman [7] (on page 331), we can extend ν by $\nu = Dd$ in Ω_μ and note that ν is a $C^3(\overline{\Omega_\mu})$ vector field. As mentioned in the book [7], we also have the following formulas

$$\begin{aligned} |D\nu| + |D^2\nu| \le C(n,\Omega), \quad &\text{in } \Omega_\mu, \\ \sum_{i=1}^n \nu^i D_j \nu^i = \sum_{i=1}^n \nu^i D_i \nu^j = \sum_{i=1}^n d_i d_{ij} = 0, \ |\nu| = |Dd| = 1, \quad &\text{in } \Omega_\mu. \end{aligned} \tag{2.6}$$

Lemma 2.1. *Denote* $v = \sqrt{1+|Du|^2}$ *and* $e_A^B = \langle e_A, \varepsilon_B\rangle$ *for* $A, B = 1, \cdots, n+1$. *We have*

$$\nabla_i v = v^2 h_{ir} \nabla_r u, \tag{2.7}$$

$$\nabla_j \nabla_i v = 2v^3 h_{jr} \nabla_r u h_{is} \nabla_s u + v^2 \nabla_r h_{ij} \nabla_r u + v h_{ir} h_{jr}, \tag{2.8}$$

$$\nabla_i u_r = v h_{is}(u_r \nabla_s u + e_s^r), \tag{2.9}$$

$$\nabla_j \nabla_i u_r = 2v^2 h_{jp} \nabla_p u h_{iq}(u_r \nabla_q u + e_q^r) + v \nabla_p h_{ij}(u_r \nabla_p u + e_p^r). \tag{2.10}$$

Proof. Note that $u = \langle X, \varepsilon_{n+1}\rangle$. Using the Gauss formula and Weingarten equation above, we obtain

$$\begin{aligned} \nabla_i v &= \nabla_i \left(-\frac{1}{\langle e_{n+1}, \varepsilon_{n+1}\rangle}\right) \\ &= \frac{1}{\langle e_{n+1}, \varepsilon_{n+1}\rangle^2} \nabla_i \langle e_{n+1}, \varepsilon_{n+1}\rangle \\ &= v^2 h_{il} \langle e_l, \varepsilon_{n+1}\rangle \\ &= v^2 h_{il} \nabla_l u. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \nabla_j \nabla_i v &= \nabla_j \left(v^2 h_{il} \langle e_l, \varepsilon_{n+1}\rangle\right) \\ &= 2v^3 h_{jr} \nabla_r u h_{is} \nabla_s u + v^2 \nabla_r h_{ij} \nabla_r u + v h_{ir} h_{jr} \end{aligned}$$

It follows that, recall $u_l = v\langle e_{n+1}, \varepsilon_l\rangle = v e_{n+1}^l$,

$$\begin{aligned} \nabla_i u_l &= \nabla_i (v \langle e_{n+1}, \varepsilon_l\rangle) = v^2 h_{ir} \nabla_r u e_{n+1}^l + v h_{ir} e_r^l \\ &= v h_{ir}(u_l \nabla_r u + e_r^l). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \nabla_j \nabla_i u_l &= \nabla_j (v h_{ir}(u_l \nabla_r u + e_r^l)) \\ &= v^2 h_{js} \nabla_s u h_{ir}(u_l \nabla_r u + e_r^l) + v \nabla_j h_{ir}(u_l \nabla_r u + e_r^l) \\ &\quad + v h_{ir}(u_l \nabla_j \nabla_r u + v \nabla_r u h_{js}(u_l \nabla_s u + e_s^l) + \nabla_j e_r^l), \end{aligned}$$

noting that

$$\nabla_j e_r^l = -h_{jr} e_{n+1}^l = -u_l \langle \nabla_j \nabla_r X, \varepsilon_{n+1}\rangle = -u_l \nabla_j \nabla_r u,$$

then, since $\nabla_j h_{ir} = \nabla_r h_{ij}$ (Codazzi equation),

$$\nabla_j \nabla_i u_l = 2v^2 h_{js} \nabla_s u h_{ir}(u_l \nabla_r u + e_r^l) + v \nabla_r h_{ij}(u_l \nabla_r u + e_r^l).$$

□

Now we give some basic properties of elementary symmetric functions, which could be found in [9].

First, we denote by $\sigma_k(\lambda|i)$ the symmetric function with $\lambda_i = 0$ and $\sigma_k(\lambda|ij)$ the symmetric function with $\lambda_i = \lambda_j = 0$.

Proposition 2.2. *Let* $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ *and* $k = 1, \cdots, n$*, then*

$$\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|i), \quad \forall 1 \le i \le n,$$
$$\sum_{i=1}^{n} \lambda_i \sigma_{k-1}(\lambda|i) = k\sigma_k(\lambda),$$
$$\sum_{i=1}^{n} \sigma_k(\lambda|i) = (n-k)\sigma_k(\lambda).$$

Recall that Garding's cone is defined as

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : \sigma_i > 0, \forall\ 1 \le i \le k\}.$$

Proposition 2.3. *Let* $\lambda \in \Gamma_k$ *and* $k \in \{1, 2, \cdots, n\}$*. Suppose that*

$$\lambda_1 \ge \cdots \ge \lambda_k \ge \cdots \ge \lambda_n,$$

then we have

$$\sigma_{k-1}(\lambda|n) \ge \cdots \ge \sigma_{k-1}(\lambda|k) \ge \cdots \ge \sigma_{k-1}(\lambda|1) > 0. \tag{2.11}$$

Then the k-curvature equation (1.1) is elliptic if the principal curvatures $\kappa \in \Gamma_k$.

3. Gradient estimate for prescribed contact angle

We consider the following k-curvature equation with the prescribed angle condition and obtain a gradient estimate of k-admissible solution. We state it again in the following theorem.

Theorem 3.1. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with* C^3 *boundary.* $f \in C^1(\Omega \times [-M_0, M_0])$ *satisfies that* $f_z \ge 0$*. Assume u is a k-admissible solution of the equation*

$$\begin{cases} \sigma_k(h_{ij}) = f(x,u), & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = \phi(x)\sqrt{1+|Du|^2}, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where ν *be the unit inner normal vector on* $\partial\Omega$ *and* $\phi \in C^3\left(\bar{\Omega}, (-1,1)\right)$*. We have*

$$\sup_{\overline{\Omega}} |Du| \le C. \tag{3.2}$$

Proof. Denote $v = \sqrt{1+|Du|^2}$ and $M_0 = \sup_{\overline{\Omega}} |u|$. Let

$$w := v - u_\nu \phi, \tag{3.3}$$
$$\psi(u) := \alpha_1(1 + M_0 + u). \tag{3.4}$$

The constant α_1 will be determined later. Fix a small $0 < \mu \le \mu_0$ and consider the auxiliary function

$$G(x) := \log\log w + \psi(u) + d, \quad x \in \overline{\Omega_\mu}. \tag{3.5}$$

There are three cases to be considered.

Case 1. $G(x)$ attains maximum at $x_0 \in \partial\Omega_\mu \cap \Omega$.

By the interior gradient estimates of Korevaar [6] and Wang [19], we have

$$\sup_{\overline{\Omega}} |Du| \le C. \tag{3.6}$$

Case 2. $G(x)$ attains maximum at $x_0 \in \partial\Omega$.

Assume $U \subset \mathbb{R}^n$ be a neighborhood of x_0. We choose a geodesic coordinate $\{x_i\}_i^{n-1}$ on $U \cap \partial\Omega$ centered at x_0. We let $\partial_{x_n} = \nu$ at x_0. In the following, we take all calculations at x_0.

Denote (b_{ij}) the second fundamental form of $\partial\Omega$ with respect to ν. We have

$$G_n = \frac{w_n}{w\log w} + \alpha_1 u_n + 1 \le 0, \tag{3.7}$$

and

$$G_j = \frac{w_j}{w\log w} + \alpha_1 u_j = 0, \quad j = 1, 2, \cdots, n-1. \tag{3.8}$$

Denote $a = w\log w$ for simplicity. Note that

$$u_n = \phi v, \tag{3.9}$$

$$w_l = v_l - u_{nl}\phi - u_n\phi_l, \quad l = 1, 2, \cdots, n, \tag{3.10}$$

$$\begin{aligned} v_n &= \frac{1}{v}\sum_{i=1}^{n-1} u_i u_{in} + \frac{u_n u_{nn}}{v} \\ &= \frac{1}{v}\sum_{i=1}^{n-1} u_i u_{in} + \phi u_{nn}. \end{aligned} \tag{3.11}$$

Choose $l = n$ in (3.10), then plug into (3.11) to get

$$\begin{aligned} w_n &= \frac{1}{v}\sum_{i=1}^{n-1} u_i u_{in} - u_n\phi_n \\ &= \frac{1}{v}\sum_{i=1}^{n-1} u_i u_{ni} - \frac{1}{v}\sum_{i=1}^{n-1} u_i b_{ij} u_j - v\phi\phi_n \\ &\ge \frac{1}{v}\sum_{i=1}^{n-1} u_i u_{ni} - Cv. \end{aligned} \tag{3.12}$$

By (3.8) and (3.10),

$$v_i = u_{ni}\phi + u_n\phi_i - \alpha_1 a u_i. \tag{3.13}$$

From the boundary data $u_n = \phi v$ and (3.13), we have

$$\begin{aligned} u_{ni} = (\phi v)_i &= \phi_i v + v_i \phi \\ &= \phi_i v - \alpha_1 a u_i \phi + u_{ni}\phi^2 + u_n \phi_i \phi. \end{aligned} \tag{3.14}$$

It follows that

$$\begin{aligned} u_{ni} &= \frac{1}{1-\phi^2}(\phi_i v - \alpha_1 a u_i \phi + v\phi_i \phi^2) \\ &\geq -\frac{\alpha_1 a \phi u_i}{1-\phi^2} - Cv. \end{aligned} \tag{3.15}$$

Plugging (3.15) into (3.12), we get

$$\begin{aligned} w_n &\geq -\frac{\alpha_1 a \phi}{(1-\phi^2)v}\sum_{i=1}^{n-1} u_i^2 - Cv \\ &= \frac{\alpha_1 a \phi}{(1-\phi^2)v} - \alpha_1 a \phi v - Cv. \end{aligned} \tag{3.16}$$

Here we use the fact

$$v^2 - 1 = \sum_{i=1}^{n-1} u_i^2 + u_n^2 = \sum_{i=1}^{n-1} u_i^2 + v^2\phi^2 \quad \text{at } x_0 \in \partial\Omega. \tag{3.17}$$

Putting (3.17) into (3.7), we have

$$\begin{aligned} 0 &\geq -\alpha_1 \phi v + \frac{\alpha_1 \phi}{(1-\phi^2)v} - \frac{Cv}{w\log w} + \alpha_1 u_n + 1 \\ &= \frac{\alpha_1 \phi}{(1-\phi^2)v} - \frac{Cv}{w\log w} + 1. \end{aligned} \tag{3.18}$$

Thus we have $v \leq C$.

Case 3. $G(x)$ attains its maximum at $x_0 \in \Omega_\mu$.

Direct computation shows that

$$\nabla_i G = \frac{\nabla_i w}{w\log w} + \alpha_1 \nabla_i u + \nabla_i d, \tag{3.19}$$

and

$$\nabla_j \nabla_i G = \frac{\nabla_j \nabla_i w}{w\log w} - \frac{\nabla_i w \nabla_j w}{(w\log w)^2}(1+\log w) + \alpha_1 \nabla_j \nabla_i u + \nabla_j \nabla_i d. \tag{3.20}$$

From (2.7)–(2.10) and (3.3), we have

$$\begin{aligned} \nabla_i w &= \nabla_i (v - u_r d_r \phi) \\ &= v^2 h_{ir} \nabla_r u - v h_{is}(u_r \nabla_s u + e_s^r) d_r \phi - u_r \nabla_i (d_r \phi) \\ &= v(v - u_r d_r \phi) h_{ir} \nabla_r u - v h_{ir} \nabla_r d \phi - u_r \nabla_i (d_r \phi) \end{aligned}$$

$$= vwh_{ir}\nabla_r u - vh_{ir}\nabla_r d\phi - u_r\nabla_i(d_r\phi), \tag{3.21}$$

and

$$\begin{aligned}
\nabla_j\nabla_i w &= \nabla_j\nabla_i v - \nabla_j\nabla_i u_r d_r\phi - \nabla_i u_r\nabla_j(d_r\phi) - \nabla_j u_r\nabla_i(d_r\phi) \\
&\quad - u_r\nabla_j\nabla_i(d_r\phi) \\
&= 2v^3 h_{jr}\nabla_r u h_{is}\nabla_s u + v^2\nabla_r h_{ij}\nabla_r u + vh_{ir}h_{jr} \\
&\quad - 2v^2 h_{jp}\nabla_p u h_{iq}(u_r\nabla_q u + e_q^r)d_r\phi - v\nabla_p h_{ij}(u_r\nabla_p u + e_p^r)d_r\phi \\
&\quad - vh_{is}(u_r\nabla_s u + e_s^r)\nabla_j(d_r\phi) - vh_{js}(u_r\nabla_s u + e_s^r)\nabla_i(d_r\phi) - u_r\nabla_j\nabla_i(d_r\phi) \\
&= 2v^2 w h_{jr}\nabla_r u h_{is}\nabla_s u + vw\nabla_r h_{ij}\nabla_r u + vh_{ir}h_{jr} \\
&\quad - 2v^2 h_{jp}\nabla_p u h_{iq}\nabla_q d\phi - v\nabla_p h_{ij}\nabla_p d\phi \\
&\quad - vh_{is}(u_r\nabla_s u + e_s^r)\nabla_j(d_r\phi) - vh_{js}(u_r\nabla_s u + e_s^r)\nabla_i(d_r\phi) - u_r\nabla_j\nabla_i(d_r\phi).
\end{aligned} \tag{3.22}$$

By selecting a suitable moving frame, we assume (h_{ij}) is diagonal at x_0. At the maximum point $x_0 \in \Omega_\mu$, from (3.19) and $\nabla G = 0$, we see that

$$-\frac{\nabla_i w}{w\log w} = \alpha_1\nabla_i u + \nabla_i d. \tag{3.23}$$

Together with (3.21), we also have

$$-w\log w(\alpha_1\nabla_i u + \nabla_i d) = vwh_{ii}\nabla_i u - vh_{ii}\nabla_i d\phi - u_r\nabla_i(d_r\phi). \tag{3.24}$$

We divide the indexes $i \in I = \{1, 2, \cdots, n\}$ into two subsets as follows.

$$J = \{i \in I : |\alpha_1\nabla_i u| \le 9\}, \tag{3.25}$$

$$K = I\backslash J = \{i \in I : |\alpha_1\nabla_i u| > 9\}. \tag{3.26}$$

For $i \in J$, recall $|\nabla d| \le 1$,

$$|\alpha_1\nabla_i u + \nabla_i d|^2 \le 100. \tag{3.27}$$

For $i \in K$, we have

$$\alpha_1|\nabla_i u| > \frac{4}{5}|\alpha_1\nabla_i u + \nabla_i d| > 6, \tag{3.28}$$

Now we assume v, w large enough, i.e., $v \ge \max\{\frac{\alpha_1}{\epsilon}, \exp\frac{2}{\epsilon}\}$, then

$$|vh_{ii}\nabla_i d\phi|| \le \epsilon|vwh_{ii}\nabla_i u|, \tag{3.29}$$

and

$$|u_r\nabla_i(d_r\phi)| \le \epsilon w\log w|\alpha_1\nabla_i u + \nabla_i d|, \tag{3.30}$$

for a small positive number ϵ. Then, using (3.24), we get

$$h_{ii} \le 0, \quad \text{for } i \in K, \tag{3.31}$$

and

$$\frac{1+\epsilon}{\log w} v|h_{ii}\nabla_i u| \ge (1-\epsilon)|\alpha_1 \nabla_i u + \nabla_i d|. \tag{3.32}$$

Thus, plugging (3.28) into (3.32), we have

$$-\alpha_1 v h_{ii}|\nabla_i u|^2 \ge \frac{3\log w}{4}|\alpha_1\nabla_i u + \nabla_i d|^2, \quad \text{for } i \in K, \tag{3.33}$$

provided we set $\epsilon = \frac{1}{100}$.

By the maximum principle, by (3.20), we have

$$\begin{aligned}
0 &\ge F^{ij}\nabla_j\nabla_i G \\
&= \frac{1}{w\log w}F^{ii}\nabla_i\nabla_i w - \frac{1+\log w}{(w\log w)^2}F^{ii}|\nabla_i w|^2 + \alpha_1 F^{ii}\nabla_i\nabla_i u + F^{ii}\nabla_i\nabla_i d \\
&\ge \frac{1}{w\log w}F^{ii}\nabla_i\nabla_i w - \frac{9\log w}{8}F^{ii}(\alpha_1\nabla_i u + \nabla_i d)^2 + \frac{\alpha_1 f}{v} - C\mathcal{F}.
\end{aligned} \tag{3.34}$$

We use the fact $\nabla_i\nabla_j u = \frac{h_{ij}}{v}$ in the last inequality and assume v large.

From (3.22) we obtain

$$\begin{aligned}
F^{ii}\nabla_i\nabla_i w &= 2v^2 w F^{ii}h_{ii}^2|\nabla_i u|^2 + vwF^{ii}\nabla_r h_{ii}\nabla_r u + vF^{ii}h_{ii}^2 - 2v^2F^{ii}h_{ii}^2\nabla_i u\nabla_i d\phi \\
&\quad - vF^{ii}\nabla_r h_{ij}\nabla_r d\phi - 2vF^{ii}h_{ii}(\nabla_i u u_r + e_i^r)\nabla_i(d_r\phi) - u_r F^{ii}\nabla_j\nabla_i(d_r\phi) \\
&= 2vF^{ii}h_{ii}\nabla_i u(vwh_{ii}\nabla_r u - vh_{ii}\nabla_i d\phi - u_r\nabla_i(d_r\phi)) + vF^{ii}h_{ii}^2 \\
&\quad + vw\langle\nabla f, \nabla u\rangle + vwf_z|\nabla u|^2 - v\phi\langle\nabla f, \nabla d\rangle \\
&\quad - 2vF^{ii}h_{ii}e_i^r\nabla_i(d_r\phi) - u_r F^{ii}\nabla_j\nabla_i(d_r\phi).
\end{aligned} \tag{3.35}$$

By (3.24), $f_z \ge 0$, $|\langle\nabla f, \nabla u\rangle| \le \frac{|Df|}{v}$, and the Cauchy-Schwartz inequality, we have

$$\begin{aligned}
F^{ii}\nabla_i\nabla_i w =&\ -2vw\log wF^{ii}h_{ii}\nabla_i u(\alpha_1\nabla_i u + \nabla_i d) + vF^{ii}h_{ii}^2 \\
&+ vw\langle\nabla f, \nabla u > -v\phi\langle\nabla f, \nabla d > -2vF^{ii}h_{ii}e_i^r\nabla_i(d_r\phi) - u_rF^{ii}\nabla_j\nabla_i(d_r\phi) \\
\ge&\ -2\alpha_1 vw\log wF^{ii}h_{ii}|\nabla_i u|^2 - 2vw\log wF^{ii}h_{ii}\nabla_i u\nabla_i d + vF^{ii}h_{ii}^2 \\
&- 2vF^{ii}h_{ii}e_i^r\nabla_i(d_r\phi) - Cv(1+\mathcal{F}) \\
\ge&\ -2\alpha_1 vw\log wF^{ii}h_{ii}|\nabla_i u|^2 - 2vw\log wF^{ii}h_{ii}\nabla_i u\nabla_i d + vF^{ii}h_{ii}^2 \\
&- \epsilon_1 vF^{ii}h_{ii}^2 - \frac{Cv}{\epsilon_1}\mathcal{F} - Cv(1+\mathcal{F}).
\end{aligned} \tag{3.36}$$

Multiplying $\nabla_i d$ with both sides of (3.24), we have

$$-w\log w(\alpha_1\nabla_i u + \nabla_i d)\nabla_i d = vwh_{ii}\nabla_i u\nabla_i d - vh_{ii}|\nabla_i d|^2\phi - u_r\nabla_i(d_r\phi)\nabla_i d.$$

It follows that, by the Cauchy-Schwartz inequality,

$$\begin{aligned}2vwF^{ii}h_{ii}\nabla_i u\nabla_i d &= 2\phi vF^{ii}h_{ii}|\nabla_i d|^2 + 2F^{ii}u_r\nabla_i(d_r\phi)\nabla_i d\\ &\quad - 2w\log wF^{ii}(\alpha_1\nabla_i u + \nabla_i d)\nabla_i d\\ &\geq -\epsilon_1\frac{v}{\log w}F^{ii}h_{ii}^2 - -\frac{\phi^2}{\epsilon_1}\log w\mathcal{F} - Cv\mathcal{F}\\ &\quad - \epsilon_2 w\log wF^{ii}(\alpha_1\nabla_i u + \nabla_i d)^2 - \frac{w\log w}{\epsilon_2}\mathcal{F}. \end{aligned}\tag{3.37}$$

Similarly, multiplying $\nabla_i u$ with both sides of (3.24), we have

$$-w\log w(\alpha_1\nabla_i u + \nabla_i d)\nabla_i u = vwh_{ii}|\nabla_i u|^2 - vh_{ii}\nabla_i d\nabla_i u\phi - u_r\nabla_i(d_r\phi)\nabla_i u.$$

By the Cauchy-Schwartz inequality, recall $|\alpha_1\nabla_i u| \leq 9$, for $i \in J$,

$$\begin{aligned}-\sum_{i\in J}2\alpha_1 vwF^{ii}h_{ii}|\nabla_i u|^2 &= -\sum_{i\in J}2\alpha_1\phi vF^{ii}h_{ii}\nabla_i d\nabla_i u - \sum_{i\in J}2\alpha_1 F^{ii}u_r\nabla_i(d_r\phi)\nabla_i u\\ &\quad + \sum_{i\in J}2\alpha_1 w\log wF^{ii}(\alpha_1\nabla_i u + \nabla_i d)\nabla_i u\\ &\geq -\epsilon_1\frac{v}{\log w}F^{ii}h_{ii}^2 - -\frac{81\phi^2}{\epsilon_1}\log w\mathcal{F} - Cv\mathcal{F}\\ &\quad - \epsilon_2 w\log wF^{ii}(\alpha_1\nabla_i u + \nabla_i d)^2 - \frac{81w\log w}{\epsilon_2}\mathcal{F}. \end{aligned}\tag{3.38}$$

Here we point out that C is independent of α_1.

Plugging (3.37) and (3.38) into (3.36), we have

$$\begin{aligned}F^{ii}\nabla_i\nabla_i w \geq &-\sum_{i\in K}2\alpha_1 vw\log wF^{ii}h_{ii}|\nabla_i u|^2 + v(1-3\epsilon_1)F^{ii}h_{ii}^2 - \frac{82\phi^2}{\epsilon_1}|\log w|^2\mathcal{F}\\ &- 2\epsilon_2 w|\log w|^2F^{ii}(\alpha_1\nabla_i u + \nabla_i d)^2 - \frac{82w|\log w|^2}{\epsilon_2}\mathcal{F} - Cv\log w\mathcal{F}\\ &\frac{Cv}{\epsilon_1}\mathcal{F} - Cv(1+\mathcal{F}). \end{aligned}\tag{3.39}$$

Set $\epsilon_1 = \frac{1}{3}$, $\epsilon_2 = \frac{1}{16}$, and assume v is sufficiently large, then

$$\begin{aligned}F^{ii}\nabla_i\nabla_i w \geq &-2\alpha_1 vw\log wF^{ii}h_{ii}|\nabla_i u|^2 - \frac{1}{8}w|\log w|^2F^{ii}(\alpha_1\nabla_i u + \nabla_i d)^2\\ &- Cw|\log w|^2\mathcal{F}, \end{aligned}\tag{3.40}$$

Putting (3.40) into (3.34), we get

$$\begin{aligned}0 &\geq F^{ij}\nabla_j\nabla_i G\\ &\geq -\sum_{i\in K}2\alpha_1 vF^{ii}h_{ii}|\nabla_i u|^2 - \frac{5\log w}{4}F^{ii}(\alpha_1\nabla_i u + \nabla_i d)^2\end{aligned}$$

$$- C \log w\mathcal{F} - C\mathcal{F}. \tag{3.41}$$

In view of (3.28), we see that

$$-\sum_{i\in I} \frac{5\log w}{4} F^{ii}(\alpha_1 \nabla_i u + \nabla_i d)^2 \geq -\sum_{i\in K} \frac{5\log w}{4} F^{ii}(\alpha_1 \nabla_i u + \nabla_i d)^2 - C\log w\mathcal{F}. \tag{3.42}$$

On the other hand, by (3.31) and (3.33),

$$-\sum_{i\in K} 2\alpha_1 v F^{ii} h_{ii} |\nabla_i u|^2 \geq \sum_{i\in K} \frac{6\log w}{4} F^{ii}(\alpha_1 \nabla_i u + \nabla_i d)^2. \tag{3.43}$$

Particularly, there is $i_0 \in K$, say $i_0 = 1$, such that $|\nabla_1 u| \geq \frac{1}{2\sqrt{n}}$ and $F^{11} \geq \frac{1}{n}\mathcal{F}$.

Plugging (3.42) and (3.43) into (3.41) and choosing $\alpha_1 = \max\{4\sqrt{n}, 16nC\}$, we have

$$\begin{aligned} 0 &\geq \frac{\log w}{4} F^{11}(\alpha_1 \nabla_1 u + \nabla_1 d)^2 - C\log w\mathcal{F} - C\mathcal{F} \\ &\geq \left(\frac{\alpha_1^2 \log w}{128n^2} - C\right)\mathcal{F}. \end{aligned} \tag{3.44}$$

Thus we have $(1 - |\phi|)v \leq w \leq \exp \frac{128n^2 C}{\alpha_1^2}$ and finish the proof.

□

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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