Advances in Mathematics 413 (2023) 108851

Contents lists available at ScienceDirect

Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim

A Liouville theorem for a class semilinear elliptic equations on the Heisenberg group

Xi-Nan Ma [a], Qianzhong Ou [b,*]

[a] *Department of Mathematics, University of Science and Technology of China, Hefei, 230026, Anhui Province, China*

[b] *School of Mathematics and Statistics, Guangxi Normal University, Guilin, 541004, Guangxi Province, China*

A R T I C L E   I N F O

Article history:
Received 10 September 2021
Received in revised form 15 June 2022
Accepted 12 December 2022
Available online xxxx
Communicated by YanYan Li

MSC:
35J61
32V20

Keywords:
Semilinear elliptic equation
Heisenberg group
Liouville theorem
CR Yamabe problem

A B S T R A C T

We obtain a Liouville type theorem for a classical semilinear subcritical elliptic equation on the Heisenberg group. A pointwise estimate near the isolated singularity is also provided. The soul of the proofs is a generalization of the Jerison-Lee's divergence identity and then an *a priori* integral estimate.

* Corresponding author.
E-mail addresses: xinan@ustc.edu.cn (X.-N. Ma), ouqzh@gxnu.edu.cn (Q. Ou).

https://doi.org/10.1016/j.aim.2022.108851

1. Introduction

In this paper, we study the following equation

$$-\triangle_{\mathbb{H}^n} u = 2n^2 u^q \quad \text{in} \quad \Omega, \tag{1.1}$$

where Ω is a domain in the Heisenberg group $\mathbb{H}^n$, and u is a real, nonnegative and smooth function defined in Ω, while $\triangle_{\mathbb{H}^n} u = \sum_{\alpha=1}^n (u_{\alpha\overline{\alpha}} + u_{\overline{\alpha}\alpha})$ is the Heisenberg Laplacian of u which will be introduced in section 2. Let $Q = 2n + 2$ be the homogeneous dimension of $\mathbb{H}^n$. Denote $q_* = \frac{Q}{Q-2}$ and $q^* = \frac{Q+2}{Q-2}$. Our main purpose in this paper is to present an entire Liouville type theorem and a pointwise estimate near the isolated singularity for solutions to (1.1) for the subcritical case $1 < q < q^*$.

The equation (1.1) studied intensively by many authors in decades is connected to the CR Yamabe problem on $\mathbb{H}^n$. Let $\mathbf{\Theta}$ be the standard contact form on $\mathbb{H}^n$, we consider another smooth contact form $\theta = u^{\frac{2}{n}}\mathbf{\Theta}$, where u is a smooth positive function in $\mathbb{H}^n$. Then the pseudo-Hermitian scalar curvature associated to the new contact form ($\mathbb{H}^n$, θ) is $R = 4n(n+1)u^{q-q^*}$ while u satisfies the equation (1.1). The CR Yamabe problem is to find such a contact form θ so that the pseudo-Hermitian scalar curvature R is a constant (i.e. $q = q^*$). The number $q^* + 1 = \frac{2Q}{Q-2}$ is the CR Sobolev embedding exponent [7]. For the equation (1.1) with $q = q^* = \frac{Q+2}{Q-2}$, by the splendid work [15] of D. Jerison and J.M. Lee, there are nontrivial solutions as follows

$$u(z,t) = C\left|t + \sqrt{-1}z \cdot \overline{z} + z \cdot \mu + \lambda\right|^{-n} \tag{1.2}$$

for some $C > 0$, $\lambda \in \mathbb{C}$, Im(λ)$> |\mu|^2/4$, and $\mu \in \mathbb{C}^n$ (where $\sqrt{-1}$ denote the imaginary unit in the complex space $\mathbb{C}$), which are the only extremals of the CR Sobolev inequality on $\mathbb{H}^n$. The CR Yamabe problem had been studied by D. Jerison and J.M. Lee in a series of fundamental works (see [14–16]). For compact, strictly pseudovonvex CR manifold, the CR Yamabe problem had been solved in case of not locally CR equivalent to sphere $\mathbf{S}^{2n+1}$ by Jerison-Lee [16] for $n \geq 2$ and Gamara [8] for $n = 1$, and in case of locally CR equivalent to $\mathbf{S}^{2n+1}$ by Gamara-Yacoub [9] for all $n \geq 1$. One can see the more recent progress in Cheng-Malchiodi-Yang [6] for the CR Yamabe problem. Using the Jerison-Lee's identity [15], Wang [21] obtained related result for a closed Einstein pseudohermitian manifold.

In fact, for the equation (1.1) with $q = \frac{Q+2}{Q-2}$, Jerison-Lee [15] obtained the uniqueness of the solutions in case of finite volume, i.e., $u \in L^{\frac{2Q}{Q-2}}(\mathbb{H}^n)$. Garofalo-Vassilev [10] also got a uniqueness result under the assumption of cylindrically symmetry on groups of Heisenberg type.

For the subcritical case $1 < q \leq q_*$, Birindelli-Dolcetta-Cutri [1] proved that the only nonnegative entire solution of (1.1) is the trivial one, where they also showed that $q = q_*$ is sharp for the nonexistence of the following differential inequality

$$-\triangle_{\mathbb{H}^n} u \geq 2n^2 u^q \quad \text{in} \quad \mathbb{H}^n. \tag{1.3}$$

There are some partial results for the subcritical case $q_* < q < q^*$, such as the solutions are cylindrical symmetry or decay at infinity in [2], and as $n > 1$, $q_* < q \leq q^* - \frac{1}{(Q-2)(Q-1)^2}$ in [22].

In this paper, we will extend the Liouville result to the whole interval of subcritical values of q. Precisely, we have

Theorem 1.1. *Let $\Omega = \mathbb{H}^n$ be the whole space and $1 < q < q^*$, then the equation (1.1) has no positive solution, namely, any nonnegative entire solution of (1.1) must be the trivial one.*

Now we state a local estimate for the isolated singularity of solutions to the equation (1.1) in $B_R(0)\backslash\{0\}$.

Theorem 1.2. *Let $\Omega = B_R(0)\backslash\{0\}$ be a punctured ball in $\mathbb{H}^n$ and $1 < q < q^*$, then any positive solution u of (1.1) satisfies:*

$$u(\xi) \leq C|\xi|^{\frac{-2}{q-1}} \quad \text{for } 0 < |\xi| \leq \frac{R}{4}, \tag{1.4}$$

with some positive constant C depending only on n and q.

The soul of the proofs of Theorem 1.1, Theorem 1.2 is an integral estimate as follows, which itself may be interesting. Let $B_{4r}(\xi_0) \subset \Omega$ be any ball centred at ξ_0 with radius $4r$, then we can prove that any positive solution u of (1.1) with $1 < q < q^*$ must satisfy:

$$\int\limits_{B_r(\xi_0)} u^{3q-q^*} \leq C\, r^{Q-2\times\frac{3q-q^*}{q-1}}, \tag{1.5}$$

where the positive constant C depends only on n and q.

For $1 < q < q^*$, we have $Q - 2 \times \frac{3q-q^*}{q-1} < 0$. So if u is a positive solution of (1.1) with $\Omega = \mathbb{H}^n$, taking $r \to +\infty$ in (1.5) we get

$$\int\limits_{\mathbb{H}^n} u^{3q-q^*} = 0. \tag{1.6}$$

This contradiction signifies directly the conclusion of Theorem 1.1. Also, we will prove Theorem 1.2 by using (1.5) combining with the Harnack inequality which was obtained by Capogna-Danielli-Garofalo (see Theorem 3.1 in [4]).

Now we compare (1.1) with the corresponding semilinear elliptic equation in Euclidean case,

$$-\triangle u = u^q \quad \text{in} \quad \mathbb{R}^n. \tag{1.7}$$

For $1 < q < \frac{n+2}{n-2}$, Gidas-Spruck [11] proved that the above equation (1.7) has no positive entire solution. The method used in [11] is the integral estimate motivated from Obata [19]. They also gave the pointwise estimates near the isolated singularity. It was proved that for $\frac{n}{n-2} < q < \frac{n+2}{n-2}$, the positive solution of (1.7) in the punctured unit ball, with a nonremovable singularity at the origin, must satisfies

$$|x|^{\frac{2}{q-1}} u(x) \to C_0(n, q) \quad \text{as } x \to 0. \tag{1.8}$$

In another paper, Caffarelli-Gidas-Spruck [3] classified all the entire solutions and the isolated singularity of (1.7) for the critical exponent $q = \frac{n+2}{n-2}$ via moving plane method. Later, Chen-Li [5] simplified the argument of Caffarelli-Gidas-Spruck and extended it to cover the case $n = 2$. One can also find a different proof by Li-Zhu [18] and Li-Zhang [17].

As in the Euclidean case [12], the Liouville type result in Theorem 1.1 would be useful to get the *a priori* boundedness in studying the semilinear equations via the blow-up analysis.

From the usual idea in [11], to get the integral estimate (1.5), there are usually two difficulties to be overcome in a noncompact domain. One is to find a suitable divergence identity, and the other is to estimate the "tail" terms after integrating by part on the divergence identity multiplied by suitable cut-off functions.

In Riemannian conformal geometry, Obata [19] found an identity to express some suitable nonnegative terms (usually with the associated geometry data) in a divergence form, and then obtained the following result: the only Riemannian metrics on the sphere that are conformal to the standard one and have constant scalar curvature are obtained from the standard metric by a conformal diffeomorphism of the sphere.

In CR conformal geometry, Jerison-Lee [15] had also found a magic identity which involved the derivative of torsion, and then got an Obata type theorem in CR geometry: if θ is a contact form associated with the standard CR structure on the sphere which has constant pseudohermitian scalar curvature, then θ is obtained from a constant multiple of the standard form $\hat{\theta}$ by a CR automorphism of the sphere. In the same paper, Jerison-Lee [15] also got the related identity to classify the extremal functions of the Sobolev inequality on Heisenberg group.

In this paper, based on our new observation, we will give a generalization of the Jerison-Lee's identity on Heisenberg group (see (4.2) for example in [15]) with a transparent proof, so that we can deduce an entire Liouville theorem for the subcritical case of the equation (1.1). This is similar to the Gidas-Spruck's [11] proof for equation (1.7) from Obata [19] identity. But in our case the matter is much more complicate, since the Jerison-Lee's identity involves extra torsion terms.

The paper is organized as follows. In section 2, we introduce some notations and give a generalization of the Jerison-Lee identity. Then, using this generalized identity, we prove the integral estimates (1.5) and Theorem 1.1 in section 3. The proof of Theorem 1.2 shall be presented in section 4.

2. A generalization of Jerison-Lee's identity

In this section we discuss the generalization of the remarkable Jerison-Lee's identity ((4.2) in [15]) on Heissenberg group $\mathbb{H}^n$ for solutions to equation (1.1) with general exponent. We adopt notations as in [15].

We shall first give a brief introduction to the Heissenberg group $\mathbb{H}^n$ and some notations. We consider $\mathbb{H}^n$ as the set $\mathbb{C}^n \times \mathbb{R}$ with coordinates (z, t) and group law $\circ$:

$$(z,t) \circ (w,s) = \big(z + w,\, t + s + 2\mathbf{Im} z^\alpha \overline{w}^\alpha\big) \quad \text{for } \xi = (z,t),\, \zeta = (w,s) \in \mathbb{C}^n \times \mathbb{R},$$

where and in the sequel, we shall use the Einstein sum with the convention: the Greek indices $1 \le \alpha, \beta, \gamma \le n$. For $\xi = (z,t) = (z_1, z_2, ..., z_n, t) \in \mathbb{C}^n \times \mathbb{R}$ as an element of $\mathbb{H}^n$, the norm $|\xi|$ is defined by $|\xi|^4 = |(z,t)|^4 = |z|^4 + t^2$, with associated distance function $d(\xi,\zeta) = |\zeta^{-1} \circ \xi|$. We will use the notation $B_r(\xi)$ for the metric ball centred at $\xi = (z,t)$ with radius $r > 0$. The Heisenberg group is a dilation group and the associated homogeneous dimension $Q = 2n + 2$ such that the volume $|B_r(\xi)| \approx r^Q$.

The CR structure of $\mathbb{H}^n$ is given by the bundle $\mathcal{H}$ spanned by the left-invariant vector fields $Z_\alpha = \partial/\partial z^\alpha + \sqrt{-1}\overline{z}^\alpha \partial/\partial t$ and $Z_{\bar{\alpha}} = \partial/\partial \overline{z}^\alpha - \sqrt{-1} z^\alpha \partial/\partial t$, $\alpha = 1, \cdots, n$. The standard (left-invariant) contact form on $\mathbb{H}^n$ is $\mathbf{\Theta} = dt + \sqrt{-1}(z^\alpha d\overline{z}^\alpha - \overline{z}^\alpha dz^\alpha)$. With respect to the standard holomorphic frame $\{Z_\alpha\}$ and dual admissible coframe $\{dz^\alpha\}$, the *Levi forms* $h_{\alpha\overline{\beta}} = 2\delta_{\alpha\overline{\beta}}$. Accordingly, for a smooth function f on $\mathbb{H}^n$, denote its derivatives by $f_\alpha = Z_\alpha f$, $f_{\alpha\overline{\beta}} = Z_{\overline{\beta}}(Z_\alpha f)$, $f_0 = \frac{\partial f}{\partial t}$, $f_{0\alpha} = Z_\alpha(\frac{\partial f}{\partial t})$, etc. We would also indicate the derivatives of functions or vector fields with indices preceded by a comma, to avoid confusion. Then as in [15] we have the following commutation formulae:

$$\begin{aligned} &f_{\alpha\beta} - f_{\beta\alpha} = 0, \qquad f_{\alpha\overline{\beta}} - f_{\overline{\beta}\alpha} = 2\sqrt{-1}\delta_{\alpha\overline{\beta}}\, f_0, \qquad f_{0\alpha} - f_{\alpha 0} = 0, \\ &f_{\alpha\beta 0} - f_{\alpha 0\beta} = 0, \qquad f_{\alpha\beta\overline{\gamma}} - f_{\alpha\overline{\gamma}\beta} = 2\sqrt{-1}\delta_{\beta\overline{\gamma}}\, f_{\alpha 0}, \cdots. \end{aligned}$$

Now we are at the point to give the generalized identity for positive solution of the equation (1.1). Let $u > 0$ be a solution of (1.1). Take $e^f = u^{\frac{1}{n}}$ and $q = q^* + \frac{p}{n}$, then the subcritical exponent $1 < q < q^*$ is corresponding to $-2 < p < 0$. It follows that f satisfies the following equation

$$-\triangle_{\mathbb{H}^n} f = 2n|\partial f|^2 + 2ne^{(2+p)f}, \tag{2.1}$$

where $|\partial f|^2 = f_\alpha f_{\overline{\alpha}}$.

As in [15], we define the tensors

$$\begin{aligned} D_{\alpha\beta} &= f_{\alpha\beta} - 2f_\alpha f_\beta, & D_\alpha &= D_{\alpha\beta} f_{\overline{\beta}}, \\ E_{\alpha\overline{\beta}} &= f_{\alpha\overline{\beta}} - \frac{1}{n} f_{\gamma\overline{\gamma}}\delta_{\alpha\overline{\beta}}, & E_\alpha &= E_{\alpha\overline{\beta}} f_\beta, \\ G_\alpha &= \sqrt{-1} f_{0\alpha} - \sqrt{-1} f_0 f_\alpha + e^{(2+p)f} f_\alpha + |\partial f|^2 f_\alpha. \end{aligned} \tag{2.2}$$

The above Jerison-Lee's tensors are also of course important in our argument, one can see [15] for the reason to introduce them.

Introduce the function $g = |\partial f|^2 + e^{(2+p)f} - \sqrt{-1} f_0$. Then the equation (2.1) can be rewritten as

$$f_{\alpha\overline{\alpha}} = -ng. \tag{2.3}$$

Moreover, we observe that

$$\begin{aligned} E_{\alpha\overline{\beta}} &= f_{\alpha\overline{\beta}} + g\delta_{\alpha\overline{\beta}}, & E_\alpha &= f_{\alpha\overline{\beta}} f_\beta + g f_\alpha, \\ D_\alpha &= f_{\alpha\beta} f_{\overline{\beta}} - 2|\partial f|^2 f_\alpha, & G_\alpha &= \sqrt{-1} f_{0\alpha} + g f_\alpha, \end{aligned} \tag{2.4}$$

and by

$$\begin{aligned} (|\partial f|^2)_{,\overline{\alpha}} =& f_\beta f_{\overline{\beta}\overline{\alpha}} + f_{\beta\overline{\alpha}} f_{\overline{\beta}} \\ =& f_\beta f_{\overline{\beta}\overline{\alpha}} + (f_{\overline{\alpha}\beta} + 2\sqrt{-1}\delta_{\beta\overline{\alpha}} f_0) f_{\overline{\beta}} \\ =& D_{\overline{\alpha}} + 2|\partial f|^2 f_{\overline{\alpha}} + E_{\overline{\alpha}} - \overline{g} f_{\overline{\alpha}} + 2\sqrt{-1} f_0 f_{\overline{\alpha}} \\ =& D_{\overline{\alpha}} + E_{\overline{\alpha}} + \overline{g} f_{\overline{\alpha}} - 2 f_{\overline{\alpha}} e^{(2+p)f}, \end{aligned} \tag{2.5}$$

we find

$$g_{\overline{\alpha}} = D_{\overline{\alpha}} + E_{\overline{\alpha}} + G_{\overline{\alpha}} + p f_{\overline{\alpha}} e^{(2+p)f}. \tag{2.6}$$

Taking conjugation one also has

$$\overline{g}_\alpha = D_\alpha + E_\alpha + G_\alpha + p f_\alpha e^{(2+p)f}. \tag{2.7}$$

In view of the above observations, now we give the crucial identity as follows.

Proposition 2.1.

$$\begin{aligned} \mathcal{M} =& \mathbf{Re} Z_{\overline{\alpha}} \Big\{ e^{2(n-1)f} \Big[\big(g + 3\sqrt{-1} f_0\big) E_\alpha \\ & + \big(g - \sqrt{-1} f_0\big) D_\alpha - 3\sqrt{-1} f_0 G_\alpha - \frac{p}{4} f_\alpha |\partial f|^4 \Big] \Big\} \end{aligned} \tag{2.8}$$

with

$$\begin{aligned} \mathcal{M} = & \ e^{(2n+p)f} \big(|E_{\alpha\overline{\beta}}|^2 + |D_{\alpha\beta}|^2 \big) \\ & + e^{2(n-1)f} \big(|G_\alpha|^2 + |G_\alpha + D_\alpha|^2 + |G_\alpha - E_\alpha|^2 + |D_{\alpha\beta} f_{\overline{\gamma}} + E_{\alpha\overline{\gamma}} f_\beta|^2 \big) \\ & + e^{2(n-1)f} \big(p e^{(2+p)f} - \frac{p}{2} |\partial f|^2 \big) \mathbf{Re} \big(f_{\overline{\alpha}} D_\alpha + f_{\overline{\alpha}} E_\alpha \big) \end{aligned}$$

$$-p(2n-1)|\partial f|^2 e^{2(n+1+p)f} - \frac{p}{4}(7n-6)|\partial f|^4 e^{(2n+p)f}$$
$$-\frac{p}{4}n|\partial f|^6 e^{2(n-1)f} - 3np|f_0|^2 e^{(2n+p)f}.$$

Remark 2.2. Note that for $p=0$, then (2.8) is exactly the remarkable identity found by Jerison and Lee (see (4.2) in [15]). For $-2<p<0$, the subcritical case, we will show by elementary computations in section 3 that the function $\mathcal{M}$ is also nonnegative. The observation (2.6) plays key role in our proof of the identity. We hope that it is helpful to find the "explanation" for the existence of such divergence identities, as expected by Jerison and Lee [15].

Proof of Proposition 2.1. Denote

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4,$$

with

$$\begin{aligned} \mathcal{L}_1 &= Z_{\overline{\alpha}}\Big\{\big(g+3\sqrt{-1}f_0\big)E_\alpha e^{2(n-1)f}\Big\}, & \mathcal{L}_2 &= Z_{\overline{\alpha}}\Big\{\big(g-\sqrt{-1}f_0\big)D_\alpha e^{2(n-1)f}\Big\}, \\ \mathcal{L}_3 &= Z_{\overline{\alpha}}\Big\{-3\sqrt{-1}f_0 G_\alpha e^{2(n-1)f}\Big\}, & \mathcal{L}_4 &= Z_{\overline{\alpha}}\Big\{-\frac{p}{4}f_\alpha|\partial f|^4 e^{2(n-1)f}\Big\}. \end{aligned} \tag{2.9}$$

First we compute $\mathcal{L}_3$. We have, by (2.4) and the commutation formulae,

$$\begin{aligned} G_{\alpha,\overline{\alpha}} &= \sqrt{-1}f_{0\alpha\overline{\alpha}} + g_{\overline{\alpha}}f_\alpha + g f_{\alpha\overline{\alpha}} \\ &= \sqrt{-1}f_{\alpha\overline{\alpha}0} + g_{\overline{\alpha}}f_\alpha + g(f_{\overline{\alpha}\alpha} + 2n\sqrt{-1}f_0) \\ &= f_\alpha g_{\overline{\alpha}} - n\sqrt{-1}g_0 - n|g|^2 + 2n\sqrt{-1}f_0 g. \end{aligned} \tag{2.10}$$

So we get

$$\begin{aligned} e^{-2(n-1)f}\mathcal{L}_3 &= e^{-2(n-1)f}Z_{\overline{\alpha}}\Big\{-3\sqrt{-1}f_0 G_\alpha e^{2(n-1)f}\Big\} \\ &= -3\sqrt{-1}f_0 G_{\alpha,\overline{\alpha}} - 3\sqrt{-1}f_{0\overline{\alpha}}G_\alpha - 6(n-1)\sqrt{-1}f_0 f_{\overline{\alpha}}G_\alpha \\ &= -3\sqrt{-1}f_0\big(f_\alpha g_{\overline{\alpha}} - n\sqrt{-1}g_0 - n|g|^2 + 2n\sqrt{-1}f_0 g\big) \\ &\quad + 3(G_{\overline{\alpha}} - \overline{g}f_{\overline{\alpha}})G_\alpha - 6(n-1)\sqrt{-1}f_0 f_{\overline{\alpha}}G_\alpha \\ &= 3|G_\alpha|^2 - 3(\overline{g} + 2(n-1)\sqrt{-1}f_0)f_{\overline{\alpha}}G_\alpha \\ &\quad - 3\sqrt{-1}f_0 f_\alpha g_{\overline{\alpha}} - 3nf_0 g_0 + 3n\sqrt{-1}f_0|g|^2 + 6n|f_0|^2 g. \end{aligned} \tag{2.11}$$

Next we compute $\mathcal{L}_1$. Also by (2.4) and the commutation formulae,

$$\begin{aligned} E_{\alpha,\overline{\alpha}} &= f_{\alpha\overline{\beta}\overline{\alpha}} f_\beta + f_{\alpha\overline{\beta}} f_{\beta\overline{\alpha}} + g_{\overline{\alpha}} f_\alpha + g f_{\alpha\overline{\alpha}} \\ &= f_{\alpha\overline{\alpha}\overline{\beta}} f_\beta + f_{\alpha\overline{\beta}} (f_{\overline{\alpha}\beta} + 2\sqrt{-1} f_0 \delta_{\beta\overline{\alpha}}) + g_{\overline{\alpha}} f_\alpha + g f_{\alpha\overline{\alpha}} \\ &= -n g_{\overline{\beta}} f_\beta + f_{\alpha\overline{\beta}} f_{\overline{\alpha}\beta} + 2\sqrt{-1} f_0 f_{\alpha\overline{\alpha}} + g_{\overline{\alpha}} f_\alpha + g f_{\alpha\overline{\alpha}} \\ &= (1-n) f_\alpha g_{\overline{\alpha}} + (E_{\alpha\overline{\beta}} - g\delta_{\alpha\overline{\beta}})(E_{\overline{\alpha}\beta} - \overline{g}\delta_{\overline{\alpha}\beta}) - n|g|^2 \\ &= |E_{\alpha\overline{\beta}}|^2 + (1-n) f_\alpha g_{\overline{\alpha}}. \end{aligned} \tag{2.12}$$

It follows that

$$\begin{aligned} e^{-2(n-1)f} \mathcal{L}_1 &= e^{-2(n-1)f} Z_{\overline{\alpha}} \Big\{ \big(g + 3\sqrt{-1} f_0\big) E_\alpha e^{2(n-1)f} \Big\} \\ &= \big(g + 3\sqrt{-1} f_0\big) E_{\alpha,\overline{\alpha}} \\ &\quad + \big(g_{\overline{\alpha}} + 3\sqrt{-1} f_{0\overline{\alpha}}\big) E_\alpha + 2(n-1)\big(g + 3\sqrt{-1} f_0\big) f_{\overline{\alpha}} E_\alpha \\ &= \big(g + 3\sqrt{-1} f_0\big)\big(|E_{\alpha\overline{\beta}}|^2 + (1-n) f_\alpha g_{\overline{\alpha}}\big) \\ &\quad + g_{\overline{\alpha}} E_\alpha + 3\big(-G_{\overline{\alpha}} + \overline{g} f_{\overline{\alpha}}\big) E_\alpha + 2(n-1)\big(g + 3\sqrt{-1} f_0\big) f_{\overline{\alpha}} E_\alpha \\ &= \big(g + 3\sqrt{-1} f_0\big) |E_{\alpha\overline{\beta}}|^2 + \big(g_{\overline{\alpha}} - 3G_{\overline{\alpha}}\big) E_\alpha \\ &\quad + \big(3\overline{g} + 2(n-1)(g + 3\sqrt{-1} f_0)\big) f_{\overline{\alpha}} E_\alpha \\ &\quad + (1-n)(g + 3\sqrt{-1} f_0) f_\alpha g_{\overline{\alpha}}. \end{aligned} \tag{2.13}$$

Now we compute $\mathcal{L}_2$. Using the commutation formulae, we compute

$$\begin{aligned} f_{\alpha\beta\overline{\alpha}} &= f_{\alpha\overline{\alpha}\beta} + 2\sqrt{-1} f_{0\alpha} \delta_{\beta\overline{\alpha}} \\ &= (f_{\overline{\alpha}\alpha} + 2n\sqrt{-1} f_0)_\beta + 2\sqrt{-1} f_{0\beta} \\ &= -n\overline{g}_\beta + 2(n+1)(G_\beta - g f_\beta) \\ &= 2(n+1) G_\beta - n\overline{g}_\beta - 2(n+1) f_\beta g. \end{aligned} \tag{2.14}$$

By this and (2.4), (2.5), it follows that

$$\begin{aligned} D_{\alpha,\overline{\alpha}} &= f_{\alpha\beta\overline{\alpha}} f_{\overline{\beta}} + f_{\alpha\beta} f_{\overline{\beta}\overline{\alpha}} - 2(|\partial f|^2)_{\overline{\alpha}} f_\alpha - 2|\partial f|^2 f_{\alpha\overline{\alpha}} \\ &= \big(2(n+1) G_\beta - n\overline{g}_\beta - 2(n+1) f_\beta g\big) f_{\overline{\beta}} \\ &\quad + (D_{\alpha\beta} + 2 f_\alpha f_\beta)(D_{\overline{\alpha}\overline{\beta}} + 2 f_{\overline{\alpha}} f_{\overline{\beta}}) \\ &\quad - 2\big(D_{\overline{\alpha}} + E_{\overline{\alpha}} + \overline{g} f_{\overline{\alpha}} - 2 f_{\overline{\alpha}} e^{(2+p)f}\big) f_\alpha + 2n|\partial f|^2 g \\ &= |D_{\alpha\beta}|^2 + 2 f_{\overline{\alpha}} D_\alpha - 2 f_\alpha E_{\overline{\alpha}} + 2(n+1) f_{\overline{\alpha}} G_\alpha - n f_{\overline{\alpha}} \overline{g}_\alpha. \end{aligned} \tag{2.15}$$

So we have

$$
\begin{aligned}
& e^{-2(n-1)f}\mathcal{L}_2 \\
= & e^{-2(n-1)f} Z_{\overline{\alpha}}\Big\{\big(g-\sqrt{-1}f_0\big)D_\alpha e^{2(n-1)f}\Big\} \\
= & \big(g-\sqrt{-1}f_0\big)D_{\alpha,\overline{\alpha}} \\
& + \big(g_{\overline{\alpha}}-\sqrt{-1}f_{0\overline{\alpha}}\big)D_\alpha + 2(n-1)\big(g-\sqrt{-1}f_0\big)f_{\overline{\alpha}}D_\alpha \\
= & \big(g-\sqrt{-1}f_0\big)\big(|D_{\alpha\beta}|^2 + 2f_{\overline{\alpha}}D_\alpha - 2f_\alpha E_{\overline{\alpha}} + 2(n+1)f_{\overline{\alpha}}G_\alpha - nf_{\overline{\alpha}}\overline{g}_\alpha\big) \\
& + \big(g_{\overline{\alpha}} + G_{\overline{\alpha}} - \overline{g}f_{\overline{\alpha}}\big)D_\alpha + 2(n-1)\big(g-\sqrt{-1}f_0\big)f_{\overline{\alpha}}D_\alpha \\
= & \big(g-\sqrt{-1}f_0\big)|D_{\alpha\beta}|^2 + (g_{\overline{\alpha}}+G_{\overline{\alpha}})D_\alpha \\
& + (2ng-\overline{g}-2n\sqrt{-1}f_0)f_{\overline{\alpha}}D_\alpha - 2(g-\sqrt{-1}f_0)f_\alpha E_{\overline{\alpha}} \\
& + 2(n+1)(g-\sqrt{-1}f_0)f_{\overline{\alpha}}G_\alpha - n(g-\sqrt{-1}f_0)f_{\overline{\alpha}}\overline{g}_\alpha.
\end{aligned}
\tag{2.16}
$$

Finally, for $\mathcal{L}_4$, by (2.3) and (2.5), a direct computation shows

$$
\begin{aligned}
e^{-2(n-1)f}\mathcal{L}_4 = & e^{-2(n-1)f}Z_{\overline{\alpha}}\Big\{-\frac{p}{4}f_\alpha|\partial f|^4e^{2(n-1)f}\Big\} \\
= & -\frac{p}{2}\big(E_{\overline{\alpha}}+D_{\overline{\alpha}}\big)f_\alpha|\partial f|^2 \\
& -\frac{p}{4}n|\partial f|^6 + \frac{p}{4}(n+2)|\partial f|^4e^{(2+p)f} - \frac{p}{4}(n+2)\sqrt{-1}f_0|\partial f|^4.
\end{aligned}
\tag{2.17}
$$

By (2.11), (2.13), (2.16) and (2.17), noticing $f_\alpha E_{\overline{\alpha}} = f_{\overline{\alpha}}E_\alpha$ (this also implies it's real), we obtain

$$
\begin{aligned}
& e^{-2(n-1)f}(\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3+\mathcal{L}_4) \\
= & \big(g-\sqrt{-1}f_0\big)|D_{\alpha\beta}|^2 + \big(g+3\sqrt{-1}f_0\big)|E_{\alpha\overline{\beta}}|^2 + 3|G_\alpha|^2 \\
& + (g_{\overline{\alpha}}+G_{\overline{\alpha}})D_\alpha + \big(g_{\overline{\alpha}}-3G_{\overline{\alpha}}\big)E_\alpha \\
& + \Big(2ng-\overline{g}-2n\sqrt{-1}f_0\Big)f_{\overline{\alpha}}D_\alpha - \frac{p}{2}|\partial f|^2D_{\overline{\alpha}}f_\alpha \\
& + \Big(2(n-2)g+3\overline{g}-\frac{p}{2}|\partial f|^2+(6n-4)\sqrt{-1}f_0\Big)\Big)f_{\overline{\alpha}}E_\alpha \\
& + \Big(2(n+1)g-3\overline{g}-(8n-4)\sqrt{-1}f_0\Big)f_{\overline{\alpha}}G_\alpha \\
& - \big((n-1)g+3n\sqrt{-1}f_0\big)f_\alpha g_{\overline{\alpha}} - n(g-\sqrt{-1}f_0)f_{\overline{\alpha}}\overline{g}_\alpha - 3nf_0g_0 \\
& + 3n\sqrt{-1}f_0|g|^2 + 6n|f_0|^2g \\
& - \frac{p}{4}n|\partial f|^6 + \frac{p}{4}(n+2)|\partial f|^4e^{(2+p)f} - \frac{p}{4}(n+2)\sqrt{-1}f_0|\partial f|^4.
\end{aligned}
\tag{2.18}
$$

Straight calculations show

$$
\begin{aligned}
g_0 = & \sqrt{-1}f_\alpha G_{\overline{\alpha}} - \sqrt{-1}f_{\overline{\alpha}}G_\alpha \\
& + 2f_0|\partial f|^2 + (2+p)f_0e^{(2+p)f} - \sqrt{-1}f_{00}.
\end{aligned}
\tag{2.19}
$$

By this and virtue of (2.6)-(2.7), we finally reach

$$
\begin{aligned}
& e^{-2(n-1)f}(\mathcal{L}_1+\mathcal{L}_2+\mathcal{L}_3+\mathcal{L}_4) \\
= & \big(g-\sqrt{-1}f_0\big)|D_{\alpha\beta}|^2+\big(g+3\sqrt{-1}f_0\big)|E_{\alpha\overline{\beta}}|^2+3|G_\alpha|^2 \\
& +(D_{\overline{\alpha}}+E_{\overline{\alpha}}+2G_{\overline{\alpha}})D_\alpha+\big(D_{\overline{\alpha}}+E_{\overline{\alpha}}-2G_{\overline{\alpha}}\big)E_\alpha \\
& +f_{\overline{\alpha}}D_\alpha\big((n-1)g-(n+2)\sqrt{-1}f_0+pe^{(2+p)f}\big) \\
& -f_\alpha D_{\overline{\alpha}}\big((n-1)g+3n\sqrt{-1}f_0+\frac{p}{2}|\partial f|^2\big) \\
& +f_{\overline{\alpha}}E_\alpha\big((4n+2)\sqrt{-1}f_0-\frac{p}{2}|\partial f|^2+pe^{(2+p)f}\big) \\
& +f_{\overline{\alpha}}G_\alpha\big((n-1)g-(4n+2)\sqrt{-1}f_0\big) \\
& +f_\alpha G_{\overline{\alpha}}\big((1-n)g-6n\sqrt{-1}f_0\big) \\
& -p(2n-1)|\partial f|^2e^{2(2+p)f}-\frac{p}{4}(7n-6)|\partial f|^4e^{(2+p)f} \\
& -\frac{p}{4}n|\partial f|^6-3np|f_0|^2e^{(2+p)f} \\
& -\frac{p}{4}(n+2)\sqrt{-1}f_0|\partial f|^4-p\sqrt{-1}f_0|\partial f|^2e^{(2+p)f} \\
& +3n\sqrt{-1}f_0|g|^2-6n\sqrt{-1}f_0|f_0|^2+3n\sqrt{-1}f_0f_{00},
\end{aligned}
\tag{2.20}
$$

where the linear terms of the tensors also can be rewritten as

$$
\begin{aligned}
& f_{\overline{\alpha}}D_\alpha\big((n-1)g-(n+2)\sqrt{-1}f_0+pe^{(2+p)f}\big) \\
& -f_\alpha D_{\overline{\alpha}}\big((n-1)g+3n\sqrt{-1}f_0+\frac{p}{2}|\partial f|^2\big) \\
& +f_{\overline{\alpha}}E_\alpha\big((4n+2)\sqrt{-1}f_0-\frac{p}{2}|\partial f|^2+pe^{(2+p)f}\big) \\
& +f_{\overline{\alpha}}G_\alpha\big((n-1)g-(4n+2)\sqrt{-1}f_0\big) \\
& +f_\alpha G_{\overline{\alpha}}\big((1-n)g-6n\sqrt{-1}f_0\big) \\
= & \big(pe^{(2+p)f}-\frac{p}{2}|\partial f|^2\big)(f_{\overline{\alpha}}D_\alpha+f_{\overline{\alpha}}E_\alpha) \\
& +\big((n-1)e^{(2+p)f}+(n-1+\frac{p}{2})|\partial f|^2\big)(f_{\overline{\alpha}}D_\alpha-f_\alpha D_{\overline{\alpha}}) \\
& -(2n+1)\sqrt{-1}f_0(f_{\overline{\alpha}}D_\alpha+f_\alpha D_{\overline{\alpha}})-(4n+2)\sqrt{-1}f_0f_{\overline{\alpha}}E_\alpha \\
& +(n-1)\big(e^{(2+p)f}+|\partial f|^2\big)(f_{\overline{\alpha}}G_\alpha-f_\alpha G_{\overline{\alpha}})-(5n+1)\sqrt{-1}f_0(f_{\overline{\alpha}}G_\alpha+f_\alpha G_{\overline{\alpha}}).
\end{aligned}
\tag{2.21}
$$

From this one can check (2.8) easily and complete the proof of Proposition 2.1. □

3. Proof of Theorem 1.1

Let's give a sketch for the proof of Theorem 1.1. We first rewrite the function $\mathcal{M}$ in (2.8) as (3.3), then we make some computation for (3.3) to get (3.6) such that all the coefficients in (3.6) are positive for $-2 < p < 0$. We choose the cut off function η as in Birindelli-Dolcetta-Cutri ((3.3), [1]). Since all the coefficients of $\mathcal{M}$ in (3.6) are positive, multiplying η^s and integrating over Ω, we first get immediately the following key inequality

$$\int\limits_{\Omega} \eta^s |\partial f|^2 e^{2(n+1+p)f} \leq C(n,p) \int\limits_{\Omega} \eta^s \mathcal{M}, \tag{3.1}$$

where $C(n,p)$ is a positive constant. Next, multiplying η^s on both sides of (2.8) and using the Cauchy inequality we obtain the inequality (3.11). Then we utilize another inequality (3.12) in Lemma 3.1 to treat (3.11), and we obtain (3.18) via the Young's inequality. Combining (3.1) and (3.18) with another Lemma 3.2, we obtain the proof of integral estimates (1.5), and hence the proof of Theorem 1.1. At the end of this section, we prove the Lemma 3.1 and Lemma 3.2.

Proof of Theorem 1.1. Let f satisfy the equation (2.3) and hence the identity (2.8). Then by $q = q^* + \frac{p}{n}$, the subcritical exponent $1 < q < q^*$ is corresponding to $-2 < p < 0$, and hence $Q - 2 \times \frac{3q-q^*}{q-1} = 2n + 2 - 2 \times \frac{2n+4+3p}{2+p} = -2 + \frac{2(n-1)p}{2+p} \leq -2$. In order to complete the proof of (1.5) and hence Theorem 1.1, we only need to prove the following inequality

$$\int\limits_{B_r(\xi_0)} e^{(2n+4+3p)f} \leq \; Cr^{2n+2-2\times\frac{2n+4+3p}{2+p}}. \tag{3.2}$$

Now we rewrite the function $\mathcal{M}$ in (2.8) as

$$\begin{aligned}
\mathcal{M} =& \big(|E_{\alpha\overline{\beta}}|^2 + |D_{\alpha\beta}|^2\big)e^{(2n+p)f} + \Big(|G_\alpha|^2 + |D_{\alpha\beta}f_{\overline{\gamma}} + E_{\alpha\overline{\gamma}}f_\beta|^2\Big)e^{2(n-1)f} \\
&+ s_0\Big(|G_\alpha + D_\alpha|^2 + |G_\alpha - E_\alpha|^2\Big)e^{2(n-1)f} \\
&+ (1-s_0)\Big(|G_\alpha + D_\alpha|^2 + |G_\alpha - E_\alpha|^2\Big)e^{2(n-1)f} \\
&+ pe^{2(n-1)f}\mathbf{Re}[f_{\overline{\alpha}}(G_\alpha + D_\alpha)]\big(e^{(2+p)f} - \frac{1}{2}|\partial f|^2\big) \\
&+ pe^{2(n-1)f}\mathbf{Re}[f_{\overline{\alpha}}(E_\alpha - G_\alpha)]\big(e^{(2+p)f} - \frac{1}{2}|\partial f|^2\big) \\
&- p(2n-1)|\partial f|^2 e^{2(n+1+p)f} - \frac{p}{4}(7n-6)|\partial f|^4 e^{(2n+p)f} \\
&- \frac{p}{4}n|\partial f|^6 e^{2(n-1)f} - 3np|f_0|^2 e^{(2n+p)f}.
\end{aligned} \tag{3.3}$$

Next, we will express $\mathcal{M}$ in suitable nonnegative terms such that all the coefficients are positive for $-2 < p < 0$. In fact, we will do it by completing a square of a binomial for the last five lines in (3.3) with suitable choice of s_0. So, first we have

$$\begin{aligned}\mathcal{M} =&\big(|E_{\alpha\overline{\beta}}|^2+|D_{\alpha\beta}|^2\big)e^{(2n+p)f}+\Big(|G_\alpha|^2+|D_{\alpha\beta}f_{\overline{\gamma}}+E_{\alpha\overline{\gamma}}f_\beta|^2\Big)e^{2(n-1)f}\\&+s_0\Big(|G_\alpha+D_\alpha|^2+|G_\alpha-E_\alpha|^2\Big)e^{2(n-1)f}\\&+e^{2(n-1)f}\Big|\sqrt{1-s_0}(G_\alpha+D_\alpha)+\frac{p}{2\sqrt{1-s_0}}f_\alpha\big(e^{(2+p)f}-\frac{1}{2}|\partial f|^2\big)\Big|^2\\&+e^{2(n-1)f}\Big|\sqrt{1-s_0}(E_\alpha-G_\alpha)+\frac{p}{2\sqrt{1-s_0}}f_\alpha\big(e^{(2+p)f}-\frac{1}{2}|\partial f|^2\big)\Big|^2\\&-\frac{p^2}{2(1-s_0)}e^{2(n-1)f}|\partial f|^2\Big(e^{2(2+p)f}+\frac{1}{4}|\partial f|^4-e^{(2+p)f}|\partial f|^2\Big)\\&-p(2n-1)|\partial f|^2e^{2(n+1+p)f}-\frac{p}{4}(7n-6)|\partial f|^4e^{(2n+p)f}\\&-\frac{p}{4}n|\partial f|^6e^{2(n-1)f}-3np|f_0|^2e^{(2n+p)f}.\end{aligned}\tag{3.4}$$

Then we treat the terms in the last three lines and get

$$\begin{aligned}\mathcal{M} =&\big(|E_{\alpha\overline{\beta}}|^2+|D_{\alpha\beta}|^2\big)e^{(2n+p)f}+\Big(|G_\alpha|^2+|D_{\alpha\beta}f_{\overline{\gamma}}+E_{\alpha\overline{\gamma}}f_\beta|^2\Big)e^{2(n-1)f}\\&+s_0\Big(|G_\alpha+D_\alpha|^2+|G_\alpha-E_\alpha|^2\Big)e^{2(n-1)f}\\&+e^{2(n-1)f}\Big|\sqrt{1-s_0}(G_\alpha+D_\alpha)+\frac{p}{2\sqrt{1-s_0}}f_\alpha\big(e^{(2+p)f}-\frac{1}{2}|\partial f|^2\big)\Big|^2\\&+e^{2(n-1)f}\Big|\sqrt{1-s_0}(E_\alpha-G_\alpha)+\frac{p}{2\sqrt{1-s_0}}f_\alpha\big(e^{(2+p)f}-\frac{1}{2}|\partial f|^2\big)\Big|^2\\&-p\Big(\frac{n}{4}+\frac{p}{8(1-s_0)}\Big)|\partial f|^6e^{2(n-1)f}-\frac{p}{4}\Big(7n-6-\frac{2p}{1-s_0}\Big)|\partial f|^4e^{(2n+p)f}\\&-p\Big(2n-1+\frac{p}{2(1-s_0)}\Big)|\partial f|^2e^{2(n+1+p)f}-3np|f_0|^2e^{(2n+p)f}.\end{aligned}\tag{3.5}$$

Now we take $0<s_0=\frac{1}{2}+\frac{p}{4n}<1$, then

$$\begin{aligned}\mathcal{M} =&\big(|E_{\alpha\overline{\beta}}|^2+|D_{\alpha\beta}|^2\big)e^{(2n+p)f}+\Big(|G_\alpha|^2+|D_{\alpha\beta}f_{\overline{\gamma}}+E_{\alpha\overline{\gamma}}f_\beta|^2\Big)e^{2(n-1)f}\\&+s_0\Big(|G_\alpha+D_\alpha|^2+|G_\alpha-E_\alpha|^2\Big)e^{2(n-1)f}\\&+e^{2(n-1)f}\Big|\sqrt{1-s_0}(G_\alpha+D_\alpha)+\frac{p}{2\sqrt{1-s_0}}f_\alpha\big(e^{(2+p)f}-\frac{1}{2}|\partial f|^2\big)\Big|^2\\&+e^{2(n-1)f}\Big|\sqrt{1-s_0}(E_\alpha-G_\alpha)+\frac{p}{2\sqrt{1-s_0}}f_\alpha\big(e^{(2+p)f}-\frac{1}{2}|\partial f|^2\big)\Big|^2\\&-p\frac{n(2n+p)}{4(2n-p)}|\partial f|^6e^{2(n-1)f}-\frac{p}{4}\Big(7n-6-\frac{8np}{2n-p}\Big)|\partial f|^4e^{(2n+p)f}\\&-p\frac{4n^2-2n+p}{2n-p}|\partial f|^2e^{2(n+1+p)f}-3np|f_0|^2e^{(2n+p)f}.\end{aligned}\tag{3.6}$$

Clearly all the coefficients in above are positive for $-2 < p < 0$ and hence $\mathcal{M} \geq 0$.

Since $B_{4r} \subset \Omega$, we can take a real smooth cut off function η such that

$$\begin{cases} \eta \equiv 1 & in\ B_r, \\ 0 \leq \eta \leq 1 & in\ B_{2r}, \\ \eta \equiv 0 & in\ \Omega \backslash B_{2r}, \\ |\partial \eta| \lesssim \frac{1}{r} & in\ \Omega, \end{cases} \tag{3.7}$$

where we use "$\lesssim$", "$\cong$" to replace "$\leq$" and "$=$" respectively, to drop out some positive constants independent of r and f.

Take a real $s > 0$ big enough. Multiplying η^s on both sides of (2.8) and integrating over Ω give

$$\begin{aligned} &\int_\Omega \eta^s \mathcal{M} \\ =&\int_\Omega \eta^s \mathbf{Re} Z_{\overline{\alpha}} \Big\{ \big[\big(D_\alpha + E_\alpha \big) (|\partial f|^2 + e^{(2+p)f}) \\ &\qquad - \sqrt{-1} f_0 \big(2D_\alpha - 2E_\alpha + 3G_\alpha \big) - \frac{p}{4} f_\alpha |\partial f|^4 \big] e^{2(n-1)f} \Big\}. \end{aligned} \tag{3.8}$$

Integrating by part and using (3.7) we get

$$\begin{aligned} &\int_\Omega \eta^s \mathcal{M} \\ =& - s \int_\Omega \eta^{s-1} \mathbf{Re} \eta_{\overline{\alpha}} \Big\{ \big[\big(D_\alpha + E_\alpha \big) (|\partial f|^2 + e^{(2+p)f}) \\ &\qquad - \sqrt{-1} f_0 \big(2D_\alpha - 2E_\alpha + 3G_\alpha \big) - \frac{p}{4} f_\alpha |\partial f|^4 \big] e^{2(n-1)f} \Big\} \\ \lesssim& \frac{1}{r} \int_\Omega \eta^{s-1} \Big\{ |D_\alpha + E_\alpha| (|\partial f|^2 + e^{(2+p)f}) e^{2(n-1)f} \\ &\qquad + |f_0| \big| 2D_\alpha - 2E_\alpha + 3G_\alpha \big| e^{2(n-1)f} + |\partial f|^5 e^{2(n-1)f} \Big\} \end{aligned} \tag{3.9}$$

Since

$$|D_\alpha + E_\alpha| \leq |D_\alpha + G_\alpha| + |E_\alpha - G_\alpha|,$$

$$\big| 2D_\alpha - 2E_\alpha + 3G_\alpha \big| \leq 2|D_\alpha + G_\alpha| + 2|E_\alpha - G_\alpha| + |G_\alpha|,$$

using the Young's inequality $ab \le \epsilon a^2 + \frac{1}{\epsilon} b^2$ in (3.9) we obtain

$$\begin{aligned}\int_\Omega \eta^s \mathcal{M} \lesssim & \epsilon \int_\Omega \eta^s \left(|D_\alpha + G_\alpha|^2 + |E_\alpha - G_\alpha|^2 + |G_\alpha|^2\right) e^{2(n-1)f} \\ & + \frac{1}{\epsilon r^2} \int_\Omega \eta^{s-2} \left(|\partial f|^4 + e^{2(2+p)f} + |f_0|^2\right) e^{2(n-1)f} \\ & + \frac{1}{r} \int_\Omega \eta^{s-1} |\partial f|^5 e^{2(n-1)f}.\end{aligned} \tag{3.10}$$

Note that in (3.6), the coefficients of all the terms of $\mathcal{M}$ are positive. By taking ϵ small, it follows that

$$\begin{aligned}\int_\Omega \eta^s \mathcal{M} \lesssim & \frac{1}{r^2} \int_\Omega \eta^{s-2} \left(|\partial f|^4 + e^{2(2+p)f} + |f_0|^2\right) e^{2(n-1)f} \\ & + \frac{1}{r} \int_\Omega \eta^{s-1} |\partial f|^5 e^{2(n-1)f}.\end{aligned} \tag{3.11}$$

To deal with the term containing $|f_0|^2$ on the right hand side of (3.11), we need the following Lemma 3.1, which will be proved at the end of this section.

Lemma 3.1.

$$\begin{aligned}\int_\Omega \eta^{s-2} |f_0|^2 e^{2(n-1)f} \lesssim & \epsilon r^2 \int_\Omega \eta^s \mathcal{M} + \int_\Omega \eta^{s-2} |\partial f|^4 e^{2(n-1)f} \\ & + \int_\Omega \eta^{s-2} |\partial f|^2 e^{(2n+p)f} + \frac{1}{r^2} \int_\Omega \eta^{s-4} |\partial f|^2 e^{2(n-1)f}.\end{aligned} \tag{3.12}$$

Now plugging (3.12) into (3.11) with small ϵ we get

$$\begin{aligned}\int_\Omega \eta^s \mathcal{M} \lesssim & \frac{1}{r^2} \int_\Omega \eta^{s-2} e^{2(n+1+p)f} \\ & + \frac{1}{r^2} \int_\Omega \eta^{s-2} |\partial f|^4 e^{2(n-1)f} + \frac{1}{r^2} \int_\Omega \eta^{s-2} |\partial f|^2 e^{(2n+p)f} \\ & + \frac{1}{r^4} \int_\Omega \eta^{s-4} |\partial f|^2 e^{2(n-1)f} + \frac{1}{r} \int_\Omega \eta^{s-1} |\partial f|^5 e^{2(n-1)f}.\end{aligned} \tag{3.13}$$

For the last term in above, using Young's inequality one gets

$$\frac{1}{r} \int_\Omega \eta^{s-1} |\partial f|^5 e^{2(n-1)f} \lesssim \epsilon \int_\Omega \eta^s |\partial f|^6 e^{2(n-1)f} + \frac{1}{r^6} \int_\Omega \eta^{s-6} e^{2(n-1)f}. \tag{3.14}$$

Similarly, one has

$$\frac{1}{r^2}\int_\Omega \eta^{s-2}|\partial f|^4 e^{2(n-1)f} \lesssim \epsilon \int_\Omega \eta^s |\partial f|^6 e^{2(n-1)f} + \frac{1}{r^6}\int_\Omega \eta^{s-6} e^{2(n-1)f}, \tag{3.15}$$

$$\frac{1}{r^2}\int_\Omega \eta^{s-2}|\partial f|^2 e^{(2n+p)f} \lesssim \epsilon \int_\Omega \eta^s |\partial f|^4 e^{(2n+p)f} + \frac{1}{r^4}\int_\Omega \eta^{s-4} e^{(2n+p)f}, \tag{3.16}$$

and

$$\frac{1}{r^4}\int_\Omega \eta^{s-4}|\partial f|^2 e^{2(n-1)f} \lesssim \epsilon \int_\Omega \eta^s |\partial f|^6 e^{2(n-1)f} + \frac{1}{r^6}\int_\Omega \eta^{s-6} e^{2(n-1)f}. \tag{3.17}$$

Inserting these into (3.13) and taking ϵ small yield

$$\begin{aligned}\int_\Omega \eta^s \mathcal{M} \lesssim & \frac{1}{r^2}\int_\Omega \eta^{s-2} e^{2(n+1+p)f} \\ & + \frac{1}{r^4}\int_\Omega \eta^{s-4} e^{(2n+p)f} + \frac{1}{r^6}\int_\Omega \eta^{s-6} e^{2(n-1)f}.\end{aligned} \tag{3.18}$$

To get the left of (3.2) to complete this proof, we need the following lemma, which will be proved also at the end of this section.

Lemma 3.2.

$$\int_\Omega \eta^s e^{(2n+4+3p)f} \lesssim \int_\Omega \eta^s |\partial f|^2 e^{2(n+1+p)f} + \frac{1}{r^2}\int_\Omega \eta^{s-2} e^{(2n+2+2p)f}. \tag{3.19}$$

Now since all the coefficients in (3.6) are positive for $-2 < p < 0$, it follows that

$$\int_\Omega \eta^s |\partial f|^2 e^{2(n+1+p)f} \le C(n,p)\int_\Omega \eta^s \mathcal{M}. \tag{3.20}$$

Combining (3.20) with (3.18)-(3.19), we have

$$\begin{aligned}&\int_\Omega \eta^s e^{(2n+4+3p)f}\\ \lesssim&\int_\Omega \eta^s|\partial f|^2 e^{2(n+1+p)f}+\frac{1}{r^2}\int_\Omega \eta^{s-2}e^{(2n+2+2p)f}\\ \lesssim&\int_\Omega \eta^s\mathcal{M}+\frac{1}{r^2}\int_\Omega \eta^{s-2}e^{(2n+2+2p)f}\\ \lesssim&\frac{1}{r^2}\int_\Omega \eta^{s-2}e^{2(n+1+p)f}\\ &+\frac{1}{r^4}\int_\Omega \eta^{s-4}e^{(2n+p)f}+\frac{1}{r^6}\int_\Omega \eta^{s-6}e^{2(n-1)f}\\ \lesssim&\epsilon\int_\Omega \eta^s e^{(2n+4+3p)f}+r^{-2\times\frac{2n+4+3p}{2+p}}\int_\Omega \eta^{s-2\times\frac{2n+4+3p}{2+p}},\end{aligned}\tag{3.21}$$

where in the last step, the Young's inequality has been used three times with different exponent pairs. Note that $0\le\eta\le 1$ in $B_{2r}(\xi_0)\subset\Omega$ and $\eta=1$ in $B_r(\xi_0)$. Therefore, by choosing $s>0$ big enough and ϵ small, we finally obtain

$$\int_{B_r(\xi_0)} e^{(2n+4+3p)f}\lesssim r^{2n+2-2\times\frac{2n+4+3p}{2+p}}.\tag{3.22}$$

This is (3.2), and hence Theorem 1.1 is proved. □

To complete this section, now we give the proofs of Lemma 3.1 and Lemma 3.2.

Proof of Lemma 3.1. Since f satisfies the equation (2.3), a straight calculation shows

$$e^{-kf}\mathbf{Re}Z_{\overline{\alpha}}\Big(\sqrt{-1}f_0f_\alpha e^{kf}\Big)=-\mathbf{Re}G_{\overline{\alpha}}f_\alpha-n|f_0|^2+|\partial f|^4+|\partial f|^2e^{(2+p)f}.\tag{3.23}$$

Multiply both sides of (3.23) by $\eta^{s-2}e^{kf}$ with $k=2(n-1)$ and integrate over Ω we have

$$\begin{aligned}&\int_\Omega \eta^{s-2}\mathbf{Re}Z_{\overline{\alpha}}\Big(\sqrt{-1}f_0f_\alpha e^{2(n-1)f}\Big)\\ =&\int_\Omega \eta^{s-2}\Big(-\mathbf{Re}G_{\overline{\alpha}}f_\alpha-n|f_0|^2+|\partial f|^4+|\partial f|^2e^{(2+p)f}\Big)e^{2(n-1)f}.\end{aligned}\tag{3.24}$$

Integrating by part, using (3.7) and arranging the terms yield

$$\begin{aligned}
n\int\limits_{\Omega}\eta^{s-2}|f_0|^2e^{2(n-1)f} &= \int\limits_{\Omega}\eta^{s-2}\big(|\partial f|^4+|\partial f|^2e^{(2+p)f}\big)e^{2(n-1)f}\\
&\quad-\int\limits_{\Omega}\eta^{s-2}\mathbf{Re}G_{\overline{\alpha}}f_\alpha e^{2(n-1)f}\\
&\quad+(s-2)\int\limits_{\Omega}\eta^{s-3}\mathbf{Re}\eta_{\overline{\alpha}}\Big(\sqrt{-1}f_0f_\alpha e^{2(n-1)f}\Big)\\
&\lesssim \int\limits_{\Omega}\eta^{s-2}\big(|\partial f|^4+|\partial f|^2e^{(2+p)f}\big)e^{2(n-1)f}\\
&\quad+\int\limits_{\Omega}\eta^{s-2}|G_{\overline{\alpha}}||\partial f|e^{2(n-1)f}\\
&\quad+\frac{1}{r}\int\limits_{\Omega}\eta^{s-3}|f_0||\partial f|e^{2(n-1)f}.
\end{aligned} \tag{3.25}$$

For the above last two terms, Young's inequality implies

$$\begin{aligned}
&\int\limits_{\Omega}\eta^{s-2}|G_{\overline{\alpha}}||\partial f|e^{2(n-1)f}+\frac{1}{r}\int\limits_{\Omega}\eta^{s-3}|f_0||\partial f|e^{2(n-1)f}\\
\le\,&\epsilon r^2\int\limits_{\Omega}\eta^{s}|G_\alpha|^2e^{2(n-1)f}+\epsilon\int\limits_{\Omega}\eta^{s-2}|f_0|^2e^{2(n-1)f}\\
&\qquad+\frac{C}{\epsilon r^2}\int\limits_{\Omega}\eta^{s-4}|\partial f|^2e^{2(n-1)f}.
\end{aligned} \tag{3.26}$$

Submitting this into (3.25) with small ϵ we get

$$\begin{aligned}
\int\limits_{\Omega}\eta^{s-2}|f_0|^2e^{2(n-1)f} &\lesssim \epsilon r^2\int\limits_{\Omega}\eta^{s}\mathcal{M}+\int\limits_{\Omega}\eta^{s-2}|\partial f|^4e^{2(n-1)f}\\
&\quad+\int\limits_{\Omega}\eta^{s-2}|\partial f|^2e^{(2n+p)f}+\frac{1}{r^2}\int\limits_{\Omega}\eta^{s-4}|\partial f|^2e^{2(n-1)f}.
\end{aligned} \tag{3.27}$$

This is just (3.12). □

The proof of Lemma 3.2 is similar to that of Lemma 3.1.

Proof of Lemma 3.2. Multiplying both sides of the equation (2.3) by $-\eta^s e^{2(n+1+p)f}$ and integrating over Ω give

$$
\begin{aligned}
n\int_\Omega \eta^s g e^{2(n+1+p)f} =& -\int_\Omega \eta^s f_{\alpha\overline{\alpha}} e^{2(n+1+p)f} \\
=&2(n+1+p)\int_\Omega \eta^s |\partial f|^2 e^{2(n+1+p)f} \\
&+ s\int_\Omega \eta^{s-1}\eta_{\overline{\alpha}} f_\alpha e^{2(n+1+p)f}.
\end{aligned}
\tag{3.28}
$$

Taking the real parts in (3.28) we get

$$
\begin{aligned}
n\int_\Omega \eta^s\big(|\partial f|^2 + e^{(2+p)f}\big)e^{2(n+1+p)f} =&2(n+1+p)\int_\Omega \eta^s |\partial f|^2 e^{2(n+1+p)f} \\
&+ s\int_\Omega \eta^{s-1}\mathbf{Re}\eta_{\overline{\alpha}} f_\alpha e^{2(n+1+p)f}.
\end{aligned}
\tag{3.29}
$$

Using (3.7) and arranging the terms yield

$$
\begin{aligned}
\int_\Omega \eta^s e^{(2n+4+3p)f} \lesssim& \int_\Omega \eta^s |\partial f|^2 e^{2(n+1+p)f} + \frac{1}{r}\int_\Omega \eta^{s-1}|\partial f| e^{2(n+1+p)f} \\
\lesssim& \int_\Omega \eta^s |\partial f|^2 e^{2(n+1+p)f} + \frac{1}{r^2}\int_\Omega \eta^{s-2} e^{2(n+1+p)f},
\end{aligned}
\tag{3.30}
$$

where in the last step, the Cauchy-Schwarz inequality has been used, and this is (3.19) as desired. □

4. Proof of Theorem 1.2

In the proof of Theorem 1.2, we use the similar method as in Euclidean case (see Theorem 3.11 in Veron [20]). Note that the dilation of the coordinates $(z, t) \in \mathbb{C}^n \times \mathbb{R}$ in Heisenberg group is defined by $(\delta z, \delta^2 t)$ $(\delta > 0)$, meanwhile the equation (1.1) and the estimates (1.4) are both invariant under the transformation $u(z,t) \longrightarrow \delta^{\frac{2}{q-1}} u(\delta z, \delta^2 t)$, so we can assume $R = 1$.

We first state the following Harnack inequality, which is a special case of that given by Capogna-Danielli-Garofalo (see Theorem 3.1 in [4]). While Capogna-Danielli-Garofalo stated the Harnack inequality with the constant C_0 depending on $||h(\xi)||_{L^s_{loc}(\Omega)}$ in their original paper, we present it here with a slight change, since one can use the similar scaling technique as in the Euclidean case (see for example page 74-75 in the book by Han-Lin [13]).

Lemma 4.1. *[4] Let $0 \le u \in C^2(\Omega)$ satisfy*

$$
\triangle_{\mathbb{H}^n} u + h(\xi)u = 0 \quad in \quad \Omega, \tag{4.1}
$$

with $h(\xi) \in L^s_{loc}(\Omega)$ for some $s > \frac{Q}{2}$. Then for any ball $B_r(\xi_0)$ with $B_{4r}(\xi_0) \subset \Omega$, there exists a constant $C_0 > 0$ depending only on n, s and $r^{2-\frac{Q}{s}}||h(\xi)||_{L^s(B_{4r}(\xi_0))}$, such that

$$\max_{B_r(\xi_0)} u \le C_0 \min_{B_r(\xi_0)} u. \tag{4.2}$$

Proof of Theorem 1.2. Rewrite the equation (1.1) as

$$\triangle_{\mathbb{H}^n} u + h(\xi)u = 0 \quad \text{in} \quad B_1(0)\backslash\{0\}, \tag{4.3}$$

with $h(\xi) = 2n^2u^{q-1}$. For any $\xi_0 \in B_{\frac{1}{2}}\backslash\{0\}$, take $r = \frac{1}{16}|\xi_0|$. Denote $\big|B_r(\xi_0)\big|$ the volume of the ball $B_r(\xi_0)$. Using the estimate (1.5) we have

$$\int\limits_{B_r(\xi_0)} h^s = (2n^2)^s \int\limits_{B_r(\xi_0)} u^{3q-q^*} \le C(n,q)\, r^{Q-2\times\frac{3q-q^*}{q-1}}, \tag{4.4}$$

with $s = \frac{3q-q^*}{q-1} > \frac{Q}{2}$ for $1 < q < q^* = \frac{Q+2}{Q-2}$. This implies $h(\xi) \in L^s_{loc}(\Omega)$ for some $s > \frac{Q}{2}$ and hence u satisfies the Harnack inequality (4.2). Moreover, using (4.4) with $B_r(\xi_0)$ replaced by $B_{4r}(\xi_0)$, we also have

$$r^{2-\frac{Q}{s}}||h(\xi)||_{L^s(B_{4r}(\xi_0))} \le C(n,q)r^{2-\frac{Q}{s}}\big[r^{Q-2\times\frac{3q-q^*}{q-1}}\big]^{\frac{1}{s}} \le C(n,q)r^{2-\frac{Q}{s}}\big[r^{Q-2s}\big]^{\frac{1}{s}} = C(n,q). \tag{4.5}$$

The estimate (4.5) implies that in the Harnack inequality (4.2) for u satisfying (4.3), the constant C_0 depends only on n and q. So combining this Harnack inequality with (1.5) we finally obtain

$$\frac{1}{C}r^Q u(\xi_0)^{3q-q^*} \le |B_r(\xi_0)|\Big[\frac{u(\xi_0)}{C_0}\Big]^{3q-q^*} \le \int\limits_{B_r(\xi_0)} u^{3q-q^*} \le C\, r^{Q-2\times\frac{3q-q^*}{q-1}}. \tag{4.6}$$

This implies

$$u(\xi) \le C|\xi|^{\frac{-2}{q-1}} \quad \text{for } \ 0 < |\xi| \le \frac{1}{4}, \tag{4.7}$$

and the proof of Theorem 1.2 is completed. □

As in [11], one is interesting the exact asymptotic behaviour of the solution u near the isolated singularity.

Acknowledgment

Partial research of the second author was done while he was visiting the Chinese University of Hong Kong in the fall of 2019. He would like to thank Prof. Zhouping Xin, Prof. Guohuan Qiu and the Institute of Mathematical Sciences of the Chinese University of Hong Kong for their warm hospitality. The authors would also like to thank the referees for their careful readings and useful suggestions. The first author was supported by National Natural Science Foundation of China (grants 11721101 and 11871255) and National Key Research and Development Project (grants SQ2020YFA070080). The second author was supported by National Natural Science Foundation of China (grants 11861016). Both the authors was supported by National Natural Science Foundation of China (grants 12141105).

References

[1] I. Birindelli, I. Capuzzo-Dolcetta, A. Cutri, Liouville theorems for semilinear equations on the Heisenberg group, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 14 (1997) 295–308.
[2] I. Birindelli, J. Prajapat, Nonlinear Liouville theorems in the Heisenberg group via the moving plane, Commun. Partial Differ. Equ. 24 (1999) 1875–1890.
[3] L. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Commun. Pure Appl. Math. 42 (1989) 271–297.
[4] L. Capogna, D. Danielli, N. Garofalo, An embedding theorem and the Harnack inequality for nonlinear subelliptic equations, Commun. Partial Differ. Equ. 18 (1993) 1765–1794.
[5] W.X. Chen, C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J. 63 (1991) 615–622.
[6] J.-H. Cheng, A. Malchiodi, P. Yang, On the Sobolev quotient of three-dimensional CR manifolds, arXiv:1904.04665.
[7] G. Folland, E.M. Stein, Estimates for the $\bar{\partial}_b$ complex and analysis on the Heisenberg group, Commun. Pure Appl. Math. 27 (1974) 429–522.
[8] N. Gamara, CR Yamabe conjecture - the case $n = 1$, J. Eur. Math. Soc. 3 (2001) 105–137.
[9] N. Gamara, R. Yacoub, CR Yamabe conjecture - the conformally flat case, Pac. J. Math. 201 (2001) 121–175.
[10] N. Garofalo, D. Vassilev, Symmetry properties of positive entire solutions of Yamabe type equations on the groups of Heisenberg type, Duke Math. J. 106 (2001) 411–448.
[11] B. Gidas, J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Commun. Pure Appl. Math. 34 (1981) 525–598.
[12] B. Gidas, J. Spruck, A priori bounds for positive solutions of nonlinear elliptic equations, Commun. Partial Differ. Equ. 6 (1981) 883–901.
[13] Q. Han, F.-H. Lin, Elliptic Partial Differential Equations, 2nd ed., Amer. Math. Soc., Providence, 2011.
[14] D. Jerison, J.M. Lee, The Yamabe problem on CR manifolds, J. Differ. Geom. 25 (1987) 167–197.
[15] D. Jerison, J.M. Lee, Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem, J. Am. Math. Soc. 1 (1988) 1–13.
[16] D. Jerison, J.M. Lee, Intrinsic CR normal coordinates and the CR Yamabe problem, J. Differ. Geom. 29 (1989) 303–343.
[17] Y.Y. Li, L. Zhang, Liouville-type theorems and Harnack-type inequalities for semilinear elliptic equations, J. Anal. Math. 90 (2003) 27–87.
[18] Y.Y. Li, M. Zhu, Uniqueness theorems through the method of moving spheres, Duke Math. J. 80 (1995) 383–417.
[19] M. Obata, The conjecture on conformal transformations of Riemannian manifolds, J. Differ. Geom. 6 (1971) 247–258.
[20] L. Veron, Singularities of Solutions of Second Order Quasilinear Equations, Pitman Research Notes in Math., vol. 353, Addison Wesley Longman Inc., 1996.
[21] X. Wang, On a remarkable formula of Jerison and Lee in CR geometry, Math. Res. Lett. 22 (2015) 279–299.
[22] L. Xu, Semi-linear Liouville theorems in the Heisenberg group via vector field methods, J. Differ. Equ. 247 (2009) 2799–2820.