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Remarks on convexity estimates for solutions of the torsion problem

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Abstract In this paper, for the solution of the torsion problem about the equation $\Delta u = -2$ with homogeneous Dirichlet boundary conditions in a bounded convex domain in $\mathbb{R}^n$, we find a superharmonic function which implies the strict concavity of $u^{\frac{1}{2}}$ and give some convexity estimates. It is a generalization of Makar-Limanov's result (Makar-Limanov (1971)) and Ma-Shi-Ye's result (Ma et al. (2012)).

Keywords convexity estimate, superharmonicity, maximum principle, torsion problem

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1 Introduction

Let Ω be a smooth, bounded convex domain in $\mathbb{R}^n$ ($n \geqslant 2$). In this paper, we consider the following torsion problem:

$$\begin{cases} \Delta u = -2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

In 1971, Makar-Limanov [12] considered the boundary value problem (1.1) in a bounded plane convex domain Ω. He introduced the function

$$P_1 = 2u \det D^2u + 2u_1u_2u_{12} - u_{11}u_2^2 - u_{22}u_1^2$$

and proved that P_1 is a superharmonic function. Then he could obtain that $u^{\frac{1}{2}}$ is strictly concave. In 1983, Korevaar [7] introduced a very useful technique to study the convexity of the solutions for a class of elliptic equations. To different extents, Kawohl [5] and Kennington [6] improved Korevaar's method, which enabled them to get that $u^{\frac{1}{2}}$ is concave in higher dimensions. In particular, Kennington [6] pointed out that the concavity number $\frac{1}{2}$ of u is sharp in the problem (1.1). Singer et al. [14] and Caffarelli and Friedman [1] introduced a new deformation technique to deal with the convexity. Caffarelli and

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Friedman [1] established the strict convexity of the solutions for some equations in 2-dimensional convex domains. Korevaar and Lewis [8] generalized the deformation method to higher dimensions, and obtained the strict concavity of $u^{\frac{1}{2}}$ in (1.1) in higher-dimensional cases.

In 2012, Ma et al. [10] found a new corresponding auxiliary function

$$P_2 = (-2)^{-n} u \det D^2 u + (-2)^{-n-1} \sum_{i,j=1}^{n} \frac{\partial \det D^2 u}{\partial u_{ij}} u_i u_j,$$

which is superharmonic modulo the gradient terms under the strict concavity assumption of $u^{\frac{1}{2}}$. So from the minimum principle, the authors got the convexity estimates for the solution of (1.1) via boundary data. Combining the deformation method, they can give a new proof of strict concavity of $v = \sqrt{u}$, and obtained the Gaussian curvature estimate for the graph of $v = \sqrt{u}$ in the problem (1.1) using the curvature of the boundary of the domain. There is much literature on studying convexity estimates for partial differential equations through finding auxiliary curvature functions. For example, Chen et al. [2] considered the Monge-Ampère equation $\det D^2 u = 1$ and Shi [13] gave the case for the Green's function.

When $n = 2$, the function P_1 introduced in [12] is superharmonic. Unfortunately, when $n > 2$, the function P_2 introduced in [10] is not really superharmonic. In this paper, we introduce a new function, which is superharmonic. Our results are as follows.

Theorem 1.1. *Let Ω be a smooth, bounded convex domain in $\mathbb{R}^n$ $(n \geqslant 2)$, and u be the solution for the problem* (1.1)*. If $v = -\sqrt{u}$ is a strictly convex function, then for*

$$\psi = (-v)^{n+2} \det D^2 v = (-2)^{-n} u \det D^2 u + (-2)^{-n-1} \sum_{i,j=1}^{n} \frac{\partial \det D^2 u}{\partial u_{ij}} u_i u_j,$$

$\varphi = \psi^{\frac{1}{n-1}}$ is a superharmonic function. Namely, the function φ perfectly satisfies the differential inequality

$$\Delta \varphi \leqslant 0 \quad \text{in } \Omega. \tag{1.2}$$

The level sets of the solution in the problem (1.1) are convex with respect to the normal direction ∇u, and the Gaussian curvature K of the level sets of the solution u can be expressed as

$$(-1)^{n-1} \sum_{i,j=1}^{n} \frac{\partial \det D^2 u}{\partial u_{ij}} u_i u_j |\nabla u|^{-(n+1)}.$$

Corollary 1.2 (See [10])**.** *Under the conditions in Theorem* 1.1*, we have the following estimate for the solution of the problem* (1.1)*:*

$$\psi = (-v)^{n+2} \det D^2 v \geqslant 2^{-(n+1)} \min_{\partial\Omega} K \min_{\partial\Omega} |\nabla u|^{n+1}, \tag{1.3}$$

where K is the Gaussian curvature of $\partial\Omega$. Moreover, the function ψ attains its minimum in Ω if and only if Ω is an ellipsoid (*ellipse*)*.*

The results in Corollary 1.2 were first obtained in [10]. We can give the proof of strict concavity of $v = \sqrt{u}$ from the estimate (1.3) by the deformation method as in [10].

Remark 1.3. When $n = 2$, 8φ is exactly P_1 introduced in [12].

Remark 1.4. For the harmonic functions u with convex level sets, Ma et al. [9] got the Gaussian curvature estimates of the level sets of u. Recently, Ma and Zhang [11] proved that $\psi = (|\nabla u|^{n-3} K)^{\frac{1}{n-1}}$ is superharmonic.

In geometric function theory and nonlinear elasticity, the superharmonicity for the logarithm of the Jacobian determinant plays an important role in studying the diffeomorphism problem (see the examples in [4]). In higher-dimensional cases, Gleason and Wolff [3] studied the diffeomorphism for the gradient mapping of the harmonic function u, and the superharmonicity for the $\log|\det D^2 u|$ is still the main ingredient in their proof.

We focus on the proofs of Theorem 1.1 and Corollary 1.2 in Section 2. The main technique in the proof of Theorem 1.1 consists of regrouping terms involving the third-order derivatives and maximizing them in each group.

2 Proofs of Theorem 1.1 and Corollary 1.2

We first give the following two lemmas.

Lemma 2.1. *Let $\boldsymbol{B}$ be an $(n-1)\times(n-1)$ symmetric matrix, $n\geqslant 2$, $\boldsymbol{x}=(x_1,\ldots,x_{n-1})$, $\boldsymbol{b}=(b_1,\ldots,b_{n-1})\in\mathbb{R}^{n-1}$ and $f(\boldsymbol{x})=\boldsymbol{xBx}^{\mathrm{T}}+2\boldsymbol{bx}^{\mathrm{T}}$. If $\boldsymbol{B}<0$, i.e., $\boldsymbol{B}$ is negative definite, then*

$$f(\boldsymbol{x})\leqslant -\boldsymbol{bB}^{-1}\boldsymbol{b}^{\mathrm{T}}.$$

Proof. Since $\boldsymbol{B}<0$, $f(\boldsymbol{x})$ has the unique critical point and takes the greatest value at this point. At the critical point, we have

$$2\boldsymbol{Bx}^{\mathrm{T}}+2\boldsymbol{b}^{\mathrm{T}}=0,$$

i.e., $\boldsymbol{x}^{\mathrm{T}}=-\boldsymbol{B}^{-1}\boldsymbol{b}^{\mathrm{T}}$. Therefore,

$$\begin{aligned}f(\boldsymbol{x})&\leqslant(-\boldsymbol{B}^{-1}\boldsymbol{b}^{\mathrm{T}})^{\mathrm{T}}\boldsymbol{B}(-\boldsymbol{B}^{-1}\boldsymbol{b}^{\mathrm{T}})+2\boldsymbol{b}(-\boldsymbol{B}^{-1}\boldsymbol{b}^{\mathrm{T}})\\&=\boldsymbol{bB}^{-1}\boldsymbol{b}^{\mathrm{T}}-2\boldsymbol{bB}^{-1}\boldsymbol{b}^{\mathrm{T}}=-\boldsymbol{bB}^{-1}\boldsymbol{b}^{\mathrm{T}}.\end{aligned}$$

This completes the proof. □

Lemma 2.2. *Let $\boldsymbol{\xi}=(\xi_1,\xi_2,\ldots,\xi_l)\in\mathbb{R}^l$ $(l\geqslant 1)$. If $\xi_i>1$ for any $i\in\{1,2,\ldots,l\}$, then we have the following inequality*:

$$\frac{(\sum_{i=1}^l\xi_i)\sum_{j=1}^l\frac{1}{\xi_j}+\sum_{j=1}^l\xi_j+9\sum_{i=1}^l\frac{1}{\xi_i}-l(l+6)}{\sum_{j=1}^l\xi_j+\sum_{j=1}^l\frac{1}{\xi_j}+4-2l}<l. \tag{2.1}$$

Proof. Because

$$\begin{aligned}&\frac{(\sum_{i=1}^l\xi_i)\sum_{j=1}^l\frac{1}{\xi_j}+\sum_{j=1}^l\xi_j+9\sum_{i=1}^l\frac{1}{\xi_i}-l(l+6)}{\sum_{j=1}^l\xi_j+\sum_{j=1}^l\frac{1}{\xi_j}+4-2l}-l\\&\quad=\frac{(\sum_{i=1}^l\xi_i)\sum_{j=1}^l\frac{1}{\xi_j}+(1-l)\sum_{j=1}^l\xi_j+(9-l)\sum_{i=1}^l\frac{1}{\xi_i}+l(l-10)}{\sum_{j=1}^l\xi_j+\sum_{j=1}^l\frac{1}{\xi_j}+4-2l}\end{aligned}$$

and

$$\sum_{j=1}^l\xi_j+\sum_{j=1}^l\frac{1}{\xi_j}+4-2l>4>0,$$

the inequality (2.1) is equivalent to

$$\left(\sum_{i=1}^l\xi_i\right)\sum_{j=1}^l\frac{1}{\xi_j}+(1-l)\sum_{j=1}^l\xi_j+(9-l)\sum_{i=1}^l\frac{1}{\xi_i}+l(l-10)<0.$$

Let

$$f(\xi_1,\ldots,\xi_l)=\left(\sum_{i=1}^l\xi_i\right)\sum_{j=1}^l\frac{1}{\xi_j}-(l-9)\sum_{j=1}^l\frac{1}{\xi_j}-(l-1)\sum_{j=1}^l\xi_j+l(l-10).$$

Then

$$f(1,\ldots,1)=0$$

and

$$\begin{aligned}\frac{\partial f}{\partial \xi_i} &= \sum_{j=1}^{l}\frac{1}{\xi_j} - \frac{1}{\xi_i^2}\sum_{j=1}^{l}\xi_j + (l-9)\frac{1}{\xi_i^2} - (l-1)\\ &= \Bigg(\sum_{\substack{1\leqslant j\leqslant l\\ j\neq i}}\frac{1}{\xi_j} - (l-1)\Bigg) + \frac{1}{\xi_i^2}\Bigg(l-9-\sum_{\substack{1\leqslant j\leqslant l\\ j\neq i}}\xi_j\Bigg).\end{aligned}$$

Since $\xi_j > 1$ for $j = 1, 2, \ldots, l$, we have

$$\sum_{\substack{1\leqslant j\leqslant l\\ j\neq i}}\frac{1}{\xi_j} - (l-1) < 0, \quad l-9-\sum_{\substack{1\leqslant j\leqslant l\\ j\neq i}}\xi_j < 0 \quad \text{and} \quad \frac{\partial f}{\partial \xi_i} < 0.$$

Thus

$$f(\xi_1, \ldots, \xi_l) < f(1, \ldots, 1) = 0,$$

and (2.1) holds. □

Now, we give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let u be the solution for the problem (1.1) and $v = -\sqrt{u}$. Then v is strictly convex from our assumption and satisfies the following problem:

$$\begin{cases}\Delta v = -\dfrac{1+|\nabla v|^2}{v} & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega.\end{cases} \tag{2.2}$$

For

$$\psi = (-v)^{n+2}\det D^2 v = (-2)^{-n}u\det D^2 u + (-2)^{-n-1}\sum_{i,j=1}^{n}\frac{\partial \det D^2 u}{\partial u_{ij}}u_i u_j,$$

we show that $\varphi = \psi^{\frac{1}{n-1}}$ satisfies the differential inequality $\Delta\varphi \leqslant 0$, which implies that φ is a superharmonic function.

Letting $a = \frac{1}{n-1}$, we have

$$\varphi_i = a((-v)^{n+2}\det D^2 v)^{a-1}((-v)^{n+2}\det D^2 v)_i$$

and

$$\begin{aligned}\Delta\varphi &= a(a-1)((-v)^{n+2}\det D^2 v)^{a-2}\sum_{i=1}^{n}((-v)^{n+2}\det D^2 v)_i^2\\ &\quad + a((-v)^{n+2}\det D^2 v)^{a-1}\Delta((-v)^{n+2}\det D^2 v).\end{aligned} \tag{2.3}$$

In order to prove (1.2) at an arbitrary point x_o, we can choose the coordinates at x_o such that the matrix $(v_{ij}(x_o))$ $(1 \leqslant i, j \leqslant n)$ is diagonal. From now on, all the calculations will be done at the fixed point x_o. Because v is strictly convex, the Hessian matrix (v_{ij}) is positive definite. Let $\lambda_i = v_{ii}$, $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\sigma_1(\boldsymbol{\lambda} \mid i) = \sum_{k=1,k\neq i}^{n}\lambda_k$ $(i = 1, 2, \ldots, n)$. Let (v^{ij}) be the inverse matrix of (v_{ij}).

Taking the first- and second-order derivatives of $\psi = (-v)^{n+2}\det D^2 v$, we have

$$\begin{aligned}&((-v)^{n+2}\det D^2 v)_i\\ &\quad = -(n+2)(-v)^{n+1}v_i\det D^2 v + (-v)^{n+2}\sum_{k,l=1}^{n}\frac{\partial\det D^2 v}{\partial v_{kl}}v_{kli}\\ &\quad = -(n+2)(-v)^{n+1}v_i\det D^2 v + (-v)^{n+2}\det D^2 v\sum_{k=1}^{n}\frac{v_{kki}}{\lambda_k}\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
&\Delta((-v)^{n+2}\det D^2v)\\
&\quad=(n+2)(n+1)(-v)^n|\nabla v|^2\det D^2v-(n+2)(-v)^{n+1}\Delta v\det D^2v\\
&\qquad-2(n+2)(-v)^{n+1}\sum_{k,l,i=1}^n v_i\frac{\partial\det D^2v}{\partial v_{kl}}v_{kli}+(-v)^{n+2}\sum_{k,l,s,t,i=1}^n\frac{\partial^2\det D^2v}{\partial v_{kl}\partial v_{st}}v_{kli}v_{sti}\\
&\qquad+(-v)^{n+2}\sum_{k,l=1}^n\frac{\partial\det D^2v}{\partial v_{kl}}\Delta v_{kl}.
\end{aligned}$$

Since $\Delta v=-\frac{1+|\nabla v|^2}{v}$, we have

$$\begin{aligned}
&\Delta((-v)^{n+2}\det D^2v)\\
&\quad=(n+2)(n+1)(-v)^n|\nabla v|^2\det D^2v-(n+2)(-v)^{n}(1+|\nabla v|^2)\det D^2v\\
&\qquad-2(n+2)(-v)^{n+1}\sum_{k,l,i=1}^n v_i\frac{\partial\det D^2v}{\partial v_{kl}}v_{kli}+(-v)^{n+2}\sum_{k,l,s,t,i=1}^n\frac{\partial^2\det D^2v}{\partial v_{kl}\partial v_{st}}v_{kli}v_{sti}\\
&\qquad+(-v)^{n+2}\sum_{k,l=1}^n\frac{\partial\det D^2v}{\partial v_{kl}}\left(\frac{1+|\nabla u|^2}{-v}\right)_{kl}\\
&\quad=n(n+2)(-v)^n|\nabla v|^2\det D^2v-(n+2)(-v)^n\det D^2v\\
&\qquad-2(n+1)(-v)^{n+1}\sum_{k,l,i=1}^n v_i\frac{\partial\det D^2v}{\partial v_{kl}}v_{kli}+(-v)^{n+2}\sum_{k,l,s,t,i=1}^n\frac{\partial^2\det D^2v}{\partial v_{kl}\partial v_{st}}v_{kli}v_{sti}\\
&\qquad+(-v)^{n+2}\sum_{k,l,i=1}^n\frac{\partial\det D^2v}{\partial v_{kl}}\left(2\frac{v_{ik}v_{il}}{-v}+4\frac{v_iv_lv_{ik}}{v^2}+\frac{1+|\nabla v|^2}{v^2}v_{kl}-2\frac{1+|\nabla v|^2}{v^3}v_kv_l\right).
\end{aligned}\tag{2.5}$$

It follows that

$$\begin{aligned}
&\Delta((-v)^{n+2}\det D^2v)\\
&\quad=n(n+2)(-v)^n|\nabla v|^2\det D^2v-(n+2)(-v)^n\det D^2v\\
&\qquad-2(n+1)(-v)^{n+1}\det D^2v\sum_{k,i=1}^n v_i\frac{v_{kki}}{\lambda_k}\\
&\qquad+(-v)^{n+2}\det D^2v\sum_{\substack{1\leqslant k,l,i\leqslant n\\k\neq l}}\frac{1}{\lambda_k\lambda_l}(v_{kki}v_{lli}-v_{kli}^2)\\
&\qquad+(-v)^{n+2}\det D^2v\sum_{k=1}^n\frac{1}{\lambda_k}\left(2\frac{\lambda_k^2}{-v}+4\frac{v_k^2\lambda_k}{v^2}+\frac{1+|\nabla v|^2}{v^2}\lambda_k-2\frac{1+|\nabla v|^2}{v^3}v_k^2\right)\\
&\quad=(-v)^{n+2}\det D^2v\left(\sum_{\substack{1\leqslant k,l,i\leqslant n\\k\neq l}}\left(\frac{v_{kki}v_{lli}}{\lambda_k\lambda_l}-\frac{v_{kli}^2}{\lambda_k\lambda_l}\right)+2(n+1)\sum_{k,i=1}^n\frac{v_i}{v}\frac{v_{kki}}{\lambda_k}\right)\\
&\qquad+(-v)^{n+2}\det D^2v\left((n^2+3n+6)\frac{|\nabla v|^2}{v^2}+2\sum_{k=1}^n\frac{\sigma_1(\boldsymbol{\lambda})}{\lambda_k}\frac{v_k^2}{v^2}\right).
\end{aligned}\tag{2.6}$$

Substituting (2.4) and (2.6) into (2.3), we have

$$\begin{aligned}
&\frac{1}{a}((-v)^{n+2}\det D^2v)^{-a}\Delta\varphi\\
&\quad=(a-1)\left((n+2)^2\frac{|\nabla v|^2}{v^2}+\sum_{k,l,i=1}^n\frac{v_{kki}v_{lli}}{\lambda_k\lambda_l}+2(n+2)\sum_{k,i=1}^n\frac{v_i}{v}\frac{v_{kki}}{\lambda_k}\right)\\
&\qquad+\sum_{\substack{1\leqslant k,l,i\leqslant n\\k\neq l}}\left(\frac{v_{kki}v_{lli}}{\lambda_k\lambda_l}-\frac{v_{kli}^2}{\lambda_k\lambda_l}\right)+2(n+1)\sum_{k,i=1}^n\frac{v_i}{v}\frac{v_{kki}}{\lambda_k}
\end{aligned}$$

$$
\begin{aligned}
&+(n^2+3n+6)\frac{|\nabla v|^2}{v^2}+2\sum_{k=1}^{n}\frac{\sigma_1(\boldsymbol{\lambda})}{\lambda_k}\frac{v_k^2}{v^2}\\
=&(a-1)\sum_{k,i=1}^{n}\frac{v_{kki}^2}{\lambda_k^2}+a\sum_{\substack{1\leqslant k,l,i\leqslant n\\ k\neq l}}\frac{v_{kki}v_{lli}}{\lambda_k\lambda_l}-\sum_{\substack{1\leqslant k,l,i\leqslant n\\ k\neq l}}\frac{v_{kli}^2}{\lambda_k\lambda_l}\\
&+(2(n+2)a-2)\sum_{k,i=1}^{n}\frac{v_i}{v}\frac{v_{kki}}{\lambda_k}+((n+2)^2a-n+2)\frac{|\nabla v|^2}{v^2}\\
&+2\sum_{k=1}^{n}\frac{\sigma_1(\boldsymbol{\lambda})}{\lambda_k}\frac{v_k^2}{v^2}.
\end{aligned}
$$

So we have the following formulas:

$$
\begin{aligned}
&\frac{1}{a}((-v)^{n+2}\det D^2v)^{-a}\Delta\varphi\\
&=(a-1)\sum_{k,i=1}^{n}\frac{v_{kki}^2}{\lambda_k^2}+a\sum_{\substack{1\leqslant k,l,i\leqslant n\\ k\neq l}}\frac{v_{kki}v_{lli}}{\lambda_k\lambda_l}-2\sum_{\substack{1\leqslant k,i\leqslant n\\ k\neq i}}\frac{v_{kki}^2}{\lambda_k\lambda_i}-\sum_{k\neq l,k\neq i,l\neq i}\frac{v_{kli}^2}{\lambda_k\lambda_l}\\
&\quad+(2(n+2)a-2)\sum_{k,i=1}^{n}\frac{v_i}{v}\frac{v_{kki}}{\lambda_k}+((n+2)^2a-n+2)\frac{|\nabla v|^2}{v^2}+2\sum_{k=1}^{n}\frac{\sigma_1(\boldsymbol{\lambda})}{\lambda_k}\frac{v_k^2}{v^2}\\
&\leqslant(a-1)\sum_{k,i=1}^{n}\frac{v_{kki}^2}{\lambda_k^2}+a\sum_{\substack{1\leqslant k,l,i\leqslant n\\ k\neq l}}\frac{v_{kki}v_{lli}}{\lambda_k\lambda_l}-2\sum_{\substack{1\leqslant k,i\leqslant n\\ k\neq i}}\frac{v_{kki}^2}{\lambda_k\lambda_i}\\
&\quad+(2(n+2)a-2)\sum_{k,i=1}^{n}\frac{v_i}{v}\frac{v_{kki}}{\lambda_k}+((n+2)^2a-n+2)\frac{|\nabla v|^2}{v^2}+2\sum_{k=1}^{n}\frac{\sigma_1(\boldsymbol{\lambda})}{\lambda_k}\frac{v_k^2}{v^2}.
\end{aligned}
$$

We claim that for $1\leqslant i\leqslant n$,

$$
\begin{aligned}
A_i:=&(a-1)\sum_{k=1}^{n}\frac{v_{kki}^2}{\lambda_k^2}+a\sum_{\substack{1\leqslant k,l\leqslant n\\ k\neq l}}\frac{v_{kki}v_{lli}}{\lambda_k\lambda_l}-2\sum_{\substack{1\leqslant k\leqslant n\\ k\neq i}}\frac{v_{kki}^2}{\lambda_k\lambda_i}\\
&+(2(n+2)a-2)\frac{v_i}{v}\sum_{k=1}^{n}\frac{v_{kki}}{\lambda_k}\\
&+((n+2)^2a-n+2)\frac{v_i^2}{v^2}+2\frac{\sigma_1(\boldsymbol{\lambda})}{\lambda_i}\frac{v_i^2}{v^2}\\
\leqslant&\ 0.
\end{aligned}
\tag{2.7}
$$

From the claim, we arrive at the conclusion that

$$
\Delta\varphi\leqslant a((-v)^{n+2}\det D^2v)^a\sum_{i=1}^{n}A_i\leqslant 0. \tag{2.8}
$$

We firstly express (2.7) in another way by the equation $\Delta v=-\frac{1+|\nabla v|^2}{v}$. Taking the first-order derivative of the equation $\Delta v=-\frac{1+|\nabla v|^2}{v}$, we have

$$
(\Delta v)_i=-\frac{(2\lambda_i+\sigma_1(\boldsymbol{\lambda}))v_i}{v},
$$

i.e.,

$$
v_{nni}=-\frac{(2\lambda_i+\sigma_1(\boldsymbol{\lambda}))v_i}{v}-\sum_{k=1}^{n-1}v_{kki}. \tag{2.9}
$$

Substituting (2.9) into (2.7), we obtain that for $1 \leqslant i \leqslant n-1$,

$$\begin{aligned} A_i = &\left(\frac{a-1}{\lambda_i^2} + \frac{a-1}{\lambda_n^2} - \frac{2}{\lambda_i\lambda_n} - \frac{2a}{\lambda_i\lambda_n}\right) v_{iii}^2 \\ &+ \sum_{\substack{1\leqslant j\leqslant n-1\\ j\neq i}} \left(\frac{a-1}{\lambda_j^2} - \frac{2}{\lambda_i\lambda_j} + \frac{a-1}{\lambda_n^2} - \frac{2}{\lambda_i\lambda_n} - \frac{2a}{\lambda_j\lambda_n}\right) v_{jji}^2 \\ &+ \sum_{\substack{1\leqslant k,j\leqslant n-1\\ k\neq j}} \left(\frac{a}{\lambda_k\lambda_j} + \frac{a-1}{\lambda_n^2} - \frac{2}{\lambda_i\lambda_n} - \frac{2a}{\lambda_j\lambda_n}\right) v_{kki}v_{jji} \\ &+ \frac{v_i}{v}\sum_{j=1}^{n-1}\left(2\left(\frac{a-1}{\lambda_n^2} - \frac{2}{\lambda_i\lambda_n} - \frac{a}{\lambda_j\lambda_n}\right)(2\lambda_i + \sigma_1(\boldsymbol{\lambda}))\right. \\ &\left. + 2((n+2)a-1)\left(\frac{1}{\lambda_j} - \frac{1}{\lambda_n}\right)\right) v_{jji} \\ &+ \left(\frac{a-1}{\lambda_n^2} - \frac{2}{\lambda_i\lambda_n}\right)(2\lambda_i + \sigma_1)^2\frac{v_i^2}{v^2} - 2((n+2)a-1)\frac{2\lambda_i + \sigma_1}{\lambda_n}\frac{v_i^2}{v^2} \\ &+ ((n+2)^2a - n + 2)\frac{v_i^2}{v^2} + 2\frac{\sigma_1(\boldsymbol{\lambda})}{\lambda_i}\frac{v_i^2}{v^2}, \end{aligned} \tag{2.10}$$

and for $i = n$,

$$\begin{aligned} A_n = &\sum_{j=1}^{n-1}\left(\frac{a-1}{\lambda_j^2} + \frac{a-1}{\lambda_n^2} - \frac{2}{\lambda_j\lambda_n} - \frac{2a}{\lambda_j\lambda_n}\right) v_{jjn}^2 \\ &+ \sum_{\substack{1\leqslant j,k\leqslant n-1\\ k\neq j}} \left(\frac{a}{\lambda_k\lambda_j} + \frac{a-1}{\lambda_n^2} - \frac{2a}{\lambda_j\lambda_n}\right) v_{jjn}v_{kkn} \\ &+ \frac{v_n}{v}\sum_{j=1}^{n-1}\left(2\left(\frac{a-1}{\lambda_n^2} - \frac{a}{\lambda_j\lambda_n}\right)(2\lambda_n + \sigma_1(\boldsymbol{\lambda})) + 2((n+2)a-1)\left(\frac{1}{\lambda_j} - \frac{1}{\lambda_n}\right)\right) v_{jjn} \\ &+ \frac{a-1}{\lambda_n^2}(2\lambda_n + \sigma_1)^2\frac{v_n^2}{v^2} - 2((n+2)a-1)\frac{2\lambda_n + \sigma_1}{\lambda_n}\frac{v_n^2}{v^2} \\ &+ ((n+2)^2a - n + 2)\frac{v_n^2}{v^2} + 2\frac{\sigma_1(\boldsymbol{\lambda})}{\lambda_n}\frac{v_n^2}{v^2}. \end{aligned} \tag{2.11}$$

From (2.10) and (2.11), we can see that for any $1 \leqslant i \leqslant n$, A_i is a quadratic polynomial about

$$\boldsymbol{x}_{[\boldsymbol{i}]} = (v_{11i}, v_{22i}, \ldots, v_{(n-1)(n-1)i})$$

and has the form

$$A_i = \boldsymbol{x}_{[\boldsymbol{i}]}\boldsymbol{B_i}\boldsymbol{x}_{[\boldsymbol{i}]}^{\mathrm{T}} + 2\boldsymbol{b}_{[\boldsymbol{i}]}\boldsymbol{x}_{[\boldsymbol{i}]}^{\mathrm{T}} + d_i.$$

Now, we prove the claim for $i = 1$ and $i = n$, and the others are the same as $i = 1$ completely.

(1) (Proof of $A_1 \leqslant 0$) Let $\boldsymbol{B_1}$ be the corresponding matrix of A_1. From (2.10), $\boldsymbol{B_1}$ has the form

$$-\boldsymbol{B_1} = \boldsymbol{E_1} + \boldsymbol{F_1} - \boldsymbol{G_1},$$

where

$$\boldsymbol{E_1} = \begin{pmatrix} \frac{1}{\lambda_1^2} & & & \\ & \frac{1}{\lambda_2^2} + \frac{2}{\lambda_1\lambda_2} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_{n-1}^2} + \frac{2}{\lambda_1\lambda_{n-1}} \end{pmatrix},$$

$$\boldsymbol{F_1}=\left(\frac{1}{\lambda_n^2}+\frac{2}{\lambda_1\lambda_n}\right)\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}(1,1,\dots,1),$$

$$\boldsymbol{G_1}=a\begin{pmatrix}\frac{1}{\lambda_1}-\frac{1}{\lambda_n}\\\frac{1}{\lambda_2}-\frac{1}{\lambda_n}\\\vdots\\\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_n}\end{pmatrix}\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_n},\frac{1}{\lambda_2}-\frac{1}{\lambda_n},\dots,\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_n}\right).$$

In order to use Lemma 2.1, we need to prove that $\boldsymbol{B_1}<0$. Firstly, letting

$$\boldsymbol{C_1}=\begin{pmatrix}\lambda_1&&&\\&\frac{\lambda_2}{\sqrt{1+2\frac{\lambda_2}{\lambda_1}}}&&\\&&\ddots&\\&&&\frac{\lambda_{n-1}}{\sqrt{1+2\frac{\lambda_{n-1}}{\lambda_1}}}\end{pmatrix},$$

we have

$$\boldsymbol{C_1}(-\boldsymbol{B_1})\boldsymbol{C_1}=\boldsymbol{I}+\boldsymbol{\nu}_{[1]}^{\mathrm{T}}\boldsymbol{\nu}_{[1]}-a\boldsymbol{\eta}_{[1]}^{\mathrm{T}}\boldsymbol{\eta}_{[1]},\tag{2.12}$$

where $\boldsymbol{I}$ is the identity matrix of order $n-1$, $\boldsymbol{\nu}_{[1]}$ and $\boldsymbol{\eta}_{[1]}$ are vectors, and

$$\begin{aligned}\boldsymbol{\nu}_{[1]}&=\sqrt{\frac{1}{\lambda_n^2}+\frac{2}{\lambda_1\lambda_n}}(1,1,\dots,1)\boldsymbol{C_1}\\&=\sqrt{\frac{1}{\lambda_n^2}+\frac{2}{\lambda_1\lambda_n}}\left(\lambda_1,\frac{\lambda_2}{\sqrt{1+2\frac{\lambda_2}{\lambda_1}}},\dots,\frac{\lambda_{n-1}}{\sqrt{1+2\frac{\lambda_{n-1}}{\lambda_1}}}\right),\end{aligned}\tag{2.13}$$

$$\begin{aligned}\boldsymbol{\eta}_{[1]}&=\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_n},\frac{1}{\lambda_2}-\frac{1}{\lambda_n},\dots,\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_n}\right)\boldsymbol{C_1}\\&=\left(1-\frac{\lambda_1}{\lambda_n},\left(1-\frac{\lambda_2}{\lambda_n}\right)\frac{1}{\sqrt{1+2\frac{\lambda_2}{\lambda_1}}},\dots,\left(1-\frac{\lambda_{n-1}}{\lambda_n}\right)\frac{1}{\sqrt{1+2\frac{\lambda_{n-1}}{\lambda_1}}}\right).\end{aligned}\tag{2.14}$$

Since $\boldsymbol{C_1}$ is positive definite and diagonal, if we obtain that $\boldsymbol{C_1}(-\boldsymbol{B_1})\boldsymbol{C_1}$ is positive definite, then $\boldsymbol{B_1}$ is negative definite. Now we prove that $\boldsymbol{C_1}(-\boldsymbol{B_1})\boldsymbol{C_1}$ is positive definite.

From (2.12)–(2.14), the first-order leading principal minor of $\boldsymbol{C_1}(-\boldsymbol{B_1})\boldsymbol{C_1}$ is

$$\left(1+\frac{\lambda_1}{\lambda_n}\right)^2-a\left(1-\frac{\lambda_1}{\lambda_n}\right)^2.$$

Thus, the first-order leading principal minor of $\boldsymbol{C_1}(-\boldsymbol{B_1})\boldsymbol{C_1}$ is greater than 0 for $a=\frac{1}{n-1}$. Moreover, for any $2\leqslant k\leqslant n-1$, from (2.12), the k-th-order leading principal minor of $\boldsymbol{C_1}(-\boldsymbol{B_1})\boldsymbol{C_1}$ is

$$\det(\boldsymbol{I_k}+\boldsymbol{\nu}_{[\boldsymbol{1k}]}^{\mathrm{T}}\boldsymbol{\nu}_{[\boldsymbol{1k}]}-a\boldsymbol{\eta}_{[\boldsymbol{1k}]}^{\mathrm{T}}\boldsymbol{\eta}_{[\boldsymbol{1k}]}),$$

where $\boldsymbol{I_k}$ is the identity matrix of order k, $\boldsymbol{\nu}_{[\boldsymbol{1k}]}$ is a vector whose components are from the first k components of $\boldsymbol{\nu}_{[1]}$, and $\boldsymbol{\eta}_{[\boldsymbol{1k}]}$ is a vector whose components are from the first k components of $\boldsymbol{\eta}_{[1]}$. Let

$$\boldsymbol{D}_{[\boldsymbol{1k}]}=\boldsymbol{I_k}+\boldsymbol{\nu}_{[\boldsymbol{1k}]}^{\mathrm{T}}\boldsymbol{\nu}_{[\boldsymbol{1k}]}.$$

Then

$$\begin{aligned}\det(\boldsymbol{I_k}+\boldsymbol{\nu}_{[\boldsymbol{1k}]}^{\mathrm{T}}\boldsymbol{\nu}_{[\boldsymbol{1k}]}-a\boldsymbol{\eta}_{[\boldsymbol{1k}]}^{\mathrm{T}}\boldsymbol{\eta}_{[\boldsymbol{1k}]}) &= \det(\boldsymbol{D}_{[\boldsymbol{1k}]}-a\boldsymbol{\eta}_{[\boldsymbol{1k}]}^{\mathrm{T}}\boldsymbol{\eta}_{[\boldsymbol{1k}]})\\ &= \det\begin{pmatrix}1 & \boldsymbol{\eta}_{[\boldsymbol{1k}]}\\ 0 & \boldsymbol{D}_{[\boldsymbol{1k}]}-a\boldsymbol{\eta}_{[\boldsymbol{1k}]}^{\mathrm{T}}\boldsymbol{\eta}_{[\boldsymbol{1k}]}\end{pmatrix}\\ &= \det\begin{pmatrix}1 & \boldsymbol{\eta}_{[\boldsymbol{1k}]}\\ a\boldsymbol{\eta}_{[\boldsymbol{1k}]}^{\mathrm{T}} & \boldsymbol{D}_{[\boldsymbol{1k}]}\end{pmatrix}\\ &= \det\begin{pmatrix}1-a\boldsymbol{\eta}_{[\boldsymbol{1k}]}\boldsymbol{D}_{[\boldsymbol{1k}]}^{-1}\boldsymbol{\eta}_{[\boldsymbol{1k}]}^{\mathrm{T}} & 0\\ a\boldsymbol{\eta}_{[\boldsymbol{1k}]}^{\mathrm{T}} & \boldsymbol{D}_{[\boldsymbol{1k}]}\end{pmatrix}\\ &= (1-a\boldsymbol{\eta}_{[\boldsymbol{1k}]}\boldsymbol{D}_{[\boldsymbol{1k}]}^{-1}\boldsymbol{\eta}_{[\boldsymbol{1k}]}^{\mathrm{T}})\det\boldsymbol{D}_{[\boldsymbol{1k}]}.\end{aligned}$$

Because $\boldsymbol{D}_{[\boldsymbol{1k}]}$ is a positive definite matrix, we only need to prove

$$1-a\boldsymbol{\eta}_{[\boldsymbol{1k}]}\boldsymbol{D}_{[\boldsymbol{1k}]}^{-1}\boldsymbol{\eta}_{[\boldsymbol{1k}]}^{\mathrm{T}}>0, \tag{2.15}$$

and then we obtain that the k-th-order leading principal minor of $\boldsymbol{C_1}(-\boldsymbol{B_1})\boldsymbol{C_1}$ is greater than 0.

From the definition of $\boldsymbol{D}_{[\boldsymbol{1k}]}$, we have

$$\boldsymbol{D}_{[\boldsymbol{1k}]}^{-1}=\boldsymbol{I_k}-\frac{\boldsymbol{\nu}_{[\boldsymbol{1k}]}^{\mathrm{T}}\boldsymbol{\nu}_{[\boldsymbol{1k}]}}{1+|\boldsymbol{\nu}_{[\boldsymbol{1k}]}|^2},$$

and then

$$1-a\boldsymbol{\eta}_{[\boldsymbol{1k}]}\boldsymbol{D}_{[\boldsymbol{1k}]}^{-1}\boldsymbol{\eta}_{[\boldsymbol{1k}]}^{\mathrm{T}}=1-a\frac{|\boldsymbol{\eta}_{[\boldsymbol{1k}]}|^2+|\boldsymbol{\eta}_{[\boldsymbol{1k}]}|^2|\boldsymbol{\nu}_{[\boldsymbol{1k}]}|^2-(\boldsymbol{\eta}_{[\boldsymbol{1k}]}\boldsymbol{\nu}_{[\boldsymbol{1k}]}^{\mathrm{T}})^2}{1+|\boldsymbol{\nu}_{[\boldsymbol{1k}]}|^2}. \tag{2.16}$$

Since

$$\begin{aligned}\boldsymbol{\nu}_{[\boldsymbol{1k}]}&=\sqrt{\frac{1}{\lambda_n^2}+\frac{2}{\lambda_1\lambda_n}}\left(\lambda_1,\frac{\lambda_2}{\sqrt{1+2\frac{\lambda_2}{\lambda_1}}},\dots,\frac{\lambda_k}{\sqrt{1+2\frac{\lambda_k}{\lambda_1}}}\right),\\ \boldsymbol{\eta}_{[\boldsymbol{1k}]}&=\left(1-\frac{\lambda_1}{\lambda_n},\left(1-\frac{\lambda_2}{\lambda_n}\right)\frac{1}{\sqrt{1+2\frac{\lambda_2}{\lambda_1}}},\dots,\left(1-\frac{\lambda_k}{\lambda_n}\right)\frac{1}{\sqrt{1+2\frac{\lambda_k}{\lambda_1}}}\right),\end{aligned}$$

it follows that

$$\begin{aligned}|\boldsymbol{\eta}_{[\boldsymbol{1k}]}|^2&=\left(1-\frac{\lambda_1}{\lambda_n}\right)^2+\sum_{j=2}^k\frac{(1-\frac{\lambda_j}{\lambda_n})^2}{1+2\frac{\lambda_j}{\lambda_1}}\\ &=\frac{1}{4}\frac{\lambda_1^2}{\lambda_n^2}\left(\left(1+2\frac{\lambda_n}{\lambda_1}\right)^2-2(k+2)\left(1+2\frac{\lambda_n}{\lambda_1}\right)+9\right)\\ &\quad+\frac{1}{4}\frac{\lambda_1^2}{\lambda_n^2}\left(\left(1+2\frac{\lambda_n}{\lambda_1}\right)^2\sum_{j=2}^k\frac{1}{1+2\frac{\lambda_j}{\lambda_1}}+\sum_{j=2}^k\left(1+2\frac{\lambda_j}{\lambda_1}\right)\right),\end{aligned} \tag{2.17}$$

$$\begin{aligned}1+|\boldsymbol{\nu}_{[\boldsymbol{1k}]}|^2&=1+\left(1+2\frac{\lambda_n}{\lambda_1}\right)\left(\frac{\lambda_1^2}{\lambda_n^2}+\sum_{j=2}^k\frac{1}{1+2\frac{\lambda_j}{\lambda_1}}\frac{\lambda_j^2}{\lambda_n^2}\right)\\ &=\frac{1}{4}\frac{\lambda_1^2}{\lambda_n^2}\left(1+2\frac{\lambda_n}{\lambda_1}\right)\left(\sum_{j=2}^k\left(1+2\frac{\lambda_j}{\lambda_1}\right)+1+2\frac{\lambda_n}{\lambda_1}\right.\\ &\quad\left.+\sum_{j=2}^k\frac{1}{1+2\frac{\lambda_j}{\lambda_1}}+\frac{1}{1+2\frac{\lambda_n}{\lambda_1}}+4-2k\right)\end{aligned} \tag{2.18}$$

and

$$|\boldsymbol{\eta}_{[\boldsymbol{1k}]}|^2|\boldsymbol{\nu}_{[\boldsymbol{1k}]}|^2-(\boldsymbol{\eta}_{[\boldsymbol{1k}]}\boldsymbol{\nu}_{[\boldsymbol{1k}]}^{\mathrm{T}})^2$$

$$
\begin{aligned}
&= \left(1+2\frac{\lambda_n}{\lambda_1}\right)\sum_{j=2}^{k}\frac{1}{1+2\frac{\lambda_j}{\lambda_1}}\left(\frac{\lambda_1}{\lambda_n}-\frac{\lambda_j}{\lambda_n}\right)^2 \\
&\quad + \frac{1}{2}\left(1+2\frac{\lambda_n}{\lambda_1}\right)\sum_{i,j=2}^{k}\frac{1}{(1+2\frac{\lambda_i}{\lambda_1})(1+2\frac{\lambda_j}{\lambda_1})}\left(\frac{\lambda_i}{\lambda_n}-\frac{\lambda_j}{\lambda_n}\right)^2 \\
&= \frac{1}{4}\frac{\lambda_1^2}{\lambda_n^2}\left(1+2\frac{\lambda_n}{\lambda_1}\right)\left(\sum_{j=2}^{k}\left(1+2\frac{\lambda_j}{\lambda_1}\right)\right)\left(\sum_{j=2}^{k}\frac{1}{1+2\frac{\lambda_j}{\lambda_1}}\right) \\
&\quad + \frac{1}{4}\frac{\lambda_1^2}{\lambda_n^2}\left(1+2\frac{\lambda_n}{\lambda_1}\right)\left(\sum_{j=2}^{k}\left(1+2\frac{\lambda_j}{\lambda_1}\right)+9\sum_{j=2}^{k}\frac{1}{1+2\frac{\lambda_j}{\lambda_1}}-(k-1)(k+5)\right). \qquad (2.19)
\end{aligned}
$$

For convenience, let $\mu_j = 1+2\frac{\lambda_j}{\lambda_1}$ for $2 \leqslant j \leqslant n$. From (2.17)–(2.19), we can conclude that

$$
\begin{aligned}
&|\boldsymbol{\eta}_{[1k]}|^2 + |\boldsymbol{\eta}_{[1k]}|^2|\boldsymbol{\nu}_{[1k]}|^2 - (\boldsymbol{\eta}_{[1k]}\boldsymbol{\nu}_{[1k]}^{\mathrm{T}})^2 \\
&\quad = \frac{1}{4}\frac{\lambda_1^2}{\lambda_n^2}\left(1+2\frac{\lambda_n}{\lambda_1}\right)\left(\left(\sum_{i=2}^{k}\mu_i+\mu_n\right)\left(\sum_{j=2}^{k}\frac{1}{\mu_j}+\frac{1}{\mu_n}\right)+\left(\sum_{i=2}^{k}\mu_i+\mu_n\right)\right) \\
&\qquad + \frac{1}{4}\frac{\lambda_1^2}{\lambda_n^2}\left(1+2\frac{\lambda_n}{\lambda_1}\right)\left(9\left(\sum_{j=2}^{k}\frac{1}{\mu_j}+\frac{1}{\mu_n}\right)-k(k+6)\right)
\end{aligned}
$$

and

$$
1+|\boldsymbol{\nu}_{[1k]}|^2 = \frac{1}{4}\frac{\lambda_1^2}{\lambda_n^2}\left(1+2\frac{\lambda_n}{\lambda_1}\right)\left(\sum_{j=2}^{k}\mu_j+\mu_n+\sum_{j=2}^{k}\frac{1}{\mu_j}+\frac{1}{\mu_n}+2(2-k)\right).
$$

Then

$$
\begin{aligned}
&\frac{|\boldsymbol{\eta}_{[1k]}|^2 + |\boldsymbol{\eta}_{[1k]}|^2|\boldsymbol{\nu}_{[1k]}|^2 - (\boldsymbol{\eta}_{[1k]}\boldsymbol{\nu}_{[1k]}^{\mathrm{T}})^2}{1+|\boldsymbol{\nu}_{[1k]}|^2} \\
&\quad = \frac{(\sum_{i=2}^{k}\mu_i+\mu_n)(\sum_{j=2}^{k}\frac{1}{\mu_j}+\frac{1}{\mu_n})+\sum_{i=2}^{k}\mu_i+\mu_n+9\sum_{j=2}^{k}\frac{1}{\mu_j}+9\frac{1}{\mu_n}-k(k+6)}{\sum_{j=2}^{k}\mu_j+\mu_n+\sum_{j=2}^{k}\frac{1}{\mu_j}+\frac{1}{\mu_n}+2(2-k)}. \qquad (2.20)
\end{aligned}
$$

Using Lemma 2.2 and (2.20), we have

$$
\frac{|\boldsymbol{\eta}_{[1k]}|^2 + |\boldsymbol{\eta}_{[1k]}|^2|\boldsymbol{\nu}_{[1k]}|^2 - (\boldsymbol{\eta}_{[1k]}\boldsymbol{\nu}_{[1k]}^{\mathrm{T}})^2}{1+|\boldsymbol{\nu}_{[1k]}|^2} < k,
$$

and then the k-th-order leading principal minor of $\boldsymbol{C_1}(-\boldsymbol{B_1})\boldsymbol{C_1}$ is greater than 0 for $a = \frac{1}{n-1} \leqslant \frac{1}{k}$.

To sum up, all the leading principal minors of $\boldsymbol{C_1}(-\boldsymbol{B_1})\boldsymbol{C_1}$ are greater than 0. Therefore, $\boldsymbol{C_1}(-\boldsymbol{B_1})\boldsymbol{C_1}$ is positive definite and $\boldsymbol{B_1} < 0$.

Furthermore, using Lemma 2.1 and (2.10), we can obtain that

$$
\begin{aligned}
A_1 &\leqslant -\boldsymbol{b}_{[1]}\boldsymbol{B_1}^{-1}\boldsymbol{b}_{[1]}^{\mathrm{T}} + \left(\frac{a-1}{\lambda_n^2}-\frac{2}{\lambda_1\lambda_n}\right)(2\lambda_1+\sigma_1)^2\frac{v_1^2}{v^2} - 2((n+2)a-1)\frac{2\lambda_1+\sigma_1}{\lambda_n}\frac{v_1^2}{v^2} \\
&\quad + ((n+2)^2a-n+2)\frac{v_1^2}{v^2} + 2\frac{\sigma_1(\boldsymbol{\lambda})}{\lambda_1}\frac{v_1^2}{v^2}.
\end{aligned}
$$

To compute $-\boldsymbol{b}_{[1]}\boldsymbol{B_1}^{-1}\boldsymbol{b}_{[1]}^{\mathrm{T}}$, letting

$$
s = (1-a|\boldsymbol{\eta}_{[1]}|^2)(1+|\boldsymbol{\nu}_{[1]}|^2) + a(\boldsymbol{\eta}_{[1]}\boldsymbol{\nu}_{[1]}^{\mathrm{T}})^2,
$$

we have $s > 0$ from (2.15) and (2.16). Let $\boldsymbol{P_1}$ be an $(n-1)\times(n-1)$ matrix and

$$
\boldsymbol{P_1} = \boldsymbol{I} - \frac{1-a|\boldsymbol{\eta}_{[1]}|^2}{s}\boldsymbol{\nu}_{[1]}^{\mathrm{T}}\boldsymbol{\nu}_{[1]} + a\frac{1+|\boldsymbol{\nu}_{[1]}|^2}{s}\boldsymbol{\eta}_{[1]}^{\mathrm{T}}\boldsymbol{\eta}_{[1]} - a\frac{\boldsymbol{\eta}_{[1]}\boldsymbol{\nu}_{[1]}^{\mathrm{T}}}{s}\boldsymbol{\eta}_{[1]}^{\mathrm{T}}\boldsymbol{\nu}_{[1]}
$$

$$- a\frac{\boldsymbol{\eta}_{[1]}\boldsymbol{\nu}_{[1]}^{\mathrm{T}}}{s}\boldsymbol{\nu}_{[1]}^{\mathrm{T}}\boldsymbol{\eta}_{[1]}. \tag{2.21}$$

By direct computations,

$$\begin{aligned}(\boldsymbol{C_1}(-\boldsymbol{B_1})\boldsymbol{C_1})\boldsymbol{P_1} &= \boldsymbol{P_1} + \boldsymbol{\nu}_{[1]}^{\mathrm{T}}\boldsymbol{\nu}_{[1]}\boldsymbol{P_1} - a\boldsymbol{\eta}_{[1]}^{\mathrm{T}}\boldsymbol{\eta}_{[1]}\boldsymbol{P_1} \\ &= \boldsymbol{I} - \frac{1-a|\boldsymbol{\eta}_{[1]}|^2}{s}\boldsymbol{\nu}_{[1]}^{\mathrm{T}}\boldsymbol{\nu}_{[1]} + a\frac{1+|\boldsymbol{\nu}_{[1]}|^2}{s}\boldsymbol{\eta}_{[1]}^{\mathrm{T}}\boldsymbol{\eta}_{[1]} \\ &\quad - a\frac{\boldsymbol{\eta}_{[1]}\boldsymbol{\nu}_{[1]}^{\mathrm{T}}}{s}\boldsymbol{\eta}_{[1]}^{\mathrm{T}}\boldsymbol{\nu}_{[1]} - a\frac{\boldsymbol{\eta}_{[1]}\boldsymbol{\nu}_{[1]}^{\mathrm{T}}}{s}\boldsymbol{\nu}_{[1]}^{\mathrm{T}}\boldsymbol{\eta}_{[1]} \\ &\quad + \frac{1-a|\boldsymbol{\eta}_{[1]}|^2}{s}\boldsymbol{\nu}_{[1]}^{\mathrm{T}}\boldsymbol{\nu}_{[1]} + a\frac{\boldsymbol{\eta}_{[1]}\boldsymbol{\nu}_{[1]}^{\mathrm{T}}}{s}\boldsymbol{\nu}_{[1]}^{\mathrm{T}}\boldsymbol{\eta}_{[1]} \\ &\quad + a\frac{\boldsymbol{\eta}_{[1]}\boldsymbol{\nu}_{[1]}^{\mathrm{T}}}{s}\boldsymbol{\eta}_{[1]}^{\mathrm{T}}\boldsymbol{\nu}_{[1]} - a\frac{1+|\boldsymbol{\nu}_{[1]}|^2}{s}\boldsymbol{\eta}_{[1]}^{\mathrm{T}}\boldsymbol{\eta}_{[1]} \\ &= \boldsymbol{I}.\end{aligned}$$

We have that $\boldsymbol{P_1}$ is the inverse matrix of $\boldsymbol{C_1}(-\boldsymbol{B_1})\boldsymbol{C_1}$, and then

$$\begin{aligned}&-\boldsymbol{B}_{\boldsymbol{1}}^{-1} = \boldsymbol{C_1}\boldsymbol{P_1}\boldsymbol{C_1}, \\ &-\boldsymbol{b}_{[1]}\boldsymbol{B}_{\boldsymbol{1}}^{-1}\boldsymbol{b}_{[1]}^{\mathrm{T}} = \boldsymbol{b}_{[1]}\boldsymbol{C_1}\boldsymbol{P_1}(\boldsymbol{b}_{[1]}\boldsymbol{C_1})^{\mathrm{T}}.\end{aligned}$$

Let

$$g_{[1]} = \sqrt{1 + 2\frac{\lambda_n}{\lambda_1}\frac{2\lambda_1 + \sigma_1(\boldsymbol{\lambda})}{\lambda_n}}$$

and

$$h_{[1]} = (n+2)a - \frac{2\lambda_1 + \sigma_1(\boldsymbol{\lambda})}{\lambda_n}a - 1.$$

From the definition of $\boldsymbol{b}_{[1]}$,

$$\boldsymbol{b}_{[1]} = \left(-g_{[1]}\sqrt{\frac{1}{\lambda_n^2} + \frac{2}{\lambda_1\lambda_n}}(1,\ldots,1) + h_{[1]}\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_n}, \ldots, \frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right)\right)\frac{v_1}{v}.$$

Combining (2.13) and (2.14), we obtain

$$\boldsymbol{b}_{[1]}\boldsymbol{C_1} = (-g_{[1]}\boldsymbol{\nu}_{[1]} + h_{[1]}\boldsymbol{\eta}_{[1]})\frac{v_1}{v}.$$

Therefore,

$$\begin{aligned}&\boldsymbol{b}_{[1]}\boldsymbol{C_1}\boldsymbol{P_1}(\boldsymbol{b}_{[1]}\boldsymbol{C_1})^{\mathrm{T}} \\ &\quad = \frac{v_1^2}{v^2}(-g_{[1]}\boldsymbol{\nu}_{[1]} + h_{[1]}\boldsymbol{\eta}_{[1]})\boldsymbol{P_1}(-g_{[1]}\boldsymbol{\nu}_{[1]} + h_{[1]}\boldsymbol{\eta}_{[1]})^{\mathrm{T}} \\ &\quad = \frac{v_1^2}{v^2}(g_{[1]}^2\boldsymbol{\nu}_{[1]}\boldsymbol{P_1}\boldsymbol{\nu}_{[1]}^{\mathrm{T}} - 2g_{[1]}h_{[1]}\boldsymbol{\eta}_{[1]}\boldsymbol{P_1}\boldsymbol{\nu}_{[1]}^{\mathrm{T}} + h_{[1]}^2\boldsymbol{\eta}_{[1]}\boldsymbol{P_1}\boldsymbol{\eta}_{[1]}^{\mathrm{T}}).\end{aligned} \tag{2.22}$$

From (2.13), (2.14) and (2.21), we obtain

$$\boldsymbol{\nu}_{[1]}\boldsymbol{P_1}\boldsymbol{\nu}_{[1]}^{\mathrm{T}} = \frac{1}{s}((1-a|\boldsymbol{\eta}_{[1]}|^2)|\boldsymbol{\nu}_{[1]}|^2 + a(\boldsymbol{\eta}_{[1]}\boldsymbol{\nu}_{[1]}^{\mathrm{T}})^2), \tag{2.23}$$

$$\boldsymbol{\eta}_{[1]}\boldsymbol{P_1}\boldsymbol{\nu}_{[1]}^{\mathrm{T}} = \frac{1}{s}\boldsymbol{\eta}_{[1]}\boldsymbol{\nu}_{[1]}^{\mathrm{T}}, \tag{2.24}$$

$$\boldsymbol{\eta}_{[1]}\boldsymbol{P_1}\boldsymbol{\eta}_{[1]}^{\mathrm{T}} = \frac{1}{s}((1+|\boldsymbol{\nu}_{[1]}|^2)|\boldsymbol{\eta}_{[1]}|^2 - (\boldsymbol{\eta}_{[1]}\boldsymbol{\nu}_{[1]}^{\mathrm{T}})^2). \tag{2.25}$$

Substituting (2.23)–(2.25) into (2.22), we have

$$\begin{aligned}\boldsymbol{b}_{[1]}\boldsymbol{C_1}\boldsymbol{P_1}(\boldsymbol{b}_{[1]}\boldsymbol{C_1})^{\mathrm{T}} &= \frac{v_1^2}{v^2}\frac{1}{s}(g_{[1]}^2((1-a|\boldsymbol{\eta}_{[1]}|^2)|\boldsymbol{\nu}_{[1]}|^2 + a(\boldsymbol{\eta}_{[1]}\boldsymbol{\nu}_{[1]}^{\mathrm{T}})^2) - 2g_{[1]}h_{[1]}\boldsymbol{\eta}_{[1]}\boldsymbol{\nu}_{[1]}^{\mathrm{T}}) \\ &\quad + \frac{v_1^2}{v^2}\frac{1}{s}h_{[1]}^2((1+|\boldsymbol{\nu}_{[1]}|^2)|\boldsymbol{\eta}_{[1]}|^2 - (\boldsymbol{\eta}_{[1]}\boldsymbol{\nu}_{[1]}^{\mathrm{T}})^2).\end{aligned}$$

From the definitions of $g_{[1]}$ and $h_{[1]}$, we can conclude that

$$
\begin{aligned}
A_1 &\leqslant \boldsymbol{b}_{[\mathbf{1}]}\boldsymbol{C}_{\mathbf{1}}\boldsymbol{P}_{\mathbf{1}}(\boldsymbol{b}_{[\mathbf{1}]}\boldsymbol{C}_{\mathbf{1}})^{\mathrm{T}} + \left(\frac{a-1}{\lambda_n^2} - \frac{2}{\lambda_1\lambda_n}\right)(2\lambda_1+\sigma_1)^2\frac{v_1^2}{v^2} - 2((n+2)a-1)\frac{2\lambda_1+\sigma_1}{\lambda_n}\frac{v_1^2}{v^2} \\
&\quad + ((n+2)^2a - n + 2)\frac{v_1^2}{v^2} + 2\frac{\sigma_1(\boldsymbol{\lambda})}{\lambda_1}\frac{v_1^2}{v^2} \\
&= \frac{v_1^2}{v^2}\frac{1}{s}(g_{[1]}^2((1-a|\boldsymbol{\eta}_{[\mathbf{1}]}|^2)|\boldsymbol{\nu}_{[\mathbf{1}]}|^2 + a(\boldsymbol{\eta}_{[\mathbf{1}]}\boldsymbol{\nu}_{[\mathbf{1}]}^{\mathrm{T}})^2) - 2g_{[1]}h_{[1]}\boldsymbol{\eta}_{[\mathbf{1}]}\boldsymbol{\nu}_{[\mathbf{1}]}^{\mathrm{T}}) \\
&\quad + \frac{v_1^2}{v^2}\frac{1}{s}h_{[1]}^2((1+|\boldsymbol{\nu}_{[\mathbf{1}]}|^2)|\boldsymbol{\eta}_{[\mathbf{1}]}|^2 - (\boldsymbol{\eta}_{[\mathbf{1}]}\boldsymbol{\nu}_{[\mathbf{1}]}^{\mathrm{T}})^2) \\
&\quad - g_{[1]}^2\frac{v_1^2}{v^2} + \frac{h_{[1]}^2}{a}\frac{v_1^2}{v^2} + \left(n+6+2\frac{\sigma_1}{\lambda_1} - \frac{1}{a}\right)\frac{v_1^2}{v^2} \\
&= \frac{v_1^2}{v^2}\frac{1}{s}\left(\frac{h_{[1]}^2}{a} - g_{[1]}^2 + \left|\frac{h_{[1]}}{\sqrt{a}}\boldsymbol{\nu}_{[\mathbf{1}]} - \sqrt{a}g_{[1]}\boldsymbol{\eta}_{[\mathbf{1}]}\right|^2\right) + \left(n+6+2\frac{\sigma_1}{\lambda_1} - \frac{1}{a}\right)\frac{v_1^2}{v^2}.
\end{aligned}
$$

Since

$$
\begin{aligned}
g_{[1]}^2 &= \left(1+2\frac{\lambda_n}{\lambda_1}\right)\frac{(2\lambda_1+\sigma_1)^2}{\lambda_n^2}, \\
\frac{h_{[1]}^2}{a} &= \left(\frac{2\lambda_1+\sigma_1}{\lambda_n} - (n+2)\right)^2 a + 2\left(\frac{2\lambda_1+\sigma_1}{\lambda_n} - (n+2)\right) + \frac{1}{a}
\end{aligned}
$$

and

$$
\begin{aligned}
\left|\frac{h_{[1]}}{\sqrt{a}}\boldsymbol{\nu}_{[\mathbf{1}]} - \sqrt{a}g_{[1]}\boldsymbol{\eta}_{[\mathbf{1}]}\right|^2 &= \left(1+2\frac{\lambda_n}{\lambda_1}\right)\left(\sqrt{a}\left(\frac{2\lambda_1+\sigma_1}{\lambda_n} - (n+2)\frac{\lambda_1}{\lambda_n}\right) + \frac{1}{\sqrt{a}}\frac{\lambda_1}{\lambda_n}\right)^2 \\
&\quad + \sum_{j=2}^{n-1}\frac{1+2\frac{\lambda_n}{\lambda_1}}{1+2\frac{\lambda_j}{\lambda_1}}\left(\sqrt{a}\left(\frac{2\lambda_1+\sigma_1}{\lambda_n} - (n+2)\frac{\lambda_j}{\lambda_n}\right) + \frac{1}{\sqrt{a}}\frac{\lambda_j}{\lambda_n}\right)^2,
\end{aligned}
$$

we obtain that

$$
\begin{aligned}
A_1 &\leqslant \frac{v_1^2}{v^2}\frac{1}{s}\left(\frac{h_{[1]}^2}{a} - g_{[1]}^2 + \left|\frac{h_{[1]}}{\sqrt{a}}\boldsymbol{\nu}_{[\mathbf{1}]} - \sqrt{a}g_{[1]}\boldsymbol{\eta}_{[\mathbf{1}]}\right|^2\right) + \left(n+6+2\frac{\sigma_1}{\lambda_1} - \frac{1}{a}\right)\frac{v_1^2}{v^2} \\
&= \left(aJ_1 + J_2 + \frac{1}{a}J_3\right)\frac{v_1^2}{v^2},
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &= \left(1+2\frac{\lambda_n}{\lambda_1}\right)\left(\frac{2\lambda_1+\sigma_1}{\lambda_n} - (n+2)\frac{\lambda_1}{\lambda_n}\right)^2 \\
&\quad + \sum_{j=2}^{n-1}\frac{1+2\frac{\lambda_n}{\lambda_1}}{1+2\frac{\lambda_j}{\lambda_1}}\left(\frac{2\lambda_1+\sigma_1}{\lambda_n} - (n+2)\frac{\lambda_j}{\lambda_n}\right)^2 \\
&\quad + \left(\frac{2\lambda_1+\sigma_1}{\lambda_n} - (n+2)\right)^2 - \left(n+6+2\frac{\sigma_1}{\lambda_1}\right)\left(1 - \frac{\lambda_1}{\lambda_n}\right)^2 \\
&\quad - \left(n+6+2\frac{\sigma_1}{\lambda_1}\right)\sum_{j=2}^{n-1}\frac{1}{1+2\frac{\lambda_j}{\lambda_1}}\left(1 - \frac{\lambda_j}{\lambda_n}\right)^2 \\
&\quad - \left(n+6+2\frac{\sigma_1}{\lambda_1}\right)\left(1+2\frac{\lambda_n}{\lambda_1}\right)\sum_{j=2}^{n-1}\frac{1}{1+2\frac{\lambda_j}{\lambda_1}}\left(\frac{\lambda_1}{\lambda_n} - \frac{\lambda_j}{\lambda_n}\right)^2 \\
&\quad - \frac{1}{2}\left(n+6+2\frac{\sigma_1}{\lambda_1}\right)\left(1+2\frac{\lambda_n}{\lambda_1}\right)\sum_{i,j=2}^{n-1}\frac{1}{(1+2\frac{\lambda_i}{\lambda_1})(1+2\frac{\lambda_j}{\lambda_1})}\left(\frac{\lambda_i}{\lambda_n} - \frac{\lambda_j}{\lambda_n}\right)^2, \\
J_2 &= 2\left(1+2\frac{\lambda_n}{\lambda_1}\right)\frac{\lambda_1}{\lambda_n}\left(\frac{2\lambda_1+\sigma_1}{\lambda_n} - (n+2)\frac{\lambda_1}{\lambda_n}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+2\sum_{j=2}^{n-1}\frac{1+2\frac{\lambda_n}{\lambda_1}}{1+2\frac{\lambda_j}{\lambda_1}}\frac{\lambda_j}{\lambda_n}\bigg(\frac{2\lambda_1+\sigma_1}{\lambda_n}-(n+2)\frac{\lambda_j}{\lambda_n}\bigg)\\
&+2\bigg(\frac{2\lambda_1+\sigma_1}{\lambda_n}-(n+2)\bigg)-\bigg(1+2\frac{\lambda_n}{\lambda_1}\bigg)\frac{(2\lambda_1+\sigma_1)^2}{\lambda_n^2}\\
&+\bigg(n+6+2\frac{\sigma_1}{\lambda_1}\bigg)\bigg(1+\bigg(1+2\frac{\lambda_n}{\lambda_1}\bigg)\bigg(\frac{\lambda_1^2}{\lambda_n^2}+\sum_{j=2}^{n-1}\frac{1}{1+2\frac{\lambda_j}{\lambda_1}}\frac{\lambda_j^2}{\lambda_n^2}\bigg)\bigg)\\
&+\bigg(1-\frac{\lambda_1}{\lambda_n}\bigg)^2+\sum_{j=2}^{n-1}\frac{1}{1+2\frac{\lambda_j}{\lambda_1}}\bigg(1-\frac{\lambda_j}{\lambda_n}\bigg)^2\\
&+\bigg(1+2\frac{\lambda_n}{\lambda_1}\bigg)\sum_{j=2}^{n-1}\frac{1}{1+2\frac{\lambda_j}{\lambda_1}}\bigg(\frac{\lambda_1}{\lambda_n}-\frac{\lambda_j}{\lambda_n}\bigg)^2\\
&+\frac{1}{2}\bigg(1+2\frac{\lambda_n}{\lambda_1}\bigg)\sum_{i,j=2}^{n-1}\frac{1}{(1+2\frac{\lambda_i}{\lambda_1})(1+2\frac{\lambda_j}{\lambda_1})}\bigg(\frac{\lambda_i}{\lambda_n}-\frac{\lambda_j}{\lambda_n}\bigg)^2,
\end{aligned}
$$

$$
J_3=\bigg(1+2\frac{\lambda_n}{\lambda_1}\bigg)\frac{\lambda_1^2}{\lambda_n^2}+\sum_{j=2}^{n-1}\frac{1+2\frac{\lambda_n}{\lambda_1}}{1+2\frac{\lambda_j}{\lambda_1}}\frac{\lambda_j^2}{\lambda_n^2}+1-1-\bigg(1+2\frac{\lambda_n}{\lambda_1}\bigg)\frac{\lambda_1^2}{\lambda_n^2}-\sum_{j=2}^{n-1}\frac{1+2\frac{\lambda_n}{\lambda_1}}{1+2\frac{\lambda_j}{\lambda_1}}\frac{\lambda_j^2}{\lambda_n^2}.
$$

We can easily get $J_1=J_2=J_3=0$, so $A_1\leqslant 0$.

(2) (Proof of $A_n\leqslant 0$) The proof of $A_n\leqslant 0$ is similar to the proof of $A_1\leqslant 0$, so some details will be skipped.

Letting $\boldsymbol{B_n}$ be the corresponding matrix of A_n, according to (2.11), we have

$$
\boldsymbol{B_n}=(b_{ij}^n)_{1\leqslant i,j\leqslant n-1},
$$

where

$$
b_{ij}^n=-\frac{1}{\lambda_n^2}-\bigg(\frac{1}{\lambda_j^2}+\frac{2}{\lambda_j\lambda_n}\bigg)\delta_{ij}+a\bigg(\frac{1}{\lambda_i}-\frac{1}{\lambda_n}\bigg)\bigg(\frac{1}{\lambda_j}-\frac{1}{\lambda_n}\bigg).
$$

$\boldsymbol{B_n}$ has the form

$$
-\boldsymbol{B_n}=\boldsymbol{E_n}+\boldsymbol{F_n}-\boldsymbol{G_n},
$$

where

$$
\boldsymbol{E_n}=\begin{pmatrix}\frac{1}{\lambda_1^2}+\frac{2}{\lambda_1\lambda_n} & & & \\ & \frac{1}{\lambda_2^2}+\frac{2}{\lambda_2\lambda_n} & & \\ & & \ddots & \\ & & & \frac{1}{\lambda_{n-1}^2}+\frac{2}{\lambda_{n-1}\lambda_n}\end{pmatrix},
$$

$$
\boldsymbol{F_n}=\frac{1}{\lambda_n^2}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}(1,1,\ldots,1),
$$

$$
\boldsymbol{G_n}=a\begin{pmatrix}\frac{1}{\lambda_1}-\frac{1}{\lambda_n}\\ \frac{1}{\lambda_2}-\frac{1}{\lambda_n}\\ \vdots\\ \frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_n}\end{pmatrix}\bigg(\frac{1}{\lambda_1}-\frac{1}{\lambda_n},\frac{1}{\lambda_2}-\frac{1}{\lambda_n},\ldots,\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_n}\bigg).
$$

Let

$$\boldsymbol{C_n} = \begin{pmatrix} \frac{\lambda_1}{\sqrt{1+2\frac{\lambda_1}{\lambda_n}}} & & & \\ & \frac{\lambda_2}{\sqrt{1+2\frac{\lambda_2}{\lambda_n}}} & & \\ & & \ddots & \\ & & & \frac{\lambda_{n-1}}{\sqrt{1+2\frac{\lambda_{n-1}}{\lambda_n}}} \end{pmatrix}.$$

Then

$$\boldsymbol{C_n}(-\boldsymbol{B_n})\boldsymbol{C_n} = \boldsymbol{I} + \boldsymbol{\nu}_{[n]}^{\mathrm{T}}\boldsymbol{\nu}_{[n]} - a\boldsymbol{\eta}_{[n]}^{\mathrm{T}}\boldsymbol{\eta}_{[n]},$$

where

$$\boldsymbol{\nu}_{[n]} = \frac{1}{\lambda_n}(1,1,\ldots,1)\boldsymbol{C_n}$$

and

$$\boldsymbol{\eta}_{[n]} = \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_n}, \frac{1}{\lambda_2} - \frac{1}{\lambda_n}, \ldots, \frac{1}{\lambda_{n-1}} - \frac{1}{\lambda_n}\right)\boldsymbol{C_n}.$$

It is easy to verify that the first leading principal minor of $\boldsymbol{C_n}(-\boldsymbol{B_n})\boldsymbol{C_n}$ is greater than 0. For $2 \leqslant k \leqslant n-1$, from a similar proof for $A_1 \leqslant 0$ above, that the k-th-order leading principal minor of $\boldsymbol{C_n}(-\boldsymbol{B_n})\boldsymbol{C_n}$ is greater than 0 is equivalent to the inequality

$$1 - a\frac{|\boldsymbol{\eta}_{[nk]}|^2 + |\boldsymbol{\eta}_{[nk]}|^2|\boldsymbol{\nu}_{[nk]}|^2 - (\boldsymbol{\eta}_{[nk]}\boldsymbol{\nu}_{[nk]}^{\mathrm{T}})^2}{1 + |\boldsymbol{\nu}_{[nk]}|^2} > 0,$$

where $\boldsymbol{\nu}_{[nk]}$ is a vector whose components are from the first k components of $\boldsymbol{\nu}_{[n]}$, and $\boldsymbol{\eta}_{[nk]}$ is a vector whose components are from the first k components of $\boldsymbol{\eta}_{[n]}$.

By direct computations, we have

$$\begin{aligned} |\boldsymbol{\eta}_{[nk]}|^2 &= \frac{1}{4}\left(\sum_{j=1}^{k}\left(1 + 2\frac{\lambda_j}{\lambda_n}\right) - 6k + 9\sum_{j=1}^{k}\frac{1}{1 + 2\frac{\lambda_j}{\lambda_n}}\right), \\ |\boldsymbol{\eta}_{[nk]}|^2|\boldsymbol{\nu}_{[nk]}|^2 - (\boldsymbol{\eta}_{[nk]}\boldsymbol{\nu}_{[nk]}^{\mathrm{T}})^2 &= \frac{1}{4}\left(\sum_{i=1}^{k}\left(1 + 2\frac{\lambda_i}{\lambda_n}\right)\right)\sum_{j=1}^{k}\frac{1}{1 + 2\frac{\lambda_j}{\lambda_n}} - \frac{k^2}{4}, \\ 1 + |\boldsymbol{\nu}_{[nk]}|^2 &= \frac{1}{4}\left(\sum_{j=1}^{k}\left(1 + 2\frac{\lambda_j}{\lambda_n}\right) + 4 - 2k + \sum_{j=1}^{k}\frac{1}{1 + 2\frac{\lambda_j}{\lambda_n}}\right). \end{aligned}$$

Let

$$\tilde{\mu_j} = 1 + 2\frac{\lambda_j}{\lambda_n}, \quad 1 \leqslant j \leqslant n-1.$$

Then

$$\begin{aligned} &\frac{|\boldsymbol{\eta}_{[nk]}|^2 + |\boldsymbol{\eta}_{[nk]}|^2|\boldsymbol{\nu}_{[nk]}|^2 - (\boldsymbol{\eta}_{[nk]}\boldsymbol{\nu}_{[nk]}^{\mathrm{T}})^2}{1 + |\boldsymbol{\nu}_{[nk]}|^2} \\ &\quad = \frac{(\sum_{i=1}^{k}\tilde{\mu_i})\sum_{j=1}^{l}\frac{1}{\tilde{\mu_j}} + \sum_{j=1}^{k}\tilde{\mu_j} + 9\sum_{i=1}^{k}\frac{1}{\tilde{\mu_i}} - k(k+6)}{\sum_{j=1}^{k}\tilde{\mu_j} + \sum_{j=1}^{k}\frac{1}{\tilde{\mu_j}} + 4 - 2k}. \end{aligned}$$

By Lemma 2.2,

$$\frac{|\boldsymbol{\eta}_{[nk]}|^2 + |\boldsymbol{\eta}_{[nk]}|^2|\boldsymbol{\nu}_{[nk]}|^2 - (\boldsymbol{\eta}_{[nk]}\boldsymbol{\nu}_{[nk]}^{\mathrm{T}})^2}{1 + |\boldsymbol{\nu}_{[nk]}|^2} < k.$$

So the k-th-order leading principal minor of $\boldsymbol{C_n}(-\boldsymbol{B_n})\boldsymbol{C_n}$ is greater than 0 for $a = \frac{1}{n-1}$.

In conclusion, all the leading principal minors of $\boldsymbol{C_n}(-\boldsymbol{B_n})\boldsymbol{C_n}$ are greater than 0, and then $\boldsymbol{C_n}(-\boldsymbol{B_n})\boldsymbol{C_n}$ is positive definite and $\boldsymbol{B_n} < 0$. By Lemma 2.1, we have

$$\begin{aligned}A_n \leqslant &-\boldsymbol{b}_{[\boldsymbol{n}]}\boldsymbol{B_n}^{-1}\boldsymbol{b}_{[\boldsymbol{n}]}^{\mathrm{T}} + \frac{a-1}{\lambda_n^2}(2\lambda_n+\sigma_1)^2\frac{v_n^2}{v^2} - 2((n+2)a-1)\frac{2\lambda_n+\sigma_1}{\lambda_n}\frac{v_n^2}{v^2}\\ &+((n+2)^2a-n+2)\frac{v_n^2}{v^2}+2\frac{\sigma_1(\boldsymbol{\lambda})}{\lambda_n}\frac{v_n^2}{v^2}.\end{aligned}$$

In the following, we calculate $-\boldsymbol{b}_{[\boldsymbol{n}]}\boldsymbol{B_n}^{-1}\boldsymbol{b}_{[\boldsymbol{n}]}^{\mathrm{T}}$. Letting

$$\tilde{s} = (1-a|\boldsymbol{\eta}_{[\boldsymbol{n}]}|^2)(1+|\boldsymbol{\nu}_{[\boldsymbol{n}]}|^2) + a(\boldsymbol{\eta}_{[\boldsymbol{n}]}\boldsymbol{\nu}_{[\boldsymbol{n}]}^{\mathrm{T}})^2$$

and

$$\begin{aligned}\boldsymbol{P_n} = \boldsymbol{I} &- \frac{1-a|\boldsymbol{\eta}_{[\boldsymbol{n}]}|^2}{\tilde{s}}\boldsymbol{\nu}_{[\boldsymbol{n}]}^{\mathrm{T}}\boldsymbol{\nu}_{[\boldsymbol{n}]} + a\frac{1+|\boldsymbol{\nu}_{[\boldsymbol{n}]}|^2}{\tilde{s}}\boldsymbol{\eta}_{[\boldsymbol{n}]}^{\mathrm{T}}\boldsymbol{\eta}_{[\boldsymbol{n}]} - a\frac{\boldsymbol{\eta}_{[\boldsymbol{n}]}\boldsymbol{\nu}_{[\boldsymbol{n}]}^{\mathrm{T}}}{\tilde{s}}\boldsymbol{\eta}_{[\boldsymbol{n}]}^{\mathrm{T}}\boldsymbol{\nu}_{[\boldsymbol{n}]}\\ &- a\frac{\boldsymbol{\eta}_{[\boldsymbol{n}]}\boldsymbol{\nu}_{[\boldsymbol{n}]}^{\mathrm{T}}}{\tilde{s}}\boldsymbol{\nu}_{[\boldsymbol{n}]}^{\mathrm{T}}\boldsymbol{\eta}_{[\boldsymbol{n}]},\end{aligned} \tag{2.26}$$

we have that $\boldsymbol{P_n}$ is the inverse matrix of $\boldsymbol{C_n}(-\boldsymbol{B_n})\boldsymbol{C_n}$ and $-\boldsymbol{B_n}^{-1} = \boldsymbol{C_n}\boldsymbol{P_n}\boldsymbol{C_n}$. Thus,

$$-\boldsymbol{b}_{[\boldsymbol{n}]}\boldsymbol{B_n}^{-1}\boldsymbol{b}_{[\boldsymbol{n}]}^{\mathrm{T}} = \boldsymbol{b}_{[\boldsymbol{n}]}\boldsymbol{C_n}\boldsymbol{P_n}(\boldsymbol{b}_{[\boldsymbol{n}]}\boldsymbol{C_n})^{\mathrm{T}}.$$

Let

$$g_{[n]} = \frac{\sigma_1(\boldsymbol{\lambda})}{\lambda_n} + 2$$

and

$$h_{[n]} = \left(n - \frac{\sigma_1(\boldsymbol{\lambda})}{\lambda_n}\right)a - 1.$$

From (2.11), we have

$$\boldsymbol{b}_{[\boldsymbol{n}]} = \left[-g_{[n]}\frac{1}{\lambda_n}(1,\dots,1) + h_{[n]}\left(\frac{1}{\lambda_1}-\frac{1}{\lambda_n},\dots,\frac{1}{\lambda_{n-1}}-\frac{1}{\lambda_n}\right)\right]\frac{v_n}{v}$$

and

$$\boldsymbol{b}_{[\boldsymbol{n}]}\boldsymbol{C_n} = (-g_{[n]}\boldsymbol{\nu}_{[\boldsymbol{n}]} + h_{[n]}\boldsymbol{\eta}_{[\boldsymbol{n}]})\frac{v_n}{v}.$$

Therefore,

$$\begin{aligned}-\boldsymbol{b}_{[\boldsymbol{n}]}\boldsymbol{B_n}^{-1}\boldsymbol{b}_{[\boldsymbol{n}]}^{\mathrm{T}} &= \boldsymbol{b}_{[\boldsymbol{n}]}\boldsymbol{C_n}\boldsymbol{P_n}(\boldsymbol{b}_{[\boldsymbol{n}]}\boldsymbol{C_n})^{\mathrm{T}}\\ &= \frac{v_n^2}{v^2}(-g_{[n]}\boldsymbol{\nu}_{[\boldsymbol{n}]} + h_{[n]}\boldsymbol{\eta}_{[\boldsymbol{n}]})\boldsymbol{P_n}(-g_{[n]}\boldsymbol{\nu}_{[\boldsymbol{n}]} + h_{[n]}\boldsymbol{\eta}_{[\boldsymbol{n}]})^{\mathrm{T}}\\ &= \frac{v_n^2}{v^2}\frac{1}{\tilde{s}}(g_{[n]}^2((1-a|\boldsymbol{\eta}_{[\boldsymbol{n}]}|^2)|\boldsymbol{\nu}_{[\boldsymbol{n}]}|^2 + a(\boldsymbol{\eta}_{[\boldsymbol{n}]}\boldsymbol{\nu}_{[\boldsymbol{n}]}^{\mathrm{T}})^2) - 2g_{[n]}h_{[n]}\boldsymbol{\eta}_{[\boldsymbol{n}]}\boldsymbol{\nu}_{[\boldsymbol{n}]}^{\mathrm{T}})\\ &\quad + \frac{v_n^2}{v^2}\frac{1}{\tilde{s}}h_{[n]}^2((1+|\boldsymbol{\nu}_{[\boldsymbol{n}]}|^2)|\boldsymbol{\eta}_{[\boldsymbol{n}]}|^2 - (\boldsymbol{\eta}_{[\boldsymbol{n}]}\boldsymbol{\nu}_{[\boldsymbol{n}]}^{\mathrm{T}})^2)\end{aligned}$$

and

$$\begin{aligned}A_n \leqslant &-\boldsymbol{b}_{[\boldsymbol{n}]}\boldsymbol{B_n}^{-1}\boldsymbol{b}_{[\boldsymbol{n}]}^{\mathrm{T}} + \frac{a-1}{\lambda_n^2}(2\lambda_n+\sigma_1)^2\frac{v_n^2}{v^2} - 2((n+2)a-1)\frac{2\lambda_n+\sigma_1}{\lambda_n}\frac{v_n^2}{v^2}\\ &+((n+2)^2a-n+2)\frac{v_n^2}{v^2}+2\frac{\sigma_1(\boldsymbol{\lambda})}{\lambda_n}\frac{v_n^2}{v^2}\\ =\ &\frac{v_n^2}{v^2}\frac{1}{\tilde{s}}(g_{[n]}^2((1-a|\boldsymbol{\eta}_{[\boldsymbol{n}]}|^2)|\boldsymbol{\nu}_{[\boldsymbol{n}]}|^2 + a(\boldsymbol{\eta}_{[\boldsymbol{n}]}\boldsymbol{\nu}_{[\boldsymbol{n}]}^{\mathrm{T}})^2) - 2g_{[n]}h_{[n]}\boldsymbol{\eta}_{[\boldsymbol{n}]}\boldsymbol{\nu}_{[\boldsymbol{n}]}^{\mathrm{T}})\\ &+ \frac{v_n^2}{v^2}\frac{1}{\tilde{s}}h_{[n]}^2((1+|\boldsymbol{\nu}_{[\boldsymbol{n}]}|^2)|\boldsymbol{\eta}_{[\boldsymbol{n}]}|^2 - (\boldsymbol{\eta}_{[\boldsymbol{n}]}\boldsymbol{\nu}_{[\boldsymbol{n}]}^{\mathrm{T}})^2)\\ &+ \frac{v_n^2}{v^2}\left(-g_{[n]}^2 + \frac{h_{[n]}^2}{a} + n + 6 + 2\frac{\sigma_1}{\lambda_n} - \frac{1}{a}\right)\\ =\ &\frac{v_n^2}{v^2}\frac{1}{\tilde{s}}\left(-g_{[n]}^2 + \frac{h_{[n]}^2}{a} + \left|\frac{h_{[n]}}{\sqrt{a}}\boldsymbol{\nu}_{[\boldsymbol{n}]} - \sqrt{a}g_{[n]}\boldsymbol{\eta}_{[\boldsymbol{n}]}\right|^2\right) + \frac{v_n^2}{v^2}\left(n+6+2\frac{\sigma_1}{\lambda_n}-\frac{1}{a}\right).\end{aligned}$$

We can get

$$A_n \leqslant 0$$

from a similar proof for $A_1 \leqslant 0$ above.

Therefore, by $A_i \leqslant 0$ for $1 \leqslant i \leqslant n$ and (2.8), we complete the proof of Theorem 1.1. □

Corollary 1.2 is the direct results of the maximum principle and the proof can be found in [10]. Here, we give the proof from Theorem 1.1, which is slightly different from that in [10].

Proof of Corollary 1.2. Since $\Delta\varphi \leqslant 0$, φ attains its minimum on the boundary $\partial\Omega$ by the standard minimum principle. Therefore, ψ attains its minimum on the boundary $\partial\Omega$ and

$$\begin{aligned}\psi &= (-v)^{n+2}\det D^2 v\\ &\geqslant \min_{\partial\Omega}(-v)^{n+2}\det D^2 v\\ &= 2^{-(n+1)}\min_{\partial\Omega}\{K|\nabla u|^{n+1}\}\\ &\geqslant 2^{-(n+1)}\min_{\partial\Omega}K\min_{\partial\Omega}|\nabla u|^{n+1},\end{aligned}$$

which is the estimate (1.3).

If φ attains its minimum in Ω, then φ is constant from the strong minimum principle, and $\Delta\varphi = 0$. From the process of the proof for Theorem 1.1, we can see that

$$\begin{aligned}&v_{ijk} = 0, \quad i \neq j, \quad j \neq k, \quad k \neq i,\\ &A_i = 0, \quad i = 1, 2, \ldots, n.\end{aligned}$$

Since $A_1 = 0$, by Lemma 2.1, we have

$$\boldsymbol{x_{[1]}} = (v_{111}, v_{221}, \ldots, v_{(n-1)(n-1)1}) = -\boldsymbol{b_{[1]}}\boldsymbol{B_1}^{-1},$$

and then

$$\boldsymbol{x_{[1]}} = -\boldsymbol{b_{[1]}}\boldsymbol{B_1}^{-1} = \boldsymbol{b_{[1]}}\boldsymbol{C_1}\boldsymbol{P_1}\boldsymbol{C_1} = \frac{v_1}{v}(-g_{[1]}\boldsymbol{\nu_{[1]}} + h_{[1]}\boldsymbol{\eta_{[1]}})\boldsymbol{P_1}\boldsymbol{C_1}.$$

From the expression of $\boldsymbol{P_1}$,

$$\begin{aligned}\boldsymbol{x_{[1]}} = \frac{1}{s}\frac{v_1}{v}&((-g_{[1]}(1 - a|\boldsymbol{\eta_{[1]}}|^2) - h_{[1]}\boldsymbol{\eta_{[1]}}\boldsymbol{\nu_{[1]}}^{\mathrm{T}})\boldsymbol{\nu_{[1]}}\\ &+ (-ag_{[1]}\boldsymbol{\eta_{[1]}}\boldsymbol{\nu_{[1]}}^{\mathrm{T}} + h_{[1]}(1 + |\boldsymbol{\nu_{[1]}}|^2))\boldsymbol{\eta_{[1]}})\boldsymbol{C_1}.\end{aligned}$$

Let

$$p = -g_{[1]}(1 - a|\boldsymbol{\eta_{[1]}}|^2) - h_{[1]}\boldsymbol{\eta_{[1]}}\boldsymbol{\nu_{[1]}}^{\mathrm{T}}$$

and

$$q = -ag_{[1]}\boldsymbol{\eta_{[1]}}\boldsymbol{\nu_{[1]}}^{\mathrm{T}} + h_{[1]}(1 + |\boldsymbol{\nu_{[1]}}|^2).$$

By direct computations, we get

$$\begin{aligned}p &= \frac{1}{4(n-1)}\frac{\lambda_1^2}{\lambda_n^2}\mu_n^{\frac{3}{2}}\bigg(\bigg(\sum_{i=2}^n \mu_i\bigg)\sum_{j=2}^n\frac{1}{\mu_j} - (n-10)\sum_{j=2}^n\frac{1}{\mu_j} - (n-2)\sum_{j=2}^n\mu_j\\ &\quad + (n-1)(n-11)\bigg),\\ q &= \frac{1}{4(n-1)}\frac{\lambda_1^2}{\lambda_n^2}\mu_n\bigg(\bigg(\sum_{i=2}^n \mu_i\bigg)\sum_{j=2}^n\frac{1}{\mu_j} - (n-10)\sum_{j=2}^n\frac{1}{\mu_j} - (n-2)\sum_{j=2}^n\mu_j\\ &\quad + (n-1)(n-11)\bigg),\\ s &= -\frac{1}{4(n-1)}\frac{\lambda_1^2}{\lambda_n^2}\mu_n\bigg(\bigg(\sum_{i=2}^n \mu_i\bigg)\sum_{j=2}^n\frac{1}{\mu_j} - (n-10)\sum_{j=2}^n\frac{1}{\mu_j} - (n-2)\sum_{j=2}^n\mu_j\\ &\quad + (n-1)(n-11)\bigg).\end{aligned}$$

Therefore, we have

$$\boldsymbol{x}_{[1]} = -\frac{v_1}{v}(\sqrt{\mu_n}\boldsymbol{\nu}_{[1]} + \boldsymbol{\eta}_{[1]})\boldsymbol{C}_1 = -\frac{v_1}{v}(3\lambda_1, \lambda_2, \ldots, \lambda_{n-1}),$$

and then

$$v_{nn1} = -\frac{(2\lambda_1 + \sigma_1(\boldsymbol{\lambda}))v_1}{v} - \sum_{k=1}^{n-1} v_{kk1} = -\frac{v_1}{v}\lambda_1.$$

A similar computation gives

$$v_{jji} = \begin{cases} -\dfrac{v_i}{v}\lambda_j, & i \neq j, \quad 2 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant n, \\ -3\dfrac{v_i}{v}\lambda_i, & i = j, \quad 2 \leqslant i \leqslant n, \quad 1 \leqslant j \leqslant n. \end{cases}$$

In conclusion, the equality $\Delta\varphi = 0$ holds if and only if for $1 \leqslant i, j, k \leqslant n$,

$$v_{jki} = \begin{cases} -\dfrac{v_i}{v}\lambda_j, & i \neq j, \quad j = k, \\ -3\dfrac{v_i}{v}\lambda_i, & i = j = k, \\ 0, & i \neq j, \quad j \neq k, \quad k \neq i. \end{cases}$$

Because of $u = v^2$,

$$u_{ijk} = 2(v_i v_{jk} + v_j v_{ik} + v_k v_{ij} + v v_{ijk}).$$

Note that the matrix (v_{ij}) is diagonal by the choice of the coordinates, and we have

$$u_{ijk} = 0, \quad 1 \leqslant i, j, k \leqslant n,$$

i.e., all the third-order derivatives of u vanish. Since

$$\Omega = \{x \in \mathbb{R}^n \mid u(x) > 0\}$$

is convex, Ω must be an ellipsoid (ellipse). On the contrary, if Ω is an ellipsoid (ellipse), then we can easily get that ψ is constant and attains its minimum in Ω naturally. □

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