THE WEAK CONVERGENCE FOR A MEASURE RELATED TO A CLASS OF CONFORMALLY INVARIANT FULLY NONLINEAR OPERATOR

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ABSTRACT. Trudinger-Wang introduced the notion of k-Hessian measure associated with k-convex functions, not necessarily continuous, and proved the weak continuity of the associated k-Hessian measure with respect to local L^1 convergence in 1999. In this paper we find a special divergence structure for the σ_k-Yamabe operator which is conformally invariant, and prove the weak continuity of the σ_k-Yamabe operator with respect to local L^1 convergence.

1. INTRODUCTION

Let (M, g) be a Riemannian manifold of dimension $n \geq 3$ with a metric g. The well-known Yamabe problem states whether there exists metrics which are pointwise conformal to g and have constant scalar curvature. The Yamabe problem was solved through the work of Yamabe[25] , Trudinger[19] , Aubin[1] and Schoen[17]. Denote Ric and R as the Ricci tensor and the scalar curvature, respectively. Then the Schouten tensor is

$$\tilde{A}^g_{ij} = \frac{1}{n-2}\left[Ric^g_{ij} - \frac{1}{2(n-1)}R^g g_{ij}\right]. \tag{1}$$

Now transform the $(0, 2)$-tensor $\tilde{A}^g_{ij}$ to a $(1, 1)$-tensor A_{ij} by $A^g = g^{-1}\tilde{A}^g$.

We are always interested in locally conformally flat (lcf) manifolds. We say a Riemannian manifold (M, g) is lcf if the metric g can be locally written as $g = v^{-2}|dx|^2$ for some smooth $v > 0$, where $|dx|^2$ is the usual Euclidean metric. Then the $(1, 1)$-tensor A_{ij} becomes

$$A_{ij}(v) = vv_{ij} - \frac{1}{2}|\nabla v|^2\delta_{ij}. \tag{2}$$

For $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, we define

$$\sigma_k(\lambda) := \sum_{1\leq i_1<\cdots<i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k}. \tag{3}$$

Let $\lambda\{A_{ij}(v)\}$ be the eigenvalues of $A_{ij}(v)$, then we define

$$S_k(v) := \sigma_k(\lambda\{A_{ij}(v)\}). \tag{4}$$

We call S_k the σ_k-Yamabe operator. For the related σ_k-Yamabe problem, there has been a lot of work, for instance, Chang- Gursky- Yang [2, 4], Gursky- Viaclovsky [12], Li- Li [14], Guan- Wang [11], Sheng- Trudinger- Wang [18], Ge-Wang[7].

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Trudinger - Wang [20] introduced the notion of k-Hessian measures, for k-convex functions defined on domains in Euclidean space, and proved the weak continuity of k-Hessian measures with respect to locally uniform convergence of functions. In the sequel paper, Trudinger - Wang [21] introduced the notion of k-Hessian measure associated with k-convex functions and proved the weak continuity of the associated k-Hessian measure with respect to local L^1 convergence. In [22], Trudinger - Wang extended to the case of mixed k-Hessian measures associated with k-tuples of k-convex functions on domains in Euclidean space, and gave some applications of the k-Hessian measure theory to the Dirichlet problem. In Dai-Trudinger-Wang [5] and Dai-Wang-Zhou [6], they generalized the Hessian measure result to higher order curvature operator with respect to almost everywhere convergence. For the Hessian measure on Heisenberg group, Trudinger-Zhang [23] proved the weak continuity of Hessian measures with respect to locally uniform convergence of functions through the monotonicity formula for k-convex functions on Heisenberg group.

On the other hand, the σ_k-Yamabe operator $S_k(v)$ is conformally invariant, which comes from the result of Chang- Gursky- Yang [3] :

Proposition 1.1 ([3]). *Let λ be any real number. Then the tensor*

$$\begin{aligned}
T &= T(g,f)\\
&= Ric(g) + \frac{n-2}{n}\nabla_g^2 f + \frac{1}{n}(\Delta_g f)g + \frac{n-2}{n^2}df\otimes df - \frac{n-2}{n^2}|\nabla f|^2 g\\
&+ \lambda\left[R(g) + \frac{2(n-1)}{n}\Delta_g f - \frac{(n-1)(n-2)}{n^2}|\nabla f|^2\right]g, \qquad (5)
\end{aligned}$$

is a pointwise conformal invariant. More precisely,

$$T(e^{2w}g, f+nw) = T(g,f). \qquad (6)$$

Letting $\lambda = -\frac{1}{2(n-1)}$, we define $\bar{T} := \frac{1}{n-2}g^{-1}T_1(g, n\log v)$ and $Y_k^m(g,v) := \sigma_k(\lambda\{\bar{T}\})$. If $g = |dx|^2$, then

$$\bar{T}(g,v) = v^{-1}v_{ij} - \frac{1}{2}v^{-2}|\nabla v|^2\delta_{ij}. \qquad (7)$$

Thus in this case we have $Y_k^m(|dx|^2, v) = v^{-2k}S_k(v)$ and for locally conformally flat manifolds, the operator $Y_k^m(g,v)$ is conformally invariant by Proposition 1.1 :

$$Y_k^m(e^{2w}|dx|^2, e^w v) = e^{-2kw}Y_k^m(|dx|^2, v). \qquad (8)$$

Our Corollary 4.1 will show $Y_k^m(|dx|^2, v)$ is a measure when v is just continuous.

In this paper, we will introduce the σ_k-Yamabe measure $S_k(v)$ and prove the weak continuity of the associated σ_k-Yamabe measure with respect to convergence in measure of v.

We recall that the Gårding's cone is defined as

$$\Gamma_k = \{\lambda\in\mathbb{R}^n : \sigma_i(\lambda)\geq 0, \forall 1\leq i\leq k\}. \qquad (9)$$

Define

$$
\begin{aligned}
&S_k^{ij} := \frac{\partial S_k}{\partial A_{ij}},\\
&\tilde{\Phi}^k(\Omega) := \Big\{u \in C^2(\Omega) : u > 0, \lambda\{A_{ij}(v)\} \in \Gamma_k\Big\},\\
&\Phi^k(\Omega) := \Big\{u \in L^1_{loc}(\Omega) : u > 0, \text{and there exists a sequence } \{u^{(m)}\} \in \tilde{\Phi}^k(\Omega)\ ,\\
&\qquad\qquad \text{such that } u^{(m)} \text{ converges to } u \text{ in } L^1_{loc}\Big\},
\end{aligned}
\tag{10}
$$

where Ω is a bounded domain in R^n. Corresponding to Theorem 1.1 in Trudinger-Wang [21], we shall establish the following characterization of σ_k-Yamabe measure on $\Phi^k(\Omega)$.

Theorem 1.1. *For any $u \in \Phi^k(\Omega)$, there exists a Borel measure $\mu_k[u]$ in Ω such that*

- *$\mu_k[u] = S_k(u)$ for $u \in C^2(\Omega)$.*
- *If $\{u^{(m)}\}$ is a sequence in $\Phi^k(\Omega)$ converging locally in measure to a function $u \in \Phi^k(\Omega)$, then the corresponding measure $\mu_k[u^{(m)}] \to \mu_k[u]$ weakly.*

Note that from well known properties of subharmonic functions and the relation between $S_m(v)$ and $v^m\sigma_m(\lambda\{v_{ij}\})$ in Lemma 2.3, as Trudinger-Wang [21] *mentioned, we have the inclusion, $\Phi^k(\Omega) \subset \Phi^1(\Omega) \subset L^1_{loc}(\Omega)$ and convergence in measure is equivalent to convergence in $L^1_{loc}(\Omega)$.*

In order to prove the above theorem, we introduce a monotonicity formula with respect to σ_k-Yamabe operator.

Theorem 1.2. *Let $u, v \in \Phi^2(\Omega) \cap C^2(\bar{\Omega})$ satisfy $u \geq v$ in Ω and $u = v$ on $\partial\Omega$. Then there exists $\Lambda < 0$, such thar for any $\alpha \geq \Lambda$, it follows that :*

$$
\int_\Omega \sum_{l=0}^{k} b_l u^\alpha |\nabla u|^{2l} S_{k-l}(u)dx \leq \int_\Omega \sum_{l=0}^{k} b_l v^\alpha |\nabla v|^{2l} S_{k-l}(v)dx, \tag{11}
$$

with

$$
\begin{cases}
b_0 = k, & (12)\\
b_l = \Big[(k+l)\alpha + kn + 2kl\Big] \cdot \dfrac{(\alpha+n)(\alpha+n+1)\cdots(\alpha+n+l-1)}{2^l l!(\alpha+n)}, \text{ for } l = 1, \cdots, k. & (13)
\end{cases}
$$

This monotonicity formula is the most important tool to obtain Theorem 1.1. It is different from the usual monotonicity formulas on k-Hessian measure by Trudinger-Wang in [20], Dai-Wang-Zhou in [6]. In some sense, it is similar to the monotonicity formula for k-convex functions on Heisenberg group by Trudinger-Zhang [23].

[16] The plan of this paper is as follows. In the next section we give various properties of σ_k-Yamabe operator, especially we obtain the relation between $S_m(v)$ and $v^m\sigma_m(\lambda\{v_{ij}\})$ in Lemma 2.3. In Section 3, we first use the special divergence structure with respect to the σ_k-Yamabe operator to get a differential identity in Lemma 3.1, which will be also used to get the upper bound estimates via Moser iteration in Section 5. Then we prove the monotonicity formulas i.e. Theorem 1.2. The new idea on this step is that we introduce the combination of $\psi(v) = \psi_k^\alpha(v) := \sum_{l=0}^{k} b_l v^\alpha |\nabla v|^{2l} S_{k-l}(v)$ to replace the usual term $\sigma_m(\lambda\{v_{ij}\})$ in Trudinger - Wang in [20]. Then we give comparison principle as a consequence of the monotonicity

formulas. In Section 4, we complete the proof of Theorem 1.1 for $k > \frac{n}{2}$. In this case we use the Hölder estimate Theorem 2.7 and integral estimate Theorem 3.1 in Trudinger-Wang [20], and we use the monotonicity formulas Theorem 1.2. The proof of the weak continuity result, Theorem 1.1 is then completed for $k > \frac{n}{2}$. In Section 5, we first obtain the interior L^∞ bound with related to the σ_k-Yamabe operator for $1 \le k \le \frac{n}{2}$ via Moser iteration. As a consequence we get a local uniform integral estimates from the differential identity Lemma 3.1. Finally, in Section 6, we use the integral estimates to complete the proof the weak continuity result, Theorem 1.1 for $1 \le k \le \frac{n}{2}$. In this step we also follow that idea from Trudinger-Wang in [20].

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2. Properties of σ_k-Yamabe operator

In this section, we will give some fundamental properties of σ_k-Yamabe operator which will be widely used in this paper. The first lemma appeared in Li-Nguyen-Wang [15], for completeness we contain its proof. In this paper we always use the notation $S_k := S_k(v) := \sigma_k(\lambda\{A_{ij}(v)\})$ and $\sigma_k := \sigma_k(v) := \sigma_k(\lambda\{v_{ij}\})$.

Lemma 2.1. [15] *If $u, v \in \tilde{\Phi}^k(\Omega)$, then $u + v \in \tilde{\Phi}^k(\Omega)$.*

Proof of Lemma 2.1. Define $w := u + v$, then

$$\begin{aligned}
A_{ij}(w) &= (u+v)(u_{ij}+v_{ij}) - \frac{1}{2}|\nabla u + \nabla v|^2\delta_{ij} \\
&= (u+v)(u^{-1}A_{ij}(u) + v^{-1}A_{ij}(v)) + \frac{1}{2}(u+v)(u^{-1}|\nabla u|^2\delta_{ij} + v^{-1}|\nabla v|^2\delta_{ij}) \\
&- \frac{1}{2}|\nabla u + \nabla v|^2\delta_{ij} \\
&= (u+v)u^{-1}A_{ij}(u) + (u+v)v^{-1}A_{ij}(v) + \frac{1}{2}\delta_{ij}(vu^{-1}|\nabla u|^2 + uv^{-1}|\nabla v|^2 - 2\nabla u \cdot \nabla v).
\end{aligned}$$

Since

$$vu^{-1}|\nabla u|^2 + uv^{-1}|\nabla v|^2 - 2\nabla u \cdot \nabla v \ge 0, \tag{14}$$

and Γ_k is convex , we get that

$$\lambda(A_{ij}(w)) \in \Gamma_k. \tag{15}$$

□

By González [8, 9, 10], we know that

Lemma 2.2. *If $v \in \tilde{\Phi}^k(\Omega)$, then for any $0 \le l \le k$, the matrices (S_l^{ij}) defined in (10) is nonnegative-definite and*

$$\sum_j \partial_j S_{l+1}^{ij} = -(n-l)S_l v_i v^{-1} + n\sum_j S_{l+1}^{ij} v_j v^{-1}. \tag{16}$$

Here we give a different proof from the one in [8].

Proof of Lemma 2.2. We prove the conclusion by induction. If $l=0$, since $S_1^{ij}=\delta_{ij}$ and $S_0=1$, the conclusion (16) holds. Then we suppose (16) holds for any $0\le l\le m$ with $0\le m\le k-1$. At this time, we have

$$\begin{aligned}
\sum_j \partial_j S_{m+1}^{ij} &= \sum_j \partial_j (S_m\delta_{ij} - S_m^{jq}A_{qi})\\
&= S_m^{jq}\partial_i A_{jq} - (\partial_j S_m^{jq}\cdot A_{qi} + S_m^{jq}\cdot\partial_j A_{qi})\\
&= S_m^{jq}(\partial_i A_{jq} - \partial_j A_{qi}) - \partial_j S_m^{jq}\cdot A_{qi}\\
&= S_m^{jq}(v_iv_{jq} - v_{pi}v_p\delta_{jq} - v_jv_{qi} + v_{pj}v_p\delta_{iq}) - \partial_j S_m^{jq}\cdot A_{qi}\\
&= S_m^{jq}(v_iv_{jq} - v_jv_{qi}) - (n-m+1)S_{m-1}v_{pi}v_p + S_m^{ji}v_{pj}v_p - \partial_j S_m^{jq}\cdot A_{qi}\\
&= S_m^{jq}\left[v^{-1}v_i\left(A_{qj}+\frac{1}{2}|\nabla v|^2\delta_{qj}\right) - v_jv_{qi}\right] - (n-m+1)S_{m-1}v_{pi}v_p + S_m^{ji}v_{pj}v_p\\
&+ \left[(n-m+1)S_{m-1}v_qv^{-1} - nS_m^{jq}v_jv^{-1}\right]A_{qi}\\
&= mv^{-1}S_mv_i + \frac{1}{2}(n-m+1)v^{-1}|\nabla v|^2S_{m-1}v_i - S_m^{jq}v_{qi}v_j\\
&- (n-m+1)S_{m-1}v_{pi}v_p + S_m^{ji}v_{pj}v_p + (n-m+1)S_{m-1}v_qv^{-1}A_{qi} - nv^{-1}S_m^{jq}v_jA_{qi}\\
&= mv^{-1}S_mv_i - nv^{-1}S_m^{jq}v_jA_{qi}\\
&= -(n-m)S_mv_iv^{-1} + nS_{m+1}^{ij}v_jv^{-1}, \qquad (17)
\end{aligned}$$

where we used the fact that the matrix $(S_l^{ij})_{i,j}$ is symmetric for any $1\le l\le k$ which means that

$$\sum_j S_{m+1}^{ji}v_j = \sum_p S_{m+1}^{ip}v_p, \qquad (18)$$

$$\Rightarrow \sum_j\left(S_m\delta_{ji} - \sum_q S_m^{jq}A_{qi}\right)v_j = \sum_p\left(S_m\delta_{ip} - \sum_j S_m^{ij}A_{jp}\right)v_p, \qquad (19)$$

$$\Rightarrow \sum_{j,q} S_m^{jq}A_{qi}v_j = \sum_{j,p} S_m^{ij}A_{jp}v_p, \qquad (20)$$

$$\Rightarrow \sum_{j,q} S_m^{jq}v_{qi}v_j = \sum_{j,p} S_m^{ij}v_{jp}v_p. \qquad (21)$$

□

Let $\lambda=(\lambda_1,\cdots,\lambda_n)$ be the eigenvalue vector of the matrix (v_{ij}) and $b:=\frac{1}{2}|\nabla v|^2$. Define

$$\sigma_k := \sigma_k(v_{ij}) = \sum_{1\le i_1<\cdots<i_k\le n}\lambda_{i_1}\cdots\lambda_{i_k}. \qquad (22)$$

Next we will show $\lambda(v_{ij})\in\Gamma_k$ if $\lambda(A_{ij}(v))\in\Gamma_k$.

Lemma 2.3. *If $v\in\tilde\Phi^k(\Omega)$, then we have*

$$S_k(v) = v^k\sigma_k - \sum_{l=1}^{k}\frac{(n-k+l)!}{l!(n-k)!}S_{k-l}b^l,\ \textit{with}\ b:=\frac{1}{2}|\nabla v|^2. \qquad (23)$$

Therefore there exists $C = C(n,l,q) > 0$, such that

$$v^l\sigma_l(v) \geq C|\nabla v|^{2q}S_{l-q}, \text{ with } 1 \leq q < l \leq k, \tag{24}$$

and

$$S_m \leq v^m\sigma_m, \text{ with } 1 \leq m \leq k. \tag{25}$$

Proof of Lemma 2.3 . Let $\mu_i = v\lambda_i - b$. Then $\lambda_i = v^{-1}(\mu_i + b)$ and

$$\begin{aligned} v^k\sigma_k &= \sigma_k(\mu_1 + b, \cdots, \mu_n + b) \\ &= S_k(v) + \sum_{l=1}^{k} \frac{(n-k+l)!}{l!(n-k)!} S_{k-l}b^l. \end{aligned} \tag{26}$$

□

3. Monotonicity formula

In this section, inspired by Lemma 3.3 in González [8], we give a special divergence structure with respect to the σ_k-Yamabe operator.

Lemma 3.1. *For any $\alpha \in \mathbb{R}$, we have*

$$\sum_{l=0}^{k-1} a_l (v^{\alpha+1}|\nabla v|^{2l}S_{k-l}^{ij}v_i)_j + \sum_{l=0}^{k} b_l v^\alpha |\nabla v|^{2l}S_{k-l} = 0, \tag{27}$$

with

$$\begin{cases} a_0 = -1, & (28) \\ b_0 = k, & (29) \\ a_l = -\dfrac{(\alpha+n+1)\cdots(\alpha+n+l)}{2^l \cdot l!}, \text{ for } l = 1, \cdots, k-1, & (30) \\ b_l = \Big[(k+l)\alpha + kn + 2kl\Big] \cdot \dfrac{(\alpha+n)(\alpha+n+1)\cdots(\alpha+n+l-1)}{2^l l!(\alpha+n)}, \text{ for } l = 1, \cdots, k. & (31) \end{cases}$$

Proof. Writing $S_{-1} = 0$. Using Lemma 2.2 and direct computation, we have

$$\begin{aligned} &(v^{\alpha+1}|\nabla v|^{2l}S_{k-l}^{ij}v_i)_j \\ =&(k+l)v^\alpha|\nabla v|^{2l}S_{k-l} + (\alpha+n+l+1)v^\alpha|\nabla v|^{2l}S_{k-l}^{ij}v_iv_j - 2lv^\alpha|\nabla v|^{2l-2}S_{k-l+1}^{ij}v_iv_j \\ &- \frac{1}{2}(n+l+1-k)v^\alpha|\nabla v|^{2l+2}S_{k-l-1}. \end{aligned} \tag{32}$$

Combining (28), (29),(30), (31) and (32), we get (27).

□

Remark 3.1. *We find that when*

$$\alpha \geq \max\left\{-n-1, -\frac{kn+2kl}{k+l}\right\} =: \Lambda, \tag{33}$$

it holds that $b_l \geq 0$ and $a_l \leq 0$. This Λ is the constant in Theorem 1.2.

From now on, let us define

$$\psi(w)=\psi_k^\alpha(w):=\sum_{l=0}^{k}B_l w^\alpha|\nabla w|^{2l}S_{k-l}(w), \tag{34}$$

where $B_l\in\mathbb{R}$ and $w\in\Phi^k(\Omega)$.

Lemma 3.2. *Let $B_l=b_l$ for $l=0,\cdots,k$ in* (29) *and* (31)*. For any $u,v\in\Phi^k(\Omega)\cap C^2(\Omega)$ and $\alpha\in\mathbb{R}$, we have*

$$\begin{aligned}\frac{\partial}{\partial t}\psi(w)=&\sum_{l=0}^{k-1}b_l\Big[w^{\alpha+1}|\nabla w|^{2l}S_{k-l}^{ij}(v-u)_i\Big]_j\\&+\sum_{l=0}^{k-1}\Big[2(l+1)b_{l+1}-(\alpha+n+l+1)b_l\Big]\Big[w^\alpha|\nabla w|^{2l}S_{k-l}^{ij}w_i(v-u)\Big]_j,\end{aligned}\tag{35}$$

where

$$w=w(x,t):=(1-t)u(x)+tv(x). \tag{36}$$

Proof of Lemma 3.2. By direct computation, we get

$$\begin{aligned}&\frac{\partial}{\partial t}(w^\alpha|\nabla w|^{2l}S_{k-l})\\=&(\alpha+k-l)w^{\alpha-1}|\nabla w|^{2l}S_{k-l}(v-u)+\frac12(n+l+1-k)w^{\alpha-1}|\nabla w|^{2l+2}S_{k-l-1}(v-u)\\&+2lw^\alpha|\nabla w|^{2l-2}S_{k-l}\nabla w\cdot\nabla(v-u)-(n+l+1-k)w^\alpha|\nabla w|^{2l}S_{k-l-1}\nabla w\cdot\nabla(v-u)\\&+w^{\alpha+1}|\nabla w|^{2l}S_{k-l}^{ij}(v-u)_{ij}.\end{aligned}\tag{37}$$

Firstly we deal with the term $w^{\alpha+1}|\nabla w|^{2l}S_{k-l}^{ij}(v-u)_{ij}$ in above formula.

$$\begin{aligned}&w^{\alpha+1}|\nabla w|^{2l}S_{k-l}^{ij}(v-u)_{ij}\\=&\Big[w^{\alpha+1}|\nabla w|^{2l}S_{k-l}^{ij}(v-u)_i\Big]_j-(\alpha+1)w^\alpha|\nabla w|^{2l}S_{k-l}^{ij}w_j(v-u)_i\\&-2lw^{\alpha+1}|\nabla w|^{2l-2}S_{k-l}^{ij}w_{jm}w_m(v-u)_i\\&-w^{\alpha+1}|\nabla w|^{2l}(v-u)_i\cdot\Big[-(n-k+l+1)w^{-1}S_{k-l-1}w_i+nw^{-1}S_{k-l}^{ij}w_j\Big]\\=&\Big[w^{\alpha+1}|\nabla w|^{2l}S_{k-l}^{ij}(v-u)_i\Big]_j-(\alpha+n+1)w^\alpha|\nabla w|^{2l}S_{k-l}^{ij}w_j(v-u)_i\\&-2lw^{\alpha+1}|\nabla w|^{2l-2}S_{k-l}^{ij}w_{jm}w_m(v-u)_i+(n-k+l+1)w^\alpha|\nabla w|^{2l}S_{k-l-1}\nabla w\cdot\nabla(v-u).\end{aligned}$$

Noting that

$$wS_{k-l}^{ij}w_{jm}=S_{k-l}^{ij}\left(A_{jm}+\frac12|\nabla w|^2\delta_{jm}\right)=S_{k-l}\delta_{im}-S_{k-l+1}^{im}+\frac12|\nabla w|^2S_{k-l}^{im}, \tag{38}$$

we obtain

$$w^{\alpha+1}|\nabla w|^{2l}S_{k-l}^{ij}(v-u)_{ij}$$

$$
\begin{aligned}
=&\Big[w^{\alpha+1}|\nabla w|^{2l}S^{ij}_{k-l}(v-u)_i\Big]_j-(\alpha+n+1)w^{\alpha}|\nabla w|^{2l}S^{ij}_{k-l}w_j(v-u)_i\\
&-2lw^{\alpha}|\nabla w|^{2l-2}w_j(v-u)_i\left(S_{k-l}\delta_{ij}-S^{ij}_{k-l+1}+\frac{1}{2}|\nabla w|^2S^{ij}_{k-l}\right)\\
&+(n-k+l+1)w^{\alpha}|\nabla w|^{2l}S_{k-l-1}\nabla w\cdot\nabla(v-u)\\
=&\Big[w^{\alpha+1}|\nabla w|^{2l}S^{ij}_{k-l}(v-u)_i\Big]_j\\
&+2lw^{\alpha}|\nabla w|^{2l-2}S^{ij}_{k-l+1}w_j(v-u)_i-(\alpha+n+l+1)w^{\alpha}|\nabla w|^{2l}S^{ij}_{k-l}w_j(v-u)_i\\
&-2lw^{\alpha}|\nabla w|^{2l-2}S_{k-l}\nabla w\cdot\nabla(v-u)+(n-k+l+1)w^{\alpha}|\nabla w|^{2l}S_{k-l-1}\nabla w\cdot\nabla(v-u).
\end{aligned}
\tag{39}
$$

Secondly we deal with the term $w^{\alpha}|\nabla w|^{2l}S^{ij}_{k-l}w_i(v-u)_j$. Using (32), we have

$$
\begin{aligned}
&w^{\alpha}|\nabla w|^{2l}S^{ij}_{k-l}w_i(v-u)_j\\
=&\Big[w^{\alpha}|\nabla w|^{2l}S^{ij}_{k-l}w_i(v-u)\Big]_j-(k+l)w^{\alpha-1}|\nabla w|^{2l}S_{k-l}(v-u)\\
&-(\alpha+n+l)w^{\alpha-1}|\nabla w|^{2l}S^{ij}_{k-l}w_iw_j(v-u)\\
&+2lw^{\alpha-1}|\nabla w|^{2l-2}S^{ij}_{k-l+1}w_iw_j(v-u)+\frac{1}{2}(n+l+1-k)w^{\alpha-1}|\nabla w|^{2l+2}S_{k-l-1}(v-u).
\end{aligned}
\tag{40}
$$

Substituting (39) and (40) into (37), we get for any $0\le l\le k$,

$$
\begin{aligned}
&\frac{\partial}{\partial t}(w^{\alpha}|\nabla w|^{2l}S_{k-l})\\
=&\Big[w^{\alpha+1}|\nabla w|^{2l}S^{ij}_{k-l}(v-u)_i\Big]_j+2l\Big[w^{\alpha}|\nabla w|^{2l-2}S^{ij}_{k-l+1}w_i(v-u)\Big]_j\\
&-(\alpha+n+l+1)\Big[w^{\alpha}|\nabla w|^{2l}S^{ij}_{k-l}w_i(v-u)\Big]_j\\
&+\Big[(\alpha+k-l)+l(n+l-k)+(\alpha+n+l+1)(k+l)\Big]w^{\alpha-1}|\nabla w|^{2l}S_{k-l}(v-u)\\
&-2l(k+l-1)w^{\alpha-1}|\nabla w|^{2l-2}S_{k-l+1}(v-u)\\
&-\frac{1}{2}(\alpha+n+l)(n+l+1-k)w^{\alpha-1}|\nabla w|^{2l+2}S_{k-l-1}(v-u)\\
&-4l(\alpha+n+l)w^{\alpha-1}|\nabla w|^{2l-2}S^{ij}_{k-l+1}w_iw_j(v-u)\\
&+2l(2l-2)w^{\alpha-1}|\nabla w|^{2l-4}S^{ij}_{k-l+2}w_iw_j(v-u)\\
&+(\alpha+n+l+1)(\alpha+n+l)w^{\alpha-1}|\nabla w|^{2l}S^{ij}_{k-l}w_iw_j(v-u).
\end{aligned}
$$

At last we obtain

$$
\frac{\partial}{\partial t}\psi=\sum_{l=0}^{k}B_l\Big[w^{\alpha+1}|\nabla w|^{2l}S^{ij}_{k-l}(v-u)_i\Big]_j
$$

$$
\begin{aligned}
&+\sum_{l=0}^{k-1}\Big[2(l+1)B_{l+1}-(\alpha+n+l+1)B_l\Big]\Big[w^{\alpha}|\nabla w|^{2l}S^{ij}_{k-l}w_i(v-u)\Big]_j\\
&+\Big\{B_0\Big[\alpha+k+(\alpha+n+1)k\Big]-2kB_1\Big\}w^{\alpha-1}S_k(v-u)\\
&+\Bigg\{B_k\Big[\alpha+kn-2k(\alpha+n+k-1)\Big]+B_{k-1}(\alpha+\frac{1}{2}n+k)(\alpha+n+k-1)\Bigg\}w^{\alpha-1}|\nabla w|^{2k}(v-u)\\
&+\sum_{l=1}^{k-1}\Bigg\{B_l\Big[\alpha+k-l+l(n+l-k)+(\alpha+n+l+1)(k+l)\Big]-2(l+1)(k+l)B_{l+1}\\
&-\frac{1}{2}(\alpha+n+l-1)(n+l-k)B_{l-1}\Bigg\}w^{\alpha-1}|\nabla w|^{2l}S_{k-l}(v-u)\\
&+\sum_{l=0}^{k-2}\Bigg[B_l(\alpha+n+l+1)(\alpha+n+l)-4(l+1)(\alpha+n+l+1)B_{l+1}\\
&+4(l+2)(l+1)B_{l+2}\Bigg]w^{\alpha-1}|\nabla w|^{2l}S^{ij}_{k-l}w_iw_j(v-u).
\end{aligned}
\tag{41}
$$

Letting $B_l=b_l$ for $l=0,\cdots,k$ in (29) and (31) , we get

$$
\begin{aligned}
\frac{\partial}{\partial t}\psi=&\sum_{l=0}^{k-1}b_l\Big[w^{\alpha+1}|\nabla w|^{2l}S^{ij}_{k-l}(v-u)_i\Big]_j\\
&+\sum_{l=0}^{k-1}\Big[2(l+1)b_{l+1}-(\alpha+n+l+1)b_l\Big]\Big[w^{\alpha}|\nabla w|^{2l}S^{ij}_{k-l}w_i(v-u)\Big]_j.
\end{aligned}
\tag{42}
$$

□

Now we can get the following monotonicity formula:

Proof of Theorem 1.2. Using Lemma 3.2, Remark 3.1 and letting $B_l=b_l$ for $l=0,\cdots,k$, we get when

$$
\frac{\partial}{\partial t}\int_\Omega\psi dx=\int_{\partial\Omega}\sum_{l=0}^{k-1}b_lw^{\alpha+1}|\nabla w|^{2l}S^{ij}_{k-l}(v-u)_i\nu_jdS\geq 0,
\tag{43}
$$

where $\nu(x)=(\nu_1,\cdots,\nu_n)$ is the outer normal vector of $\partial\Omega$ at $x\in\partial\Omega$. □

Using monotonicity formula and (41) , we can get one comparison principle directly. But the proof is different from the Hessian measure case in [20] and [21].

Theorem 3.1. *Let $B_0 = k$ and $B_l \geq 0$ for $1 \leq l \leq k$. Suppose $\{B_i\}_{i=0}^k$ satisfies:*

$$\begin{cases} B_0\Big[\alpha + k + (\alpha+n+1)k\Big] - 2kB_1 < 0, & (44)\\ B_k\Big[\alpha + kn - 2k(\alpha+n+k-1)\Big] + B_{k-1}(\alpha + \frac{1}{2}n + k)(\alpha+n+k-1) \leq 0, & (45)\\ B_l\Big[\alpha + k - l + l(n+l-k) + (\alpha+n+l+1)(k+l)\Big] - 2(l+1)(k+l)B_{l+1} & \\ -\frac{1}{2}(\alpha+n+l-1)(n+l-k)B_{l-1} \leq 0 \quad \textit{for } 1 \leq l \leq k-1, & (46)\\ B_l(\alpha+n+l+1)(\alpha+n+l) - 4(l+1)(\alpha+n+l+1)B_{l+1} & \\ +4(l+2)(l+1)B_{l+2} \leq 0 \quad \textit{for } 0 \leq l \leq k-2. & (47)\end{cases}$$

If $u, v \in C^2(\bar{\Omega}) \cap \Phi^k(\Omega)$ satisfy

$$\begin{cases} \sum_{l=0}^{k} B_l v^\alpha |\nabla v|^{2l} S_{k-l}(v) \leq \sum_{l=0}^{k} B_l u^\alpha |\nabla u|^{2l} S_{k-l}(u) \textit{ in } \Omega, & (48)\\ v \geq u \textit{ on } \partial\Omega. & (49)\end{cases}$$

Then $v \geq u$ in Ω.

Proof of Theorem 3.1. Define $\Omega_1 := \{x \in \Omega : u(x) > v(x)\}$ and $\psi(u) = \sum_{l=0}^k B_l u^\alpha |\nabla u|^{2l} S_{k-l}(u)$. Suppose the set Ω_1 is not empty. Then we know that

$$\begin{cases} u > v \text{ in } \Omega_1, & (50)\\ v = u \text{ on } \partial\Omega_1. & (51)\end{cases}$$

Let $w := (1-t)u + tv$. Then combining (41) and (48) and using conditions (45), (46), (47), we get

$$\begin{aligned} 0 &\geq \int_{\Omega_1} \psi(v)dx - \int_{\Omega_1} \psi(u)dx \\ &\geq \int_0^1 \frac{\partial}{\partial t}\int_{\Omega_1} \psi(w)dxdt \\ &\geq \int_0^1 \int_{\Omega_1} \sum_{l=0}^{k} B_l \Big[w^{\alpha+1}|\nabla w|^{2l} S_{k-l}^{ij}(v-u)_i\Big]_j \\ &+ \sum_{l=0}^{k-1}\Big[2(l+1)B_{l+1} - (\alpha+n+l+1)B_l\Big]\Big[w^\alpha |\nabla w|^{2l} S_{k-l}^{ij} w_i (v-u)\Big]_j \\ &+ \Big\{B_0\Big[\alpha + k + (\alpha+n+1)k\Big] - 2kB_1\Big\}(v-u)w^{\alpha-1}S_k dxdt \\ &= \int_0^1 \int_{\partial\Omega_1} \sum_{l=0}^{k} B_l w^{\alpha+1}|\nabla w|^{2l} S_{k-l}^{ij}(v-u)_i \nu_j dSdt \\ &+ \int_0^1 \int_{\Omega_1} \Big\{B_0\Big[\alpha + k + (\alpha+n+1)k\Big] - 2kB_1\Big\}(v-u)w^{\alpha-1}S_k dxdt \end{aligned}$$

$$\geq \int_0^1 \int_{\Omega_1} \Big\{ B_0 \Big[\alpha + k + (\alpha + n + 1)k \Big] - 2kB_1 \Big\} (v-u) w^{\alpha-1} S_k dxdt.$$

By condition (44) , we obtain

$$\int_0^1 \int_{\Omega_1} (v-u) w^{\alpha-1} S_k dxdt = 0. \tag{52}$$

Since $w, u - v > 0$ in Ω, we deduce that $S_k(w) = 0$ in Ω_1. By the proof of Lemma 2.1 and the fact that $\sigma_k^{\frac{1}{k}}$ is concave, we get

$$vu^{-1}|\nabla u|^2 + uv^{-1}|\nabla v|^2 - 2\nabla u \cdot \nabla v = 0 \text{ in } \Omega_1, \tag{53}$$
$$\Rightarrow |\nabla \log u - \nabla \log v| = 0 \text{ in } \Omega_1, \tag{54}$$
$$\Rightarrow u \equiv v \text{ in } \Omega_1, \tag{55}$$

which is impossible. Therefore the set Ω_1 is empty. □

Such $\{B_i\}_{i=0}^k$ exists for some special α. For example, letting $B_0 = k$, $B_l = 0$ for $1 \leq l \leq k$, $k < \frac{n}{2}$ and $\alpha = -n$, then the conditions (44) - (47) are all satisfied.

Corollary 3.1. *If $n > 2k$ and $u, v \in C^2(\bar{\Omega}) \cap \Phi^k(\Omega)$ satisfy*

$$\begin{cases} v^{-n} S_k(v) \leq u^{-n} S_k(u) \textit{ in } \Omega, & (56) \\ v \geq u \textit{ on } \partial\Omega. & (57) \end{cases}$$

Then $v \geq u$ in Ω.

4. Locally uniform convergence for $n < 2k$

In this section, we will prove Theorem 1.1 for the case $n < 2k$. Firstly by Lemma 2.3 and Theorem 2.7 in [21], we get:

Lemma 4.1. *For $n < 2k$, we have $\Phi^k(\Omega) \subset C^{0,\alpha}(\Omega)$ for $\alpha = 2 - \frac{n}{k}$, and for any $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega$ with $u \in \Phi^k(\Omega)$,*

$$\|u\|_{C^{0,\alpha}(\Omega_2)} \leq C \int_{\Omega_1} u dx. \tag{58}$$

So in this case, $u^{(m)}$ converges to u in $L^1_{loc}(\Omega)$ means that $u^{(m)}$ converges to u in $C^0_{loc}(\Omega)$. Then we have the following result:

Theorem 4.1. *For any $u \in \Phi^k(\Omega)$, there exists a Borel measure $\mu_k[u]$ in Ω such that*

- *$\mu_{k,\alpha}[u] = u^\alpha S_k(u)$ for $u \in C^2(\Omega)$.*
- *If $\{u^{(m)}\}$ is a sequence in $\Phi^k(\Omega)$ converging to u locally uniformly in Ω, then the corresponding measure $\mu_k[u^{(m)}] \to \mu_k[u]$ weakly.*

Recalling the identity (23) and Lemma 2.3, together with the Lemma 2.2 in [20] and Theorem 3.1 in [21], we get:

Lemma 4.2. *Let $u \in \Phi^k(\Omega) \cap C^2(\Omega)$ satisfy that $|u| \le M$ in $\Omega_1 \subset\subset \Omega$. Then*

$$\int_{\Omega_2} \sum_{l=0}^{k} |\nabla u|^{2l} S_{k-l} dx \le C(osc_{\Omega_1} u)^{2k}, \text{ if } \Omega_2 \subset\subset \Omega_1. \tag{59}$$

Next, let us introduce a special convex function in $B_2 := \{x \in \mathbb{R}^n : |x| \le 2\}$. Define

$$\begin{aligned} &\eta = C, \text{ if } |x| \le 1, \\ &\eta = (r-1)^3 + C, \text{ if } 1 \le |x| \le 2, \end{aligned} \tag{60}$$

where $C > 0$ is a constant, we know $\eta \in C^2(\mathbb{R}^n)$.

Lemma 4.3. *There exists C_0 depending only on n, such that the matrix*

$$A_{ij}(\eta) = \eta\eta_{ij} - \frac{1}{2}|\nabla\eta|^2\delta_{ij}, \tag{61}$$

is nonnegative-definite in B_2 for any $C > C_0$.

Proof of Lemma 4.3. Recall that

$$\begin{aligned} r_i &= \frac{x_i}{r}, \\ r_{ij} &= \frac{1}{r}\delta_{ij} - \frac{x_i x_j}{r^3}, \end{aligned}$$

with $r = |x|$. Then we know that when $r > 1$

$$\begin{aligned} \eta_i &= 3(r-1)^2\frac{x_i}{r}, \\ \eta_{ij} &= 6(r-1)\frac{x_i x_j}{r^2} + 3(r-1)^2\left(\frac{1}{r}\delta_{ij} - \frac{x_i x_j}{r^3}\right) \\ &= \frac{3(r-1)(r+1)}{r^3}x_i x_j + 3(r-1)^2\frac{1}{r}\delta_{ij}, \end{aligned}$$

and

$$\begin{aligned} A_{ij}(\eta) &= \Big[(r-1)^3 + C\Big] \cdot \Big[\frac{3(r-1)(r+1)}{r^3}x_i x_j + 3(r-1)^2\frac{1}{r}\delta_{ij}\Big] - \frac{9}{2}(r-1)^4\delta_{ij} \\ &= \Big[(r-1)^3 + C\Big] \cdot \frac{3(r-1)(r+1)}{r^3}x_i x_j + \Big[\frac{(r-1)^3 + C}{r} - \frac{3}{2}(r-1)^2\Big]3(r-1)^2\delta_{ij}. \end{aligned}$$

So if we let $C > 3$, then the matrix $(A_{ij}(\eta))$ is always nonnegative-definite in B_2. $\square$

Now we can give the proof for $n < 2k$:

Proof of Theorem 4.1. We divide this proof into two steps.

- Step1: Firstly we can follow the idea in [20] and [24] to prove the weak convergence of ψ_k^α, which is defined by (34). Suppose $u \in \Phi^k(\Omega)$, $\{u^{(m)}\} \subset \Phi^k(\Omega) \cap C^2(\Omega)$ and $u^{(m)} \to u$ locally uniformly in Ω. By Lemma 4.2 and letting $B_l = b_l$, the integrals

$$\int_{\Omega'} \psi_k^\alpha(u^{(m)})dx \tag{62}$$

are uniformly bounded for any subdomain $\Omega' \subset\subset \Omega$ (the bound also depends on α). Hence there is a subsequnce $\{\psi_k^\alpha(u^{(m_p)})\}$ that converges weakly to a Borel measure $\mu_k^\alpha[u]$. Firstly we need to prove that the measure $\mu_k^\alpha[u]$ is uniquely determined by the function u. Assume there exist two sequences $\{u^{(m)}\}$, $\{v^{(m)}\} \subset \Phi^k(\Omega)\cap C^2(\Omega)$ which both converge to u locally uniformly, but the corresponding sequences $\{\psi_k^\alpha(u^{(m)})\}$ and $\{\psi_k^\alpha(v^{(m)})\}$ weakly converge to Borel measure $\tilde{\mu}_1$ and $\tilde{\mu}_2$, respectively. Let $B_R = B_R(x_0) \Subset \Omega$ and fix some $\sigma \in (0,1)$. Define

$$\begin{aligned} &\eta = \tilde{C}, \text{ if } |x-x_0| \le \sigma R, \\ &\eta = \frac{(r-\sigma R)^3}{(R-\sigma R)^3} + \tilde{C}, \text{ if } \sigma R \le |x| \le R. \end{aligned} \tag{63}$$

Then by Lemma 4.3, we know that when $\tilde{C}$ is large enough, then the matrix

$$A_{ij}(\eta) = \eta\eta_{ij} - \frac{1}{2}|\nabla\eta|^2\delta_{ij}, \tag{64}$$

is nonnegative-definite in B_R. So $\eta \in \Phi^k(B_R)$. For fixed $\varepsilon > 0$, it then follows from the uniform convergence of $\{u^{(m)}\}$ and $\{v^{(m)}\}$ on $\bar{B}_R$, that

$$-\frac{\varepsilon}{2} \le u^{(m)} - v^{(m)} \le \frac{\varepsilon}{2} \text{ in } \bar{B}_R, \tag{65}$$

for sufficiently large m. Hence

$$u^{(m)} + \varepsilon\left(\frac{1}{2} + \tilde{C}\right) \le v^{(m)} + \varepsilon\eta \text{ on } \partial\bar{B}_R. \tag{66}$$

Define

$$\Omega_m := \left\{x \in B_R : u^{(m)} + \varepsilon\left(\frac{1}{2} + \tilde{C}\right) > v^{(m)} + \varepsilon\eta\right\}. \tag{67}$$

Without loss of generality, we may assume that $\partial\Omega_m$ is sufficiently smooth so that from Lemma 1.2, when $\alpha \ge 0$,

$$\int_\Omega \psi_k^\alpha\left(u^{(m)} + \varepsilon\left(\frac{1}{2} + \tilde{C}\right)\right)dx \le \int_\Omega \psi_k^\alpha(v^{(m)} + \varepsilon\eta)dx. \tag{68}$$

Using $\alpha \ge 0$ and Lemma 2.3, and expanding $S_k\left(u^m + \varepsilon\left(\frac{1}{2} + \tilde{C}\right)\right)$ as the sum of mixed k-Hessian operators, we get that

$$\int_\Omega \psi_k^\alpha(u^{(m)})dx \le \int_\Omega \psi_k^\alpha\left(u^{(m)} + \varepsilon\left(\frac{1}{2} + \tilde{C}\right)\right)dx. \tag{69}$$

Recalling Lemma 4.2 and expanding $\psi_k^\alpha(v^{(m)} + \varepsilon\eta)$ as the sum of mixed k-Hessian operators,

$$\int_\Omega \psi_k^\alpha(v^{(m)} + \varepsilon\eta)dx \le \int_\Omega \psi_k^\alpha(v^{(m)})dx + \varepsilon C, \tag{70}$$

where the constant C depends on $n, k, \sigma, u, R, \alpha$. Since $\eta = \tilde{C}$ in $B_{\sigma R}$, by the definition of Ω_m, we have $B_{\sigma R} \subset \Omega_m$ and hence

$$\int_{B_{\sigma R}} \psi_k^\alpha(u^{(m)})dx \le \int_{B_{\sigma R}} \psi_k^\alpha(v^{(m)})dx + \varepsilon C. \tag{71}$$

Letting $\varepsilon \to 0$ and $m \to \infty$, we then obtain

$$\tilde{\mu}_1(B_{\sigma R}) \le \tilde{\mu}_2(B_{\sigma R}). \tag{72}$$

By interchanging $\{u^{(m)}\}$ and $\{v^{(m)}\}$, we have $\tilde{\mu}_1(B_{\sigma R}) = \tilde{\mu}_2(B_{\sigma R})$.

- Step2: By the conclusions of Step 1 , we get the weak convergence of the Borel measure

$$\psi_k^\alpha(u) = \sum_{l=0}^{k} b_l(\alpha) u^\alpha |\nabla u|^{2l} S_{k-l}(u). \tag{73}$$

By the locally uniform convergence of $u^{(m)}$, we get the weak convergence of the Borel measure

$$\varphi_k^\alpha(u) := \sum_{l=0}^{k} b_l(\alpha) |\nabla u|^{2l} S_{k-l}(u). \tag{74}$$

Using (31), we find that for any $0 \le l \le k$, $b_l \approx C_l \alpha^l$ when $\alpha > 0$ is large enough. So there always exist $\alpha_0, \alpha_1, \cdots, \alpha_k > 0$, such that

$$\det(H_{ij}) \ne 0, \tag{75}$$

where

$$H_{ij} = b_j(\alpha_i), \text{ with } i, j = 0, \cdots, k. \tag{76}$$

So for any $0 \le l \le k$, we get the weak convergence of the Borel measure

$$|\nabla u|^{2l} S_{k-l}(u). \tag{77}$$

$\square$

By the above proof, in fact we have already got:

Corollary 4.1. *For any $u \in \Phi^k(\Omega)$, $\alpha \in \mathbb{R}$, $0 \le l \le k$ and $n < 2k$, there exists a Borel measure $\mu_{k,l,\alpha}[u]$ in Ω such that*

- $\mu_{k,l,\alpha}[u] = u^\alpha |\nabla u|^{2l} S_{k-l}(u)$ *for* $u \in C^2(\Omega)$.
- *If $\{u^{(m)}\}$ is a sequence in $\Phi^k(\Omega)$ converging locally in measure to a function $u \in \Phi^k(\Omega)$, then the corresponding measure $\mu_{k,l,\alpha}[u^{(m)}] \to \mu_{k,l,\alpha}[u]$ weakly.*

Define

$$\begin{aligned}\Psi^k(\Omega) := \Big\{ & u \in C^0(\bar{\Omega}) : u > 0, \text{and there exists a sequence } \{u^{(m)}\} \in \tilde{\Phi}^k(\Omega) \ , \\ & \text{such that } u^{(m)} \text{ converges to } u \text{ uniformly in } \bar{\Omega} \Big\}.\end{aligned} \tag{78}$$

Then using Corollary 4.1, we can extend the result in Corollary 3.1 to $\Psi^k(\Omega)$.

Corollary 4.2. *If $n > 2k$ and $u, v \in \Psi^k(\Omega)$ satisfy*

$$\begin{cases} v^{-n} S_k(v) \le u^{-n} S_k(u) \text{ in } \Omega, & (79) \\ v \ge u \text{ on } \partial\Omega. & (80) \end{cases}$$

Then $v \ge u$ in Ω.

5. L^∞ ESTIMATE

In this section, we will give interior L^∞ bound with respect to the σ_k-Yamabe operator and prove Lemma 5.1. We use the idea from González [10] to use the Moser iteration, see also Li-Nguyen-Wang [15] the related works.

Lemma 5.1. *If $v \in \Phi^k(B_3)$, for $1 \leq k \leq \frac{n}{2}$ there exists a positive constant C which only depends on n, k , such that we have*

$$\sup_{B_1} v \leq C\|v\|_{L^1(B_2)}. \tag{81}$$

Proof of Lemma 5.1. By Lemma 4.1, we only need to consider the case that $n \geq 2k$. The proof is based on standard Moser iteration. It is similar to the ones in [9] and [10] by González. By Remark 3.1 , we know that when $\alpha > \Lambda$,

$$b_l > 0, \text{ for any } 0 \leq l \leq k. \tag{82}$$

By Lemma 3.1, we get

$$\begin{aligned} &\sum_{l=0}^{k-1} a_l(v^{\alpha+1}|\nabla v|^{2l}S^{ij}_{k-l}v_i)_j + \sum_{l=0}^{k} b_l v^\alpha |\nabla v|^{2l} S_{k-l} = 0, \\ \Rightarrow &\sum_{l=0}^{k} b_l v^\alpha |\nabla v|^{2l} S_{k-l} = -\sum_{l=0}^{k-1} a_l(v^{\alpha+1}|\nabla v|^{2l}S^{ij}_{k-l}v_i)_j. \end{aligned} \tag{83}$$

Suppose $\frac{1}{2} \leq r < R \leq 2$. Let cut-off function $\eta \equiv 1$ in B_r and $\eta \equiv 0$ in $\mathbb{R}^n \backslash B_R$. When $\alpha > \Lambda$ and $\delta > 4k$, we have $b_l > 0$ and

$$\int \sum_{l=0}^{k} b_l v^\alpha |\nabla v|^{2l} S_{k-l}\eta^\delta dx \leq C \int \sum_{l=0}^{k-1} a_l v^{\alpha+1}|\nabla v|^{2l} \left|S^{ij}_{k-l}v_i(\eta^\delta)_j\right| dx. \tag{84}$$

Since S^{ij}_{k-l} is nonnegative-definite, it follows that for any $0 \leq l \leq k-1$,

$$\begin{aligned} &\int a_l v^{\alpha+1}|\nabla v|^{2l} S^{ij}_{k-l}v_i(\eta^\delta)_j dx \\ \leq &\int C|a_l|(R-r)^{-1}v^{\alpha+1}|\nabla v|^{2l+1}S_{k-l-1}\eta^{\delta-1}dx. \end{aligned} \tag{85}$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\int (R-r)^{-1}v^{\alpha+1}|\nabla v|^{2l+1}S_{k-l-1}\eta^{\delta-1}dx \\ \leq &\int [C(R-r)^{-2}v^{\alpha+2}|\nabla v|^{2l}S_{k-l-1}\eta^{\delta-2} + C^{-1}v^\alpha|\nabla v|^{2l+2}S_{k-l-1}\eta^\delta]dx. \end{aligned} \tag{86}$$

Because we are using the method of Moser iteration, it is necessary to explain the case when α tends to $+\infty$. There are two different cases depending on α.

- When $\Lambda < \alpha < 2n$, substitute (85) and (86) into (84):

$$\int \sum_{l=0}^{k} v^\alpha |\nabla v|^{2l} S_{k-l}\eta^\delta dx \leq \sum_{l=0}^{k-1} C(R-r)^{-2} \int v^{\alpha+2}|\nabla v|^{2l}S_{k-l-1}\eta^{\delta-2}dx. \tag{87}$$

So by induction, we finally get

$$\int \sum_{l=0}^{k} v^{\alpha}|\nabla v|^{2l} S_{k-l}\eta^{\delta} dx \leq C(R-r)^{-2k}\int v^{\alpha+2k}\eta^{\delta-2k}dx. \tag{88}$$

- When $\alpha \geq 2n$, we have $\alpha+n+l \leq 2\alpha$ and

$$|a_l| = \frac{(\alpha+n+1)\cdots(\alpha+n+l)}{2^l \cdot l!} \leq \frac{\alpha^l}{l!}, \text{ for } l = 0,1,\cdots,k-1, \tag{89}$$

and

$$b_l \geq \frac{(k+l)\alpha^l}{2^l l!}, \text{ for } l = 0,1,\cdots,k. \tag{90}$$

In this case, the same as above argument , by (84) we get

$$\begin{aligned}
&\int \sum_{l=0}^{k} \alpha^l v^{\alpha}|\nabla v|^{2l} S_{k-l}\eta^{\delta} dx \\
\leq &C\int \sum_{l=0}^{k-1} a_l v^{\alpha+1}|\nabla v|^{2l} S_{k-l}^{ij} v_i(\eta^{\delta})_j dx \\
\leq &\sum_{l=0}^{k-1} C|a_l| \int (R-r)^{-1} v^{\alpha+1}|\nabla v|^{2l+1} S_{k-l-1}\eta^{\delta-1}dx \\
\leq &\sum_{l=0}^{k-1} C\alpha^l \int (R-r)^{-1} v^{\alpha+1}|\nabla v|^{2l+1} S_{k-l-1}\eta^{\delta-1}dx \\
\leq &\sum_{l=0}^{k-1} [C\alpha^{l-1}\int (R-r)^{-2}v^{\alpha+2}|\nabla v|^{2l}S_{k-l-1}\eta^{\delta-2}dx + C^{-1}\alpha^{l+1}\int v^{\alpha}|\nabla v|^{2l+2}S_{k-l-1}\eta^{\delta}dx],
\end{aligned} \tag{91}$$

the last inequality follows from Cauchy-Schwarz inequality. Then we deduce that

$$\int \sum_{l=0}^{k} \alpha^l v^{\alpha}|\nabla v|^{2l} S_{k-l}\eta^{\delta} dx \leq \sum_{l=0}^{k-1} C\alpha^{l-1}\int (R-r)^{-2}v^{\alpha+2}|\nabla v|^{2l}S_{k-l-1}\eta^{\delta-2}dx. \tag{92}$$

By induction, we finally get that

$$\int \sum_{l=0}^{k} \alpha^l v^{\alpha}|\nabla v|^{2l} S_{k-l}\eta^{\delta} dx \leq C\alpha^{-k}\int (R-r)^{-2k}v^{\alpha+2k}\eta^{\delta-2k}dx, \tag{93}$$

it folows that

$$\alpha^{2k}\int v^{\alpha}|\nabla v|^{2k}\eta^{\delta}dx \leq C(R-r)^{-2k}\int v^{\alpha+2k}\eta^{\delta-2k}dx. \tag{94}$$

Let $\delta = 4k$, then

$$\begin{aligned}
&\int \left|\nabla(v^{\frac{\alpha+2k}{2k}}\eta^2)\right|^{2k} \leq C(R-r)^{-2k}\int v^{\alpha+2k}\eta^{2k}, \\
\Rightarrow &\|\nabla(v^{\frac{\alpha+2k}{2k}}\eta^2)\|_{L^{2k}(B_R)} \leq C(R-r)^{-1}\|v^{\frac{\alpha+2k}{2k}}\|_{L^{2k}(B_R)}.
\end{aligned}$$

By Sobolev inequality, it holds that

$$\|v^{\frac{\alpha+2k}{2k}}\|_{L^{\frac{2kn}{n-2k}}(B_r)} \le C_1\|\nabla(v^{\frac{\alpha+2k}{2k}}\eta^2)\|_{L^{2k}(B_R)} \le C(R-r)^{-1}\|v^{\frac{\alpha+2k}{2k}}\|_{L^{2k}(B_R)},$$

$$\Rightarrow \|v\|_{L^{\frac{(\alpha+2k)n}{n-2k}}(B_r)} \le C^{\frac{2k}{\alpha+2k}}(R-r)^{-\frac{2k}{\alpha+2k}}\|v\|_{L^{\alpha+2k}(B_R)}.$$

Following the proof of Theorem 4.1 in [13] and using the fact that

$$\|v\|_{L^{p_1}(B_1)} \le C\|v\|_{L^{p_2}(B_1)}, \text{ if } p_2 > p_1 > 0, \tag{95}$$

we finally get that

$$\sup_{B_1} v \le C\|v\|_{L^1(B_2)}. \tag{96}$$

□

Now we can give the locally uniform bound:

Lemma 5.2. *Let $u \in \Phi^k(B_3) \cap C^2(B_3)$ and $n \ge 2k$. Then for $\alpha > \Lambda$ and $\alpha \ge -2k$, there exists a positive constant C which only depends on n, k, α such that*

$$\int_{B_1} \sum_{l=0}^{k} u^\alpha |\nabla u|^{2l} S_{k-l} dx \le C\|u\|^{2k+\alpha}_{L^1(B_2)}. \tag{97}$$

Proof of Lemma 5.2. Let cut-off function $\eta \equiv 1$ in B_1 and $\eta \equiv 0$ in $B_3 \backslash B_2$. By Lemma 3.1

$$\begin{aligned}
&\int \sum_{l=0}^{k} b_l u^\alpha |\nabla u|^{2l} S_{k-l}\eta^\delta dx\\
=&-\int \sum_{l=0}^{k-1} a_l (u^{\alpha+1}|\nabla u|^{2l} S^{ij}_{k-l} u_i)_j \eta^\delta dx\\
=&\int \sum_{l=0}^{k-1} a_l u^{\alpha+1}|\nabla u|^{2l} S^{ij}_{k-l} u_i (\eta^\delta)_j dx\\
\le& C\int \sum_{l=0}^{k-1} u^{\alpha+1}|\nabla u|^{2l+1} S_{k-l-1}\eta^{\delta-1} dx.
\end{aligned}$$

Using Cauchy-Schwarz inequality and Lemma 2.3, we get

$$\Rightarrow \int \sum_{l=0}^{k} b_l u^\alpha |\nabla u|^{2l} S_{k-l}\eta^\delta dx \le C\int \sum_{l=0}^{k-1} u^{\alpha+2}|\nabla u|^{2l} S_{k-l-1}\eta^{\delta-2} dx,$$

$$\Rightarrow \int u^{\alpha+k}\sigma_k \eta^\delta dx + \int \sum_{l=0}^{k} b_l u^\alpha |\nabla u|^{2l} S_{k-l}\eta^\delta dx \le C\int u^{\alpha+k+1}\sigma_{k-1}\eta^{\delta-2} dx + C\int u^{\alpha+2k}\eta^{\delta-2k} dx.$$

By iteration and Lemma 5.1 , we get if $\delta = 4k$,

$$\begin{aligned}
\Rightarrow \int \sum_{l=0}^{k} u^\alpha |\nabla u|^{2l} S_{k-l}\eta^\delta &\le C\int u^{\alpha+k+2}\sigma_{k-2}\eta^{\delta-4} dx + C\int u^{\alpha+2k}\eta^{\delta-2k} dx\\
&\le \cdots
\end{aligned}$$

$$
\begin{aligned}
&\leq C \int u^{\alpha+2k}\eta^{\delta-2k}dx \\
&\leq C\|u\|^{2k+\alpha}_{L^1(B_2)}.
\end{aligned}
$$

□

6. Weak continuity for $n \geq 2k$.

In this section, we will prove Theorem 1.1 when $n \geq 2k$.

Proof of Theorem 1.1. We divide the proof into two steps. Letting $B_l = b_l$, suppose $u \in \Phi^k(\Omega)$, $\{u^{(m)}\} \subset \Phi^k(\Omega) \cap C^2(\Omega)$ and $u^{(m)}$ converge locally in measure to u.

- Step1: In this step, we will show that for any $\alpha \geq 1-2k$, it follows that $\psi_k^\alpha(u^{(m)}) \to \psi_k^\alpha(u)$ weakly. Our proof follows the idea of Trudinger-Wang in [21]. Define

$$
w = w(x,t) := (1-t)u^{(m_1)}(x) + tu^{(m_2)}(x), \tag{98}
$$

and let $\eta \equiv 1$ in $B_r(y)$ and $\eta \equiv 0$ in $\Omega \backslash B_{2r}(y)$ with $B_{3r}(y) \subset\subset \Omega$. Letting $B_l = b_l$ and using Lemma 3.2:

$$
\begin{aligned}
&\int_{B_{2r(y)}} \eta\psi_k^\alpha(u^{(m_2)}) - \eta\psi_k^\alpha(u^{(m_1)})dx \\
=&\int_0^1\int_{B_{2r(y)}} \eta\frac{\partial}{\partial t}\psi_k^\alpha(w)dxdt \\
=&\int_0^1\int_{B_{2r(y)}} \eta\Bigg\{\sum_{l=0}^{k-1} b_l\Big[w^{\alpha+1}|\nabla w|^{2l}S_{k-l}^{ij}(u^{(m_2)}-u^{(m_1)})_i\Big]_j \\
&+\sum_{l=0}^{k-1}\Big[2(l+1)b_{l+1}-(\alpha+n+l+1)b_l\Big]\Big[w^\alpha|\nabla w|^{2l}S_{k-l}^{ij}w_i(u^{(m_2)}-u^{(m_1)})\Big]_j\Bigg\}dxdt \\
=&\int_0^1\int_{B_{2r(y)}} \Bigg\{\sum_{l=0}^{k-1} b_l(\eta_j\cdot w^{\alpha+1}|\nabla w|^{2l}S_{k-l}^{ij})_i(u^{(m_2)}-u^{(m_1)}) \\
&-\sum_{l=0}^{k-1}\Big[2(l+1)b_{l+1}-(\alpha+n+l+1)b_l\Big]\eta_j\cdot w^\alpha|\nabla w|^{2l}S_{k-l}^{ij}w_i(u^{(m_2)}-u^{(m_1)})\Bigg\}dxdt.
\end{aligned} \tag{99}
$$

By Lemma 2.2, we have

$$
\left|\sum_i \partial_i(S_{k-l}^{ij})\right| \leq Cw^{-1}|\nabla w|S_{k-l-1}. \tag{100}
$$

Then by direct cpmputation , we find that

$$
\begin{aligned}
&\Big[w^{\alpha+1}|\nabla w|^{2l}S^{ij}_{k-l}\eta_j\Big]_i\\
=&(\alpha+1)w^{\alpha}|\nabla w|^{2l}S^{ij}_{k-l}\eta_j w_i+2lw^{\alpha+1}|\nabla w|^{2l-2}S^{ij}_{k-l}\eta_j w_{im}w_m+w^{\alpha+1}|\nabla w|^{2l}S^{ij}_{k-l}\eta_{ij}\\
&+w^{\alpha+1}|\nabla w|^{2l}\eta_j\cdot\partial_i(S^{ij}_{k-l})\\
=&(\alpha+1)w^{\alpha}|\nabla w|^{2l}S^{ij}_{k-l}\eta_j w_i+2lw^{\alpha}|\nabla w|^{2l-2}S^{ij}_{k-l}\eta_j A_{im}w_m+lw^{\alpha}|\nabla w|^{2l}S^{ij}_{k-l}\eta_j w_i+w^{\alpha+1}|\nabla w|^{2l}S^{ij}_{k-l}\eta_{ij}\\
&+w^{\alpha+1}|\nabla w|^{2l}\eta_j\cdot\partial_i(S^{ij}_{k-l})\\
=&(\alpha+1)w^{\alpha}|\nabla w|^{2l}S^{ij}_{k-l}\eta_j w_i+2lw^{\alpha}|\nabla w|^{2l-2}\eta_j w_m(S_{k-l}\delta_{mj}-S^{mj}_{k-l+1})+lw^{\alpha}|\nabla w|^{2l}S^{ij}_{k-l}\eta_j w_i\\
&+w^{\alpha+1}|\nabla w|^{2l}S^{ij}_{k-l}\eta_{ij}+w^{\alpha+1}|\nabla w|^{2l}\eta_j\cdot\partial_i(S^{ij}_{k-l})\\
\le& Cw^{\alpha}|\nabla w|^{2l+1}S_{k-l-1}+Clw^{\alpha}|\nabla w|^{2l-1}S_{k-l}+Cw^{\alpha+1}|\nabla w|^{2l}S_{k-l-1},
\end{aligned}
\tag{101}
$$

and

$$
w^{\alpha}|\nabla w|^{2l}S^{ij}_{k-l}w_i\eta_j\le Cw^{\alpha}|\nabla w|^{2l+1}S_{k-l-1}. \tag{102}
$$

So by Cauchy-Schwarz inequality, we get that

$$
\begin{aligned}
&\int \eta\psi^{\alpha}_k(u^{(m_1)})-\eta\psi^{\alpha}_k(u^{(m_2)})\\
\le&\sum_{l=0}^{k-1}\int_0^1\int_{B_{2r(y)}}\left[Cw^{\alpha}|\nabla w|^{2l+1}S_{k-l-1}(u^{(m_2)}-u^{(m_1)})^{+}+Cw^{\alpha+1}|\nabla w|^{2l}S_{k-l-1}(u^{(m_2)}-u^{(m_1)})^{+}\right]dxdt\\
\le&\sum_{l=0}^{k-1}\int_0^1\int_{B_{2r(y)}}\left[Cw^{\alpha+1}|\nabla w|^{2l}S_{k-l-1}(u^{(m_2)}-u^{(m_1)})^{+}+Cw^{\alpha-1}|\nabla w|^{2l+2}S_{k-l-1}(u^{(m_2)}-u^{(m_1)})^{+}\right]dxdt.
\end{aligned}
$$

Fix $\varepsilon\in(0,1)$ and N so that for

$$
O_\varepsilon:=\{x\in B_{2r}(y)||u^{(m_2)}-u^{(m_1)}|>\varepsilon\}, \tag{103}
$$

we have $|O_\varepsilon|<\varepsilon$ if $m_1,m_2\ge N$. Besides, we can suppose

$$
\sup_m\int_{B_{3r}(y)}u^{(m)}\le K, \tag{104}
$$

for some fixed constant K. We then have

$$
\begin{aligned}
&\int \eta\psi^{\alpha}_k(u^{(m_1)})-\eta\psi^{\alpha}_k(u^{(m_2)})dx\\
\le&\sum_{l=0}^{k-1}\int_0^1\int_{B_{2r(y)}}[Cw^{\alpha+1}|\nabla w|^{2l}S_{k-l-1}(u^{(m_2)}-u^{(m_1)}-\varepsilon)^{+}+C\varepsilon w^{\alpha+1}|\nabla w|^{2l}S_{k-l-1}\\
&+Cw^{\alpha-1}|\nabla w|^{2l+2}S_{k-l-1}(u^{(m_2)}-u^{(m_1)}-\varepsilon)^{+}+C\varepsilon w^{\alpha-1}|\nabla w|^{2l+2}S_{k-l-1}]dxdt.
\end{aligned}
\tag{105}
$$

Since $n\ge 2k$ and $\alpha\ge 1-2k$, we get $\alpha>\Lambda-1$. So using Lemma 5.2, we have

$$
\int_0^1\int_{B_{2r(y)}}w^{\alpha+1}|\nabla w|^{2l}S_{k-l-1}dxdt\le CK^{2k+\alpha-1},
$$

and

$$\begin{aligned}&\int_0^1\int_{B_{2r(y)}} w^{\alpha+1}|\nabla w|^{2l}S_{k-l-1}(u^{(m_2)}-u^{(m_1)}-\varepsilon)^+dxdt\\ \leq&\int_0^1\int_{O_\varepsilon} w^{\alpha+1}|\nabla w|^{2l}S_{k-l-1}(u^{(m_2)}-u^{(m_1)}-\varepsilon)^+dxdt\\ \leq& CK\int_0^1\int_{O_\varepsilon} w^{\alpha+1}|\nabla w|^{2l}S_{k-l-1}dxdt.\end{aligned}$$

Then it follows that for $m_1, m_2 \geq N' \geq N$, for a further constant N' depending on ε, r,

$$\int_0^1\int_{B_{2r(y)}} w^{\alpha+1}|\nabla w|^{2l}S_{k-l-1}(u^{(m_2)}-u^{(m_1)}-\varepsilon)^+dxdt \leq C\varepsilon. \tag{106}$$

So far, we obtain that

$$\int \eta\psi_k^\alpha(u^{(m_1)}) - \eta\psi_k^\alpha(u^{(m_2)})dx \leq C\varepsilon. \tag{107}$$

This means that $\psi_k^\alpha(u^{(m)})$ converges weakly to a Borel measure in Ω.

- Step2: Note that

$$\begin{aligned}&\int_{B_{2r(y)}}[\eta(u^{(m_1)})^\gamma\psi_k^\alpha(u^{(m_1)}) - \eta(u^{(m_2)})^\gamma\psi_k^\alpha(u^{(m_2)})]dx\\ =&\int_0^1\int_{B_{2r(y)}}\eta\frac{\partial}{\partial t}\Big[w^\gamma\psi_k^\alpha(w)\Big]dxdt\\ =&\int_0^1\int_{B_{2r(y)}}\underbrace{w^\gamma\frac{\partial}{\partial t}\psi_k^\alpha(w)\eta}_{\mathbb{H}_1}+\underbrace{\gamma w^{\gamma-1}\psi_k^\alpha(w)\cdot(u^{(m_2)}-u^{(m_1)})\eta}_{\mathbb{H}_2}dxdt.\end{aligned}$$

Choose $\gamma > -2k-\alpha+1$, then by (33) we know that $\gamma > \Lambda-\alpha+1$. Using the same arguments as Step1, we deal with the term $\mathbb{H}_1$ similar to (99) and the term $\mathbb{H}_2$ similar to (105) So we get $(u^{(m)})^\gamma\psi_k^\alpha(u^{(m)})$ converges weakly to a Borel measure in Ω. So for any $\bar\gamma > 1-2k$ and $\alpha \geq 1-2k$, we get the weak convergence of the Borel measure

$$\varphi_k^{\alpha,\bar\gamma}(u) := \sum_{l=0}^k b_l(\alpha)u^{\bar\gamma}|\nabla u|^{2l}S_{k-l}(u). \tag{108}$$

Since for any $0 \leq l \leq k$, $b_l \approx C_l\alpha^l$, there always exist $\alpha_0, \alpha_1, \cdots, \alpha_k > 0$, such that

$$\det(H_{ij}) \neq 0, \tag{109}$$

where

$$H_{ij} = b_j(\alpha_i), \text{ with } i, j = 0, \cdots, k. \tag{110}$$

So for any $0 \leq l \leq k$ and $\bar{\gamma} > 1 - 2k$, we get the weak convergence of the Borel measure

$$u^{\bar{\gamma}}|\nabla u|^{2l} S_{k-l}(u). \tag{111}$$

□

By the above proof and combining Corollary 4.1 , in fact we have already got:

Corollary 6.1. *For any $u \in \Phi^k(\Omega)$, $\alpha > 1 - 2k$, $0 \leq l \leq k$, there exists a Borel measure $\mu_{k,l,\alpha}[u]$ in Ω such that*

- $\mu_{k,l,\alpha}[u] = u^\alpha |\nabla u|^{2l} S_{k-l}(u)$ *for* $u \in C^2(\Omega)$.
- *If $\{u^{(m)}\}$ is a sequence in $\Phi^k(\Omega)$ converging locally in measure to a function $u \in \Phi^k(\Omega)$, then the corresponding measure $\mu_{k,l,\alpha}[u^{(m)}] \to \mu_{k,l,\alpha}[u]$ weakly.*

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