Abstract of thesis entitled

Synthesis of Dynamic Systems with Markovian Characteristics

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The thesis is concerned with the synthesis of dynamic systems with Markovian characteristics. Two classes of systems are considered: Markovian jump linear systems and networked control systems. Three types of problems are investigated. They are (a) robust stabilization, robust $H_2$ control, robust $H_\infty$ control and robust $H_\infty$ filter design of uncertain Markovian jump linear systems; (b) stabilization of Markovian jump linear systems with delayed system information; and (c) stabilization of networked control systems with bounded time delay and packet loss.

For problem (a), the system under consideration is a continuous-time Markovian jump linear system. The uncertainties are norm bounded in the system matrices and element-wise bounded in the mode transition rate matrix. An improved model of the mode transition rate uncertainties is described, which enables the use of the probability constraints on the rows of the mode transition rate matrix to reduce the conservatism of the results. Sufficient conditions are given for the designs of the robust stabilizing controllers, robust $H_2$ controllers, robust $H_\infty$ controllers and fixed-order robust $H_\infty$
For problem (b), the system under consideration is a discrete-time Markovian jump linear system. Both the system state and the system mode are time delayed. The delay in the system state is time-varying and bounded, while that in the system mode is constant. The delay in the latter case manifests as a constant mismatch of the modes between the system and the controller. The resulting closed-loop system is shown to be a time-varying delayed Markovian jump linear system with extended operation modes. A sufficient condition is provided for the design of the controller with the delayed information.

For problem (c), two techniques are developed for the stabilization of networked control systems. One is to use packet-loss dependent Lyapunov functions. In this case, the process to be controlled is a linear discrete-time system. The communication network has the bounded packet-loss behavior characterized either by an arbitrary process or by a Markovian chain. The stabilization problem is tackled. The other technique is to employ a special zero-order hold. In this case, the zero-order hold has a logic to apply the newest control input data to control the process. Under this framework, the networked control system with both time delay and packet loss is discretized to a discrete-time system with input delay. Several sufficient conditions are given for the design of the networked controllers.

All the obtained conditions are given in terms of either linear matrix inequalities only or linear matrix inequalities with equality constraints. Numerical examples and simulations are used to demonstrate that (a) it is necessary to consider the uncertainties in the mode transition rate matrix; (b) the results developed in the thesis are more powerful than existing ones; (c) the delay in the system mode is critical for the control of Markovian jump linear systems with delayed information; and (d) the networked controllers can be effectively designed by the developed theories.
Synthesis of Dynamic Systems with Markovian Characteristics

by

Junlin Xiong

A thesis submitted in partial fulfilment of the requirements for
the Degree of Doctor of Philosophy
at The University of Hong Kong.

August 2007
Declaration

I declare that this thesis represents my own work, except where due acknowledgement is made, and that it has not been previously included in a thesis, dissertation or report submitted to this University or to any other institution for a degree, diploma or other qualifications.

Signed

Junlin Xiong
I wish to express my deep thanks to my supervisor, Professor James Lam, for his enlightening guidance, invaluable discussions and insightful ideas throughout the years. What I have benefited most from being his student is probably his rigorous and diligent attitude to scientific research.

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Notations

\( \mathbb{N} \) set of natural numbers
\( \mathbb{Z}_+ \) set of non-negative integers
\( \mathbb{R}^+ \) set of positive real numbers
\( \mathbb{R}^n \) \( n \)-dimensional Euclidean space
\( \mathbb{R}^{m \times n} \) set of \( m \times n \) real matrices
\( S^{n \times n} \) set of \( n \times n \) real symmetric positive definite matrices
\( L^2_\mathcal{F}[0, \infty) \) space of square integrable functions on \([0, \infty)\)
\( (\Omega, \mathcal{F}, P) \) a complete probability space

| \( |x| \) | absolute value of the number \( x \) |
| \( \|x\| \) | Euclidean norm of the vector \( x \) |
| \( \|A\| \) | induced two norm of the matrix \( A \) |
| \( \|\rho\|_2 \) | \( \left( \int_0^\infty \|\rho(t)\|^2 \, dt \right)^{1/2} \) if \( \rho(\cdot) \in L^2_{\mathcal{F}}[0, \infty) \) or \\
| | \( \left( \int_0^\infty E\left(\|\rho(t)\|^2\right) \, dt \right)^{1/2} \) if \( \rho(\cdot) \) is a stochastic process |
| \( \|G\|_{H_2} \) | \( H_2 \) norm of the operator \( G \) |
| \( \|G\|_{H_\infty} \) | \( H_\infty \) norm of the operator \( G \) |

\( \inf(\cdot) \) greatest lower bound
\( \delta(\cdot) \) Dirac delta
\( \delta(\cdot, \cdot) \) Kronecker delta
\( \lceil x \rceil \) integer \( n \) such that \( n - 1 < x \leq n \)
\( \Pr(\cdot) \) mathematical probability operator
\( \mathbb{E}(\cdot) \) mathematical expectation operator

\( \text{diag}(A_1, \ldots, A_n) \) diagonal matrix with \( A_1, \ldots, A_n \) on the diagonal
\( \text{trace}(A) \) trace of the matrix \( A \)
\( \text{rank}(A) \) rank of the matrix \( A \)
\( \lambda_{\text{min}}(A) \) minimum of the eigenvalues of the real symmetric matrix \( A \)
$A^T$  
transpose of the matrix $A$

$A^\perp$  
orthogonal complement matrix of the matrix $A$

$A^\perp_2$  
$A^\perp_2 A^\perp_2 = A$ with $A$ positive definite

$A^{-1}$  
inverse of the matrix $A$

$A < B$  
$A - B$ is negative definite

$A \leq B$  
$A - B$ is negative semi-definite

$I$  
identity matrix of appropriate dimensions

$I_n$  
n $\times$ n identity matrix

$0_{m \times n}$  
m $\times$ n zero matrix

\[
\begin{bmatrix}
A_{11} & \bullet \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\]

$\begin{bmatrix}
A_{11} & \bullet \\
A_{11} & A_{12}
\end{bmatrix}
\begin{bmatrix}
A_{11} & A_{12} \\
A_{12} & A_{22}
\end{bmatrix}$

:=  
assigned to (used in algorithms)

≜  
defined as

≡  
identically equals

□  
end of proof
Acronyms

MJLS  Markovian jump linear system
DMJLS discrete-time Markovian jump linear system
LMI  linear matrix inequality
CCL  cone complementarity linearization
ZOH  zero-order hold
NCS  networked control system
Introduction

1.1 Background

The ultimate goal of control engineering is, given a physical system, to determine the control input to the system, based on the available information, such that the output of interest behaves in a desirable way, regardless of the unknown system initial conditions, the unknown system disturbance inputs and the unknown system modeling errors. Stability, disturbance attenuation and robustness are three elements of control specifications.

Stability means that the steady state of a system is insensitive to the initial conditions of the system. Roughly speaking, a system is stable in the sense of Lyapunov, if, for any initial condition sufficiently close to zero, its time trajectory remains near to zero. The main limitation of this concept is that the system trajectory is not required to converge to zero. Very often, it is simply not enough. If a system is stable and furthermore its trajectory tends to zero as time approaches to infinity, the system is said to be asymptotically stable. Asymptotic stability is the desirable property in most applications. The principal weakness of this concept is that asymptotic stability says nothing about how fast the trajectory approaches zero. A stronger form of asymptotic stability is referred to as exponential stability. On one hand, stability analysis and stabilization synthesis are the most fundamental concepts in control engineering, because
it is unlikely that a physical system remains staying in its operating point all the time. The system needs to be stabilized before any other sophisticated performance specifications are considered. On the other hand, stability is far from sufficient by itself for a successful control system design.

The performance of a system is often measured by the attenuation ability against the disturbance inputs, and frequently quantified by some norms related to the system response and the disturbance input. For example, the $H_2$ norm of a system is a measurement to quantify the output variance of the system with white noises as the exogenous input. Also, the $H_2$ norm could be thought of as the measurement of the average output energy over impulsive inputs. Another popular tool to quantify the system performance is the $H_\infty$ norm, which can be termed as the $L_2$-induced norm as well. Briefly speaking, the $H_\infty$ norm of a system is the maximum gain of the system in the energy sense. In other words, the $H_\infty$ norm of the system specifies the worst-case effect on the energy of the system output when the disturbance is an arbitrary signal with bounded energy. The philosophy of both control and filtering problems in control engineering is to make these norms small in the presence of the unknown system disturbance inputs. The optimal $H_2$ control problem, for instance, is to design a control system achieving the minimal $H_2$ norm. The sub-optimal $H_\infty$ filtering problem is to design a filter such that the $H_\infty$ norm of the error system is less than a given level.

Robustness is another essential specification of control systems. Its goal is to make the system performance insensitive to the modeling errors. To analyze a complicated physical system in practice, an over-bounded model is often constructed to include the physical system as a member. The over-bounded model generally consists of two parts: a nominal model and an admissible uncertainty set. The nominal model is used to incorporate the major properties of the physical system and of a simple form, hence easier for analysis and synthesis. The most common nominal model is linear time-invariant systems. For systems subject to random abrupt changes in their parameters, Markovian jump linear systems are an appropriate model class. Time-delay systems
and networked control systems are other mathematical models suitable to model dynamic systems with inherent time delay and information loss. Time delay means that the system information is available with a delay, while information loss implies that the system information may be lost from time to time. The admissible uncertainty set is used to capture the un-modeled, higher-order dynamics of the complicated physical system. Two common models of the admissible uncertainty set are the norm-bounded uncertainty set and the polytope-type uncertainty set. The main task of robust control is therefore to design a controller, based on the nominal model and the admissible uncertainty set, such that not only the nominal closed-loop model achieves the required performance but also the performance maintains over the admissible uncertainty set. As a result, the original physical system has the same performance as well. This method may introduce conservatism, but does provide a systematic technique for the analysis and synthesis of complicated systems.

To achieve the desirable control specifications mentioned above, the predominant approach is the use of feedback compensation. The design of a control system is often carried out in the state-space framework, and can be generally decomposed into two stages. In the first stage, conditions are determined under which the control system satisfies the specified properties for all admissible initial conditions, disturbance inputs and modeling errors. In the second stage, a controller is sought so that the closed-loop system satisfies these conditions. Solutions to both the analysis and the synthesis problems are often related to the solvability of certain matrix equations (e.g., Lyapunov or Riccati equations) or some matrix inequalities (e.g., linear matrix inequalities). Solving a Lyapunov or Riccati equation is straightforward, while solving linear matrix inequalities needs iterations. However, a system of coupled linear matrix inequalities can be solved efficiently as well, while a system of coupled Riccati equations is hard to solve. Nowadays, expressing conditions in terms of linear matrix inequalities is very popular as many sophisticated algorithms have been developed to solve the coupled linear matrix inequalities.
CHAPTER 1. INTRODUCTION

This thesis studies the synthesis of dynamic systems with Markovian characteristics. Two classes of systems are considered. The first class of systems is dynamic systems with Markovian jumping parameters, and referred to as Markovian jump systems. A Markovian jump linear system consists of a group of linear time-invariant subsystems, and the changes of the system parameters among these subsystems are governed by a Markov process/chain. The other class of systems is dynamic systems with Markovian signal transmissions, and referred to as networked control systems. In this case, the system itself is a linear time-invariant system, but the exchange of the signals among the control components is through a shared communication network. Both time delay and packet loss may occur during the network transmission. The process of packet loss may be Markovian. To sum up, the following synthesis problems will be studied:

• Given an uncertain Markovian jump linear system, design a controller such that the closed-loop system is robustly mean square stable, and has certain robust $H_2$ and $H_\infty$ performances; design a robust filter with some $H_\infty$ filtering level.

• Given a Markovian jump linear system, design a controller such that the closed-loop system is mean square stable, where time delays exist in the transmissions of the system state and the system mode.

• Given a linear time-invariant system, design a controller such that the networked control system is asymptotically stable, where time delay and packet loss may exist during the network transmission of the system state.

1.2 Literature Review

This section reviews the state of the art of the research about dynamic systems with Markovian jumping parameters and signal transmissions. The literature review is divided into three parts. The first part reviews Markovian jump linear systems. The
second part reviews Markovian jump linear systems with time delay. The third part is the review of networked control systems. Each part begins with a description of the system to be reviewed, then moves on to various topics of interest, and finally ends with the motivation of the relevant research problem(s).

1.2.1 Markovian Jump Linear Systems

This subsection reviews Markovian jump linear systems. The scope of the review is restricted to the following four topics: stability, $H_2$ control, $H_{\infty}$ control and filter design. After the reviews, the first research problem will be formulated.

Loosely speaking, a Markovian jump system is a hybrid system composed of a finite number of subsystems. The state vector of the hybrid system has two components: system state and system mode. Within a fixed mode, the system evolves as a linear dynamic system characterized by that subsystem. The jumps among the subsystems are described by a Markov process/chain, and characterize the random changes of system parameters. Markovian jump linear systems provide a suitable mathematical model for dynamic systems subject to random, abrupt and significant changes in parameters. In practice, the changes of system parameters may be due to, for instance, random failure of the components, sudden disturbances and variations of the environment, changes of the subsystem interconnections and abrupt variations of the operating point on a non-linear plant. Over the years, Markovian jump linear systems have continually attracted the attention of system theorists.

Stability

The stability properties of Markovian jump linear systems have been studied by using a stochastic Lyapunov approach in the context of stochastic theory. A necessary and sufficient condition for mean square stability has been established in terms of the solvability of a set of coupled algebraic Riccati equations [34] or a set of coupled linear matrix inequalities [12]. Moreover, it has been shown that all the second moment sta-
bility properties (e.g., the stochastic stability, the asymptotic mean square stability and the exponential mean square stability) are equivalent, and sufficient for the almost sure stability [28].

An uncertain Markovian jump linear system is often described by a family of linear state-space systems containing parameter uncertainties. The polytope-type uncertainty set and the norm-bounded uncertainty set provide two different models about the uncertainty forms widely adopted in the study of robust control.

For systems with norm-bounded uncertainties, the nominal values of system parameters are fixed and known, and the uncertain part is represented as matrix multiplications of unknown time-varying matrices and known constant matrices. The unknown matrices denote the uncertainties, and are assumed to be norm bounded, while the known matrices characterize how the uncertainties enter the nominal values of system parameters. The robust stabilization problem of continuous-time Markovian jump linear systems with norm-bounded uncertainties was studied in [4], where a sufficient condition was proposed in terms of coupled algebraic Riccati equations.

For systems with polytope-type uncertainties, system parameters are only known to be in a matrix polytope. The vertices of the matrix polytope are known and constant. The number of vertices is finite. The parameters of the system are expressed as an unknown convex combination of the vertices. The robust stabilization problem of discrete-time Markovian jump linear systems with such uncertainties was considered in [18].

In addition, a special feature of Markovian jump linear systems is that uncertainties may exist in the mode transition rate matrix for continuous-time Markovian jump linear systems [4, 27] or the mode transition probability matrix for discrete-time ones [18]. In [36], sufficient conditions were obtained for the robust stochastic stability of Markovian jump systems in terms of the upper bounds of the perturbed transition rates and probabilities.
Although stability is always the first requirement of any control system design, satisfactory performance is also necessary for the success of the design. The performance of a system is generally measured by the values of the quadratic cost function or the $H_2$ norm or the $H_\infty$ norm of the system.

For linear quadratic control problems, a quadratic cost function is used as the criterion for system performance. The objective of system design is thus to construct an optimal controller that minimizes the cost function [1, 24]. The $H_2$ norm of Markovian jump linear systems is defined according to the system responses with respect to the unit impulse inputs [13, 25]. It has been shown that the $H_2$ norms of both continuous-time [13] and discrete-time [25] Markovian jump linear systems can be calculated from the solutions of the coupled observability and controllability Gramians, which are coupled Riccati equations. However, because of the difficulties inherent in solving coupled Riccati equations, the $H_2$ norm is alternately approximated by using a convex optimization procedure, which minimizes an upper bound of the $H_2$ norm subject to a set of coupled linear matrix inequality constraints [17].

Another common performance criterion is the $H_\infty$ norm. To define the $H_\infty$ norm of Markovian jump linear systems, the system is viewed as an operator from the energy-bounded disturbance inputs to the corresponding energy-bounded outputs. The $H_\infty$ norm is thus defined as the square root of the worst ratio between the energy of the outputs and the energy of the disturbance inputs [19, 51]. Bounded real lemmas have been established as a sufficient condition for continuous-time Markovian jump linear systems [19] and as a necessary and sufficient condition for discrete-time Markovian jump linear systems [51].

Sometimes, sub-optimal controllers are constructed via solving a set of coupled linear matrix inequalities with reduced computational effort. Sometimes two or more performance criteria are combined together to achieve enhanced performance. For example, the mixed $H_2/H_\infty$ control is one of the many. Specifically, the mixed $H_2/H_\infty$
control problem is to find a stabilizing controller that minimizes an upper bound of the $H_2$ norm under the restriction that the $H_\infty$ norm is less than a pre-specified value [16].

**Filtering**

Filtering problems arise out of the need to estimate unmeasurable system variables. According to the nature of the exogenous input signals, filtering techniques may be categorized into two classes: stochastic filtering and $H_\infty$ filtering.

A classical approach to stochastic filtering is the Kalman filtering technique. This filtering technique assumes that the exogenous input signals in the state equation and the measurement equation are stationary Gaussian noises. The properties of the Gaussian noises need to be known. When the jumping parameter (i.e., the system mode) of a Markovian jump linear system is accessible to filter implementation, mode-dependent estimators can be constructed from the solution of two sets of coupled Riccati equations, and the covariance of the estimation error is guaranteed to be within a certain bound [55]. When the jumping parameter is not accessible at all, an optimal mode-independent filter can be designed via a convex optimization procedure, which minimizes an upper bound of the expected value of the estimation error [14]. In [15], the limit of the error covariance matrix of the linear minimum mean square error estimator was proved to be the unique solution of a stationary Riccati equation. Moreover, a time-invariant, stable, sub-optimal filter can be constructed from the unique solution.

The $H_\infty$ filtering technique is also very popular for signal estimation. Compared to Kalman filtering, the exogenous input signals are assumed to be energy bounded rather than Gaussian. Hence no statistical assumptions are needed. This filtering approach is very appropriate to applications where the statistics of the noise signals are not exactly known. Moreover, robust $H_\infty$ filters tend to be more robust than Kalman filters in the presence of system uncertainties. In particular, the $H_\infty$ filtering problem is concerned with the design of estimators ensuring not only that the error system is asymptotically stable, but also that the $H_\infty$ norm of the error system is below a prescribed level. A
continuous-time $H_\infty$ filter was reported in [20] and a discrete-time one in [22]. A robust $H_\infty$ filter for systems with parameters having uncertainties is described in [21]. For uncertain Markovian jump systems with mode-dependent time delay, a robust $H_\infty$ filtering technique was proposed in [64]. If the jumping parameter is not accessible, a robust mode-independent $H_\infty$ filter was described in [23], where polytope-type uncertainties exist in either the transition rate matrix or the system matrices.

It is worth noticing that the filters mentioned above [20–23, 64] are all of full order. That is, the order of the designed filter equals that of the original system. However, the implementation of full-order filters may be impractical when large-scale systems are involved or real-time estimation is desired or fast data processing is necessary. Reduced-order filters are more suitable for such cases. A filter of reduced order is a filter whose order is less than that of the original system. In the literature, a fixed-order filter indicates that the filter is of a given order, the given order may be equal to or less than the order of the original system. Reduced-order filtering techniques, concerning the energy-to-peak performance in discrete-time domain [42], the $H_\infty$ performance of either Markovian jump linear systems [58] or time-delay Markovian jump linear systems [41] in the continuous-time domain, have been studied.

**Motivation and Research Problem**

The uncertainties considered by most researchers exist in the system matrices only, and the mode transition rate matrix (or the mode transition probability matrix for discrete-time cases) is assumed to be precisely known *a priori*. However, it should be more reasonable that only the estimated values of the mode transition rate matrix are available and estimation errors exist. In other words, the mode transition rate matrix has uncertainties, too. Naturally, the uncertainties in the mode transition rate matrix may lead to instability or at least degraded performance of the system as the uncertainties in the system matrices may do. Therefore, it is necessary to handle the uncertainties in the mode transition rate matrix in addition to the uncertainties in the system matrices.
Fortunately, two different types of descriptions about the uncertainties in the mode transition rate matrix have been proposed. The first one is the polytope-type description, where either the mode transition rate matrix of continuous-time Markovian jump linear systems, or the mode transition probability matrix of discrete-time Markovian jump linear systems, is assumed to be in a convex matrix polytope with known vertices [13, 18, 27]. The other type is described in an element-wise way, in which every element of the mode transition rate matrix is upper-bounded [4, 44].

The element-wise uncertainty description can be reformulated into an equivalent polytope-type description, but the number of vertices of the matrix polytope will be extremely large if the system has more than three operation modes. Therefore, it makes sense to study the element-wise uncertainties directly. Moreover, the elements of the mode transition rate matrix are often estimated individually in practice, and the error bounds of the estimation are provided at the same time. Hence, the element-wise description is more natural than the polytope-type description.

The robust stabilization problem was studied in [4, 44], where the system has element-wise uncertainties in the mode transition rate matrix. However, the technique used in [4, 44] is quite simple, and the results tend to be conservative. In addition, an important fact that the sum of the uncertainties along the rows of the transition rate matrix must be zero is ignored in this technique.

So a research problem is raised as follows:

**Problem 1** How to improve current techniques via exploiting the probability constraints on the rows of the mode transition rate matrix?

### 1.2.2 Markovian Jump Linear Systems with Time Delay

Ordinary differential/difference equations alone may satisfactorily describe a dynamic system, if the future evolution of the system state depends solely on its current state, but often fail to describe dynamic systems whose future evolutions not only depend on the current state, but also on the past states. Such systems are often satisfactorily modeled
by functional differential/difference equations with after-effects, and are referred to as time-delay systems. Since time delay may be constant or time-varying, the literature review in this subsection is done according to the types of time delay, and the focus is placed on the stabilization and $H_\infty$ control problems of Markovian jump linear systems with time delay. After the review, we are fully motivated to state our second research problem.

**Constant Delay**

The simplest model of time delay for Markovian jump linear systems is that all the subsystems share a constant time delay. The solutions to the stabilization and $H_\infty$ control problems of such time-delay Markovian jump linear systems are often given as delay-independent conditions. Delay-independent conditions imply that the conditions do not include any information about the time delay, hence can guarantee that the system has the desired performance for any finite constant delay. The stochastic stability of continuous-time Markovian jump linear systems was analyzed in [2], and a sufficient condition was provided in the form of an inequality of matrix eigenvalues concerning the system matrices and the mode transition rate matrix. The robust stabilization and $H_\infty$ control problems were studied in [8, 54] as well, and sufficient conditions were presented based on a set of coupled linear matrix inequalities for the design of state-feedback controllers.

A major disadvantage of delay-independent conditions is that they are far too conservative when the time delay is small. To reduce the conservatism of the delay-independent conditions, some information about the time delay can be used to obtain the so-called delay-dependent conditions. Much effort in this research direction is devoted to seek less conservative conditions. An effective way to reduce the conservatism is to introduce slack matrices in the derived conditions. In [9], a full-order dynamic output-feedback controller was constructed to stabilize the system based upon a delay-dependent condition, where the descriptor system approach [29] was used to introduce
slack matrices. The $H_\infty$ control problem was investigated in [6] as well, and a positive semi-definite matrix [46] was used to introduce slack matrices. By using some identities containing slack matrices [30, 65], new delay-dependent results were obtained in the form of linear matrix inequalities, and the robust stabilization and $H_\infty$ control problems were dealt with in [62]. Instead of studying the robust stochastic stability of Markovian jump linear systems, the authors of [56] studied the robust exponential stabilization problem of time-delay Markovian jump linear systems. For delay-dependent results, the use of an optimization procedure is often recommended to determine the maximum allowable range of the constant time delay such that the system has the desired performance for any constant delay in that range.

Some researchers have observed that requiring all of the subsystems to have the same constant time delay may be too stringent for a Markovian jump linear system, and thought each subsystem having its own constant time delay as being more reasonable. In other words, the time delay should depend on system modes. Two types of delay-dependent conditions for Markovian jump linear systems with mode-dependent time delay have been proposed. The first type depends on the difference between the values of the largest and smallest time delays [3, 11], and the value of the time delay of each subsystem needs not to be known. The other type requires knowing the time-delay values of all the subsystems, and the conditions are related to each value of the time delays [10, 11, 43]. The robust stabilization and $H_\infty$ control problems were studied for discrete-time Markovian jump linear systems with mode-dependent time delays in [3, 10] and continuous-time ones in [11, 43].

**Time-Varying Delay**

A more complicated model of the time delay in Markovian jump linear systems is a common time-varying delay for all the subsystems. In the continuous-time domain, the time-varying delay changes in an interval, and its derivative is less than one. The stabilization and $H_\infty$ control problems have been studied via either state-feedback con-
Motivation and Research Problem

The basic idea of the control schemes in the references mentioned above is that the control signal is constructed according to the current system state and the current jumping parameter (i.e., the system mode).

On one hand, the accessibility of the jumping parameter is critical to the control of Markovian jump systems. Basically, there are three cases: (a) the jumping parameter is fully accessible. Then mode-dependent controllers can be used to issue the control inputs effectively according to the current system state and system mode. This case has been considered by most of the researchers; (b) the jumping parameter is not accessible at all. In this case, a mode-independent controller can be used to issue the control inputs based upon the current system state only, some researchers have considered this case; (c) the jumping parameter is accessible but with a time delay, no researchers have considered this situation so far.

On the other hand, the stabilization of delay Markovian jump linear systems with time delay is achieved by making the unstable plant stable enough so that the delayed terms can be tolerated. In other words, to stabilize a time-delay Markovian jump linear system, the delay-free system must be stabilized first [10, 43]. However, it is not always possible in some situations, for instance, when the current system state cannot be accessed instantly, to stabilize the unstable delay-free system.

To make the concept clear, let us consider the situation as shown in Figure 1.1, where the system itself is neither stable nor time delayed. However, time delays exist in the channels from the system to the controller. Such delays often arise from the measurements and the network transmissions of signals. In this case, the overall closed-loop system is a time-delay system with the delay-free system unstable. Moreover, the current system state and the current system mode are only accessible with time delay. The control techniques developed in the references above are no longer...
Consequently, a research problem arises naturally:

**Problem 2** How to achieve efficient control of Markovian jump linear systems with delayed system information?

### 1.2.3 Networked Control Systems

In this subsection, a general architecture of networked control systems is described first, then the literature review is accomplished by surveying two important topics of networked control systems. Finally, the third research problem is stated.

Networked control systems are feedback control systems with the control loops closed via a shared communication network as shown in Figure 1.2. It should be pointed out that the use of the network among the control components is mainly motivated by lower cost, easier maintenance and higher reliability, but often at the expense of the performance of the closed-loop systems [59]. As shown in Figure 1.3, a general
modern control system uses dedicated point-to-point connections for information exchange by contrast. These point-to-point connections are often required to be reliable enough so that signals are guaranteed to reach their destinations instantly. Networked control systems have been applied to many fields such as automobiles, aircraft and HVAC systems [59] since the system components may allow to be spatially isolated from each other, operated in an asynchronous manner, and communicated over a wide or local area via both wired and wireless links.

The analysis and synthesis of networked control systems are much more complicated than those of general modern control systems. Conventional control theories must be re-examined before they are applied to a networked setting. Many methodologies to deal with the control problems of networked control systems have been proposed, and may be classified into the following three categories: (a) control systems are firstly designed in the traditional way without considering the network. Then a performance level that the network should satisfy is determined so that the networked control systems maintain their performances when the network is inserted. Finally, a lot of effort goes into the design and scheduling of the communication network to meet the required network performance level; (b) the characteristic of the network is given in advance as communication constraints. The main effort is then placed on the design of control components under these communication constraints such that the networked control system has the desirable performance; (c) the communication network and the control components are co-designed.

In a typical networked control system setting, the plant, the sensor and the actuator are located on one side of the communication network, while the controller is located

![Figure 1.3](image-url)
remotely on the other side (see Figure 1.2). Because of the unreliable and random nature of the network for signal transmissions, new interesting and challenging problems arise. Network-induced time delay and packet loss are two essential issues deserving careful consideration in a networked control system design.

In the literature, network-induced time delay and packet loss can be handled separately or simultaneously. So, the literature review in this subsection is divided into three parts. The first part reviews the techniques developed for networked control systems with time delay only. In this case, the network uses a reliable transport protocol. As a result, packet loss can be avoided while time delay is significant. The techniques for networked control systems with packet loss only are reviewed in the second part. In this case, the network tends to drop packets to shorten the delay of the successful transmissions. As a result, the effect of time delay is negligible but packet loss must be considered. The third part reviews the techniques for networked control systems where neither time delay nor packet loss can be ignored.

**Time Delay Only**

There are several situations where time delay may occur. Firstly, sampling and coding of physical signals take time. Also the controller consumes time in order to calculate control signals. Secondly, accessing delay almost always exists because a network node has to wait to send out packets when the network is busy. Thirdly, transmission delay always exists due to the limited bit rate of the communication network. Two differences between networked control systems and non-networked time-delay systems are that: (a) the measured outputs of networked control systems must be sampled; (b) the overall time delay of networked control systems can be further divided into the delays from sensors to controllers and from controllers to actuators. There are generally three types of network-induced time delay.

The first one is constant time delay, which is achieved by fixing every time delay to the upper bound value. Long data buffers are often necessary in this case [37].
However, the use of the data buffer makes the arrived signal to be used at the maximum time delay, and results in an augmented closed-loop system.

The second type is random time-varying delay. In the continuous-time domain, the delay is time-varying, and characterized by a continuous random variable. In [47], all the time delays were assumed to be independent over the full horizon, and their probability distributions were known \textit{a priori}. Under such assumptions, the continuous-time system was discretized according to the time delays from the sensor to the controller and from the controller to the actuator. The optimal linear quadratic Gaussian problem was studied using dynamic programming techniques. A major drawback of this approach is that the total time delay is restricted to be less than one sampling period. The authors of [32] followed the research line of [47], and successfully relaxed this restriction to longer time delays by using the state augmentation technique. Another way of relaxing this restriction is via adopting the delta operator representation of the system [31]. However, a major disadvantage of the methods in [31,32] is that the obtained conditions are concerned about matrices of large dimensions due to the augmentation technique, and become intractable if time delay is large enough.

In the discrete-time domain, time delay can be modeled either by a white discrete random variable with Bernoulli distribution [66] or by a discrete-time Markov chain with known transition probabilities [72]. In [66], time delay was restricted to shorter than two sampling periods. Only two possibilities exist: (a) if the delay is less than one sampling period, the influence of time delay is ignored, and the system evolves as a delay-free system; (b) if the delay is greater than one sampling period, the system is considered to be a unit-delay system. A Bernoulli sequence was used to indicate if the delay is zero or unity. The closed-loop system was finally translated into a delay-free system with stochastic parameters. In the mean square sense, an observer-based control scheme was proposed for the $H_{\infty}$ control problem of networked control systems. In [72], the total time delay was decomposed into a sensor-to-controller delay and a controller-to-actuator delay. The random time delays are modeled as two independent
discrete-time Markov chains, and treated separately. Through augmenting the system state, the networked control system was written as a delay-free Markovian jump linear system. A necessary and sufficient condition was established for the design of the delay-dependent controller such that the closed-loop system is stochastically stable.

The third type is deterministic time-varying delay. In [37], the controller-to-actuator delay was fixed at its upper bound value via a data buffer, while the sensor-to-controller delay could be changed arbitrarily within an upper bound. Also, through the augmentation technique, the closed-loop system was written as a switched system with arbitrary switching. Dynamic output-feedback controllers were designed for the $H_{\infty}$ control of the system in the discrete-time domain. In [48], the total time delay was unknown and time-varying, but could be decomposed into two parts: fixed part and varying part. The fixed part was assumed to be an unknown integer multiple of the sampling period. The varying part was assumed to be unknown, but upper-bounded by one sampling period. The system was discretized to an uncertain system with time delay. The idea of [48] is that the parameters involving the varying part of the delay in the discretized system are treated as system uncertainties. A stabilization technique was developed in the discrete-time domain based upon the assumption that the parameter uncertainties are of norm-bounded form.

**Packet Loss Only**

Packet loss results in both the controller receiving less measurement signals and the plant receiving less control commands. Here are some practical reasons why packets are often lost during the transmission through the network. Firstly, before being sent to the network, packets may be purposely dropped by the senders according to the scheduling algorithms of the network (e.g., try-once-discard policy [60], which discards data as long as the network is unavailable). Secondly, even if packets are sent into the network, they may also be lost due to network congestion. In this case, the network drops out packets to reduce the queue size in the path, and informs the senders
to reduce their transmission rates. Thirdly, after arriving at their destinations, packets may be discarded by the receivers either because these packets contain fatal errors introduced during the transmission or because they reach the destinations too late (e.g., later than their successors). In these cases, the receivers usually drop the newly arrived but information-old packets, and continuously use the most recent ones [50, 52] or just zeros [49] or estimated values [40].

A network with packet loss can be modeled by a switch with two states. In [73], the controller was designed to stabilize the plant without considering the network. Design effort went into determining the level of the successful transition rate of the network under which the system remains stable. A Bernoulli process provides a more natural model for the network with packet loss. In this case, a networked control system can be converted into a two-state Markovian jump system. The techniques developed for Markovian jump linear systems become applicable if the probability distribution of the Bernoulli process is known. In [50, 52], dynamic output-feedback controllers were designed such that networked control systems are mean square stable and have $H_\infty$ gains below a certain value. In [40], the networked control system was shown to be equivalent to a linear time-invariant system in the sense that both systems generate the same power spectral densities. A packet-loss compensator was determined via an optimization formulation that minimizes the output power.

In [57], the discrete-time Kalman filtering technique was re-examined in the networked setting. A linear discrete-time system was considered. The arrival of the observation was modeled by a Bernoulli process. A critical value for the arrival probability of the observation update was proved to exist such that if the network performance level is greater than the critical value, then the mean state covariance of the estimation error is bounded for all initial conditions; otherwise, the covariance diverges for some initial conditions.

The variance-constrained filtering problem of a class of uncertain stochastic systems was studied in [61], where the measurements are not consecutive, but contain
missing observations. Although not in the networked control system setting, the filtering problem of [61] can be regarded as a variance-constrained filtering problem of networked control systems with packet loss. A full-order filter was designed in [61] such that the error state of the filtering process is mean square bounded and the steady-state variance of the estimation error of each state is not more than the individual prescribed bound.

The packet-loss process of the network can also be modeled by an arbitrary switching signal taking values in a finite set. Then networked control systems can be converted into switched systems with arbitrary switching. A stabilizing state-feedback controller was designed based on the theory of linear switched systems in [69].

**Both Time Delay and Packet Loss**

Both network-induced time delay and packet loss can be considered in a unified framework through the viewpoint of either switched systems or input-delay systems. In [39], the controller has a receiving buffer which contains the most recently arrived data packet from the sensor. The buffer is periodically read by the controller at a higher frequency than the sampling frequency. Consequently, all the probabilities that time delay and packet loss may occur can be classified into a finite number of patterns. The networked control system is accordingly discretized to a switched system with a switching signal indicating the patterns. The switching of the discretized system is arbitrary. The strength of this approach comes from the solid theoretical results contained in the literature of switched systems.

Networked control systems with both time delay and packet loss can also be modeled by dynamic systems with time-varying input delay. The input delay takes values in an interval and its derivative is not strictly less than one. Under this framework, many techniques developed for time-delay systems can be borrowed with slight modifications. For the continuous-time case, the discretization of the networked control system is generally not needed for controller design. The stabilization problem [68]
and the robust $H_\infty$ control problem [70] have been studied in the continuous-time domain. For the discrete-time plants, the stabilization problem has also been investigated when time delay changes arbitrarily [68], or randomly [72].

Motivation and Research Problem

On one hand, the motivation of the research problem in this subsection is partially inspired by [50] and [68]. In [50], the effect of packet loss in a network was modeled as a Bernoulli process. The closed-loop system was then converted into a Markovian jump linear system with two modes. As a result, the general theory for Markovian jump linear systems, which has been developed for any finite number of operation modes, was not fully used. Therefore, improved results may be obtained if the theory of Markovian jump systems are fully applied. In [69], the network experiences arbitrary but finite packet loss. The state trajectory was studied directly. A switching signal was used to indicate different state responses for different packet-loss patterns. Although the idea is interesting, a quadratic Lyapunov function method was adopted. Consequently, if one could construct a new Lyapunov function that may depend on the packet-loss process, less conservative conditions are likely to be obtained in the same way that delay-dependent conditions are obtained for time-delay systems. Moreover, the network in [50, 68] only exists between the sensor and the controller.

On the other hand, the actuator is commonly configured to be an event-driven zero-order hold in most of the networked control system settings. Consequently, the inputs take effect as soon as they arrive at the zero-order hold. However, the event-driven zero-order hold complicates the analysis and synthesis of the system if the problem is studied in the discrete-time domain. The reason is that the zero-order hold is no longer synchronized with the sensor. In addition, during each sampling period, the number of the arrived packets is time-varying (maybe one or more or even zero). Therefore, the zero-order hold should have a logic to enable the use of the most recent control input packet, or something computed based on several packages that have arrived previously,
to the plant. To sum up, a smart (or special) zero-order hold should be used when networked control systems are studied in the discrete-time domain.

When a network exists not only between the sensor and the controller but also between the controller and the actuator, the following research problem(s) is raised:

**Problem 3** *How to control a plant over the network in the discrete-time domain by finding a new packet-loss dependent Lyapunov function or configuring a smart zero-order hold?*

### 1.3 Thesis Outline

- Chapters 2–4 provide a solution to Problem 1. The robust stabilization, robust $H_2$ control, robust $H_\infty$ control and fixed-order $H_\infty$ filter design are studied for continuous-time Markovian jump linear systems. The uncertainties under consideration are norm-bounded uncertainties in the system matrices and element-wise uncertainties in the mode transition rate matrix. By exploiting the probability constraints on the rows of the mode transition rate matrix, new sufficient conditions are established to solve these synthesis problems in terms of the solvability of linear matrix inequalities with equality constraints. To design the controllers and the filters effectively, globally convergent algorithms involving convex optimizations are suggested. Finally, numerical examples and comparisons with existing results are used to illustrate the effectiveness of the developed results.

- Chapter 5 studies Problem 2. The stabilization problem is considered for discrete-time Markovian jump linear systems. The delay in the system state may be time-varying. The delay in the jumping parameter is manifested as a constant mismatch of the modes between the controller and the system. A sufficient condition is proposed for the design of the controller such that the closed-loop system is stochastically stable. Numerical simulations are used to illustrate the
developed theory.

- Chapters 6–7 investigate Problem 3. In Chapter 6, two types of packet-loss process are introduced: arbitrary packet-loss process and Markovian packet-loss process. Stabilization conditions are established via a packet-loss dependent Lyapunov function approach. Although the main focus is placed on the packet-loss issue, these conditions are extended to the unit time-delay case as well. In Chapter 7, time delay and packet loss are considered simultaneously. Zero-order hold is configured to be both time-driven and event-driven. A logic is used by the zero-order hold to choose the most recent control input packet as well. Thus the continuous-time process can be discretized to a discrete-time system with input delay. Several sufficient conditions for constructing the stabilizing controllers are given based upon the Lyapunov theory. In both Chapters 6 and 7, numerical examples and simulations are used to demonstrate the developed results.

1.4 Contributions

The contributions of the thesis are summarized as follows:

- It is shown that the uncertainties in the mode transition rate matrix can destabilize a Markovian jump linear system even if the nominal system can be stabilized.

- New techniques for robust stabilization, robust $H_2$ control, robust $H_{\infty}$ control and fixed-order robust $H_{\infty}$ filter design are given for continuous-time Markovian jump linear systems. Numerical examples and theoretical comparisons show that these new techniques are less conservative than existing ones.

- New stabilization conditions are presented for discrete-time Markovian jump linear systems with delayed system information. Moreover, it is demonstrated that the delay in the jumping parameter is critical for the control task.
• New stabilization results of networked control systems with packet loss are obtained through the use of packet-loss dependent Lyapunov functions.

• A general framework of networked control systems by adopting a special zero-order hold is proposed. New conditions are found for the stabilization of networked control systems with both time delay and packet loss.
Robust Stabilization of Uncertain MJLS

Starting from this chapter and up to Chapter 4, improved techniques will be presented to handle uncertain Markovian jump linear systems. This chapter studies the robust stabilization problem. Uncertainties are assumed to be norm-bounded in the system matrices and element-wise bounded in the mode transition rate matrix. The main objective of this chapter is to design a state-feedback controller such that the closed-loop system is robustly mean square stable over all the admissible uncertainties.

The organization of the chapter is as follows. Section 2.1 formulates the robust stabilization problem to be solved. The element-wise uncertainties in the mode transition rate matrix are described in details, and compared with other models of the uncertain mode transition rate matrix. In Section 2.2, the robust stability property of the system is analyzed. The obtained result is also compared with existing ones. A controller design procedure is provided in Section 2.3 and Section 2.4. Section 2.5 presents an example. A summary is given in Section 2.6.
CHAPTER 2. ROBUST STABILIZATION OF UNCERTAIN MJLS

2.1 Problem Formulation

Consider the following continuous-time Markovian jump linear systems defined on a complete probability space \((\Omega, \mathcal{F}, P)\):

\[
\dot{x}(t) = \hat{A}(\hat{r}(t))x(t) + \hat{B}(\hat{r}(t))u(t) \tag{2.1}
\]

where \(t \geq 0\) is the time, \(x(t) \in \mathbb{R}^n\) is the system state, \(u(t) \in \mathbb{R}^m\) is the control input. The mode jumping process \(\{\hat{r}(t) : t \geq 0\}\) is a continuous-time, discrete-state and homogeneous Markov process with right continuous trajectories. \(\hat{r}(t)\) takes values in a finite state space \(S \equiv \{1, 2, \ldots, s\}\), and has the stationary mode transition probabilities

\[
\Pr(\hat{r}(t + \delta t) = j \mid \hat{r}(t) = i) = \begin{cases} \hat{\pi}_{ij}\delta t + o(\delta t), & \text{if } j \neq i \\ 1 + \hat{\pi}_{ii}\delta t + o(\delta t), & \text{if } j = i \end{cases}
\]

where \(\delta t > 0\) satisfies \(\lim_{\delta t \to 0} \frac{o(\delta t)}{\delta t} = 0\). For each \(i, j \in S\), if \(j \neq i\) then \(\hat{\pi}_{ij} \geq 0\) is the transition rate from mode \(i\) at time \(t\) to mode \(j\) at time \(t + \delta t\); if \(j = i\) then \(\hat{\pi}_{ii} \triangleq -\sum_{j=1,j\neq i}^s \hat{\pi}_{ij}\).

For a brief presentation, we denote \(\hat{A}_i \triangleq \hat{A}(\hat{r}(t) = i)\) and \(\hat{B}_i \triangleq \hat{B}(\hat{r}(t) = i)\) for all \(i \in S\). \(\hat{A}_i\) and \(\hat{B}_i\) are not precisely known \textit{a priori}, and have the following norm-bounded uncertainties:

\[
\begin{bmatrix} \hat{A}_i & \hat{B}_i \end{bmatrix} = \begin{bmatrix} A_i & B_i \end{bmatrix} + E_i F_i \begin{bmatrix} H_{ai} & H_{bi} \end{bmatrix} \tag{2.2a}
\]

where \(A_i\) and \(B_i\) are the nominal values of \(\hat{A}_i\) and \(\hat{B}_i\), respectively; \(F_i\) denotes the uncertainties in the system matrices \(\hat{A}_i\) and \(\hat{B}_i\), and satisfies the norm-bounded condition:

\[F_i^T F_i \leq I\]

Matrices \(E_i, H_{ai}\) and \(H_{bi}\) describe the way that the uncertainties in \(F_i\) enter the system.
matrices. Moveover, matrices $A_i$, $B_i$, $E_i$, $H_{ai}$ and $H_{bi}$ are known constant real matrices of appropriate dimensions.

The mode transition rate matrix $\hat{\Pi} \triangleq (\hat{\pi}_{ij}) \in \mathbb{R}^{s \times s}$ is also not exactly known \textit{a priori}, and has the element-wise bounded uncertainties:

$$\hat{\Pi} = \Pi + \Delta \Pi$$ (2.2b)

where $\Pi \triangleq (\pi_{ij}) \in \mathbb{R}^{s \times s}$ with $\pi_{ij}$ being the nominal value of $\hat{\pi}_{ij}$. For each $i, j \in S$, if $j \neq i$ then $\pi_{ij} \geq 0$ denotes the estimated transition rate for mode $i$ to mode $j$; if $j = i$ then $\pi_{ii} \triangleq - \sum_{j=1, j \neq i}^{s} \pi_{ij}$. $\Delta \Pi \triangleq (\Delta \pi_{ij}) \in \mathbb{R}^{s \times s}$, where $\Delta \pi_{ij} \triangleq \hat{\pi}_{ij} - \pi_{ij}$ is the bounded uncertainty in $\hat{\pi}_{ij}$, and could be thought of as the estimation error in practice. The uncertainty set $\Delta \Pi$ is described by

$$\left\{ \begin{array}{l}
|\Delta \pi_{ij}| \leq \varepsilon_{ij}, \quad 0 \leq \varepsilon_{ij} \leq \pi_{ij}, \quad \text{for all } i, j \in S, j \neq i \\
\Delta \pi_{ii} \triangleq - \sum_{j=1, j \neq i}^{s} \Delta \pi_{ij}, \quad \text{for all } i \in S.
\end{array} \right. $$ (2.2c)

\textbf{Remark 2.1} A slightly different description of $\Delta \Pi$ is

$$|\Delta \pi_{ij}| \leq \varepsilon_{ij}, \quad 0 \leq \varepsilon_{ij} \leq \pi_{ij}, \quad \text{for all } i, j \in S. $$ (2.3)

Such a model was considered in [4, 44, 53]. A crucial difference between (2.2c) and (2.3) is that $\varepsilon_{ii}$ is not defined in (2.2c) for all $i \in S$. If the probability constraints on the rows of $\hat{\Pi}$ and $\Pi$:

$$\sum_{j=1}^{s} (\pi_{ij} + \Delta \pi_{ij}) = 0 \quad \text{and} \quad \sum_{j=1}^{s} \pi_{ij} = 0$$

are considered, then one has

$$\sum_{j=1}^{s} \Delta \pi_{ij} = 0, \quad \text{that is,} \quad \Delta \pi_{ii} = - \sum_{j=1, j \neq i}^{s} \Delta \pi_{ij}. $$ (2.4)
Hence
\[ \varepsilon_{ii} = \sum_{j=1, j \neq i}^{s} \varepsilon_{ij}. \] (2.5)

Obviously, the model in (2.3) ignores the probability constraints in (2.4) and the upper bound relationship in (2.5).

**Remark 2.2** Another description of the uncertain mode transition rates is the polytope-type model. In this model, the mode transition rate matrix \( \hat{\Pi} \) is unknown, but belongs to a fixed matrix polytope:
\[ \hat{\Pi} \triangleq \sum_{l=1}^{L} \alpha_l \Pi^{(l)} \] (2.6)
where \( \alpha_l \geq 0, l = 1, 2, \ldots, L \), satisfy \( \sum_{l=1}^{L} \alpha_l = 1 \), and
\[ \Pi^{(l)} \triangleq (\pi^{(l)}_{ij}) \]
for \( l = 1, 2, \ldots, L \) are the vertices of the polytope. Each of them is a well-defined mode transition rate matrix. All of the vertices \( \Pi^{(l)} \) are known, while none of the coefficients \( \alpha_l \) are known. From an application point of view, element-wise description given in (2.2b) and (2.2c) is more natural than polytope-type description shown in (2.6).

**Remark 2.3** The element-wise description in (2.2b)–(2.2c) can be reformulated into the polytope-type description in (2.6) by introducing \( L = 2^{s(s-1)} \) vertex matrices. For example, suppose the uncertain Markovian jump system has two operation modes (i.e., \( s = 2 \)) and the element-wise uncertainties in (2.2b)–(2.2c) are
\[ \hat{\Pi} = \begin{bmatrix} -\pi_{12} - \Delta \pi_{12} & \pi_{12} + \Delta \pi_{12} \\ \pi_{21} + \Delta \pi_{21} & -\pi_{21} - \Delta \pi_{21} \end{bmatrix} \]
with
\[
\begin{cases}
|\Delta \pi_{12}| \leq \varepsilon_{12}, & 0 \leq \varepsilon_{12} \leq \pi_{12} \\
|\Delta \pi_{21}| \leq \varepsilon_{21}, & 0 \leq \varepsilon_{21} \leq \pi_{21}.
\end{cases}
\]
The equivalent polytope-type description is

\[ \hat{\Pi} = \sum_{l=1}^{4} \alpha_l \Pi^{(l)} \]

with \( \alpha_l \geq 0, l = 1, 2, 3, 4 \), satisfying \( \sum_{l=1}^{4} \alpha_l = 1 \), where the \( L = 2^{(2-1)} = 4 \) vertex matrices are

\[
\begin{align*}
\Pi^{(1)} &= \begin{bmatrix} -\pi_{12} + \varepsilon_{12} & \pi_{12} - \varepsilon_{12} \\ \pi_{21} - \varepsilon_{21} & -\pi_{21} + \varepsilon_{21} \end{bmatrix}, \\
\Pi^{(2)} &= \begin{bmatrix} -\pi_{12} - \varepsilon_{12} & \pi_{12} + \varepsilon_{12} \\ \pi_{21} - \varepsilon_{21} & -\pi_{21} + \varepsilon_{21} \end{bmatrix}, \\
\Pi^{(3)} &= \begin{bmatrix} -\pi_{12} + \varepsilon_{12} & \pi_{12} - \varepsilon_{12} \\ \pi_{21} + \varepsilon_{21} & -\pi_{21} - \varepsilon_{21} \end{bmatrix}, \\
\Pi^{(4)} &= \begin{bmatrix} -\pi_{12} - \varepsilon_{12} & \pi_{12} + \varepsilon_{12} \\ \pi_{21} + \varepsilon_{21} & -\pi_{21} - \varepsilon_{21} \end{bmatrix}.
\end{align*}
\]

However, if the system has four operation modes, that is, \( s = 4 \), the total number of the vertex matrices reaches 4096 (\( L = 2^{4\times3} = 4096 \)), which is too large to translate the element-wise uncertainties in (2.2b)–(2.2c) into the polytope-type uncertainties in (2.6). As a result, the technique developed in [13, 27] becomes impractical. This drawback of the polytope-type uncertainty description in (2.6) motivates us to investigate the element-wise uncertainty description in (2.2b)–(2.2c) directly.

Let \( x(t; x_0, \hat{r}_0) \) be the trajectory of the system state of (2.1) from any initial system state \( x_0 \doteq x(0) \in \mathbb{R}^n \) and any initial operation mode \( \hat{r}_0 \doteq \hat{r}(0) \in S \). We have the following definition and proposition concerning mean square stability.

**Definition 2.1** [17] The nominal Markovian jump system of (2.1) with \( u(t) \equiv 0 \) is said to be mean square stable if

\[
\lim_{t \to \infty} \mathbb{E} \left( \|x(t; x_0, \hat{r}_0)\|^2 \right) = 0
\]

for any initial conditions \( x_0 \in \mathbb{R}^n \) and \( \hat{r}_0 \in S \).

**Proposition 2.1** [17] The nominal Markovian jump system of (2.1) with \( u(t) \equiv 0 \) is
mean square stable if and only if there exist matrices $P_i \in \mathbb{S}^{n \times n}, i \in S$, such that

$$A_i^T P_i + P_i A_i + \sum_{j=1}^{s} \pi_{ij} P_j < 0$$

for all $i \in S$.

The objective of this chapter is to develop new techniques for the stability analysis and stabilization synthesis of the uncertain Markovian jump linear system. To achieve this, we first introduce the concept of robust mean square stability for uncertain system (2.1) based upon Proposition 2.1.

**Definition 2.2** Uncertain Markovian jump system (2.1) with $u(t) \equiv 0$ is said to be robustly mean square stable if there exist matrices $P_i \in \mathbb{S}^{n \times n}, i \in S$, such that the inequality

$$\hat{A}_i^T P_i + P_i \hat{A}_i + \sum_{j=1}^{s} \hat{\pi}_{ij} P_j < 0$$

holds for all $i \in S$ over the admissible uncertainties in (2.2).

To obtain the main results, the following lemmas are useful.

**Lemma 2.1 (Schur Complement Equivalence)** [5] Given real constant matrices $Q_1, Q_2$ and $Q_3$, where $Q_1 = Q_1^T$ and $0 < Q_2 = Q_2^T$. Then the following inequality

$$Q_1 + Q_3 Q_2^{-1} Q_3^T < 0$$

holds if and only if

$$\begin{bmatrix} Q_1 & Q_3 \\ Q_3^T & -Q_2 \end{bmatrix} < 0.$$ 

**Lemma 2.2** [63] Given real matrices $Q$, $E$ and $H$ of appropriate dimensions with $Q = Q^T$. Then the inequality

$$Q + E F H + (E F H)^T < 0$$

holds.
holds for all $F$ satisfying $F^TF \leq I$ if and only if there exists a real number $\lambda \in \mathbb{R}^+$ such that

$$Q + \lambda H^T H + \frac{1}{\lambda} EE^T < 0.$$  

### 2.2 Robust Stability Analysis

The goal of robust stability analysis in this section is to develop a new criterion for testing the robust mean square stability of system (2.1) over admissible uncertainties (2.2).

**Theorem 2.1** Uncertain Markovian jump system (2.1) with $u(t) \equiv 0$ is robustly mean square stable, if there exist matrices $P_i \in \mathbb{S}^{n \times n}$, and scalars $\lambda_i \in \mathbb{R}^+$, $\lambda_{ij} \in \mathbb{R}^+$, $i, j \in \mathcal{S}$, $j \neq i$, such that

$$
\begin{bmatrix}
\Phi_{ii} & P_i E_i & \Gamma_i \\
E_i^T P_i & -\lambda_i I & 0 \\
I_i^T & 0 & -\Lambda_i
\end{bmatrix} < 0 
$$

(2.7)

for all $i \in \mathcal{S}$, where

$$\Phi_{ii} = A_i^T P_i + P_i A_i + \sum_{j=1}^{s} \pi_{ij} P_j + \frac{1}{4} \sum_{j=1, j\neq i}^{s} \lambda_{ij} E_{ij}^2 I + \lambda_i H_{ai}^T H_{ai},$$

$$\Gamma_i = \begin{bmatrix}
P_i - P_1 & \cdots & P_i - P_{i-1} & P_i - P_{i+1} & \cdots & P_i - P_s
\end{bmatrix},$$

$$\Lambda_i = \text{diag}(\lambda_1 I, \ldots, \lambda_{i-1} I, \lambda_{i+1} I, \ldots, \lambda_s I).$$

**Proof:** Consider uncertain system (2.1) with $u(t) \equiv 0$ and the admissible uncertainties in (2.2), we have $\hat{A}_i = A_i + E_i F_i H_{ai}$ and $\hat{\pi}_{ij} = \pi_{ij} + \Delta \pi_{ij}$ for all $i, j \in \mathcal{S}$. According to Definition 2.2, uncertain Markovian jump system (2.1) is robustly mean square stable if there exist matrices $P_i \in \mathbb{S}^{n \times n}$, $i \in \mathcal{S}$, such that

$$(A_i + E_i F_i H_{ai})^T P_i + P_i (A_i + E_i F_i H_{ai}) + \sum_{j=1}^{s} (\pi_{ij} + \Delta \pi_{ij}) P_j < 0$$

for all $i \in \mathcal{S}$. 


From (2.2c), we have

$$\Delta \pi_{ij} = - \sum_{j=1, j \neq i}^{S} \Delta \pi_{ij},$$

and hence

$$\sum_{j=1}^{S} (\Delta \pi_{ij} P_j) = \sum_{j=1, j \neq i}^{S} (\Delta \pi_{ij} P_j) + \Delta \pi_{ii} P_i = \sum_{j=1, j \neq i}^{S} (\Delta \pi_{ij} P_j) - \sum_{j=1, j \neq i}^{S} (\Delta \pi_{ij} P_i) = \sum_{j=1, j \neq i}^{S} \left[ \Delta \pi_{ij} (P_j - P_i) \right].$$

Thus, the above inequality given on the previous page can be re-written as:

$$A_i^T P_i + P_i A_i + \sum_{j=1}^{S} \pi_{ij} P_j + H_{ai}^T F_i E_i^T P_i + P_i E_i F_i H_{ai} + \sum_{j=1, j \neq i}^{S} \left[ \frac{1}{2} \Delta \pi_{ij} (P_j - P_i) + \frac{1}{2} \Delta \pi_{ij} (P_j - P_i)^2 \right] < 0.$$  \hspace{1cm} (2.8)

Using Lemma 2.2, this inequality holds for all $F_i$ satisfying $F_i^T F_i \leq I$ and all $\Delta \pi_{ij}$ satisfying $|\Delta \pi_{ij}| \leq \varepsilon_{ij}$, if there exist real numbers $\lambda_i \in \mathbb{R}^+$, $\lambda_{ij} \in \mathbb{R}^+$, $i, j \in S$, $j \neq i$, such that

$$A_i^T P_i + P_i A_i + \sum_{j=1}^{S} \pi_{ij} P_j + \lambda_i H_{ai}^T H_{ai} + \frac{1}{\lambda_i} P_i E_i E_i^T P_i + \sum_{j=1, j \neq i}^{S} \left[ \frac{\lambda_{ij}}{4} \varepsilon_{ij}^2 I + \frac{1}{\lambda_{ij}} (P_j - P_i)^2 \right] < 0,$$

which is equivalent to inequality (2.7) in view of Schur complement equivalence. \hspace{1cm} \square

Remark 2.4 Inequalities (2.7) are linear in $P_i$, $\lambda_i$, $\lambda_{ij}$ for $i, j \in S$, $j \neq i$, and thus the standard linear matrix inequality techniques [5] can be employed to check the robust mean square stability of uncertain Markovian jump system (2.1).

Remark 2.5 If no uncertainties exist in the mode transition rate matrix, that is, $\varepsilon_{ij} = 0$ for all $i, j \in S$, $j \neq i$, then Theorem 2.1 reduces to Theorem 2 of [55].

In the following, Theorem 2.1 is compared with the results in [13, 27]. To simplify the arguments, it is supposed that no uncertainties exist in the system matrices. In
this case, Theorem 2.1 reduces to Corollary 2.1, and the results in [13, 27] is stated in Proposition 2.2 (see Theorem 3.3 in [27] and Proposition 5 in [13]).

**Corollary 2.1** Uncertain Markovian jump system (2.1) with uncertainties (2.2b)–(2.2c) is robustly mean square stable if there exist matrices \( P_i \in \mathbb{S}^{n \times n}, \lambda_{ij} \in \mathbb{R}^+, i, j \in S, j \neq i, \) such that

\[
\begin{bmatrix}
\Phi_{2i} & \Gamma_i \\
\Gamma_i^T & -\Lambda_i
\end{bmatrix} < 0
\]  

for all \( i \in S, \) where

\[
\Phi_{2i} = A_i^T P_i + P_i A_i + \sum_{j=1}^{s} \pi_{ij} P_j + \frac{1}{4} \sum_{j=1, j \neq i}^{s} \lambda_{ij} \varepsilon_{ij}^2 I
\]

and \( \Gamma_i \) and \( \Lambda_i \) are given in Theorem 2.1.

**Proposition 2.2** [13, 27] Uncertain Markovian jump system (2.1) with uncertainties (2.6) is robustly mean square stable if there exist matrices \( P_i \in \mathbb{S}^{n \times n}, i \in S, \) such that

\[
A_i P_i + P_i A_i^T + \sum_{j=1}^{s} \pi_{ij}^{(l)} P_j < 0, \quad l = 1, 2, \ldots, L
\]

for all \( i \in S. \)

When Corollary 2.1 is used to test the robust stability of uncertain system (2.1), it is necessary to check the solvability of a \((s + 1)sn \times (s + 1)sn\) linear matrix inequality with respect with to \( \frac{n(n+1)}{2} s + s(s - 1) \) scalar variables. However, when Proposition 2.2 is used, a \((L+1)sn \times (L+1)sn\) linear matrix inequality with respect with to \( \frac{n(n+1)}{2} s \) scalar variables should be solved, where \( L = 2^{s(s-1)}. \) For instance, when \( n = 2 \) and \( s = 4, \) to test the robust stability of the uncertain system, it is needed to test the solvability of a \(40 \times 40\) linear matrix inequality system with respect to \(24\) scalar variables according to Corollary 2.1. To apply Proposition 2.2, the element-wise uncertainty in (2.2b)–(2.2c) is first translated into the polytope-type uncertainty in (2.6). The number of the
vertex matrices of the polytope is 4096 (see Remark 2.3). Then, it is needed to test the solvability of a $32776 \times 32776$ linear matrix inequality system with respect to 12 scalar variables. Therefore, Corollary 2.1 is much easier to solve in this case.

To compare Corollary 2.1 with the result in [4, 44, 53], Proposition 2.3 is given by using the technique in [4, 44, 53] (see Theorem 3 in [4] and Theorem 6.3 in [44]).

**Proposition 2.3** [4, 44] Uncertain Markovian jump system (2.1) with uncertainties (2.2b) and (2.3) is robustly mean square stable if there exist matrices $P_i \in \mathbb{S}^{n \times n}$, $i \in \mathcal{S}$, such that

$$A_i^TP_i + P_iA_i + \sum_{j=1}^{s}(\pi_{ij} + \varepsilon_{ij})P_j < 0$$

(2.10)

for all $i \in \mathcal{S}$.

On one hand, inequality (2.10) is equivalent to

$$A_i^TP_i + P_iA_i + \sum_{j=1, j \neq i}^{s} \pi_{ij}(P_j - P_i) + \sum_{j=1, j \neq i}^{s} \varepsilon_{ij}(P_j + P_i) < 0$$

in view of (2.5) and $\pi_{ii} = -\sum_{j=1, j \neq i}^{s} \pi_{ij}$. On the other hand, inequality (2.9) is equivalent to

$$A_i^TP_i + P_iA_i + \sum_{j=1, j \neq i}^{s} \pi_{ij}(P_j - P_i) + \sum_{j=1, j \neq i}^{s} \left[ \frac{1}{4} \lambda_{ij}\varepsilon_{ij}^2I + \frac{1}{\lambda_{ij}}(P_j - P_i)^2 \right] < 0.$$ 

By comparing the two inequalities above, it is not straightforward to say which term,

$$\sum_{j=1, j \neq i}^{s} \varepsilon_{ij}(P_j + P_i)$$

or

$$\sum_{j=1, j \neq i}^{s} \left[ \frac{1}{4} \lambda_{ij}\varepsilon_{ij}^2I + \frac{1}{\lambda_{ij}}(P_j - P_i)^2 \right],$$

is less conservative. Hence, it is difficult to conclude which one (Corollary 2.1 or Proposition 2.3) is theoretically better in the sense of conservatism.
2.3 Robust Stability Synthesis

This section handles the robust stabilization problem of uncertain Markovian jump linear systems. We aim at designing a state-feedback controller such that the resulting closed-loop system is robustly mean square stable over the admissible uncertainties.

Consider the state-feedback control law

$$ u(t) = K(\hat{r}(t))x(t) \quad (2.11) $$

where $K_i \doteq K(\hat{r}(t) = i) \in \mathbb{R}^{m \times n}$, $i \in S$, are the control gains to be determined.

The closed-loop system is

$$ \dot{x}(t) = \left([A(\hat{r}(t)) + B(\hat{r}(t))K(\hat{r}(t))] + E(\hat{r}(t))F(\hat{r}(t)) \times [H_a(\hat{r}(t)) + H_b(\hat{r}(t))K(\hat{r}(t))]]\right) x(t). \quad (2.12) $$

The following result solves the robust stabilization problem for system (2.1) with the admissible uncertainties in (2.2).

**Theorem 2.2** Consider uncertain Markovian jump system (2.1), there exists a state-feedback control law (2.11) such that closed-loop system (2.12) is robustly mean square stable, if there exist matrices $P_i \in \mathbb{S}^{n \times n}$, $X_i \in \mathbb{S}^{n \times n}$, $V_i \in \mathbb{S}^{n \times n}$, $Z_i \in \mathbb{S}^{n \times n}$, $Y_i \in \mathbb{R}^{m \times n}$, $\alpha_i \in \mathbb{R}^+$, $\lambda_{ij} \in \mathbb{R}^+$, $i, j \in S$, $j \neq i$, satisfying the coupled linear matrix inequalities

$$ \begin{bmatrix} \Phi_{3i} & (H_{ai}X_i + H_{bi}Y_i)^T & X_i \\ H_{ai}X_i + H_{bi}Y_i & -\alpha_i I & 0 \\ X_i & 0 & -Z_i \end{bmatrix} < 0 \quad (2.13) $$

$$ \begin{bmatrix} \Phi_{4i} & \Gamma_i \\ \Gamma_i^T & -A_i \end{bmatrix} \leq 0 \quad (2.14) $$
with equality constraints

\[ PX_i = I, \quad V_i Z_i = I \quad (2.15) \]

for all \( i \in S \), where

\[
\Phi_{3i} = (A_i X_i + B_i Y_i)^T + (A_i X_i + B_i Y_i) + \alpha_i E_i E_i^T,
\]
\[
\Phi_{4i} = -V_i + \sum_{j=1}^s \pi_{ij} P_j + \frac{1}{4} \sum_{j=1,j\neq i}^s \lambda_{ij} \xi_{ij}^2 I,
\]

and \( \Gamma_i \) and \( \Lambda_i \) are given in Theorem 2.1. In this case, controller (2.11) is given by \( K_i = Y_i P_i, \ i \in S \).

**Proof:** Let \( V_i \in \mathbb{R}^{n \times n} \) such that

\[
\sum_{j=1}^s \pi_{ij} P_j + \sum_{j=1,j\neq i}^s \left[ \frac{\lambda_{ij}}{4} \xi_{ij}^2 I + \frac{1}{\lambda_{ij}} (P_i - P_j)^2 \right] \leq V_i.
\]

This inequality is equivalent to (2.14) in view of Schur complement equivalence. Consider inequality (2.8), closed-loop system (2.12) is robustly mean square stable if the inequality

\[
(A_i + B_i K_i)^T P_i + P_i (A_i + B_i K_i) + \frac{1}{\lambda_i} P_i E_i E_i^T P_i
\]
\[+ \lambda_i (H_{ai} + H_{bi} K_i)^T (H_{ai} + H_{bi} K_i) + V_i < 0
\]

holds for all \( i \in S \).

Now, pre- and post-multiplying both sides of the above inequality by \( P_i^{-1} \) and applying the change of variables: \( X_i \equiv P_i K_i, \ Z_i \equiv V_i^{-1}, \ Y_i \equiv K_i X_i, \ \alpha_i \equiv \frac{1}{\lambda_i}, \) we have

\[
(A_i X_i + B_i Y_i)^T + (A_i X_i + B_i Y_i) + X_i Z_i^{-1} X_i
\]
\[+ \alpha_i E_i E_i^T + \frac{1}{\alpha_i} (H_{ai} X_i + H_{bi} Y_i)^T (H_{ai} X_i + H_{bi} Y_i) < 0
\]
which is equivalent to (2.13) by using Schur complement equivalence again. This completes the proof. □

2.4 Computational Method

Although the solution set of Theorem 2.2 is not convex due to the equality constraints in (2.15), the sequential linear programming matrix method (SLPMM) developed in [38] can be used to solve such non-convex problems effectively.

Firstly, for computational purposes, a sufficiently small scalar $\beta \in \mathbb{R}^+$ is introduced and inequality (2.13) is changed to

$$
\begin{bmatrix}
\Phi_{3i} + \beta I & (H_{a_i}X_i + H_{b_i}Y_i)^T X_i \\
H_{a_i}X_i + H_{b_i}Y_i & -\alpha_i I & 0 \\
X_i & 0 & -Z_i
\end{bmatrix} \leq 0.
$$

(2.16)

Secondly, equality constraints (2.15) are weakened to the semi-definite programming conditions:

$$
\begin{bmatrix}
P_i & I \\
I & X_i
\end{bmatrix} \succeq 0, \quad \begin{bmatrix}
V_i & I \\
I & Z_i
\end{bmatrix} \succeq 0.
$$

(2.17)

Note that the equality constraints in (2.15) correspond to the boundary of the convex sets defined by (2.17).

Finally, the sequential linear programming matrix method can be employed to find a solution to Theorem 2.2. The solution to the robust stabilization problem (RSP) is summarized below.

**Algorithm RSP** For a given precision $\delta \in \mathbb{R}^+$, let $k_{\text{max}}$ be the maximum number of iterations and $\beta \in \mathbb{R}^+$ be a sufficiently small number.

1. Determine $P_i^{(0)}, X_i^{(0)}, V_i^{(0)}, Z_i^{(0)}, Y_i^{(0)}, \alpha_i^{(0)},$ and $A_{ij}^{(0)}, i, j \in \mathcal{S}, j \neq i,$ satisfying (2.14), (2.16) and (2.17). Let $k := 0.$
(2) Solve the convex optimization problem

\[
\min \sum_{i=1}^{S} \text{trace} \left( P_i X_i^{(k)} + P_i^{(k)} X_i + V_i Z_i^{(k)} + V_i^{(k)} Z_i \right)
\]

subject to (2.14), (2.16) and (2.17) for all \( i \in S \).

with respect to the variables \( P_i, X_i, V_i, Z_i, Y_i, \alpha_i, \) and \( \lambda_{ij}, i, j \in S, j \neq i \).

(3) Let

\[
T_i^{(k)} := P_i, \quad L_i^{(k)} := X_i, \quad U_i^{(k)} := V_i, \quad R_i^{(k)} := Z_i
\]

for all \( i \in S \).

(4) If

\[
\left| \sum_{i=1}^{S} \text{trace} \left( T_i^{(k)} X_i^{(k)} + P_i^{(k)} X_i + U_i^{(k)} Z_i^{(k)} + V_i^{(k)} R_i^{(k)} \right) \right| - 2 \sum_{i=1}^{S} \text{trace} \left( P_i^{(k)} X_i^{(k)} + V_i^{(k)} Z_i \right) < \delta
\]

then go to Step (7), otherwise go to Step (5).

(5) Compute \( \theta^* \in [0, 1] \) by solving

\[
\min_{\theta \in [0,1]} \sum_{i=1}^{S} \text{trace} \left( [P_i^{(k)} + \theta (T_i^{(k)} - P_i^{(k)})] [X_i^{(k)} + \theta (L_i^{(k)} - X_i^{(k)})] \right)
\]

\[
+ \left[ V_i^{(k)} + \theta (U_i^{(k)} - V_i^{(k)}) \right] [Z_i^{(k)} + \theta (R_i^{(k)} - Z_i^{(k)})] \right).
\]

(6) Let

\[
P_i^{(k+1)} := P_i^{(k)} + \theta^* (T_i^{(k)} - P_i^{(k)}), \quad X_i^{(k+1)} := X_i^{(k)} + \theta^* (L_i^{(k)} - X_i^{(k)}),
\]

\[
V_i^{(k+1)} := V_i^{(k)} + \theta^* (U_i^{(k)} - V_i^{(k)}), \quad Z_i^{(k+1)} := Z_i^{(k)} + \theta^* (R_i^{(k)} - Z_i^{(k)}),
\]
for all $i \in S$, and $k := k + 1$. If $k < k_{\text{max}}$, then go to Step (2), otherwise go to Step (7).

(7) If
\[ \sum_{i=1}^{s} \text{trace} \left( P_i^{(k)} X_i^{(k)} + V_i^{(k)} Z_i^{(k)} \right) = 2sn \]
then a solution is found, otherwise a solution cannot be found by this algorithm.

**Remark 2.6** Algorithm RSP is similar to the SLPMM procedure proposed in [38] for the design of suboptimal static $H_2/H_{\infty}$ output-feedback controllers. Moreover, if Step (5) in Algorithm RSP is skipped by setting $\theta^* = 0$, Algorithm RSP becomes similar to the cone complementarity linearization (CCL) algorithm in [26]. As explained in [38], Algorithm RSP always generates a strictly decreasing sequence of the objective function
\[ f(k) = \sum_{i=1}^{s} \text{trace} \left( P_i^{(k)} X_i^{(k)} + V_i^{(k)} Z_i^{(k)} \right). \]
That is,
\[ \sum_{i=1}^{s} \text{trace} \left( P_i^{(k+1)} X_i^{(k+1)} + V_i^{(k+1)} Z_i^{(k+1)} \right) < \sum_{i=1}^{s} \text{trace} \left( P_i^{(k)} X_i^{(k)} + V_i^{(k)} Z_i^{(k)} \right). \]
Thus, the sequence $\{f(k) : k = 0, 1, \ldots\}$ always converges to some $f^* \geq 2sn$. If $f^* = 2sn$, then the corresponding optimal values $P_i^*, X_i^*, V_i^*, Z_i^*, Y_i^*, \alpha_i^*, \lambda_{ij}^*$, $i, j \in S$, $j \neq i$, are a solution to Theorem 2.2. In addition, the sequence
\[ \left\{ (P_i^{(k)}, X_i^{(k)}, V_i^{(k)}, Z_i^{(k)}, Y_i^{(k)}, \alpha_i^{(k)}, \lambda_{ij}^{(k)}) : k = 0, 1, \ldots \right\} \]
generated by Algorithm RSP is bounded for all $i, j \in S$, $j \neq i$.

## 2.5 Numerical Examples

To illustrate the usefulness and flexibility of the developed theory, a simulation example is presented. Attention is focused on the design of a robust stabilizing controller
for an uncertain Markovian jump system.

Consider a uncertain Markovian jump system (2.1) with two operation modes. The system data and the initial condition are as follows:

\[
A_1 = \begin{bmatrix}
0.1769 & 0.7843 \\
0.9266 & 0.1363
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
0.5478 & 0.1279 \\
0.6160 & 0.9657
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
0.2995 \\
0.4471
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0.7417 \\
0.7957
\end{bmatrix},
\]

\[
\Pi = \begin{bmatrix}
-6.7000 & 6.7000 \\
6.9180 & -6.9180
\end{bmatrix}, \quad x_0 = \begin{bmatrix}
1 \\
-1
\end{bmatrix},
\]

\[
\hat{r}_0 = 1.
\]

The uncertainties in the mode transition rate matrix \( \hat{\Pi} \) are such that \(|\Delta \pi_{12}| \leq \varepsilon_{12} \) with \( \varepsilon_{12} \triangleq \frac{\pi_{12}}{2} \), and \(|\Delta \pi_{21}| \leq \varepsilon_{21} \) with \( \varepsilon_{21} \triangleq \frac{\pi_{21}}{2} \). The uncertain system given above is not mean square stable for all admissible uncertainties in the mode transition rate matrix. The instability of the nominal open-loop system is illustrated in Figure 2.1.

Without considering the uncertainties, a stabilizing controller for the nominal system can be obtained by Theorem 2 of [55] as

\[
K_1 = \begin{bmatrix}
-5.8205 & -7.1201
\end{bmatrix}, \quad K_2 = \begin{bmatrix}
-2.7969 & -3.7968
\end{bmatrix}.
\]

The corresponding numerical solution to Theorem 2 of [55] is

\[
X_1 = \begin{bmatrix}
0.0544 & -0.0072 \\
-0.0072 & 0.0490
\end{bmatrix}, \quad X_2 = \begin{bmatrix}
0.0514 & -0.0024 \\
-0.0024 & 0.0420
\end{bmatrix},
\]

\[
Y_1 = \begin{bmatrix}
-0.2655 & -0.3070
\end{bmatrix}, \quad Y_2 = \begin{bmatrix}
-0.1349 & -0.1529
\end{bmatrix}.
\]

Applying this controller makes the resulting closed-loop system mean square stable as illustrated in Figure 2.2. However, the closed-loop system turns out to be unstable
(see Figure 2.3) if there exist mode transition rate uncertainties, say, $\Delta \pi_{12} = \varepsilon_{12}$ and $\Delta \pi_{21} = -\varepsilon_{21}$. The stability property of the closed-loop system is illustrated by the shaded regions in the probability domain in Figure 2.4. Hence, it is necessary to consider the uncertainties in $\hat{\Pi}$ when designing controllers.

Using Algorithm RSP, a robust controller (2.11) could be achieved such that the closed-loop system is robustly mean square stable over all admissible uncertainties $|\pi_{12}| \leq \varepsilon_{12}$ and $|\pi_{21}| \leq \varepsilon_{21}$ (see Figure 2.5, where $\Delta \pi_{12} = \varepsilon_{12}$ and $\Delta \pi_{21} = -\varepsilon_{21}$). To compute with Algorithm RSP for this example, it is chosen that $\delta = 10^{-10}$, $k_{\text{max}} = 100$ and $\beta = 0.01$. One set of the solution is

\[
P_1 = \begin{bmatrix} 3.1258 & -3.2132 \\ -3.2132 & 6.5887 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3.1363 & -3.2961 \\ -3.2961 & 6.6446 \end{bmatrix},
\]

\[
X_1 = \begin{bmatrix} 0.6415 & 0.3129 \\ 0.3129 & 0.3044 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0.6661 & 0.3304 \\ 0.3304 & 0.3144 \end{bmatrix},
\]

\[
V_1 = \begin{bmatrix} 1.0973 & -0.0165 \\ -0.0165 & 1.2363 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0.4621 & -0.1150 \\ -0.1150 & 1.2419 \end{bmatrix},
\]

\[
Z_1 = \begin{bmatrix} 0.9115 & 0.0121 \\ 0.0121 & 0.8091 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 2.2149 & 0.2051 \\ 0.2051 & 0.8242 \end{bmatrix},
\]

\[
Y_1 = \begin{bmatrix} -2.3278 & -1.2479 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} -1.2714 & -1.2955 \end{bmatrix},
\]

\[
\lambda_{12} = 0.0711, \quad \lambda_{21} = 0.0337.
\]

The obtained controller is

\[
K_1 = \begin{bmatrix} -3.2667 & -0.7420 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.2829, & -4.4178 \end{bmatrix}.
\]

It can be verified that $\|P_1 X_1 - I\|_2 = 5.2148 \times 10^{-12}$, $\|P_2 X_2 - I\|_2 = 5.2138 \times 10^{-12}$, $\|V_1 Z_1 - I\|_2 = 5.2018 \times 10^{-12}$, $\|V_2 Z_2 - I\|_2 = 5.2023 \times 10^{-12}$. Therefore, the equality constraints in (2.15) are numerically satisfied.
Finally, the stability of the closed-loop system controlled by our controller should be verified by Corollary 2.1 or Proposition 2.2 or Proposition 2.3 numerically. As a comparison, Corollary 2.1 successfully indicates that the closed-loop system is robustly mean square stable over all admissible uncertainties $|\Delta \pi_{12}| \leq 3.35$ and $|\Delta \pi_{21}| \leq 3.459$. One set of the solution to Corollary 2.1 is

$$P_1 = \begin{bmatrix} 1.6238 & -1.4506 \\ -1.4506 & 2.3275 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1.6872 & -1.5546 \\ -1.5546 & 2.3344 \end{bmatrix},$$

$$\lambda_{12} = 0.1489, \quad \lambda_{21} = 0.1057.$$

However, Proposition 2.2 and Proposition 2.3 cannot give an indication on the robust mean square stability of the closed-loop system since MATLAB fails to find a feasible solution to the linear matrix inequality systems they defined.

### 2.6 Summary

This chapter has studied the robust stabilization problem for a class of uncertain Markovian jump linear systems. A new criterion for testing the robust stability of such systems has been established in terms of coupled linear matrix inequalities. A sufficient condition has also been proposed for the design of the robust state-feedback controllers such that the system is stabilized over all the admissible uncertainties. Moreover, a globally convergent algorithm involving convex optimization has been presented to help controller design. A numerical example is used to illustrate that the constructed controller could stabilize the system over the uncertainties in the mode transition rate matrix.
Figure 2.1  Initial condition response of the nominal open-loop system

Figure 2.2  Initial condition response of the nominal closed-loop system
Figure 2.3  Initial condition response of the uncertain closed-loop system

Figure 2.4  Stability regions of the probability domain
Figure 2.5  Initial condition response of the uncertain closed-loop system
Robust $H_2$ and $H_\infty$ Control of Uncertain MJLS

This chapter continues to solve Problem 1 (cf. page 10), and considers robust $H_2$ control and robust $H_\infty$ control of uncertain Markovian jump linear systems. In this chapter, uncertainties exist in the system matrices and the mode transition rate matrix. State-feedback controllers are designed such that the closed-loop system is not only robustly mean square stable but also satisfies certain $H_2$ or $H_\infty$ performance specification.

The chapter is organized as follows. Section 3.1 studies the robust $H_2$ control problem, and the robust $H_\infty$ control problem is investigated in Section 3.2. Section 3.3 is a summary which ends the chapter.

3.1 Robust $H_2$ Control

In this section, we aim at designing a linear state-feedback controller such that, over all the admissible uncertainties, the closed-loop system is robustly mean square stable and the $H_2$ norm of the closed-loop system is no more than a prescribed upper bound. The organization of this section is as follows. The robust $H_2$ control problem is formulated in Section 3.1.1. Some preliminary results are also provided to help the development of the main results. Section 3.1.2 deals with the robust $H_2$ analysis problem. A compar-
ison with existing results is given as well. The robust $H_2$ synthesis problem is tackled in Section 3.1.3, where a CCL-type algorithm is given to help the design of the controllers. In Section 3.1.4, an example is used to numerically illustrate the developed theory.

### 3.1.1 Problem Formulation

Consider the following class of uncertain Markovian jump linear systems defined on a complete probability space $(\Omega, \mathcal{F}, P)$:

$$
\begin{align*}
\dot{x}(t) &= \hat{A}(\hat{r}(t))x(t) + \hat{B}(\hat{r}(t))u(t) + \hat{B}_w(\hat{r}(t))w(t) \\
z(t) &= \hat{C}(\hat{r}(t))x(t) + \hat{D}(\hat{r}(t))u(t), \quad t \geq 0
\end{align*}
$$

(3.1)

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $w(t) \in \mathbb{R}^{n_w}$ is the disturbance input, $z(t) \in \mathbb{R}^{n_z}$ is the regulated output. The mode jumping process \{\hat{r}(t) : t \geq 0\} is a continuous-time, discrete-state, homogeneous Markov process with right continuous trajectories. \hat{r}(t) takes values in a finite state space $S \triangleq \{1, 2, \ldots, s\}$, and has the mode transition probabilities

$$
\Pr (\hat{r}(t + \delta t) = j | \hat{r}(t) = i) = \begin{cases} \\
\hat{\pi}_{ij}\delta t + o(\delta t), & \text{if } j \neq i \\
1 + \hat{\pi}_{ii}\delta t + o(\delta t), & \text{if } j = i
\end{cases}
$$

where $\delta t > 0$ and $\lim_{\delta t \to 0} \frac{o(\delta t)}{\delta t} = 0$; $\hat{\pi}_{ij} \geq 0$ denotes the switching rate from mode $i$ to mode $j$ for $i, j \in S$, $j \neq i$; $\hat{\pi}_{ii} \triangleq -\sum_{j=1, j\neq i}^s \hat{\pi}_{ij}$ for all $i \in S$.

The initial condition of the system state is $x_0 \triangleq x(0)$. The initial probability distribution of $\hat{r}_0 \triangleq \hat{r}(0)$ is given by $(\mu_1, \mu_2, \ldots, \mu_s)$ in such a way that $\mu_i \triangleq \Pr (\hat{r}_0 = i) \geq 0$ for $i \in S$, and $\sum_{i=1}^s \mu_i = 1$.

System matrices $\hat{A}_i \triangleq \hat{A}(\hat{r}(t) = i)$, $\hat{B}_i \triangleq \hat{B}(\hat{r}(t) = i)$, $\hat{B}_{wi} \triangleq \hat{B}_w(\hat{r}(t) = i)$, $\hat{C}_i \triangleq \hat{C}(\hat{r}(t) = i)$ and $\hat{D}_i \triangleq \hat{D}(\hat{r}(t) = i)$, $i \in S$, are appropriately dimensioned constant real
matrices, and of norm-bounded uncertainties:

\[
\begin{bmatrix}
\hat{A}_i & \hat{B}_i \\
\end{bmatrix} = \begin{bmatrix} A_i & B_i \end{bmatrix} + E_{ai} F_{ai} \begin{bmatrix} H_{ai} & H_{bi} \end{bmatrix}
\]  
(3.2a)

\[
\hat{B}_{ui} = B_{ui} + E_{bu_i} F_{bu_i} H_{bu_i}
\]  
(3.2b)

\[
\begin{bmatrix}
\hat{C}_i & \hat{D}_i \\
\end{bmatrix} = \begin{bmatrix} C_i & D_i \end{bmatrix} + E_{ci} F_{ci} \begin{bmatrix} H_{ci} & H_{di} \end{bmatrix}
\]  
(3.2c)

with

\[
F_{ai}^T F_{ai} \leq I, \quad F_{bu_i}^T F_{bu_i} \leq I, \quad F_{ci}^T F_{ci} \leq I
\]  
(3.2d)

where matrices \(A_i, B_i, B_{ui}, C_i, D_i, E_{ai}, H_{ai}, H_{bi}, E_{bu_i}, H_{bu_i}, E_{ci}, H_{ci}\) and \(H_{di}\) are known constant real matrices of appropriate dimensions; matrices \(F_{ai}, F_{bu_i}\) and \(F_{ci}\) are the uncertainties in the system matrices.

Mode transition rate matrix \(\hat{\Pi} \doteq (\hat{\pi}_{ij}) \in \mathbb{R}^{\mathcal{S} \times \mathcal{S}}\) is also not precisely known and has element-wise uncertainties:

\[
\hat{\Pi} = \Pi + \Delta \Pi
\]  
(3.2e)

where \(\Pi \doteq (\pi_{ij})\) is a known mode transition rate matrix satisfying \(\pi_{ij} \geq 0\) for all \(i, j \in \mathcal{S}, j \neq i\); \(\pi_{ii} \doteq -\sum_{j=1, j \neq i}^{s} \pi_{ij}\) for all \(i \in \mathcal{S}\). \(\Delta \Pi \doteq (\Delta \pi_{ij})\) is the uncertainty in \(\hat{\Pi}\) and of element-wise form:

\[
\begin{cases}
|\Delta \pi_{ij}| \leq 2\varepsilon_{ij}, & 0 \leq 2\varepsilon_{ij} \leq \pi_{ij}, \quad \text{for all } i, j \in \mathcal{S}, j \neq i \\
\Delta \pi_{ii} \doteq -\sum_{j=1, j \neq i}^{s} \Delta \pi_{ij}, & \text{for all } i \in \mathcal{S}.
\end{cases}
\]  
(3.2f)

The following definition generalizes the \(H_2\)-norm concept of continuous-time deterministic systems to the stochastic Markovian jump case.

**Definition 3.1** \([13]\) Consider the nominal Markovian jump system of (3.1) with \(u(t) \equiv 0\); let \(G_{zw}\) denote the operator from \(w(t)\) to \(z(t)\). Then the \(H_2\) norm of operator \(G_{zw}\) is
defined as

\[ \|G_{zw}\|_{H^2}^2 = \sum_{k=1}^{n_w} \sum_{i=1}^{s} \mu_i \|z_{k,i}\|_2^2 \]

where \( z_{k,i} \) represents the output given by (3.1) when

(a) \( w(t) = e_k \delta(t) \), where \( \delta(t) \) is unit impulse, and \( e_k \) the \( n_w \)-dimensional unit vector formed by 1 at the \( k \)-th position and zeros elsewhere;

(b) \( x_0 = 0 \);

(c) the initial probability distribution of \( \hat{r}_0 \) is \( (\mu_1, \mu_2, \ldots, \mu_s) \).

The following proposition shows that the \( H^2 \) norm of the nominal system of (3.1) can be calculated precisely in terms of a set of coupled algebraic Riccati equations.

Proposition 3.1 [13] The nominal Markovian jump system of (3.1) with \( u(t) \equiv 0 \) is mean square stable and has \( H^2 \) performance

\[ \|G_{zw}\|_{H^2}^2 = \sum_{i=1}^{s} \mu_i \text{trace}(B_{wi}^T P_i B_{wi}) \]

if matrices \( P_i \in \mathbb{S}^{n \times n} \), \( i \in S \), are the unique solution to the coupled algebraic Riccati equations

\[ A_i^T P_i + P_i A_i + \sum_{j=1}^{s} \pi_{ij} P_j + C_i^T C_i = 0 \]

for all \( i \in S \).

Based on Proposition 3.1, we introduce the following definition for uncertain system (3.1).

Definition 3.2 For a prescribed scalar \( \gamma_{H^2} \in \mathbb{R}^+ \), uncertain Markovian jump linear system (3.1) with \( u(t) \equiv 0 \) is said to be robustly mean square stable and has robust \( H^2 \)
CHAPTER 3. ROBUST $H_2$ AND $H_\infty$ CONTROL OF UNCERTAIN MJLS

performance $\|G_{zw}\|_{H_2} < \gamma_{H_2}$, if there exist matrices $P_i \in \mathbb{S}^{n \times n}$, $i \in S$, such that

$$\sum_{i=1}^{s} \mu_i \text{trace} \left( \hat{B}_{ai}^T P_i \hat{B}_{ai} \right) < \gamma_{H_2}^2$$

(3.3)

$$\hat{A}_i^T P_i + P_i \hat{A}_i + \sum_{j=1}^{s} \hat{r}_{ij} P_j + \hat{C}_i^T \hat{C}_i < 0$$

(3.4)

hold for all $i \in S$ over all the admissible uncertainties in (3.2).

Now, consider a state-feedback control law

$$u(t) = K(\hat{r}(t))x(t)$$

(3.5)

where $K_i \triangleq K(\hat{r}(t) = i) \in \mathbb{R}^{n_u \times n}$, $i \in S$, are to be designed.

Substituting state-feedback controller (3.5) into system (3.1) yields the corresponding closed-loop system

$$\begin{cases}
\dot{x}(t) = \hat{A}_{ci}(\hat{r}(t))x(t) + \hat{B}_{ci}(\hat{r}(t))w(t) \\
z(t) = \hat{C}_{ci}(\hat{r}(t))x(t), \quad t \geq 0
\end{cases}$$

(3.6)

where

$$\hat{A}_{cli} \triangleq \hat{A}_{ci}(\hat{r}(t) = i) = (A_i + B_i K_i) + E_{ai} F_{ai} (H_{ai} + H_{bi} K_i),$$

$$\hat{C}_{cli} \triangleq \hat{C}_{ci}(\hat{r}(t) = i) = (C_i + D_i K_i) + E_{ci} F_{ci} (H_{ci} + H_{di} K_i),$$

for all $i \in S$.

To obtain the main results of this section, the following lemmas will be used.

**Lemma 3.1** Given any real number $\alpha \in \mathbb{R}$ and any square matrix $Q \in \mathbb{R}^{n \times n}$. Then the matrix inequality

$$\alpha \left( Q + Q^T \right) \leq \alpha^2 T + QT^{-1}Q^T$$

(3.7)
holds for any matrix $T \in \mathbb{S}^{n \times n}$.

**Proof:** Inequality (3.7) follows from the inequality

$$0 \leq (\alpha T^\frac{1}{2} - QT^{-\frac{1}{2}})(\alpha T^\frac{1}{2} - QT^{-\frac{1}{2}})^T = \alpha^2 T + QT^{-1}Q^T - \alpha(Q + Q^T)$$

immediately. $\Box$

The following lemma describes a common technique for dealing with the norm-bounded uncertainties. As it appears that the result has not been presented as a lemma in the literature, it is given here for self-containedness.

**Lemma 3.2** Given real matrices $Q$, $P$, $D$, $E$ and $H$ of appropriate dimensions with $Q = Q^T < 0$ and $P = P^T > 0$. Then

$$Q + (D + EFH)^T P (D + EFH) < 0$$

holds for all $F$ satisfying $F^T F \leq I$ if and only if one of the following conditions holds:

- (a) there exists a real number $\lambda \in \mathbb{R}^+$ such that

$$\begin{bmatrix} Q + \lambda H^T H + D^T PD & D^T PE \\ E^T PD & -\lambda I + E^T PE \end{bmatrix} < 0;$$

- (b) there exists a real number $\alpha \in \mathbb{R}^+$ such that

$$\begin{bmatrix} Q & D^T & H^T \\ D & -P^{-1} + \alpha EE^T & 0 \\ H & 0 & -\alpha I \end{bmatrix} < 0.$$ 

**Proof:** We first prove part (a). In view of Schur complement equivalence, inequal-
ity (3.8) is equivalent to
\[
\begin{bmatrix}
Q & (D + EFH)^T \\
D + EFH & -P^{-1}
\end{bmatrix} < 0,
\]
which can be rewritten as
\[
\begin{bmatrix}
Q & D^T \\
D & -P^{-1}
\end{bmatrix} + \begin{bmatrix} 0 \\ E \end{bmatrix} F \begin{bmatrix} H \\ 0 \end{bmatrix} + \begin{bmatrix} H^T \\ 0 \end{bmatrix} F^T \begin{bmatrix} 0 & E^T \end{bmatrix} < 0.
\]

Using Lemma 2.2, the above inequality holds for all \( F \) satisfying \( F^T F \leq I \) if, and only if, there exists a real number \( \lambda \in \mathbb{R}^+ \) such that
\[
\begin{bmatrix}
Q + \lambda H^T H & D^T \\
D & -P^{-1} + \frac{1}{\lambda} EE^T
\end{bmatrix} < 0.
\]  
(3.11)

Applying Schur complement equivalence again, we conclude that inequality (3.11) is equivalent to
\[
\begin{bmatrix}
Q + \lambda H^T H & 0 \\
D & -P^{-1} & E \\
0 & E^T & -\lambda I
\end{bmatrix} < 0.
\]

Pre- and post-multiplying both sides of above inequality by
\[
\begin{bmatrix}
I & 0 & 0 \\
0 & 0 & I \\
0 & I & 0
\end{bmatrix}
\]
we have
\[
\begin{bmatrix}
Q + \lambda H^T H & 0 & D^T \\
0 & -\lambda I & E^T \\
D & E & -P^{-1}
\end{bmatrix} < 0,
\]
which is equivalent to (3.9) in view of Schur complement equivalence. This completes
the proof of part (a).

To prove part (b), by defining \( \alpha = \frac{1}{\lambda} \), inequality (3.11) is equivalent to inequality (3.10) by Schur complement equivalence. This completes the proof of part (b). □

In the sequel, the robust \( H_2 \) performance analysis problem is addressed first in terms of coupled linear matrix inequalities, then the associated synthesis problem is dealt with in terms of the solvability of a set of coupled linear matrix inequalities with equality constraints, which can be solved using the sequential linear programming method developed in [38].

### 3.1.2 Robust \( H_2 \) Performance Analysis

The goal of this subsection is to develop a criterion for testing the robust \( H_2 \) performance of the uncertain Markovian jump system in (3.1) over the uncertainties in (3.2).

A new criterion is stated in the following theorem in terms of coupled linear matrix inequalities.

**Theorem 3.1** For a prescribed scalar \( \gamma_{H_2} \in \mathbb{R}^+ \), uncertain Markovian jump system (3.1) with \( u(t) \equiv 0 \) is robustly mean square stable and satisfies \( \|G_{zw}\|_{H_2} < \gamma_{H_2} \) over all the admissible uncertainties in (3.2), if there exist matrices \( P_i \in \mathbb{S}^{n \times n} \), \( T_{ij} \in \mathbb{S}^{n \times n} \), \( W_i \in \mathbb{S}^{n_w \times n_w} \) and scalars \( \lambda_{ai} \in \mathbb{R}^+ \), \( \lambda_{bwi} \in \mathbb{R}^+ \), \( \lambda_{ci} \in \mathbb{R}^+ \), \( i, j \in S, j \neq i \), such that the coupled linear matrix inequalities

\[
\sum_{i=1}^{s} \mu_i \text{trace}(W_i) < \gamma_{H_2}^2 \quad (3.12)
\]

\[
\begin{bmatrix}
-W_i + \lambda_{bwi}H_{bwi}^T H_{bwi} + B_{wi}^T P_i B_{wi} & B_{wi}^T P_i E_{bwi} \\
E_{bwi}^T P_i B_{wi} & -\lambda_{bwi} I + E_{bwi}^T P_i E_{bwi}
\end{bmatrix} < 0 \quad (3.13)
\]

\[
\begin{bmatrix}
\Phi_{ti} & C_{ti} & P_{tai} & \Gamma_{ti} \\
C_{ti}^T & -\lambda_{ci} I + E_{ci}^T E_{ci} & 0 & 0 \\
E_{ci}^T P_{ti} & 0 & -\lambda_{ai} I & 0 \\
\Gamma_{ti}^T & 0 & 0 & -\Lambda_{ai}
\end{bmatrix} < 0 \quad (3.14)
\]
hold for all \(i \in \mathcal{S}\), where

\[
\Phi_{ii} = A_i^T P_i + P_i A_i + C_i^T C_i + \sum_{j=1}^{s} \pi_{ij} P_j + \sum_{j=1, j \neq i}^{s} \epsilon_{ij}^2 T_{ij} + \lambda_{ai} H_{ai}^T H_{ai} + \lambda_{ci} H_{ei}^T H_{ei},
\]

\[
\Gamma_{ii} = \begin{bmatrix} P_i - P_1 & P_i - P_2 & \cdots & P_i - P_{i-1} & P_i - P_{i+1} & \cdots & P_i - P_s \end{bmatrix},
\]

\[
\Lambda_{ii} = \text{diag} \left( T_{i1}, T_{i2}, \ldots, T_{i(i-1)}, T_{i(i+1)}, \ldots, T_{is} \right).
\]

**Proof:** On one hand, according to Definition 3.2, inequality (3.3) holds if, and only if, there exist matrices \(W_i \in \mathbb{S}^{n \times n}, i \in \mathcal{S}\), such that (3.12) and

\[
\hat{B}_{ai}^T P_i \hat{B}_{ai} < W_i
\]

hold. In view of (3.2b), the above inequality is

\[
-W_i + (B_{ai} + E_{bui} F_{bui} H_{bui})^T P_i (B_{ai} + E_{bui} F_{bui} H_{bui}) < 0.
\]

Applying part (a) of Lemma 3.2, this inequality holds for all \(F_{bui}\) satisfying \(F_{bui}^T F_{bui} \leq I\) if, and only if, there exists a real number \(\lambda_{bui} \in \mathbb{R}^+\) such that (3.13) holds.

On the other hand, because \(\Delta \pi_{ii} = -\sum_{j=1, j \neq i}^{s} \Delta \pi_{ij}\), we have

\[
\sum_{j=1}^{s} \Delta \pi_{ij} P_j = \sum_{j=1, j \neq i}^{s} \Delta \pi_{ij} \left( P_j - P_i \right)
\]

\[
= \sum_{j=1, j \neq i}^{s} \left[ \frac{1}{2} \Delta \pi_{ij} \left( P_j - P_i \right) + \frac{1}{2} \Delta \pi_{ij} \left( P_j - P_i \right) \right]
\]

\[
\leq \sum_{j=1, j \neq i}^{s} \left[ \frac{1}{2} \Delta \pi_{ij} \right] T_{ij} + \left( P_i - P_j \right) T_{ij}^{-1} \left( P_i - P_j \right)
\]

\[
\leq \sum_{j=1, j \neq i}^{s} \left[ \epsilon_{ij}^2 T_{ij} + \left( P_i - P_j \right) T_{ij}^{-1} \left( P_i - P_j \right) \right]
\]

for any matrix \(T_{ij} \in \mathbb{S}^{n \times n}\) in view of Lemma 3.1. In addition, note that \(\hat{\pi}_{ij} = \pi_{ij} + \Delta \pi_{ij}\).
inequality (3.4) holds if
\[
\hat{A}_i^T P_i + P_i \hat{A}_i + \sum_{j=1}^{s} \pi_{ij} P_j + \sum_{j=1, j \neq i}^{s} \left[ \varepsilon_{ij}^2 T_{ij} + \left( P_i - P_j \right) \left( P_i - P_j \right)^{-1} \left( P_i - P_j \right) \right] + \hat{C}_i^T \hat{C}_i < 0.
\]

Also, noting that \( \hat{A}_i = A_i + E_{ai}F_{ai}H_{ai} \), \( \hat{C}_i = C_i + E_{ci}F_{ci}H_{ci} \) and according to Lemma 2.2, the above inequality holds for all \( F_{ai} \) satisfying \( F_{ai}^T F_{ai} \leq I \) if, and only if, there exists a real number \( \lambda_{ai} \in \mathbb{R}^+ \), such that
\[
\Xi_{1i} + \left( C_i + E_{ci}F_{ci}H_{ci} \right)^T \left( C_i + E_{ci}F_{ci}H_{ci} \right) < 0 \tag{3.15}
\]
where
\[
\Xi_{1i} = A_i^T P_i + P_i A_i + \sum_{j=1}^{s} \pi_{ij} P_j + \lambda_{ai} H_{ai}^T H_{ai} + \frac{1}{\lambda_{ai}} P_i E_{ai} E_{ai}^T P_i + \sum_{j=1, j \neq i}^{s} \left[ \varepsilon_{ij}^2 T_{ij} + \left( P_i - P_j \right) \left( P_i - P_j \right)^{-1} \left( P_i - P_j \right) \right].
\]

In view of part (a) of Lemma 3.2 again, we would conclude that inequality (3.15) holds for all \( F_{ci} \) satisfying \( F_{ci}^T F_{ci} \leq I \) if, and only if, there exists a real number \( \lambda_{ci} \in \mathbb{R}^+ \) such that
\[
\begin{bmatrix}
\Xi_{1i} + \lambda_{ci} H_{ci}^T H_{ci} + C_i^T C_i & C_i^T E_{ci} \\
E_{ci}^T C_i & -\lambda_{ci} I + E_{ci}^T E_{ci}
\end{bmatrix} < 0
\]
which is equivalent to (3.14) by Schur complement equivalence. This completes the proof. \( \square \)

Theorem 3.1 can be shown to be less conservative than those in \([4, 44, 53]\) for the treatment of the element-wise uncertainties in (3.2e) and (3.2f) of the mode transition rate matrix (see also Remark 2.1).

**Remark 3.1** Suppose there exists at least one \( \varepsilon_{ij} > 0, j \neq i \). The bounding technique
of the matrix inequality used in [4, 44, 53] is

\[
\sum_{j=1}^{s} \Delta \pi_{ij} P_j \leq \sum_{j=1}^{s} 2 \varepsilon_{ij} P_j.
\] (3.16)

The bounding technique used in Theorem 3.1 is

\[
\sum_{j=1}^{s} \Delta \pi_{ij} P_j \leq \sum_{j=1, j \neq i}^{s} \left[ \varepsilon_{ij}^2 T_{ij} + \left( P_i - P_j \right) T_{ij}^{-1} \left( P_i - P_j \right) \right]
\] (3.17)

for any \( T_{ij} \in \mathbb{S}^{n \times n} \). For those \( \varepsilon_{ij} > 0, j \neq i \), if \( T_{ij} \equiv \frac{1}{\varepsilon_{ij}} \left( P_i + P_j \right) \) is chosen, then

\[
\varepsilon_{ij}^2 T_{ij} + \left( P_i - P_j \right) T_{ij}^{-1} \left( P_i - P_j \right)
= \varepsilon_{ij} \left( P_i + P_j \right) + \varepsilon_{ij} \left( P_i + P_j - 2P_j \right) \left( P_i + P_j \right)^{-1} \left( P_i + P_j - 2P_j \right)
= \varepsilon_{ij} \left( P_i + P_j \right) + \varepsilon_{ij} \left[ P_i + P_j + 4P_j \left( P_i + P_j \right)^{-1} P_j - 4P_j \right]
= 2\varepsilon_{ij} \left( P_i + P_j \right) + 4\varepsilon_{ij} P_j \left( \left( P_i + P_j \right)^{-1} - P_j^{-1} \right) P_j
< 2\varepsilon_{ij} \left( P_i + P_j \right).
\]

For those \( \varepsilon_{ij} = 0 \), let \( T_{ij} \equiv \frac{1}{\alpha} I \) with \( \alpha \in \mathbb{R}^+ \) sufficiently small so that

\[
\sum_{j=1, j \neq i}^{s} \left[ \varepsilon_{ij}^2 T_{ij} + \left( P_i - P_j \right) T_{ij}^{-1} \left( P_i - P_j \right) \right] < \sum_{j=1, j \neq i}^{s} 2\varepsilon_{ij} \left( P_i + P_j \right) = \sum_{j=1}^{s} 2\varepsilon_{ij} P_j.
\]

Here \( \varepsilon_{ii} = \sum_{j=1, j \neq i}^{s} \varepsilon_{ij} \) was used. Therefore, the bounding technique in (3.17) is less conservative than the one in (3.16) as long as there exist uncertainties.

### 3.1.3 Robust \( H_2 \) Controller Design

The objective of this subsection is to design a state-feedback controller (3.5) such that the closed-loop system in (3.6) is robustly mean square stable and satisfies a prescribed level of \( H_2 \) performance. The following result provides a solution to the robust \( H_2 \) control problem for uncertain system (3.1) with uncertain mode transition rate matrix.
Theorem 3.2 Consider the uncertain Markovian jump system given in (3.1); for a prescribed scalar $\gamma_{H_2} \in \mathbb{R}^+$, there exists a state-feedback controller (3.5) such that the closed-loop system in (3.6) is robustly mean square stable and has robust $H_2$ performance $\|G_{zu}\|_{H_2} < \gamma_{H_2}$ over all the admissible uncertainties in (3.2), if there exist matrices $P_i \in \mathbb{S}^{n \times n}$, $X_i \in \mathbb{S}^{n \times n}$, $V_i \in \mathbb{S}^{n \times n}$, $Z_i \in \mathbb{S}^{n \times n}$, $T_{ij} \in \mathbb{S}^{n \times n}$, $W_i \in \mathbb{S}^{n_u \times n_u}$, $Y_i \in \mathbb{R}^{n_u \times n}$ and scalars $\alpha_{ai} \in \mathbb{R}^+$, $\alpha_{bwi} \in \mathbb{R}^+$, $\alpha_{ci} \in \mathbb{R}^+$, $i, j \in S$, $j \neq i$, such that the coupled linear matrix inequalities

$$\sum_{i=1}^{S} \mu_i \text{trace} (W_i) < \gamma_{H_2}^2 \quad (3.18)$$

$$\begin{bmatrix}
-W_i & B_{wi}^T & H_{bwi}^T \\
B_{wi} & -X_i + \alpha_{bwi} E_{bwi} E_{bwi}^T & 0 \\
H_{bwi} & 0 & -\alpha_{bwi} I
\end{bmatrix} < 0 \quad (3.19)$$

$$\begin{bmatrix}
\Phi_{2i} & \bullet & \bullet & \bullet & \bullet \\
C_i X_i + D_i Y_i & -I + \alpha_{ci} E_{ci} E_{ci}^T & \bullet & \bullet & \bullet \\
H_{ci} X_i + H_{di} Y_i & 0 & -\alpha_{ci} I & \bullet & \bullet < 0 \quad (3.20)
\end{bmatrix}$$

$$\begin{bmatrix}
\Phi_{3i} & \Gamma_{1i} \\
\Gamma_{1i}^T & -\Lambda_{1i}
\end{bmatrix} \leq 0 \quad (3.21)$$

with equality constraints

$$P_i X_i = I, \quad V_i Z_i = I \quad (3.22)$$

hold for all $i \in S$, where

$$\Phi_{2i} = (A_i X_i + B_i Y_i) + (A_i X_i + B_i Y_i)^T + \alpha_{ai} E_{ai} E_{ai}^T,$$

$$\Phi_{3i} = -V_i + \sum_{j=1}^{S} \pi_{ij} P_j + \sum_{j=1, j \neq i}^{S} \epsilon_{ij}^2 T_{ij},$$
and \( \Gamma_{1i} \) and \( \Lambda_{1i} \) are given in Theorem 3.1. In this case, a controller (3.5) is given by \( K_i = Y_i P_i, \ i \in S \).

**Proof:** Firstly, linear matrix inequality (3.12) is the same as linear matrix inequality (3.18).

In view of Lemma 3.2, linear matrix inequality (3.13) is equivalent to linear matrix inequality (3.19) with \( X_i = P_i^{-1} \) and \( \alpha_{bui} = \frac{1}{\lambda_{bu}} \).

Next, consider closed-loop system (3.6), and denote \( \bar{A}_i = A_i + B_i K_i, \bar{C}_i = C_i + D_i K_i, \bar{H}_{ai} = H_{ai} + H_{bi} K_i \) and \( \bar{H}_{ci} = H_{ci} + H_{di} K_i \), then replace matrices \( A_i, C_i, H_{ai}, H_{ci} \) in inequality (3.15) with matrices \( \bar{A}_i, \bar{C}_i, \bar{H}_{ai}, \bar{H}_{ci} \), respectively, we have

\[
\bar{A}_i^T P_i + P_i \bar{A}_i + \frac{1}{\lambda_{ai}} P_i E_{ai} E_{ai}^T P_i + \lambda_{ai} \bar{H}_{ai}^T \bar{H}_{ai} + \left( \bar{C}_i + E_{ci} F_{ci} \bar{H}_{ci} \right)^T \left( \bar{C}_i + E_{ci} F_{ci} \bar{H}_{ci} \right) + \sum_{j=1}^s \pi_{ij} P_j + \sum_{j=1, j \neq i}^s \left[ \varepsilon_{ij}^2 T_{ij} + (P_i - P_j) T_{ij}^{-1} (P_i - P_j) \right] < 0.
\] (3.23)

Let \( V_i \in \mathbb{S}^{n \times n} \) such that

\[
\sum_{j=1}^s \pi_{ij} P_j + \sum_{j=1, j \neq i}^s \left[ \varepsilon_{ij}^2 T_{ij} + (P_i - P_j) T_{ij}^{-1} (P_i - P_j) \right] \leq V_i
\]

which is equivalent to (3.21) in view of Schur complement equivalence.

Also, inequality (3.23) is equivalent to

\[
\bar{A}_i^T P_i + P_i \bar{A}_i + V_i + \frac{1}{\lambda_{ai}} P_i E_{ai} E_{ai}^T P_i + \lambda_{ai} \bar{H}_{ai}^T \bar{H}_{ai} + \left( \bar{C}_i + E_{ci} F_{ci} \bar{H}_{ci} \right)^T \left( \bar{C}_i + E_{ci} F_{ci} \bar{H}_{ci} \right) < 0.
\]

Now, pre- and post-multiplying both sides of the above inequality by \( X_i \), and applying the change of variables: \( Z_i = V_i^{-1}, Y_i = K_i X_i \) and \( \alpha_{ai} = \frac{1}{\lambda_{ai}} \), we obtain

\[
[(C_i X_i + D_i Y_i) + E_{ci} F_{ci} (H_{ci} X_i + H_{di} Y_i)]^T [(C_i X_i + D_i Y_i) + E_{ci} F_{ci} (H_{ci} X_i + H_{di} Y_i)] < -\varepsilon_{2i}
\]
where

\[
\Xi_2i = (A_iX_i + B_iY_i) + (A_iX_i + B_iY_i)^T + X_iZ^{-1}X_i + \alpha_{ai}E_{ai}E_{ai}^T \\
+ \frac{1}{\alpha_{ai}}(H_{ai}X_i + H_{bi}Y_i)^T (H_{ai}X_i + H_{bi}Y_i).
\]

According to part (b) of Lemma 3.2, the above inequality holds for all \(F_{ci}\) satisfying \(F_{ci}^TF_{ci} \leq I\) if, and only if, there exists a real number \(\alpha_{ci} \in \mathbb{R}^+\) such that

\[
\begin{bmatrix}
\Xi_{2i} & (C_iX_i + D_iY_i)^T & (H_{ci}X_i + H_{di}Y_i)^T \\
C_iX_i + D_iY_i & -I + \alpha_{ci}E_{ci}E_{ci}^T & 0 \\
H_{ci}X_i + H_{di}Y_i & 0 & -\alpha_{ci}I
\end{bmatrix} < 0
\]

which is equivalent to (3.20) in view of Schur complement equivalence. This completes the proof. \(\square\)

In the case when the mode transition rate matrix is known exactly, we do not need to introduce the additional variables \(V_i, Z_i, i \in S\). Hence the equality constraints in (3.22) are no longer needed for the design of the robust \(H_2\) controller. The corresponding result is stated in the following corollary in terms of coupled linear matrix inequalities, and can be proved in a manner similar to that of Theorem 3.2. It is also worthy noting that the condition is necessary and sufficient since Lemma 3.1 is no longer needed for the proof.

**Corollary 3.1** Consider the uncertain Markovian jump system given in (3.1) with the mode transition rate matrix known exactly; for a prescribed scalar \(\gamma_{H_2} \in \mathbb{R}^+\), there exists a state-feedback controller (3.5) such that closed-loop system (3.6) is robustly mean square stable and has robust \(H_2\) performance \(\|G_{zw}\|_{H_2} < \gamma_{H_2}\) over the uncertainties in (3.2a)–(3.2c) if, and only if, there exist matrices \(X_i \in \mathbb{S}^{n_n}\), \(W_i \in \mathbb{S}^{n_w\times n_w}\), \(Y_i \in \mathbb{R}^{n_u\times n}\) and scalars \(\alpha_{ai} \in \mathbb{R}^+, \alpha_{bi} \in \mathbb{R}^+, \alpha_{ci} \in \mathbb{R}^+, i \in S\), such that the coupled
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linear matrix inequalities

$$\sum_{i=1}^s \mu_i \text{trace} (W_i) < \gamma_{H_2}^2$$

$$\begin{bmatrix}
-W_i & B_{wi}^T & H_{bwi}^T \\
B_{wi} & -X_i + \alpha_{bwi} E_{bwi} E_{bwi}^T & 0 \\
H_{bwi} & 0 & -\alpha_{bwi} I
\end{bmatrix} < 0$$

$$\begin{bmatrix}
\Phi_{4i} & \bullet & \bullet & \bullet & \bullet \\
C_i Y_i + D_i Y_i & -I + \alpha_{ci} E_{ci} E_{ci}^T & \bullet & \bullet & \bullet \\
H_{ci} X_i + H_{di} Y_i & 0 & -\alpha_{ci} I & \bullet & \bullet \\
H_{ai} X_i + H_{bi} Y_i & 0 & 0 & -\alpha_{ai} I & \bullet \\
\Gamma_{2i}^T & 0 & 0 & 0 & -\Lambda_{2i}
\end{bmatrix} < 0$$

hold for all $i \in S$, where

$$\Phi_{4i} = (A_i X_i + B_i Y_i) + (A_i X_i + B_i Y_i)^T + \pi_{ii} X_i + \alpha_{ai} E_{ai} E_{ai}^T,$$

$$\Gamma_{2i} = \begin{bmatrix}
\sqrt{\pi_{ii}} X_i & \sqrt{\pi_{i2}} X_i & \cdots & \sqrt{\pi_{i(i-1)}} X_i & \sqrt{\pi_{i(i+1)}} X_i & \cdots & \sqrt{\pi_{is}} X_i
\end{bmatrix},$$

$$\Lambda_{2i} = \text{diag} (X_1, X_2, \ldots, X_{i-1}, X_{i+1}, \ldots, X_s).$$

In this case, controller (3.5) is given by $K_i = Y_i X_i^{-1}$, $i \in S$.

To solve the conditions in Theorem 3.2 effectively, an algorithm similar to Algorithm RSP can be used. The equality constraints in (3.22) are relaxed to

$$\begin{bmatrix}
P_i & I \\
I & X_i
\end{bmatrix} \geq 0,$$

$$\begin{bmatrix}
V_i & I \\
I & Z_i
\end{bmatrix} \geq 0.$$  \hspace{1cm} (3.24)

The strict inequalities in (3.18), (3.19) and (3.20) are replaced with

$$\sum_{i=1}^s \mu_i \text{trace} (W_i) + \beta \leq \gamma_{H_2}^2$$  \hspace{1cm} (3.25)
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\[
\begin{bmatrix}
-W_i + \beta I & B_{ai}^T & H_{bui}^T \\
B_{ai} & -X_i + \alpha_{bui}E_{bui}E_{bui}^T & 0 \\
H_{bui} & 0 & -\alpha_{bui}I
\end{bmatrix} \leq 0 \quad (3.26)
\]

and

\[
\begin{bmatrix}
\Phi_{2i} + \beta I & \bullet & \bullet & \bullet & \bullet \\
C_iX_i + D_iY_i & -I + \alpha_{ci}E_{ci}^TE_{ci} & \bullet & \bullet & \bullet \\
H_{ei}X_i + H_{di}Y_i & 0 & -\alpha_{ci}I & \bullet & \bullet \\
H_{ai}X_i + H_{bi}Y_i & 0 & 0 & -\alpha_{ci}I & \bullet \\
X_i & 0 & 0 & 0 & -Z_i
\end{bmatrix} \leq 0 \quad (3.27)
\]

respectively for a sufficiently small number $\beta \in \mathbb{R}^+$. The solution to the robust $H_2$ control problem (RH2P) is summarized below.

**Algorithm RH2P** For a given precision $\delta \in \mathbb{R}^+$, let $k_{\text{max}}$ be the maximum number of iterations and a sufficiently small number $\beta \in \mathbb{R}^+$ be given.

1. Determine $P_i^{(0)}, X_i^{(0)}, V_i^{(0)}, Z_i^{(0)}, T_{ij}^{(0)}, W_i^{(0)}, Y_i^{(0)}, \alpha_{ai}^{(0)}, \alpha_{bi}^{(0)}, i, j \in S, j \neq i$, satisfying (3.21), (3.24), (3.25), (3.26) and (3.27). Let $k := 0$.

2. Solve the following convex optimization problem for the variables $P_i, X_i, V_i, Z_i, T_{ij}, W_i, Y_i, \alpha_{ai}, \alpha_{bi}, i, j \in S, j \neq i$:

\[
\min \sum_{i=1}^S \text{trace} \left( P_iX_i^{(k)} + P_i^{(k)}X_i + V_iZ_i^{(k)} + V_i^{(k)}Z_i \right)
\]

subject to (3.21), (3.24), (3.25), (3.26) and (3.27) for all $i \in S$.

3. Let $T_i^{(k)} := P_i, L_i^{(k)} := X_i, U_i^{(k)} := V_i$ and $R_i^{(k)} := Z_i$ for all $i \in S$.

4. If

\[
\sum_{i=1}^S \text{trace} \left( T_i^{(k)}X_i^{(k)} + P_i^{(k)}L_i^{(k)} + U_i^{(k)}Z_i^{(k)} + V_i^{(k)}R_i^{(k)} \right)
\]
\[ -2 \sum_{i=1}^{s} \text{trace} \left( P_i^{(k)} X_i^{(k)} + V_i^{(k)} Z_i^{(k)} \right) < \delta \]

then go to Step (7), otherwise go to Step (5).

(5) Compute \( \theta^* \in [0, 1] \) by solving

\[
\min_{\theta \in [0, 1]} \sum_{i=1}^{s} \text{trace} \left( \left[ P_i^{(k)} + \theta \left( T_i^{(k)} - P_i^{(k)} \right) \right] \left[ X_i^{(k)} + \theta \left( L_i^{(k)} - X_i^{(k)} \right) \right] \right.
\]

\[
+ \left[ V_i^{(k)} + \theta \left( U_i^{(k)} - V_i^{(k)} \right) \right] \left[ Z_i^{(k)} + \theta \left( R_i^{(k)} - Z_i^{(k)} \right) \right] \right).
\]

(6) Let

\[
P_i^{(k+1)} := P_i^{(k)} + \theta^* \left( T_i^{(k)} - P_i^{(k)} \right), \quad X_i^{(k+1)} := X_i^{(k)} + \theta^* \left( L_i^{(k)} - X_i^{(k)} \right),
\]

\[
V_i^{(k+1)} := V_i^{(k)} + \theta^* \left( U_i^{(k)} - V_i^{(k)} \right), \quad Z_i^{(k+1)} := Z_i^{(k)} + \theta^* \left( R_i^{(k)} - Z_i^{(k)} \right),
\]

for all \( i \in \mathcal{S} \), and \( k := k + 1 \). If \( k < k_{\text{max}} \), then go to Step (2), otherwise go to Step (7).

(7) If \( \sum_{i=1}^{s} \text{trace} \left( P_i^{(k)} X_i^{(k)} + V_i^{(k)} Z_i^{(k)} \right) = 2sn \), then a solution is found, otherwise a solution cannot be found by this algorithm.

### 3.1.4 Numerical Examples

In this subsection, a numerical example is presented to illustrate the usefulness and flexibility of the developed theory. Attention is focused on designing a robust \( H_2 \) controller such that the closed-loop system is robustly stable and has guaranteed \( H_2 \) performance with respect to the uncertainties in the mode transition rate matrix.

It is assumed that the system under consideration has two operation modes, and uncertainties exist in the mode transition rate matrix only. The system data of (3.1) are
as follows:

\[
A_1 = \begin{bmatrix} 0 & 0.1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0.1 \\ 0 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.9 \\ -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix},
\]

\[
C_1 = \begin{bmatrix} 1 & -0.1 \\ 0 & 1 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

\[
\Pi = \begin{bmatrix} -1.9 & 1.9 \\ 10 & -10 \end{bmatrix}, \quad B_{w1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_{w2} = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

\[
\epsilon_{12} = 0.9, \quad \epsilon_{21} = 4, \quad \mu_1 = 0.5, \quad \mu_2 = 0.5.
\]

Suppose the requirement for the controller given in (3.5) is that the closed-loop system given in (3.6) is robustly mean square stable and has robust \(H_2\) performance \(\|G_{zw}\|_{H_2} < \gamma_{H_2}\) with \(\gamma_{H_2} \triangleq 2\) over all possible uncertainties \(\Delta \pi_{12} \in [-1.8, 1.8]\) and \(\Delta \pi_{21} \in [-8, 8]\). The uncertain system given above is not mean square stable for all the admissible uncertainties in the mode transition rate matrix.

If the uncertainties are ignored, then Corollary 3.1 can be used to obtain the controller. A solution to Corollary 3.1 is as follows:

\[
X_1 = \begin{bmatrix} 2.7721 & -2.1463 \\ -2.1463 & 2.3428 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1.8839 & -0.7832 \\ -0.7832 & 2.0859 \end{bmatrix},
\]

\[
Y_1 = \begin{bmatrix} -14.7217 & 15.2169 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} -12.4819 & -15.0692 \end{bmatrix},
\]

\[
W_1 = 2.9163, \quad W_2 = 2.7685.
\]

The controller is given by

\[
K_1 = \begin{bmatrix} -0.9693 & 5.6072 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -11.4095 & -11.5081 \end{bmatrix}.
\]

This controller is denoted by \(K_{d=0}\). Applying controller \(K_{d=0}\), the resulting nominal closed-loop system becomes mean square stable and has the \(H_2\) norm \(\gamma_{H_2} = 0.6022\).
according to Proposition 3.1 with associated Gramian matrices

\[
P_1 = \begin{bmatrix} 0.9028 & 0.7742 \\ 0.7742 & 0.8526 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.4156 & 0.1794 \\ 0.1794 & 0.2616 \end{bmatrix}.
\]

However, this controller cannot guarantee the \(H_2\) performance or even the stability of the closed-loop system, if uncertainty exists. Let us consider the case when \(\Delta \pi_{12} = -1.3\) and \(\Delta \pi_{21} = 6\), the closed-loop system remains mean square stable but has a largely degraded \(H_2\) performance as \(\gamma_{H_2}^* = 12.4895\). The associated Gramian matrices are given by

\[
P_1 = \begin{bmatrix} 16.5255 & 17.6342 \\ 17.6342 & 19.0324 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 5.9375 & 6.3536 \\ 6.3536 & 7.1234 \end{bmatrix}.
\]

Moreover, in the case of \(\Delta \pi_{12} = -1.4\) and \(\Delta \pi_{21} = 6\), the closed-loop system becomes mean square unstable. The shaded regions in the probability domain in Figure 3.1 illustrate different characteristics of the stability and performance of the closed-loop system.

Fortunately, Algorithm RH2P can be employed here to construct a more powerful controller such that the closed-loop system is robustly mean square stable and preserves the desired \(H_2\) performance over all the admissible uncertainties in the mode transition rate matrix. To compute with Algorithm RH2P for this example, it is chosen that \(\delta = 10^{-10}\), \(k_{\text{max}} = 100\) and \(\beta = 0.01\). One set of solutions is

\[
P_1 = \begin{bmatrix} 3.1503 & 3.1648 \\ 3.1648 & 3.9220 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3.1113 & 3.1157 \\ 3.1157 & 4.1554 \end{bmatrix},
\]

\[
V_1 = \begin{bmatrix} 0.1499 & 0.3961 \\ 0.3961 & 1.9990 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0.7043 & 0.7469 \\ 0.7469 & 1.1198 \end{bmatrix},
\]

\[
X_1 = \begin{bmatrix} 1.6763 & -1.3527 \\ -1.3527 & 1.3465 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1.2900 & -0.9672 \\ -0.9672 & 0.9658 \end{bmatrix},
\]

\[
V_1 = \begin{bmatrix} 0.1499 & 0.3961 \\ 0.3961 & 1.9990 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0.7043 & 0.7469 \\ 0.7469 & 1.1198 \end{bmatrix}.
\]

Moreover, in the case of \(\Delta \pi_{12} = -1.4\) and \(\Delta \pi_{21} = 6\), the closed-loop system becomes mean square unstable. The shaded regions in the probability domain in Figure 3.1 illustrate different characteristics of the stability and performance of the closed-loop system.

Fortunately, Algorithm RH2P can be employed here to construct a more powerful controller such that the closed-loop system is robustly mean square stable and preserves the desired \(H_2\) performance over all the admissible uncertainties in the mode transition rate matrix. To compute with Algorithm RH2P for this example, it is chosen that \(\delta = 10^{-10}\), \(k_{\text{max}} = 100\) and \(\beta = 0.01\). One set of solutions is

\[
P_1 = \begin{bmatrix} 3.1503 & 3.1648 \\ 3.1648 & 3.9220 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3.1113 & 3.1157 \\ 3.1157 & 4.1554 \end{bmatrix},
\]

\[
V_1 = \begin{bmatrix} 0.1499 & 0.3961 \\ 0.3961 & 1.9990 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0.7043 & 0.7469 \\ 0.7469 & 1.1198 \end{bmatrix},
\]

\[
X_1 = \begin{bmatrix} 1.6763 & -1.3527 \\ -1.3527 & 1.3465 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1.2900 & -0.9672 \\ -0.9672 & 0.9658 \end{bmatrix}.
\]


\[ Z_1 = \begin{bmatrix} 14.0088 & -2.7760 \\ -2.7760 & 1.0503 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 4.8517 & -3.2359 \\ -3.2359 & 3.0512 \end{bmatrix}, \]

\[ T_{12} = \begin{bmatrix} 0.1382 & 0.3021 \\ 0.3021 & 0.9602 \end{bmatrix}, \quad T_{21} = \begin{bmatrix} 0.0098 & 0.0080 \\ 0.0080 & 0.1079 \end{bmatrix}, \]

\[ Y_1 = \begin{bmatrix} -5.3504 & 6.1334 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} -0.3319 & -5.7722 \end{bmatrix}, \]

\[ W_1 = 3.1603, \quad W_2 = 4.8197. \]

It can be verified that \( \| P_1 X_1 - I \| = 2.6615 \times 10^{-12}, \| P_2 X_2 - I \| = 2.6547 \times 10^{-12}, \)
\( \| V_1 Z_1 - I \| = 2.6523 \times 10^{-12}, \| V_2 Z_2 - I \| = 2.7112 \times 10^{-12}. \)
Therefore, the equality constraints in (3.22) are satisfied. Thus the controller is obtained as

\[ K_1 = \begin{bmatrix} 2.5553 & 7.1221 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -19.0172 & -25.0202 \end{bmatrix}. \]

This controller is denoted by \( K_{\Delta 0}. \) Applying controller \( K_{\Delta 0}, \) the resulting nominal closed-loop system is mean square stable and has the \( H_2 \) norm \( \gamma_{H_2}^{*} = 0.8516 \) with associated Gramian matrices

\[ P_1 = \begin{bmatrix} 1.4684 & 1.1822 \\ 1.1822 & 1.1857 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.5673 & 0.1767 \\ 0.1767 & 0.1937 \end{bmatrix}. \]

To contrast with controller \( K_{\Delta 0}, \) let us consider the same case when \( \Delta \pi_{12} = -1.3 \)
and \( \Delta \pi_{21} = 6, \) the closed-loop system remains mean square stable and achieves the guaranteed \( H_2 \) performance \( \gamma_{H_2}^{*} = 1.3120. \) The associated Gramian matrices are given by

\[ P_1 = \begin{bmatrix} 2.0662 & 1.9052 \\ 1.9052 & 2.0315 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.7886 & 0.4434 \\ 0.4434 & 0.4612 \end{bmatrix}. \]

In the case that \( \Delta \pi_{12} = -1.4 \) and \( \Delta \pi_{21} = 6, \) the closed-loop system remains mean square stable, and has guaranteed \( H_2 \) performance \( \gamma_{H_2}^{*} = 1.3570. \) The associated Gramian
matrices are given by

\[ P_1 = \begin{bmatrix} 2.1316 & 1.9843 \\ 1.9843 & 2.1254 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.8044 & 0.4619 \\ 0.4619 & 0.4820 \end{bmatrix}. \]

Table 3.1 gives a numerical comparison of the \( H_2 \) performance of the two controllers (\( K_d=0 \) and \( K_d≠0 \)) on some points of the probability domain.

### 3.2 Robust \( H_\infty \) Control

This section is concerned with the robust \( H_\infty \) control problem. The objective is to design a state-feedback control law such that the closed-loop system is robustly mean square stable and has a prescribed \( H_\infty \) performance level over the admissible uncertainties. The organization of this section is as follows. Section 3.2.1 formulates the problem to be solved. Section 3.2.2 develops a solution to the formulated robust \( H_\infty \) control problem. A numerical example is used to illustrate the developed theory in Section 3.2.3.

#### 3.2.1 Problem Formulation

Consider the following class of Markovian jump linear systems defined on a complete probability space \( (\Omega, \mathcal{F}, P) \):

\[
\begin{cases}
\dot{x}(t) = \hat{A}(\hat{r}(t))x(t) + \hat{B}_1(\hat{r}(t))u(t) + \hat{B}_2(\hat{r}(t))w(t) \\
z(t) = \hat{C}(\hat{r}(t))x(t) + \hat{D}_1(\hat{r}(t))u(t) + \hat{D}_2(\hat{r}(t))w(t)
\end{cases}
\] (3.28)

where \( x(t) \in \mathbb{R}^n \) is the system state, \( u(t) \in \mathbb{R}^m \) is the control input, \( w(t) \in \mathbb{R}^p \) is the non-zero disturbance input that belongs to \( \mathbb{L}_2^p[0, \infty) \), \( z(t) \in \mathbb{R}^q \) is the regulated output, \( t \geq 0 \) is the time. The mode jumping process \( \{\hat{r}(t) : t \geq 0\} \) is a continuous-time, discrete-state and homogeneous Markov process on the probability space. \( \hat{r}(t) \) takes values in a finite
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mode space $S \triangleq \{1, 2, \ldots, s\}$, and has the mode transition probabilities

$$\Pr(\hat{r}(t + \delta t) = j \mid \hat{r}(t) = i) = \begin{cases} \hat{\pi}_{ij} \delta t + o(\delta t), & \text{if } j \neq i \\ 1 + \hat{\pi}_{ii} \delta t + o(\delta t), & \text{if } j = i \end{cases}$$

where $\delta t > 0$ and $\lim_{\delta t \to 0} \frac{o(\delta t)}{\delta t} = 0$; $\hat{\pi}_{ij} \geq 0$ denotes the switching rate from mode $i$ at time $t$ to mode $j$ at time $t + \delta t$ for all $i, j \in S, j \neq i$; $\hat{\pi}_{ii} \triangleq -\sum_{j=1,j\neq i}^{s} \hat{\pi}_{ij}$ for all $i \in S$.

Whenever $i \in S$, matrices $\hat{A}_i \triangleq \hat{A}(\hat{r}(t) = i)$, $\hat{B}_{1i} \triangleq \hat{B}_1(\hat{r}(t) = i)$, $\hat{B}_{2i} \triangleq \hat{B}_2(\hat{r}(t) = i)$, $\hat{C}_i \triangleq \hat{C}(\hat{r}(t) = i)$, $\hat{D}_{1i} \triangleq \hat{D}_1(\hat{r}(t) = i)$ and $\hat{D}_{2i} \triangleq \hat{D}_2(\hat{r}(t) = i)$ are assumed to be not known precisely, but have the norm-bounded uncertainties as follows:

$$\begin{bmatrix} \hat{A}_i & \hat{B}_{1i} & \hat{B}_{2i} \\ \hat{C}_i & \hat{D}_{1i} & \hat{D}_{2i} \end{bmatrix} = \begin{bmatrix} A_i & B_{1i} & B_{2i} \\ C_i & D_{1i} & D_{2i} \end{bmatrix} + \begin{bmatrix} E_{0i} \\ E_{1i} \end{bmatrix} F_i \begin{bmatrix} H_{0i} & H_{1i} & H_{2i} \end{bmatrix}$$ (3.29a)

where matrices $A_i, B_{1i}, B_{2i}, C_i, D_{1i}, D_{2i}, E_{0i}, E_{1i}, H_{0i}, H_{1i}$ and $H_{2i}$ are known constant real matrices of appropriate dimensions, while $F_i$, which denotes the uncertainties in the system matrices, is unknown and satisfies $F_i^T F_i \leq I$.

In addition, mode transition rate matrix $\hat{\Pi} \triangleq (\hat{\pi}_{ij})$ is also assumed to be not exactly known and has element-wise uncertainties

$$\hat{\Pi} = \Pi + \Delta \Pi$$ (3.29b)

with $\Pi \triangleq (\pi_{ij})$ satisfying $\pi_{ij} \geq 0$ for $i, j \in S, j \neq i$, and $\pi_{ii} \triangleq -\sum_{j=1,j\neq i}^{s} \pi_{ij}$ for $i \in S$, where $\pi_{ij}$ denotes the estimated value of $\hat{\pi}_{ij}$ in practice. $\Delta \Pi \triangleq (\Delta \pi_{ij}) = (\hat{\pi}_{ij} - \pi_{ij})$, where $|\Delta \pi_{ij}| \leq 2\varepsilon_{ij}, 0 \leq 2\varepsilon_{ij} \leq \pi_{ij}$ for all $i, j \in S, j \neq i$; $\Delta \pi_{ii} \triangleq -\sum_{j=1,j\neq i}^{s} \Delta \pi_{ij}$ for all $i \in S$.

Consider the state-feedback control law

$$u(t) = K(\hat{r}(t))x(t)$$ (3.30)
where $K_i \triangleq K(\hat{r}(t) = i) \in \mathbb{R}^{m \times n}$, $i \in S$, are the control gains to be designed.

Substituting (3.30) into (3.28) yields the closed-loop system

$$
\begin{align*}
\dot{x}(t) &= \hat{A}_{cli}(\hat{r}(t))x(t) + \hat{B}_2(\hat{r}(t))w(t) \\
z(t) &= \hat{C}_{cli}(\hat{r}(t))x(t) + \hat{D}_2(\hat{r}(t))w(t)
\end{align*}
$$

(3.31)

where

$$
\begin{align*}
\hat{A}_{cli} &= (A_i + B_1K_i) + E_{0i}F_i (H_{0i} + H_{1i}K_i) \\
\hat{C}_{cli} &= (C_i + D_1K_i) + E_{1i}F_i (H_{0i} + H_{1i}K_i)
\end{align*}
$$

for all $i \in S$.

The objective of this section is to tackle the robust $H_\infty$ performance analysis and synthesis problems for the uncertain Markovian jump system in (3.28) with the admissible uncertainties in (3.29).

**Definition 3.3** [71] Consider the nominal Markovian jump system of uncertain system (3.28) with $u(t) \equiv 0$; let $G_{zw}$ denote the operator from $w(t)$ to $z(t)$. Then the $H_\infty$ norm of operator $G_{zw}$ is defined as $\|G_{zw}\|_{H_\infty} \triangleq \inf \gamma$ such that

$$
\|z\|_2 < \gamma \|w\|_2
$$

holds for all non-zero processes $w(\cdot) \in \mathbb{L}_2^w[0, \infty)$.

**Proposition 3.2** [8] Given a prescribed scalar $\gamma_{H_\infty} \in \mathbb{R}^+$, the nominal Markovian jump system of uncertain system (3.28) with $u(t) \equiv 0$ is mean square stable and has $H_\infty$ performance $\|G_{zw}\|_{H_\infty} < \gamma_{H_\infty}$, if there exist matrices $P_i \in \mathbb{S}^{n \times n}$, $i \in S$, such that coupled linear matrix inequalities

$$
\begin{bmatrix}
A_i^T P_i + P_i A_i + \sum_{j=1}^i \pi_{ij} P_j + C_i^T C_i & P_i B_{2i} + C_i^T D_{2i} \\
B_{2i}^T P_i + D_{2i}^T C_i & -\gamma_{H_\infty}^2 I + D_{2i}^T D_{2i}
\end{bmatrix} < 0
$$
hold for all $i \in S$.

**Definition 3.4** For a prescribed scalar $\gamma_{H_\infty} \in \mathbb{R}^+$, uncertain Markovian jump linear system (3.28) with $u(t) \equiv 0$ is said to be robustly mean square stable and has robust $H_\infty$ performance $\|G_{zw}\|_{H_\infty} < \gamma_{H_\infty}$, if there exist matrices $P_i \in \mathbb{S}^{n \times n}, i \in S$, such that

$$
\begin{bmatrix}
\hat{A}_i^T P_i + P_i \hat{A}_i + \sum_{j=1}^{i} \hat{\pi}_{ij} P_j & P_i \hat{B}_2i & \hat{C}_i^T \\
\hat{B}_2i P_i & -\gamma^2_{H_\infty} I & \hat{D}_2i \\
\hat{C}_i & \hat{D}_2i & -I
\end{bmatrix} < 0
$$

holds for all $i \in S$ over all the admissible uncertainties in (3.29).

### 3.2.2 Robust $H_\infty$ Control

In this subsection, the robust $H_\infty$ performance analysis is first proposed, then the associated synthesis is dealt with in terms of the solvability of a set of coupled linear matrix inequalities with equality constraints, which can be tackled effectively by some well-developed algorithms.

The following theorem provides a new criterion to test the robust $H_\infty$ performance level of the uncertain Markovian jump system in (3.28) over all the admissible uncertainties in (3.29) in terms of coupled linear matrix inequalities.

**Theorem 3.3** For a prescribed scalar $\gamma_{H_\infty} \in \mathbb{R}^+$, uncertain Markovian jump system (3.28) with $u(t) \equiv 0$ is robustly mean square stable and satisfies $\|G_{zw}\|_{H_\infty} < \gamma_{H_\infty}$ over uncertainties (3.29), if there exist matrices $P_i \in \mathbb{S}^{n \times n}, T_{ij} \in \mathbb{S}^{n \times n}$, and $\lambda_i \in \mathbb{R}^+$, $i, j \in S, j \neq i$, such that the coupled linear matrix inequalities

$$
\begin{bmatrix}
\Phi_{1i} & \Gamma_{1i} & P_i E_{0i} + C_i^T E_{1i} & \Gamma_{2i} \\
\Gamma_{1i}^T & \Phi_{2i} & D_{2i}^T E_{1i} & 0 \\
E_{0i}^T P_i + E_{1i}^T C_i & E_{1i}^T D_{2i} & -\lambda_i I + E_{1i}^T E_{1i} & 0 \\
\Gamma_{2i}^T & 0 & 0 & -\Lambda_i
\end{bmatrix} < 0
$$

(3.32)
According to Definition 3.4, for a prescribed scalar \( \gamma_{H_{\infty}} \in \mathbb{R}^+ \), uncertain Markovian jump system (3.28) with \( u(t) \equiv 0 \) is robustly mean square stable and has \( H_{\infty} \) performance \( \|G_{sv}\|_{H_{\infty}} < \gamma_{H_{\infty}} \), if there exist matrices \( P_i \in \mathbb{S}^{n \times n}, i \in S \), such that

\[
\begin{bmatrix}
\Phi_{1i} & \Phi_{2i} \\
\Gamma_{1i} & \Gamma_{2i}
\end{bmatrix} =
\begin{bmatrix}
A_i^T P_i + P_i A_i + \sum_{j=1}^{s} \pi_{ij} P_j + \sum_{j=1, j \neq i}^{s} \epsilon_{ij}^2 T_{ij} + C_i^T C_i + \lambda_i H_{0i}^T H_{0i} \\
-P_i B_{2i} + C_i^T D_{2i} + \lambda_i H_{0i}^T H_{2i},
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Phi_{1i} & \Phi_{2i} \\
\Gamma_{1i} & \Gamma_{2i}
\end{bmatrix} =
\begin{bmatrix}
P_i - P_1 & \cdots & P_i - P_{i-1} & P_i - P_{i+1} & \cdots & P_i - P_s
\end{bmatrix},
\]

\[
\Lambda_i = \text{diag}(T_{1i}, \ldots, T_{(i-1)}, T_{(i+1)}, \ldots, T_{is}).
\]

**Proof:** According to Definition 3.4, for a prescribed scalar \( \gamma_{H_{\infty}} \in \mathbb{R}^+ \), uncertain Markovian jump system (3.28) with \( u(t) \equiv 0 \) is robustly mean square stable and has \( H_{\infty} \) performance \( \|G_{sv}\|_{H_{\infty}} < \gamma_{H_{\infty}} \), if there exist matrices \( P_i \in \mathbb{S}^{n \times n}, i \in S \), such that

\[
\Xi_{1i} \doteq \begin{bmatrix}
\hat{A}_i^T P_i + P_i \hat{A}_i + \sum_{j=1}^{s} \pi_{ij} P_j + P_i \hat{B}_{2i} & \hat{C}_i^T \\
\hat{B}_{2i}^T P_i & -\gamma_{H_{\infty}}^2 I & \hat{D}_{2i}^T \\
\hat{C}_i & \hat{D}_{2i} & -I
\end{bmatrix} < 0
\]

for all \( i \in S \). Considering that the structure of the uncertainties in (3.29), we have

\[
\Xi_{1i} = \begin{bmatrix}
A_i^T P_i + P_i A_i + \sum_{j=1}^{s} \pi_{ij} P_j & \sum_{j=1, j \neq i}^{s} \epsilon_{ij}^2 T_{ij} \\
B_{2i}^T P_i & -\gamma_{H_{\infty}}^2 I & D_{2i}^T \\
C_i & D_{2i} & -I
\end{bmatrix} + \sum_{i=1, j \neq i}^{s} \left[ \begin{bmatrix}
P_i E_{0i} \\
E_{1i}
\end{bmatrix} \\
0 \\
F_i H_{0i}^T \\
0 \\
H_{2i}^T \\
F_i^T \\
0 \\
E_{1i}
\right] T_{ij} \left[ \begin{bmatrix}
P_i E_{0i} \\
E_{1i}
\end{bmatrix} \\
0 \\
F_i H_{0i}^T \\
0 \\
H_{2i}^T \\
F_i^T \\
0 \\
E_{1i}
\right] \leq \begin{bmatrix}
A_i^T P_i + P_i A_i + \sum_{j=1}^{s} \pi_{ij} P_j & \sum_{j=1, j \neq i}^{s} \epsilon_{ij}^2 T_{ij} \\
B_{2i}^T P_i & -\gamma_{H_{\infty}}^2 I & D_{2i}^T \\
C_i & D_{2i} & -I
\end{bmatrix} + \frac{1}{\lambda_i} \begin{bmatrix}
P_i E_{0i} \\
E_{1i}
\end{bmatrix} T_{ij} \left[ \begin{bmatrix}
P_i E_{0i} \\
E_{1i}
\end{bmatrix} \\
0 \\
F_i H_{0i}^T \\
0 \\
H_{2i}^T \\
F_i^T \\
0 \\
E_{1i}
\right].
\]
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\[
\begin{bmatrix}
\sum_{j=1,j\neq i}^s \left[ \varepsilon_{ij}^2 T_{ij} + (P_i - P_j)^{-1} (P_i - P_j) \right] & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 
\end{bmatrix}
\]

for any $\lambda_i \in \mathbb{R}^+$ and any $T_{ij} \in \mathbb{S}^{n \times n}$. In view of Schur complement equivalence, $\Xi_{1i} < 0$ holds if inequality

\[
\begin{bmatrix}
\Xi_{2i} + \lambda_i H_{0i}^T H_{0i} & P_i B_{2i} + \lambda_i H_{0i}^T H_{2i} & C_i^T & P_i E_{0i} \\
B_{2i}^T P_i + \lambda_i H_{2i}^T H_{0i} & -\lambda_i I + \lambda_i H_{2i}^T H_{2i} & D_{2i}^T & 0 \\
C_i & D_{2i} & -I & E_{1i} \\
E_{0i}^T P_i & 0 & E_{1i}^T & -\lambda I 
\end{bmatrix} < 0
\]

holds, where

\[
\Xi_{2i} = A_i^T P_i + P_i A_i + \sum_{j=1}^s \pi_{ij} P_j + \sum_{j=1,j\neq i}^s \left[ \varepsilon_{ij}^2 T_{ij} + (P_i - P_j)^{-1} (P_i - P_j) \right].
\]

Pre- and post-multiplying both sides of the above inequality by

\[
\begin{bmatrix}
I & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & I & 0 
\end{bmatrix}
\]

we have

\[
\begin{bmatrix}
\Xi_{2i} + \lambda_i H_{0i}^T H_{0i} & P_i B_{2i} + \lambda_i H_{0i}^T H_{2i} & P_i E_{0i} & C_i^T \\
B_{2i}^T P_i + \lambda_i H_{2i}^T H_{0i} & -\lambda_i I + \lambda_i H_{2i}^T H_{2i} & 0 & D_{2i}^T \\
E_{0i}^T P_i & 0 & E_{1i}^T & -\lambda I \\
C_i & D_{2i} & E_{1i} & -I 
\end{bmatrix} < 0
\]

which is equivalent to (3.32) based on Schur complement equivalence again. This
completes the proof.

In the following, we provide a solution to the robust $H_\infty$ control problem for uncertain system (3.28) in terms of coupled linear matrix inequalities with equality constraints.

**Theorem 3.4** Consider uncertain Markovian jump system (3.28); for a prescribed scalar $\gamma_{H_\infty} \in \mathbb{R}^+$, there exists a state-feedback controller (3.30) such that closed-loop system (3.31) is robustly mean square stable and has $H_\infty$ performance $\|G_zw\|_{H_\infty} < \gamma_{H_\infty}$ over the uncertainties in (3.29), if there exist matrices $P_i \in \mathbb{S}^{n \times n}$, $X_i \in \mathbb{S}^{n \times n}$, $V_i \in \mathbb{S}^{m \times n}$, $Z_i \in \mathbb{S}^{n \times n}$, $Y_i \in \mathbb{R}^{m \times n}$, $T_{ij} \in \mathbb{S}^{n \times n}$, and $\alpha_i \in \mathbb{R}^+$, $i, j \in S$, $j \neq i$, such that the coupled linear matrix inequalities

$$
\begin{bmatrix}
\Phi_{3i} & \Gamma_{3i} & \Gamma_{4i} & X_i \\
\Gamma_{3i}^T & \Phi_{4i} & D_{2i}H_{2i}^T & 0 \\
\Gamma_{4i}^T & H_{2i}D_{2i}^T & -\alpha_i I + H_{2i}H_{2i}^T & 0 \\
X_i & 0 & 0 & -Z_i
\end{bmatrix} < 0 \quad (3.33)
$$

with equality constraints

$$
\begin{bmatrix}
\Phi_{5i} & \Gamma_{2i} \\
\Gamma_{2i}^T & -\Lambda_i
\end{bmatrix} \leq 0 \quad (3.34)
$$

hold for all $i \in S$, where

$$
\begin{align*}
\Phi_{3i} &= (A_iX_i + B_{1i}Y_i)^T + (A_iX_i + B_{1i}Y_i) + B_{2i}B_{2i}^T + \alpha_i E_{0i}E_{0i}^T, \\
\Phi_{4i} &= -\lambda_{H_\infty}^2 I + D_{2i}D_{2i}^T + \alpha_i E_{1i}E_{1i}^T, \\
\Phi_{5i} &= -V_i + \sum_{j=1}^s \pi_{ij}P_j + \sum_{j=1,j\neq i}^s \varepsilon_{ij}^2 T_{ij}, \\
\Gamma_{3i} &= (C_iX_i + D_{1i}Y_i)^T + B_{2i}D_{2i}^T + \alpha_i E_{0j}E_{1j}^T,
\end{align*}
$$
\( \Gamma_{2i} = (H_{0i}X_i + H_{1i}Y_i)^T + B_2H_{2i}^T, \)

\( \Gamma_{2i} \) and \( \Lambda_i \) are given in Theorem 3.3. In this case, a controller (3.30) is given by \( K_i = Y_iP_i, \ i \in S. \)

Proof: Consider closed-loop system (3.31); let \( \tilde{A}_i = A_i + B_1K_i, \ \tilde{C}_i = C_i + D_1K_i \) and \( \tilde{H}_0i = H_0i + H_1K_i. \) Similar to the argument of Theorem 3.3, we have that the closed-loop system (3.31) is robustly mean square stable and has robust \( H_\infty \) performance \( \|G_{zw}\|_{H_\infty} < \gamma_{H_\infty}, \) if there exist matrices \( P_i \in \mathbb{S}^{n \times n}, \ T_{ij} \in \mathbb{S}^{n \times n}, \) and \( \alpha_{ij} \in \mathbb{R}^+, \ i, j \in S, \ j \neq i, \) such that

\[
\begin{bmatrix}
\tilde{A}_i^TP_i + P_i\tilde{A}_i + \sum_{j=1}^{s} \pi_{ij}P_j & P_iB_{2i} & \tilde{C}_i^T \\
B_{2i}^TP_i & -I & D_{2i}^T \\
\tilde{C}_i & D_{2i} & -\gamma_{H_\infty}^2I
\end{bmatrix}
\begin{bmatrix}
\sum_{j=1,j \neq i}^{s} \left[ \varepsilon_{ij}^2T_{ij} + \left( P_i - P_j \right)T_{ij}^{-1}\left( P_i - P_j \right) \right] & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
P_iE_{0i} & P_iE_{0i}^T \\
E_{1i} & E_{1i}
\end{bmatrix}
+ \frac{1}{\alpha_{ij}} \begin{bmatrix}
H_{0i}^T \\
H_{2i}^T
\end{bmatrix}
< 0 \tag{3.36}
\]

holds for all \( i \in S. \)

Let \( V_i \in \mathbb{S}^{n \times n} \) such that

\[
\sum_{j=1}^{s} \pi_{ij}P_j + \sum_{j=1,j \neq i}^{s} \left[ \varepsilon_{ij}^2T_{ij} + \left( P_i - P_j \right)T_{ij}^{-1}\left( P_i - P_j \right) \right] \leq V_i.
\]

This inequality is equivalent to (3.34) in view of Schur complement equivalence.

Now, pre- and post-multiplying both sides of (3.36) by \( \text{diag}(P_i^{-1}, I, I) \) and applying the change of variables: \( X_i \approx P_i^{-1}, \ Z_i \approx V_i^{-1} \) and \( Y_i \approx K_iP_i^{-1}, \) we have that (3.36) holds
if
\[
\begin{bmatrix}
\Xi_{3i} & B_{2i} (C_i X_i + D_{1i} Y_i)^T \\
B_{2i}^T & -I & D_{2i}^T \\
C_i X_i + D_{1i} Y_i & D_{2i} & -\lambda_{H_\infty}^2 I
\end{bmatrix} + \alpha_i \begin{bmatrix}
E_{0i} \\
E_{1i}
\end{bmatrix} < 0
\]
holds, where
\[
\Xi_{3i} = (A_i X_i + B_{2i} Y_i)^T + (A_i X_i + B_{2i} Y_i) + X_i Z_i^{-1} X_i.
\]

The above inequality is equivalent to
\[
\begin{bmatrix}
\Xi_{3i} + \alpha_i E_{0i} E_{0i}^T & \bullet & \bullet & \bullet \\
B_{2i}^T & -I & \bullet & \bullet \\
C_i X_i + D_{1i} Y_i + \alpha_i E_{1i} E_{1i}^T & D_{2i} & -\lambda_{H_\infty}^2 I + \alpha_i E_{1i} E_{1i}^T & \bullet \\
H_{0i} X_i + H_{1i} Y_i & H_{2i} & 0 & -\alpha_i I
\end{bmatrix} < 0
\]
which is further equivalent to
\[
\begin{bmatrix}
\Xi_{3i} + \alpha_i E_{0i} E_{0i}^T & \bullet & \bullet & \bullet \\
C_i X_i + D_{1i} Y_i + \alpha_i E_{1i} E_{1i}^T & D_{2i} & -\lambda_{H_\infty}^2 I + \alpha_i E_{1i} E_{1i}^T & \bullet \\
H_{0i} X_i + H_{1i} Y_i & 0 & -\alpha_i I & \bullet \\
B_{2i}^T & D_{2i}^T & H_{2i}^T & -I
\end{bmatrix} < 0.
\]

The above inequality is equivalent to (3.33) in view of Schur complement equivalence. This completes the proof. \qed

To solve the set of the coupled linear matrix inequalities in (3.33)–(3.34) with the equality constraints in (3.35) effectively, we modify (3.33) with a sufficiently small
number $\beta \in \mathbb{R}^+$ to

$$
\begin{bmatrix}
\Phi_{3i} + \beta I & \Gamma_{3i} & \Gamma_{4i} & X_i \\
\Gamma_{3i}^T & \Phi_{4i} & D_{2i}^T H_{2i}^T & 0 \\
\Gamma_{4i}^T & H_{2i} D_{2i}^T & -\alpha_i I + H_{2i} H_{2i}^T & 0 \\
X_i & 0 & 0 & -Z_i
\end{bmatrix} \leq 0 \quad (3.37)
$$

and weaken the equality constraints in (3.35) to the semi-definite programming conditions:

$$
\begin{bmatrix}
P_i & I \\
I & X_i
\end{bmatrix} \geq 0, \quad \begin{bmatrix}
V_i & I \\
I & Z_i
\end{bmatrix} \geq 0. \quad (3.38)
$$

Now, we are ready to employ the sequential linear programming method [38] to solve the robust $H_\infty$ control problem ($R H_{\infty} P$), which is summarized below.

**Algorithm $R H_{\infty} P$** For a given precision $\delta \in \mathbb{R}^+$, let $k_{\text{max}}$ be the maximum number of iterations and a sufficiently small number $\beta \in \mathbb{R}^+$ be given.

1. Determine $P_i^{(0)}, X_i^{(0)}, V_i^{(0)}, Z_i^{(0)}, Y_i^{(0)}, T_{ij}^{(0)}$ and $\alpha_i^{(0)}$, $i, j \in S$, $j \neq i$ satisfying (3.34), (3.37) and (3.38). Let $k := 0$.

2. Solve the convex optimization problem

$$
\min \sum_{i=1}^s \text{trace} \left( P_i X_i^{(k)} + P_i^{(k)} X_i + V_i Z_i^{(k)} + V_i^{(k)} Z_i \right)
$$

subject to (3.34), (3.37) and (3.38) for all $i \in S$.

for the variables $P_i, X_i, V_i, Z_i, Y_i, T_{ij}, \alpha_i, i, j \in S$, $j \neq i$.

3. Let $T_i^{(k)} := P_i, L_i^{(k)} := X_i, U_i^{(k)} := V_i$ and $R_i^{(k)} := Z_i$ for all $i \in S$. 

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If
\[
\left| \sum_{i=1}^{s} \text{trace} \left( T_{i}(k)X_{i}(k) + P_{i}(k)L_{i}(k) + U_{i}(k)Z_{i}(k) + V_{i}(k)R_{i}(k) \right) \right| - 2 \sum_{i=1}^{s} \text{trace} \left( P_{i}(k)X_{i}(k) + V_{i}(k)Z_{i}(k) \right) < \delta
\]

then go to Step (7), otherwise go to Step (5).

(5) Compute \( \theta^* \in [0, 1] \) by solving
\[
\min_{\theta \in [0, 1]} \sum_{i=1}^{s} \text{trace} \left( \left[ P_{i}(k) + \theta \left( T_{i}(k) - P_{i}(k) \right) \right] \left[ X_{i}(k) + \theta \left( L_{i}(k) - X_{i}(k) \right) \right] \right) + \left[ V_{i}(k) + \theta \left( U_{i}(k) - V_{i}(k) \right) \right] \left[ Z_{i}(k) + \theta \left( R_{i}(k) - Z_{i}(k) \right) \right].
\]

(6) Let
\[
P_{i}^{(k+1)} := P_{i}^{(k)} + \theta^* \left( T_{i}(k) - P_{i}(k) \right), \quad X_{i}^{(k+1)} := X_{i}^{(k)} + \theta^* \left( L_{i}(k) - X_{i}(k) \right),
\]
\[
V_{i}^{(k+1)} := V_{i}^{(k)} + \theta^* \left( U_{i}(k) - V_{i}(k) \right), \quad Z_{i}^{(k+1)} := Z_{i}^{(k)} + \theta^* \left( R_{i}(k) - Z_{i}(k) \right),
\]
for all \( i \in S \), and \( k := k + 1 \). If \( k < k_{\max} \), then go to Step (2), otherwise go to Step (7).

(7) If \( \sum_{i=1}^{s} \text{trace} \left( P_{i}^{(k)}X_{i}(k) + V_{i}^{(k)}Z_{i}(k) \right) = 2sn \), then a solution is found, otherwise a solution cannot be found by this algorithm.

3.2.3 Numerical Example

In this subsection, we present a simple numerical example to illustrate the usefulness of the proposed theory. Attention is mainly focused on the design of a robust \( H_\infty \) state-feedback controller such that the closed-loop system is robustly mean square stable and has the desired \( H_\infty \) performance level over the admissible uncertainties in the mode transition rate matrix.
It is assumed that the system under consideration has two operation modes and has uncertainties only in the mode transition rate matrix. The system data of (3.28) are as follows.

Mode 1:

\[
A_1 = \begin{bmatrix} 0.5 & -1 \\ 0 & -1 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \\
C_1 = \begin{bmatrix} 0.5 & 0 \\ 0.1 & 0.5 \end{bmatrix}, \quad D_{11} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

Mode 2:

\[
A_2 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}, \\
C_2 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad D_{22} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

The measured mode transition rate matrix and the error bounds are

\[
\Pi = \begin{bmatrix} -2 & 2 \\ 10 & -10 \end{bmatrix}, \quad \varepsilon_{12} = 0.5, \quad \varepsilon_{21} = 3.
\]

Suppose that a controller of form (3.30) is desired such that the closed-loop system is robustly mean square stable and has guaranteed $H_\infty$ performance level $\gamma_{H_\infty} \triangleq 0.5$ over all possible uncertainties $\Delta \pi_{12} \in [-1, 1]$ and $\Delta \pi_{21} \in [-6, 6]$. The uncertain system given above is not mean square stable for all the admissible uncertainties in the mode transition rate matrix.

To compute with Algorithm RH$_\infty$P for this example, it is chosen that $\delta = 10^{-10}$. 
$k_{\text{max}} = 100$ and $\beta = 0.01$. One set of solutions is

\[ P_1 = \begin{bmatrix} 3.2955 & -2.1522 \\ -2.1522 & 3.0661 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3.3038 & -2.2124 \\ -2.2124 & 3.4371 \end{bmatrix}, \]

\[ X_1 = \begin{bmatrix} 0.5603 & 0.3933 \\ 0.3933 & 0.6022 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0.5320 & 0.3425 \\ 0.3425 & 0.5114 \end{bmatrix}, \]

\[ V_1 = \begin{bmatrix} 0.9564 & -0.1518 \\ -0.1518 & 1.1139 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0.0287 & -0.0124 \\ -0.0124 & 0.0188 \end{bmatrix}, \]

\[ Z_1 = \begin{bmatrix} 1.0687 & 0.1456 \\ 0.1456 & 0.9176 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 48.9253 & 32.4144 \\ 32.4144 & 74.7296 \end{bmatrix}, \]

\[ T_{12} = \begin{bmatrix} 1.8799 & -0.0626 \\ -0.0626 & 0.7438 \end{bmatrix}, \quad T_{21} = \begin{bmatrix} 0.0062 & -0.0342 \\ -0.0342 & 0.2072 \end{bmatrix}, \]

\[ Y_1 = \begin{bmatrix} -0.5027 & -0.0904 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} -0.5913 & -0.6436 \end{bmatrix}. \]

It can be verified that $\|P_1 X_1 - I\| = 5.5375 \times 10^{-12}$, $\|P_2 X_2 - I\| = 5.5430 \times 10^{-12}$, $\|V_1 Z_1 - I\| = 5.5427 \times 10^{-12}$ and $\|V_2 Z_2 - I\| = 5.5407 \times 10^{-12}$. Therefore, the equality constraints in (3.35) are satisfied. Finally, the state-feedback controller can be obtained from the above solution as

\[ K_1 = \begin{bmatrix} -1.4619 & 0.8046 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.5295 & -0.9040 \end{bmatrix}. \]

This controller can stabilize the system and the robust $H_\infty$ performance of the closed-loop system on some points in the uncertainty domain is given in Table 3.2.

### 3.3 Summary

This chapter has studied the robust $H_2$ and $H_\infty$ control problems for a class of uncertain Markovian jump linear systems. Attention has been focused on the design of the robust
$H_2$ and $H_\infty$ controllers. The solutions to the addressed problems are related to a set of coupled linear matrix inequalities with equality constraints. Algorithms involving convex optimization have been suggested to design the controllers effectively. The developed theory has been illustrated by numerical examples as well.
CHAPTER 3. ROBUST $H_2$ AND $H_\infty$ CONTROL OF UNCERTAIN MJLS

<table>
<thead>
<tr>
<th>$\Delta \pi_{12}$</th>
<th>$\Delta \pi_{21}$</th>
<th>$K_{\Delta=0}$</th>
<th>$K_{\Delta \neq 0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.8</td>
<td>8</td>
<td>unstable</td>
<td>1.6108</td>
</tr>
<tr>
<td>0</td>
<td>8</td>
<td>0.7967</td>
<td>0.9556</td>
</tr>
<tr>
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<td>8</td>
<td>0.4143</td>
<td>0.7045</td>
</tr>
<tr>
<td>-1.8</td>
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<td>unstable</td>
<td>1.4626</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0.6022</td>
<td>0.8516</td>
</tr>
<tr>
<td>1.8</td>
<td>0</td>
<td>0.3488</td>
<td>0.6352</td>
</tr>
<tr>
<td>-1.8</td>
<td>-8</td>
<td>unstable</td>
<td>1.2827</td>
</tr>
<tr>
<td>0</td>
<td>-8</td>
<td>0.4880</td>
<td>0.7875</td>
</tr>
<tr>
<td>1.8</td>
<td>-8</td>
<td>0.3424</td>
<td>0.6169</td>
</tr>
</tbody>
</table>

Table 3.1 $H_2$ performance of two controllers

<table>
<thead>
<tr>
<th>$\Delta \pi_{12}$</th>
<th>$\Delta \pi_{21}$</th>
<th>$H_\infty$ norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>6</td>
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<tr>
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<td>0</td>
<td>0.3986</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.4093</td>
</tr>
<tr>
<td>-1</td>
<td>-6</td>
<td>0.4159</td>
</tr>
<tr>
<td>0</td>
<td>-6</td>
<td>0.4272</td>
</tr>
<tr>
<td>1</td>
<td>-6</td>
<td>0.4345</td>
</tr>
</tbody>
</table>

Table 3.2 $H_\infty$ performance of the closed-loop system

Figure 3.1 Performance regions of the probability domain
Chapter 4

Robust $H_\infty$ Filtering of Uncertain MJLS

In the last two chapters, Problem 1 (cf. page 10) has been investigated in the context of robust stabilization, robust $H_2$ control and robust $H_\infty$ control, respectively. This chapter continues to study this problem in the context of fixed-order robust $H_\infty$ filter design of uncertain Markovian jump linear systems. The goal of the work reported in this chapter is to design a general dynamic filter of fixed-order such that, over all the admissible uncertainties, the filtering error system is robustly mean square stable and the $H_\infty$ norm of the filtering error system is less than a prescribed level.

The chapter is organized as follows. Section 4.1 formulates the fixed-order robust $H_\infty$ filtering problem to be solved and provides some preliminary results needed to develop the main results. The main results of this chapter are presented in Section 4.2. A criterion for analyzing the filtering performance of a given filter is first established. Then a filter design method is provided for the design of the fixed-order robust $H_\infty$ filter. Several special cases are also stated in corollaries. Section 4.3 gives a numerical comparison of the performance of the filters designed by various methods. Section 4.4 ends this chapter and gives a summary of Chapters 2–4.
CHAPTER 4. ROBUST H\(_{\infty}\) FILTERING OF UNCERTAIN MJLS

4.1 Problem Formulation

Consider the following class of uncertain Markovian jump linear systems defined on a complete probability space \((\Omega, \mathcal{F}, P)\):

\[
\begin{align*}
\dot{x}(t) &= \hat{A}(\hat{r}(t)) x(t) + \hat{B}(\hat{r}(t)) w(t) \\
z(t) &= \hat{C}_z(\hat{r}(t)) x(t) + \hat{D}_z(\hat{r}(t)) w(t) \\
y(t) &= \hat{C}_y(\hat{r}(t)) x(t) + \hat{D}_y(\hat{r}(t)) w(t)
\end{align*}
\] (4.1)

where \(x(t) \in \mathbb{R}^n, t \geq 0\), is the system state, \(w(t) \in \mathbb{R}^{n_w}\) is the exogenous non-zero disturbance that belongs to \(L_{\infty}^{2w}[0, \infty)\), \(z(t) \in \mathbb{R}^{n_z}\) is the signal to be estimated, \(y(t) \in \mathbb{R}^{n_y}\) is the measured output used to estimate the signal \(z(t)\). The mode jumping process \(\hat{r}(t) : t \geq 0\) is a continuous-time, discrete-state, homogeneous Markov process on the probability space. \(\hat{r}(t)\) takes values in a finite state space \(S \triangleq \{1, 2, \ldots, s\}\), and has the mode transition probabilities

\[
\Pr(\hat{r}(t + \delta t) = j | \hat{r}(t) = i) = \begin{cases} 
\hat{r}_{ij} \delta t + o(\delta t), & \text{if } j \neq i \\
1 + \hat{r}_{ii} \delta t + o(\delta t), & \text{if } j = i
\end{cases}
\]

where \(\delta t > 0\) and \(\lim_{\delta t \to 0} \frac{o(\delta t)}{\delta t} = 0\); \(\hat{r}_{ij} \geq 0\) denotes the switching rate from mode \(i\) at time \(t\) to mode \(j\) at time \(t + \delta t\) for \(i, j \in S, j \neq i\); \(\hat{r}_{ii} = -\sum_{j=1, j \neq i}^{s} \hat{r}_{ij}\) for all \(i \in S\).

In each mode \(i \in S\), system matrices \(\hat{A}_i \triangleq \hat{A}(\hat{r}(t) = i), \hat{B}_i \triangleq \hat{B}(\hat{r}(t) = i), \hat{C}_z \triangleq \hat{C}_z(\hat{r}(t) = i), \hat{D}_z \triangleq \hat{D}_z(\hat{r}(t) = i), \hat{C}_y \triangleq \hat{C}_y(\hat{r}(t) = i), \hat{D}_y \triangleq \hat{D}_y(\hat{r}(t) = i)\) and mode transition rate matrix \(\hat{\Pi} \triangleq (\hat{r}_{ij}) \in \mathbb{R}^{s \times s}\) are unknown constant real matrices, but of the following forms of uncertainties, respectively:

\[
\begin{align*}
\hat{A}_i &= A_i + E_{ai} F_{ai} H_{ai}, \text{ with } F_{ai}^T F_{ai} \leq I \quad (4.2a) \\
\hat{B}_i &= B_i + E_{bi} F_{bi} H_{bi}, \text{ with } F_{bi}^T F_{bi} \leq I \quad (4.2b) \\
\hat{C}_z &= C_z + E_{cz} F_{cz} H_{cz}, \text{ with } F_{cz}^T F_{cz} \leq I \quad (4.2c)
\end{align*}
\]
\[ \hat{\Pi} = \Pi + \Delta \Pi, \text{ with } |\Delta \pi_{ij}| \leq 2 \epsilon_{ij} \text{ for all } i, j \in \mathcal{S}, j \neq i \] (4.2g)

where \( 0 \leq 2 \epsilon_{ij} \leq \pi_{ij} \), matrices \( A_i, B_i, C_i, D_i, C_yi, D_yi, E_{ai}, H_{ai}, E_{bi}, H_{bi}, E_{czi}, H_{czi}, E_{dzi}, H_{dzi}, E_{cyi}, H_{cyi}, E_{dyi}, H_{dyi} \) and \( \Pi \triangleq (\pi_{ij}) \) are known constant real matrices of appropriate dimensions. Moreover, matrix \( \Pi \) satisfies

\[
\begin{cases}
\pi_{ij} \geq 0, & \text{for all } i, j \in \mathcal{S}, j \neq i \\
\pi_{ii} \triangleq - \sum_{j=1, j \neq i}^{S} \pi_{ij}, & \text{for all } i \in \mathcal{S}
\end{cases}
\]

and can be thought of as the measured value of matrix \( \hat{\Pi} \) in practice.

Matrices \( F_{ai}, F_{bi}, F_{czi}, F_{dzi}, F_{cyi} \) and \( F_{dyi}, i \in \mathcal{S} \), denote the norm-bounded uncertainties in the system matrices. \( \Delta \Pi \triangleq (\Delta \pi_{ij}) = (\hat{\pi}_{ij} - \pi_{ij}) \) is the element-wise uncertainty in the mode transition rate matrix. The probability constraint on each row of \( \Delta \Pi \) is \( \sum_{j=1}^{S} \Delta \pi_{ij} = 0 \), which ensures \( \sum_{j=1}^{S} (\pi_{ij} + \Delta \pi_{ij}) = 0 \) and also implies that the uncertainty bound of \( \Delta \pi_{ii} \) must be \( \epsilon_{ii} = \sum_{j=1, j \neq i}^{S} \epsilon_{ij} \).

The uncertainties defined in (4.2) are called the admissible uncertainties of uncertain system (4.1). In addition, uncertain system (4.1) is assumed to be robustly mean square stable over all the admissible uncertainties in (4.2).

We now consider a general fixed-order filter

\[
\begin{align*}
\dot{x}_f(t) &= A_f(\hat{\tau}(t))x_f(t) + B_f(\hat{\tau}(t))y(t) \\
z_f(t) &= C_f(\hat{\tau}(t))x_f(t) + D_f(\hat{\tau}(t))y(t)
\end{align*}
\] (4.3)

where \( x_f(t) \in \mathbb{R}^{n_f} \) is the filter state with \( 0 \leq n_f \leq n \), the filter output \( z_f(t) \in \mathbb{R}^{n_z} \) is an estimation of the signal \( z(t) \). Matrices \( A_{fi} \triangleq A_f(\hat{\tau}(t) = i), B_{fi} \triangleq B_f(\hat{\tau}(t) = i), C_{fi} \triangleq C_f(\hat{\tau}(t) = i) \) and \( D_{fi} \triangleq D_f(\hat{\tau}(t) = i) \) are to be determined for each mode \( i \in \mathcal{S} \).
The filtering error is defined as \( e(t) = z(t) - z_f(t) \).

Connecting filter (4.3) to uncertain system (4.1) yields the filtering error system

\[
\begin{align*}
\dot{x}_e(t) &= \hat{A}_e(\hat{\rho}(t))x_e(t) + \hat{B}_e(\hat{\rho}(t))w(t) \\
e(t) &= \hat{C}_e(\hat{\rho}(t))x_e(t) + \hat{D}_e(\hat{\rho}(t))w(t)
\end{align*}
\]

(4.4)

where \( x_e(t) = \begin{bmatrix} x^T(t) & x^T_f(t) \end{bmatrix}^T \) is the state of the error system. For all \( i \in S \), we have

\[
\begin{align*}
\hat{A}_{ei} &= \begin{bmatrix} \hat{A}_i & 0 \\ B_{fi} \hat{C}_{yi} & A_{fi} \end{bmatrix}, & \hat{B}_{ei} &= \begin{bmatrix} \hat{B}_i \\ B_{fi} \hat{D}_{yi} \end{bmatrix}, \\
\hat{C}_{ei} &= \begin{bmatrix} \hat{C}_{zi} - D_{fi} \hat{C}_{yi} & -C_{fi} \end{bmatrix}, & \hat{D}_{ei} &= \hat{D}_{zi} - D_{fi} \hat{D}_{yi}.
\end{align*}
\]

(4.5a)

For filtering error system (4.4), we have the following definitions and proposition.

**Definition 4.1** [17] The nominal Markovian jump filtering error system of (4.4) with \( w(t) \equiv 0 \) is said to be mean square stable if

\[
\lim_{t \to \infty} E \left( \| x_e(t) \|^2 \right) = 0
\]

for any initial conditions \( x_e(0) \in \mathbb{R}^{n+n_f} \) and \( \hat{\rho}(0) \in S \).

**Definition 4.2** [71] Consider the nominal Markovian jump filtering error system of (4.4); let \( G_{ew} \) denote the operator from the exogenous energy-bounded disturbance \( w(t) \) to the stochastic filtering error \( e(t) \). Then the \( H_\infty \) norm of operator \( G_{ew} \) is defined as \( \| G_{ew} \|_{H_\infty} \equiv \inf_\gamma \) such that

\[
\| e \|_2 < \gamma \| w \|_2
\]

for all non-zero processes \( w(\cdot) \in L_2^n [0, \infty) \) for zero initial system state \( x_e(0) = 0 \) and any initial system mode \( \hat{\rho}(0) \in S \).

**Proposition 4.1** [8] Given a prescribed scalar \( \gamma_{H_\infty} \in \mathbb{R}^+ \), the nominal Markovian jump filtering error system of (4.4) is mean square stable and has \( H_\infty \) performance
\[ \|G_{ew}\|_{\infty} < \gamma_{\infty} \] if there exist matrices \( P_i \in \mathbb{S}^{(n+n_f)\times(n+n_f)} \), \( i \in S \), such that

\[
\begin{bmatrix}
A_{ei}^T P_i + P_i A_{ei} + \sum_{j=1}^{s} \pi_{ij} P_j & P_i B_{ei} & C_{el}^T \\
B_{el}^T P_i & -\gamma_{\infty}^2 I & D_{el}^T \\
C_{ei} & D_{ei} & -I
\end{bmatrix} < 0
\]

holds for all \( i \in S \), where \( A_{ei}, B_{ei}, C_{ei}, D_{ei} \) and \( \pi_{ij} \) are the nominal values of \( \hat{A}_{ei}, \hat{B}_{ei}, \hat{C}_{ei}, \hat{D}_{ei} \) and \( \hat{\pi}_{ij} \), respectively, for \( i, j \in S \).

Based upon Proposition 4.1, we introduce the following definition.

**Definition 4.3** Given a prescribed scalar \( \gamma_{\infty} \in \mathbb{R}^+ \), filter (4.3) is said to be a robust \( H_\infty \) filter of fixed-order \( n_f \) with level \( \gamma_{\infty} \) for uncertain system (4.1), if there exist matrices \( P_i \in \mathbb{S}^{(n+n_f)\times(n+n_f)} \), \( i \in S \), such that

\[
\begin{bmatrix}
\hat{A}_{ei}^T P_i + P_i \hat{A}_{ei} + \sum_{j=1}^{s} \hat{\pi}_{ij} P_j & P_i \hat{B}_{ei} & \hat{C}_{el}^T \\
\hat{B}_{el}^T P_i & -\gamma_{\infty}^2 I & \hat{D}_{el}^T \\
\hat{C}_{ei} & \hat{D}_{ei} & -I
\end{bmatrix} < 0
\] (4.6)

holds for all \( i \in S \) over the admissible uncertainties in (4.2).

The objective of this chapter is to design a robust \( H_\infty \) filter of form (4.3) of order \( n_f \) with a prescribed level \( \gamma_{\infty} \in \mathbb{R}^+ \) for uncertain Markovian jump linear systems (4.1) over all the admissible uncertainties in (4.2).

The following lemma is used to obtain the results.

**Lemma 4.1** [33] Let \( W = W^T \in \mathbb{R}^{n \times n} \), \( U \in \mathbb{R}^{n \times m} \) and \( V \in \mathbb{R}^{k \times n} \) be three given matrices that satisfy \( \text{rank}(U) < n \) and \( \text{rank}(V) < n \). Then the linear matrix inequality

\[
W + UGV + (UGV)^T < 0
\]
has a solution $G \in \mathbb{R}^{m \times k}$ if, and only if, the following two linear matrix inequalities

$$U^T W (U^T)^T < 0 \quad \text{and} \quad (V^T)^T W [(V^T)^T]^T < 0$$

hold.

### 4.2 Fixed-Order Robust $H_\infty$ Filter Design

In this section, we first present a result for analyzing the robust $H_\infty$ filtering level when filter (4.3) is given, then establish a sufficient condition for constructing the desired robust $H_\infty$ filter of form (4.3). Finally, an effective algorithm is suggested to solve the proposed conditions.

The following result gives a criterion for testing the robust $H_\infty$ filtering level of filter (4.3) for uncertain Markovian jump linear systems (4.1) over all the admissible uncertainties in (4.2) in terms of coupled linear matrix inequalities.

**Theorem 4.1** Given a prescribed scalar $\gamma_{H_\infty} \in \mathbb{R}^+$, filter (4.3) is a robust $H_\infty$ filter of order $n_f$ with level $\gamma_{H_\infty}$ for uncertain system (4.1), if there exist matrices $P_i \in \mathbb{S}^{(n+n_f)\times(n+n_f)}$, $T_{ij} \in \mathbb{S}^{(n+n_f)\times(n+n_f)}$, and scalars $\lambda_{ai} \in \mathbb{R}^+$, $\lambda_{bi} \in \mathbb{R}^+$, $\lambda_{czi} \in \mathbb{R}^+$, $\lambda_{dzi} \in \mathbb{R}^+$, $\lambda_{czi} \in \mathbb{R}^+$, $\lambda_{dzi} \in \mathbb{R}^+$, $i, j \in \mathcal{S}$, $j \neq i$, satisfying the coupled linear matrix inequalities

$$\begin{bmatrix} \Phi_{1i} & P_i \tilde{B}_i & P_i \Gamma_{1i} & \Gamma_{2i} \\ \tilde{B}_i^T P_i & \Phi_{2i} & 0 & 0 \\ \Gamma_{1i}^T P_i & 0 & -\Lambda_{1i} & 0 \\ \Gamma_{2i}^T & 0 & 0 & -\Lambda_{2i} \end{bmatrix} + \begin{bmatrix} \tilde{C}_i^T & \tilde{D}_i^T \\ \tilde{D}_i & \tilde{G}_i \end{bmatrix} < 0$$

for all $i \in \mathcal{S}$, where

$$\Phi_{1i} = \tilde{A}_i^T P_i + P_i \tilde{A}_i + \sum_{j=1}^s p_{ij} P_j + \sum_{j=1, j \neq i}^s \varepsilon_{ij} T_{ij}$$

$$+ N_1 (\lambda_{ai} H_{ai}^T H_{ai} + \lambda_{czi} H_{czi}^T H_{czi} + \lambda_{dzi} H_{dzi}^T H_{dzi}) N_1^T,$$
\[
\Phi_{2i} = -\gamma_{H_{\infty}}^2 I + \lambda_{bi} H_{bi}^T H_{bi} + \lambda_{dzi} H_{dzi}^T H_{dzi} + \lambda_{dgy} H_{dgy}^T H_{dgy},
\]
\[
\Gamma_i = \begin{bmatrix}
N_1 E_{ai} & N_1 E_{bi} & N_2 B_{fi} E_{czy} & N_2 B_{fi} E_{dgy} & 0 & 0
\end{bmatrix},
\]
\[
\Gamma_{2i} = \begin{bmatrix}
P_i - P_1 & \cdots & P_i - P_{i-1} & P_i - P_{i+1} & \cdots & P_i - P_1
\end{bmatrix},
\]
\[
\Gamma_{3i} = \begin{bmatrix}
0 & 0 & -D_{fi} E_{czy} & -D_{fi} E_{dgy} & E_{czy} & E_{dzi}
\end{bmatrix},
\]
\[
\Lambda_{1i} = \text{diag}(\lambda_{ai} I, \lambda_{bi} I, \lambda_{cyi} I, \lambda_{dgi} I, \lambda_{cyzi} I, \lambda_{dzi} I),
\]
\[
\Lambda_{2i} = \text{diag}(T_{i1}, \ldots, T_{i(i-1)}, T_{i(i+1)}, \ldots, T_{is}),
\]
\[
\tilde{A}_i = \begin{bmatrix}
A_i & 0 \\
B_{fi} C_{yi} & A_{fi}
\end{bmatrix},
\]
\[
\tilde{B}_i = \begin{bmatrix}
B_i \\
B_{fi} D_{yi}
\end{bmatrix},
\]
\[
\tilde{C}_i = \begin{bmatrix}
C_{zi} - D_{fi} C_{yi} & -C_{fi}
\end{bmatrix},
\]
\[
\tilde{D}_i = D_{zi} - D_{fi} D_{yi},
\]
\[
N_1 = \begin{bmatrix}
I_n \\
0_{n \times n_f}
\end{bmatrix},
\]
\[
N_2 = \begin{bmatrix}
0_{n \times n_f} \\
I_{nf}
\end{bmatrix}.
\]

**Proof:** According to Definition 4.3 and Schur complement equivalence, given a prescribed scalar \(\gamma_{H_{\infty}} \in \mathbb{R}^+\), filter (4.3) is a robust \(H_{\infty}\) filter of order \(n_f\) with level \(\gamma_{H_{\infty}}\) for uncertain system (4.1) if, and only if, there exist matrices \(P_i \in \mathbb{S}^{(n+n_f) \times (n+n_f)}, i \in S\), such that inequalities (4.6) hold for all \(i \in S\) over the admissible uncertainties in (4.2).

Since the matrices \(\tilde{A}_{ei}, \tilde{B}_{ei}, \tilde{C}_{ei}\) and \(\tilde{D}_{ei}\) given in (4.5) can be written as

\[
\tilde{A}_{ei} = \tilde{A}_i + N_1 E_{ai} F_{ai} H_{ai} N_1^T + N_2 B_{fi} E_{czy} F_{czy} H_{czy} N_1^T
\]
\[
\tilde{B}_{ei} = \tilde{B}_i + N_1 E_{bi} F_{bi} H_{bi} + N_2 B_{fi} E_{dgy} F_{dgy} H_{dgy}
\]
\[
\tilde{C}_{ei} = \tilde{C}_i + E_{czy} F_{czy} H_{czy} N_1^T - D_{fi} E_{czy} F_{czy} H_{czy} N_1^T
\]

Proof: According to Definition 4.3 and Schur complement equivalence, given a prescribed scalar \(\gamma_{H_{\infty}} \in \mathbb{R}^+\), filter (4.3) is a robust \(H_{\infty}\) filter of order \(n_f\) with level \(\gamma_{H_{\infty}}\) for uncertain system (4.1) if, and only if, there exist matrices \(P_i \in \mathbb{S}^{(n+n_f) \times (n+n_f)}, i \in S\), such that inequalities (4.6) hold for all \(i \in S\) over the admissible uncertainties in (4.2).

Since the matrices \(\tilde{A}_{ei}, \tilde{B}_{ei}, \tilde{C}_{ei}\) and \(\tilde{D}_{ei}\) given in (4.5) can be written as

\[
\tilde{A}_{ei} = \tilde{A}_i + N_1 E_{ai} F_{ai} H_{ai} N_1^T + N_2 B_{fi} E_{czy} F_{czy} H_{czy} N_1^T
\]
\[
\tilde{B}_{ei} = \tilde{B}_i + N_1 E_{bi} F_{bi} H_{bi} + N_2 B_{fi} E_{dgy} F_{dgy} H_{dgy}
\]
\[
\tilde{C}_{ei} = \tilde{C}_i + E_{czy} F_{czy} H_{czy} N_1^T - D_{fi} E_{czy} F_{czy} H_{czy} N_1^T
\]
\[ \hat{D}_{ei} = \bar{D}_i + E_{dzi}F_{dzi}H_{dzi} - D_{fi}E_{dgi}F_{dgi}H_{dgi} \]

through expanding the uncertainty terms, inequality (4.6) can be rewritten as

\[
\begin{bmatrix}
\bar{A}_i^TP_i + P_i\bar{A}_i + \sum_{j=1}^l \pi_{ij}P_j & P_i\bar{B}_i & \bar{C}_i^T \\
\bar{B}_i^TP_i & -\gamma_{H_{\infty}}^2I & \bar{D}_i^T \\
\bar{C}_i & \bar{D}_i & -I \\
\end{bmatrix}
+ \begin{bmatrix}
P_iN_1E_{ai} & N_1H_{ai}^T & N_1H_{ai}^T & P_iN_1E_{ai}^T \\
0 & F_{ai} & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
+ \begin{bmatrix}
P_iN_1E_{bi} & 0 & 0 & P_iN_1E_{bi}^T \\
0 & F_{bi} & H_{bi}^T & F_{bi}^T \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
+ \begin{bmatrix}
0 & N_1H_{dzi}^T & N_1H_{dzi}^T & 0 \\
0 & F_{dzi} & 0 & 0 \\
0 & 0 & 0 & E_{dzi} \\
\end{bmatrix}
+ \begin{bmatrix}
P_iN_2B_{fi}E_{cyl} & N_1H_{cyl}^T & N_1H_{cyl}^T & P_iN_2B_{fi}E_{cyl}^T \\
0 & F_{cyl} & 0 & 0 \\
-D_{fi}E_{cyl} & 0 & 0 & -D_{fi}E_{cyl} \\
\end{bmatrix}
+ \begin{bmatrix}
P_iN_2B_{fi}E_{dgi} & 0 & 0 & P_iN_2B_{fi}E_{dgi}^T \\
0 & F_{dgi} & H_{dgi}^T & F_{dgi}^T \\
-D_{fi}E_{dgi} & 0 & 0 & -D_{fi}E_{dgi} \\
\end{bmatrix}
< 0.

(4.8)
From Lemma 3.1, we have

\[
\sum_{j=1}^{s} \Delta \pi_{ij} P_j = \sum_{j=1, j \neq i}^{s} \Delta \pi_{ij} (P_j - P_i) \\
\leq \sum_{j=1, j \neq i}^{s} \left( \frac{1}{2} \Delta \pi_{ij} \right)^2 T_{ij} + (P_i - P_j) T_{ij}^{-1} (P_i - P_j) \\
\leq \sum_{j=1, j \neq i}^{s} \left[ \varepsilon_{ij}^2 T_{ij} + (P_i - P_j) T_{ij}^{-1} (P_i - P_j) \right] \tag{4.9}
\]

for any matrix \( T_{ij} \in \mathbb{S}^{(n+n_f) \times (n+n_f)} \). Then in view of Lemma 2.2, inequality (4.8) holds over all the admissible uncertainties in (4.2) if there exist real numbers \( \lambda_{ai} \in \mathbb{R}^+ \), \( \lambda_{bi} \in \mathbb{R}^+ \), \( \lambda_{czi} \in \mathbb{R}^+ \), \( \lambda_{cyi} \in \mathbb{R}^+ \) and \( \lambda_{dgi} \in \mathbb{R}^+ \) such that

\[
\begin{bmatrix}
\Xi_{1i} & P_i \tilde{B}_i & C_i^T \\
\tilde{B}_i^T P_i & \Phi_{2i} & \tilde{D}_i^T \\
\tilde{C}_i & \tilde{D}_i & \Xi_{2i}
\end{bmatrix} + \frac{1}{\lambda_{czi}} \begin{bmatrix}
P_i N_2 B_{fj} E_{cyi} & 0 & P_i N_2 B_{fj} E_{cyi}
0 & -D_{fj} E_{cyi} & 0
-P_{fj} E_{dji} & -D_{fj} E_{dji}
\end{bmatrix} < 0
\]

where

\[
\Xi_{1i} = \tilde{A}_i^T P_i + P_i \tilde{A}_i + \sum_{j=1}^{s} \pi_{ij} P_j + \sum_{j=1, j \neq i}^{s} \left[ \varepsilon_{ij}^2 T_{ij} + (P_i - P_j) T_{ij}^{-1} (P_i - P_j) \right] + N_1 \left( \lambda_{ai} H_{ai}^T H_{ai} + \lambda_{czi} H_{czi}^T H_{czi} + \lambda_{cyi} H_{cyi}^T H_{cyi} \right) N_1^T + P_i N_1 \left( \frac{1}{\lambda_{ai}} E_{ai} E_{ai}^T + \frac{1}{\lambda_{bi}} E_{bi} E_{bi}^T \right) N_1^T P_i,
\]

\[
\Xi_{2i} = -I + \frac{1}{\lambda_{czi}} E_{czi} E_{czi}^T + \frac{1}{\lambda_{dzi}} E_{dzi} E_{dzi}^T.
\]
This inequality is equivalent to

\[
\begin{bmatrix}
\Phi_{1i} & P_i \bar{B}_i & \bar{C}_i^T & P_i \Gamma_{1i} & \Gamma_{2i} \\
\bar{B}_i^T P_i & \Phi_{2i} & \bar{D}_i^T & 0 & 0 \\
\bar{C}_i & \bar{D}_i & -I & \Gamma_{3i} & 0 \\
\Gamma_{1i}^T P_i & 0 & \Gamma_{3i}^T & -\Lambda_{1i} & 0 \\
\Gamma_{2i}^T & 0 & 0 & -\Lambda_{2i} & 0
\end{bmatrix} < 0
\]

in view of Schur complement equivalence.

Now, pre- and post-multiplying both sides of the inequality above by

\[
\begin{bmatrix}
I & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & I & 0 \\
0 & 0 & I & 0 & 0
\end{bmatrix}
\]

and its transpose, respectively, we obtain

\[
\begin{bmatrix}
\Phi_{1i} & P_i \bar{B}_i & P_i \Gamma_{1i} & \Gamma_{2i} & \bar{C}_i^T \\
\bar{B}_i^T P_i & \Phi_{2i} & 0 & 0 & \bar{D}_i^T \\
\Gamma_{1i}^T P_i & 0 & -\Lambda_{1i} & 0 & \Gamma_{3i}^T < 0 \\
\Gamma_{2i}^T & 0 & 0 & -\Lambda_{2i} & 0 \\
\bar{C}_i & \bar{D}_i & \Gamma_{3i} & 0 & -I
\end{bmatrix}
\]

This inequality is equivalent to (4.7). The proof is completed.

The following theorem provides a solution to the fixed-order robust $H_\infty$ filtering problem for uncertain Markovian jump linear systems in terms of coupled linear matrix inequalities and equality constraints.

**Theorem 4.2** Given a prescribed scalar $\gamma_{H_\infty} \in \mathbb{R}^+$, there exists a robust $H_\infty$ filter (4.3)
of order $n_f$ with level $\gamma_{H_\infty}$ for uncertain system (4.1), if there exist matrices $P_i \in \mathbb{S}^{(n+n_f)\times(n+n_f)}$, $X_i \in \mathbb{S}^{(n+n_f)\times(n+n_f)}$, $V_i \in \mathbb{S}^{(n+n_f)\times(n+n_f)}$, $Z_i \in \mathbb{S}^{(n+n_f)\times(n+n_f)}$, $T_{ij} \in \mathbb{S}^{(n+n_f)\times(n+n_f)}$, and scalars $\lambda_{ai} \in \mathbb{R}^+$, $\alpha_{ai} \in \mathbb{R}^+$, $\lambda_{bi} \in \mathbb{R}^+$, $\lambda_{czi} \in \mathbb{R}^+$, $\alpha_{czi} \in \mathbb{R}^+$, $\lambda_{cyi} \in \mathbb{R}^+$, $\alpha_{cyi} \in \mathbb{R}^+$, $i, j \in \mathcal{S}, j \neq i$, such that the coupled linear matrix inequalities

$$
\begin{bmatrix}
\Phi_{3i} & B_i & \Gamma_{4i} & \Pi_i T_{X_i} \\
B_i^T & \Phi_{2i} & 0 & 0 \\
\Gamma_{4i}^T & 0 & -\Lambda_{3i} & 0 \\
X_i N_1 & 0 & 0 & -Z_i
\end{bmatrix} < 0 \quad (4.10)
$$

$$
\begin{bmatrix}
-V_i + \sum_{j=1, j\neq i}^s (\pi_{ij} P_j + \varepsilon_{ij}^2 T_{ij}) & \Gamma_{2i} \\
\Gamma_{2i}^T & -\Lambda_{2i}
\end{bmatrix} \leq 0 \quad (4.11)
$$

$$
\begin{bmatrix}
\Theta_{i} & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
\Phi_{4i} & \Gamma_{5i} \\
\Gamma_{5i}^T & \Phi_{5i}
\end{bmatrix}
\begin{bmatrix}
\Theta_{i} & 0 \\
0 & I
\end{bmatrix} < 0 \quad (4.12)
$$

with equality constraints

$$
\lambda_{ai} \alpha_{ai} = 1, \quad \lambda_{czi} \alpha_{czi} = 1, \quad \lambda_{cyi} \alpha_{cyi} = 1 \quad (4.13a)
$$

$$
P_i X_i = I, \quad V_i Z_i = I \quad (4.13b)
$$

hold for all $i \in \mathcal{S}$, where

$$
\Phi_{3i} = N_i^T X_i N_1 A_i^T + A_i N_i^T X_i N_1 + \pi_{ii} N_i^T X_i N_1 + \alpha_{ai} E_i E_i^T,
$$

$$
\Gamma_{4i} = \begin{bmatrix}
E_{bi} & N_i^T X_i N_i H_{ai}^T & N_i^T X_i N_i H_{czi}^T & N_i^T X_i N_i H_{czi}^T
\end{bmatrix},
$$

$$
\Lambda_{3i} = \text{diag} \left( \lambda_{bi} I, \alpha_{ai} I, \alpha_{czi} I, \alpha_{czi} I \right),
$$

$$
\Theta_{i} = \begin{bmatrix}
C_{yi} & D_{yi} & E_{czi} & E_{dzi}
\end{bmatrix}^T.
$$
and $\Phi_{2i}$, $\Gamma_{2i}$, $\Lambda_{2i}$ and $N_1$ are given in Theorem 4.1. In this case, by substituting the solution of (4.10)–(4.13) into

$$
\Psi_i + \Psi_{Li} \Psi_{Ri} + (\Psi_{Li} J_i \Psi_{Ri})^T < 0 \tag{4.14}
$$

for all $i \in S$, where

$$
\Psi_i = \begin{bmatrix}
\Phi_{7i} & N_1^T P_i B_i & N_1 C_{zi}^T & 0 & 0 \\
B_i^T N_1^T P_i & \Phi_{2i} & D_{zi}^T & 0 & 0 \\
C_{zi} N_1^T & D_{zi} & \Phi_{3i} & 0 & 0 \\
0 & 0 & 0 & -\lambda_{czi} I & 0 \\
0 & 0 & 0 & 0 & -\lambda_{dzi} I
\end{bmatrix}
$$
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\[
\Psi_{li} = \begin{bmatrix}
P_i N_2 & 0 \\
0 & 0 \\
0 & -I \\
0 & 0 \\
0 & 0
\end{bmatrix},
\]

\[
J_i = \begin{bmatrix}
A_{fi} & B_{fi} \\
C_{fi} & D_{fi}
\end{bmatrix},
\]

\[
\Psi_{Ri} = \begin{bmatrix}
N^T_2 & 0 & 0 & 0 & 0 \\
C_{yi} N^T_1 & D_{yi} & 0 & E_{cyi} & E_{dyi}
\end{bmatrix},
\]

\[
\Phi_{7i} = N_1 A^T_i N^T_1 P_i + P_i N_1 A_i N^T_1 + \sum_{j=1}^s \pi_j P_j
\]

\[
+ \sum_{j=1, j\neq i}^s \left[ \varepsilon^2_{ij} T_{ij} + (P_i - P_j) T_{ij}^{-1} (P_i - P_j) \right]
\]

\[
+ P_i N_1 \left( \frac{1}{\lambda_{ai}} E_{ai} E^T_{ai} + \frac{1}{\lambda_{bi}} E_{bi} E^T_{bi} \right) N^T_1 P_i
\]

\[
+ N_1 \left( \lambda_{ai} H^T_{ai} H_{ai} + \lambda_{czi} H^T_{czi} H_{czi} + \lambda_{cyi} H^T_{cyi} H_{cyi} \right) N^T_1,
\]

\[
\Phi_{8i} = -I + \frac{1}{\lambda_{czi}} E_{czi} E^T_{czi} + \frac{1}{\lambda_{dzi}} E_{dzi} E^T_{dzi},
\]

and $N_2$ is given in Theorem 4.1, one filter of form (4.3) can be obtained by solving the coupled linear matrix inequalities in (4.14).

**Proof:** Because the matrices $\hat{A}_{ei}, \hat{B}_{ei}, \hat{C}_{ei}$ and $\hat{D}_{ei}$ given in (4.5) can also be expressed as

\[
\hat{A}_{ei} = N_1 A_i N^T_1 + N_1 E_{ai} F_{ai} H_{ai} N^T_1 + N_5 J_i \hat{C}_{yi} + N_5 J_i N_3 E_{cyi} F_{cyi} H_{cyi} N^T_1
\]

\[
\hat{B}_{ei} = N_1 B_i + N_1 E_{bi} F_{bi} H_{bi} + N_5 J_i N_3 D_{yi} + N_5 J_i N_3 E_{dyi} F_{dyi} H_{dyi}
\]

\[
\hat{C}_{ei} = C_{zi} N^T_1 + E_{czi} F_{czi} H_{czi} N^T_1 + N_4 J_i \hat{C}_{yi} + N_4 J_i N_3 E_{cyi} F_{cyi} H_{cyi} N^T_1
\]

\[
\hat{D}_{ei} = D_{zi} + E_{dzi} F_{dzi} H_{dzi} + N_4 J_i N_3 D_{yi} + N_4 J_i N_3 E_{dyi} F_{dyi} H_{dyi}
\]
where

\[
\begin{align*}
N_3 & = \begin{bmatrix} 0_{n \times n_f} & I_n \end{bmatrix}, \\
N_4 & = \begin{bmatrix} 0_{n \times n_f} & -I_n \end{bmatrix}, \\
N_5 & = \begin{bmatrix} 0 \bigg| 0_{n \times n_i} \\ I_n \bigg| 0 \end{bmatrix} = \begin{bmatrix} N_2 \ 0 \end{bmatrix}, \\
\bar{C}_{yi} & = \begin{bmatrix} 0 \bigg| I_n \end{bmatrix} = \begin{bmatrix} N_{2}^{T} \\ C_{yi}N_1^{T} \end{bmatrix},
\end{align*}
\]

and \( J_i \) is given in Theorem 4.2, inequality (4.6) can be expanded as

\[
\begin{align*}
& \left[ N_1 A_i^{T} N_1^{T} P_i + P_i N_1 A_i N_1^{T} \right. \\
& \quad + \left. P_i N_1 B_i N_1 C_{zi}^{T} \right] \\
& \quad + \sum_{j=1}^{s} \left( \pi_{ij} + \Delta \pi_{ij} \right) P_j \\
& \quad + \begin{bmatrix} \gamma_{H_{\infty}}^{-2} & I \\
D_{zi}^{T} & -I \end{bmatrix} \\
& \quad + \begin{bmatrix} 0 \bigg| 0 \\ 0 \bigg| \sum_{j=1}^{s} \left( \pi_{ij} + \Delta \pi_{ij} \right) P_j \\
0 \bigg| 0 \end{bmatrix}
\end{align*}
\]
Applying Lemma 2.2 and the bounding technique in (4.9), the above inequality holds over all the admissible uncertainties in (4.2) if there exist real numbers \( \lambda_{a_{i}} \in \mathbb{R}^{+}, \lambda_{b_{i}} \in \mathbb{R}^{+}, \lambda_{c_{zi}} \in \mathbb{R}^{+}, \lambda_{d_{zi}} \in \mathbb{R}^{+}, \lambda_{c_{yi}} \in \mathbb{R}^{+}, \lambda_{d_{yi}} \in \mathbb{R}^{+}, \) and matrices \( T_{ij} \in \mathbb{S}^{(n+n_f) \times (n+n_f)}, j \in S, j \neq i, \) such that

\[
\begin{bmatrix}
P_{i}N_{5} & \begin{bmatrix} 0 & N_{1}H_{yi}^{T} & N_{1}H_{yi}^{T} \end{bmatrix}^{T} + \begin{bmatrix} 0 & 0 & F_{c_{yi}}^{T}E_{c_{yi}}^{T}N_{3}^{T}N_{1}\end{bmatrix}^{T} \\
0 & 0 & 0 & 0 \\
N_{4} & N_{4} & N_{4} & 0
\end{bmatrix}^{T} < 0.
\]

\[
\begin{bmatrix}
P_{i}N_{5} & \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^{T} + \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^{T} \\
0 & 0 & 0 & 0 \\
N_{4} & N_{4} & N_{4} & 0
\end{bmatrix}^{T} < 0.
\]

Applying Lemma 2.2 and the bounding technique in (4.9), the above inequality holds over all the admissible uncertainties in (4.2) if there exist real numbers \( \lambda_{a_{i}} \in \mathbb{R}^{+}, \lambda_{b_{i}} \in \mathbb{R}^{+}, \lambda_{c_{zi}} \in \mathbb{R}^{+}, \lambda_{d_{zi}} \in \mathbb{R}^{+}, \lambda_{c_{yi}} \in \mathbb{R}^{+}, \lambda_{d_{yi}} \in \mathbb{R}^{+}, \) and matrices \( T_{ij} \in \mathbb{S}^{(n+n_f) \times (n+n_f)}, j \in S, j \neq i, \) such that

\[
\begin{bmatrix}
\Phi_{y_{j}} & P_{i}N_{1}B_{i} & N_{1}C_{yi}^{T} \\
B_{i}^{T}N_{1}^{T}P_{i} & \Phi_{y_{j}} & D_{yi}^{T} \\
C_{yi}N_{1}^{T} & D_{yi} & \Phi_{y_{j}}
\end{bmatrix}
\begin{bmatrix}
P_{i}N_{2} & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
N_{2}^{T} & 0 & 0 \\
0 & 0 & 0 \\
C_{yi}N_{1}^{T} & D_{yi} & 0
\end{bmatrix}^{T} + \begin{bmatrix}
N_{2}^{T} & 0 & 0 \\
0 & 0 & 0 \\
C_{yi}N_{1}^{T} & D_{yi} & 0
\end{bmatrix}^{T} + \begin{bmatrix}
P_{i}N_{2} & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}^{T} < 0.
\]

\[
\begin{bmatrix}
P_{i}N_{2} & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}^{T} + \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}^{T} < 0.
\]
+ \frac{1}{\lambda_{di}} \begin{bmatrix} P_i N_2 & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix} J_i \begin{bmatrix} 0 \\ E_{di} \\ E_{di} \end{bmatrix} \begin{bmatrix} P_i N_2 & 0 \\ 0 & 0 \\ 0 & -I \end{bmatrix}^T < 0.

In view of Schur complement equivalence, the above inequality is equivalent to (4.14).

Now, according to Lemma 4.1, inequality (4.14) is solvable for $J_i$ if, and only if, the following two matrix inequalities hold:

\begin{align}
\Psi_{Li}^T \Psi_{Li} (\Psi_{Li}^T)^T &< 0; \\
(\Psi_{Ri}^T)^T \Psi_{Ri} (\Psi_{Ri}^T)^T &< 0.
\end{align}

Note that matrix inequalities (4.15)–(4.16) cannot be solved easily since they are not linear matrix inequalities. Therefore, in the sequel, we try to translate (4.15) and (4.16) into the form of linear matrix inequalities with equality constraints, which can be solved easily using algorithms developed in [26, 38]. To this end, we firstly have

\begin{equation}
(\Psi_{Li})^T = \begin{bmatrix} N_1^T P_{i}^{-1} & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & I \end{bmatrix}.
\end{equation}

Note that $N_1^T N_1 = I_n, \lambda_{ci} \in \mathbb{R}^+, \lambda_{di} \in \mathbb{R}^+$ and define $X_i = P_{i}^{-1}$, inequality (4.15) is equivalent to

\begin{equation}
\begin{bmatrix} \Xi_{3i} & B_i \\ B_i^T & \Phi_{2i} \end{bmatrix} < 0
\end{equation}
where
\[
\Xi_{3i} = N_1^T X_i N_1 A_i^T + A_i N_1^T X_i N_1 + \sum_{j=1, j \neq i}^S \left[ \pi_{ij} P_j + \bar{e}_{ij}^2 T_{ij} + \left( P_i - P_j \right) T_{ij}^{-1} \left( P_i - P_j \right) \right] X_i N_1 \\
+ N_1^T X_i \left( \alpha_{ai} E_{ai} E_{ai}^T + \frac{1}{\lambda_{bi}} E_{bi} E_{bi}^T \right) N_1^T X_i N_1.
\]

Now, let \( V_i \in S^{(n+n_f) \times (n+n_f)} \) such that
\[
\sum_{j=1, j \neq i}^S \left[ \pi_{ij} P_j + \bar{e}_{ij}^2 T_{ij} + \left( P_i - P_j \right) T_{ij}^{-1} \left( P_i - P_j \right) \right] \leq V_i
\]
and define \( \alpha_{ai} \leq \frac{1}{\lambda_{ai}}, \alpha_{czi} \leq \frac{1}{\lambda_{czi}}, \alpha_{czi} \leq \frac{1}{\lambda_{czi}} \) and \( Z_i \leq V_i^{-1} \). Then, inequality (4.17) is equivalent to (4.10)–(4.11) with equality constraints (4.13).

Secondly, we have
\[
\begin{pmatrix}
C_{yi}^T \\
D_{yi}^T \\
E_{czi}^T \\
E_{dgi}^T
\end{pmatrix} = \begin{pmatrix}
N_1^T \Phi_{yi} N_1 \\
B_1^T N_1^T P_i N_1 \Phi_{zi} \\
0 \\
0
\end{pmatrix} \begin{pmatrix}
N_1^T 0 0 0 0 \\
0 I 0 0 0 \\
0 0 0 I 0 \\
0 0 0 0 I
\end{pmatrix}.
\]

Then inequality (4.16) is equivalent to
\[
\begin{bmatrix}
C_{yi}^T \\
D_{yi}^T \\
E_{czi}^T \\
E_{dgi}^T \\
0 I
\end{bmatrix} = \begin{bmatrix}
N_1^T \Phi_{yi} N_1 \\
B_1^T N_1^T P_i N_1 \Phi_{zi} \\
0 \\
0 \\
C_{zi} D_{zi} \Phi_{zi}
\end{bmatrix} \begin{bmatrix}
C_{yi}^T \\
D_{yi}^T \\
E_{czi}^T \\
E_{dgi}^T \\
0 I
\end{bmatrix} < 0.
\]

In view of Schur complement equivalence and a congruence transformation, the above
inequality is equivalent to (4.11)–(4.12). This completes the first part of the proof.

To end the proof, note that (4.10)–(4.13) being solvable means that (4.15)–(4.16) hold, which further implies that a filter of form (4.3) can be obtained by solving (4.14) after the substitution of the solution of (4.10)–(4.13) into (4.14) in view of Lemma 4.1. This completes the proof. □

In the case when the filter (4.3) is reduced to a static filter (i.e., \( n_f = 0 \)) of form

\[
z_f(t) = D_f(\hat{r}(t))y(t),
\]

(4.18)

the conditions in Theorem 4.2 can be simplified as LMI conditions. The simplified version of Theorem 4.2 for \( n_f = 0 \) is stated in the following corollary and can be proved similarly to that of Theorem 4.2.

**Corollary 4.1** Given a prescribed scalar \( \gamma_{H\infty} \in \mathbb{R}^+ \), there exists a robust \( H_{\infty} \) filter (4.18) with level \( \gamma_{H\infty} \) for uncertain system (4.1), if there exist matrices \( P_i \in \mathbb{S}^{n \times n} \), \( T_{ij} \in \mathbb{S}^{n \times n} \), and scalars \( \lambda_{ai} \in \mathbb{R}^+ \), \( \lambda_{bi} \in \mathbb{R}^+ \), \( \lambda_{czi} \in \mathbb{R}^+ \), \( \lambda_{dzi} \in \mathbb{R}^+ \), \( \lambda_{cyi} \in \mathbb{R}^+ \), \( \lambda_{dyi} \in \mathbb{R}^+ \), \( i, j \in \mathcal{S} \), \( j \neq i \), such that the coupled linear matrix inequalities

\[
\begin{bmatrix}
\Phi_{3i} & P_i \Phi_{2i} & P_i \Phi_{1} & P_i \Phi_{2i} & \Gamma_{2i} \\
B_i^T P_i & \Phi_{2i} & 0 & 0 & 0 \\
E_{ai}^T P_i & 0 & -\lambda_{ai} I & 0 & 0 \\
E_{bi}^T P_i & 0 & 0 & -\lambda_{bi} I & 0 \\
\Gamma_{2i}^T & 0 & 0 & 0 & -\lambda_{2i}
\end{bmatrix} < 0
\]

and

\[
\begin{bmatrix}
\Theta_i & 0 & \Phi_{4i} & \Phi_{5i} \\
0 & I & \Phi_{4i}^T & \Phi_{5i}^T \\
0 & 0 & \Theta_i & I
\end{bmatrix} < 0
\]

hold for all \( i \in \mathcal{S} \), where

\[
\Phi_{3i} = A_i^T P_i + P_i A_i + \sum_{j=1}^{s} \pi_{ij} P_{ij} + \sum_{j=1, j \neq i}^{s} \varepsilon_{ij}^T T_{ij} + \lambda_{ai} H_{ai}^T H_{ai} + \lambda_{czi} H_{czi}^T H_{czi} + \lambda_{cyi} H_{cyi}^T H_{cyi},
\]

(4.18)
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\[
\bar{\Phi}_{4i} = \begin{bmatrix}
\Phi_{3i} + C_{zi}^T C_{zi} & P_i B_i + C_{zi}^T D_{zi} & 0 & 0 \\
\Phi_{2i} + D_{zi}^T D_{zi} & 0 & 0 & 0 \\
\Phi_{5i} & -\lambda_{cji} I & 0 & 0 \\
& & -\lambda_{dji} I & 0
\end{bmatrix},
\]

\[
\bar{\Gamma}_{5i} = \begin{bmatrix}
P_i E_{ai} & P_i E_{bi} & C_{zi} E_{czi} & C_{zi}^T E_{dzi} & \Gamma_{2i} \\
0 & 0 & D_{zi}^T E_{czi} & D_{zi}^T E_{dzi} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
-\lambda_{ai} I & 0 & 0 & 0 & 0 \\
0 & -\lambda_{bi} I & 0 & 0 & 0 \\
0 & 0 & -\lambda_{czi} I + E_{czi}^T E_{czi} & E_{czi}^T E_{dzi} & 0 \\
0 & 0 & E_{dzi}^T E_{czi} & -\lambda_{dzi} I + E_{dzi}^T E_{dzi} & 0 \\
0 & 0 & 0 & 0 & -\Lambda_{2i}
\end{bmatrix},
\]

and $\Phi_{2i}$, $\Gamma_{2i}$, $\Lambda_{2i}$ and $\Theta_i$ are given in Theorem 4.2. In this case, one filter of form (4.18) can be obtained by solving the coupled linear matrix inequalities

\[
\bar{\Psi}_i + \bar{\Psi}_{Li} D_{fi} \bar{\Psi}_{Ri} + (\bar{\Psi}_{Li} D_{fi} \bar{\Psi}_{Ri})^T < 0
\]

for all $i \in S$, where

\[
\bar{\Psi}_i = \begin{bmatrix}
\bar{\Phi}_{li} & P_i B_i & C_{zi}^T & 0 & 0 \\
B_i^T P_i & \Phi_{2i} & D_{zi}^T & 0 & 0 \\
C_{zi} & D_{zi} & \Phi_{5i} & 0 & 0 \\
0 & 0 & 0 & -\lambda_{cji} I & 0 \\
0 & 0 & 0 & 0 & -\lambda_{dji} I
\end{bmatrix},
\]

\[
\bar{\Psi}_{Li} = \begin{bmatrix}
0 & 0 & -I & 0 & 0
\end{bmatrix}^T,
\]

\[
\bar{\Psi}_{Ri} = \begin{bmatrix}
C_{gi} & D_{gi} & 0 & E_{cji} & E_{dji}
\end{bmatrix},
\]
\[ \Phi_{\bar{i}} = A^T P_i + P_i A_i + \sum_{j=1}^{s} \pi_{ij} P_j + \sum_{j=1, j \neq i}^{s} \left[ e_{ij}^2 T_{ij} + \left( P_i - P_j \right) T_{ij}^{-1} \left( P_i - P_j \right) \right] 
+ P_i \left( \frac{1}{\lambda_{ai}} E_{ai} E_{ai}^T + \frac{1}{\lambda_{bi}} E_{bi} E_{bi}^T \right) P_i + \lambda_{ai} H_{ai}^T H_{ai} + \lambda_{czl} H_{czl}^T H_{czl} + \lambda_{cyl} H_{cyl}^T H_{cyl}, \]

and \( \Phi_{\bar{i}} \) is given in Theorem 4.2.

In the case when the mode transition rate matrix is known exactly, we can obtain a simplified result for constructing fixed-order robust \( H_\infty \) filter (4.3), which is stated in the following corollary and can be proved similarly to that of Theorem 4.2. However, the condition is necessary and sufficient since the bounding technique in (4.9) is no longer needed in the proof.

**Corollary 4.2** Consider uncertain Markovian jump linear system (4.1) with mode transition rate matrix known exactly; for a prescribed scalar \( \gamma_{H_\infty} \in \mathbb{R}^+ \), there exists a robust \( H_\infty \) filter (4.3) of order \( n_f \) with level \( \gamma_{H_\infty} \) if, and only if, there exist matrices \( P_i \in \mathbb{S}^{(n+n_f) \times (n+n_f)} \), \( X_i \in \mathbb{S}^{(n+n_f) \times (n+n_f)} \) and scalars \( \lambda_{ai} \in \mathbb{R}^+, \alpha_{ai} \in \mathbb{R}^+, \lambda_{bi} \in \mathbb{R}^+, \lambda_{czl} \in \mathbb{R}^+, \alpha_{czl} \in \mathbb{R}^+, \lambda_{cyl} \in \mathbb{R}^+, \alpha_{cyl} \in \mathbb{R}^+, \lambda_{dyl} \in \mathbb{R}^+, i \in \mathcal{S} \), such that the coupled linear matrix inequalities

\[
\begin{bmatrix}
\Phi_{3i} & B_i & \Gamma_{4i} & \Gamma_{6i} \\
B_i^T & \Phi_{2i} & 0 & 0 \\
\Gamma_{4i}^T & 0 & -\Lambda_{3i} & 0 \\
\Gamma_{6i}^T & 0 & 0 & -\Lambda_{4i}
\end{bmatrix} < 0 \quad (4.19)
\]

\[
\begin{bmatrix}
\Theta_i & 0 & \Phi_{4i} & \Gamma_{5i} \\
0 & I & \Gamma_{5i}^T & \Phi_{5i}
\end{bmatrix} < 0 \quad (4.20)
\]

with equality constraints

\[
P_i X_i = I, \quad \lambda_{ai} \alpha_{ai} = 1, \quad \lambda_{czl} \alpha_{czl} = 1, \quad \lambda_{cyl} \alpha_{cyl} = 1 \quad (4.21)
\]
hold for all $i \in S$, where

$$
\Gamma_{6i} = \begin{bmatrix}
\sqrt{\pi_i}N_1^TX_i & \cdots & \sqrt{\pi_{i(i-1)}}N_1^TX_i & \sqrt{\pi_{i(i+1)}}N_1^TX_i & \cdots & \sqrt{\pi_{is}}N_1^TX_i \\
\end{bmatrix},
$$

$$
\Lambda_{4i} = \text{diag}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_s),
$$

$$
\tilde{\Phi}_{4i} = \begin{bmatrix}
\Phi_{6i} & N_1^TP_1B_i+C^TD_{ij} & 0 & 0 \\
\Phi_{2i}+D_{ij}^TD_{ij} & 0 & 0 \\
\Phi_{3i} & -\lambda_{cji}I & 0 \\
\Phi_{5i} & -\lambda_{dzi}I \\
\end{bmatrix},
$$

$$
\tilde{\Phi}_{6i} = A_i^TN_1^TP_1N_1 + N_1^TP_1N_1A_i + N_1^T\left(\sum_{j=1}^{s}\pi_{ij}P_j\right)N_1 + C_{ij}^TC_{ij}
$$

$$
+ \lambda_{ai}H_{ai}^TH_{ai} + \lambda_{ci}H_{cji}^TH_{cji} + \lambda_{cyi}H_{cyi}^TH_{cyi},
$$

and $\Phi_{2i}$, $\Phi_{3i}$, $\Phi_{5i}$, $\Lambda_{3i}$, $\Gamma_{4i}$, $\Gamma_{5i}$ and $\Theta_i$ are given in Theorem 4.2. In this case, one filter of form (4.3) can be obtained by solving the linear matrix inequalities in (4.14) with $\Phi_{7i}$ replaced by $\tilde{\Phi}_{7i}$ for all $i \in S$, where

$$
\tilde{\Phi}_{7i} = N_1A_i^TN_1^TP_1 + P_1N_1A_iN_1^T + \sum_{j=1}^{s}\pi_{ij}P_j
$$

$$
+ P_1N_1\left(\frac{1}{\lambda_{ai}}E_{ai}E_{ai}^T + \frac{1}{\lambda_{bi}}E_{bi}E_{bi}^T\right)N_1^TP_i
$$

$$
+ N_1\left(\lambda_{ai}H_{ai}^TH_{ai} + \lambda_{ci}H_{cji}^TH_{cji} + \lambda_{cyi}H_{cyi}^TH_{cyi}\right)N_1^T.
$$

To solve the coupled linear matrix inequalities in (4.10)–(4.12) with the equality constraints in (4.13) effectively, we firstly choose a sufficiently small number $\beta \in \mathbb{R}^+$, then replace $\Phi_{3i}$ in (4.10) with $\Phi_{3i} + \beta I$, $\Phi_{6i}$ in (4.12) with $\Phi_{6i} + \beta I$, respectively, and change “<” to “≤” in both (4.10) and (4.12). The modified versions of (4.10) and (4.12) will be denoted by $\overline{(4.10)}$ and $\overline{(4.12)}$, respectively. Finally, the equality
constraints in (4.13) are relaxed to
\[
\begin{bmatrix}
\lambda_{ai} & 1 \\
1 & \alpha_{ai}
\end{bmatrix} \geq 0, \quad \begin{bmatrix}
\lambda_{czi} & 1 \\
1 & \alpha_{czi}
\end{bmatrix} \geq 0, \quad \begin{bmatrix}
\lambda_{cyi} & 1 \\
1 & \alpha_{cyi}
\end{bmatrix} \geq 0 \tag{4.22a}
\]
\[
\begin{bmatrix}
P_i & I \\
I & X_i
\end{bmatrix} \geq 0, \quad \begin{bmatrix}
V_i & I \\
I & Z_i
\end{bmatrix} \geq 0. \tag{4.22b}
\]

Now, the optimization algorithms developed in [26, 38] can be employed to solve this non-convex problem. The solution to the fixed-order robust $H_\infty$ filtering problem (FRFP) is summarized below.

**Algorithm FRFP**  For a desired precision $\delta \in \mathbb{R}^+$, let $k_{\text{max}}$ be the maximum number of iterations, and a sufficiently small number $\beta \in \mathbb{R}^+$ be given. Define two decision variables $W_i \doteq \text{diag}(P_i, V_i, \lambda_{ai}, \lambda_{czi}, \lambda_{cyi})$ and $U_i \doteq \text{diag}(X_i, Z_i, \alpha_{ai}, \alpha_{czi}, \alpha_{cyi})$.

1. Determine $W_i^{(0)}$, $U_i^{(0)}$, $\lambda_{ai}^{(0)}$, $\lambda_{bi}^{(0)}$, $\lambda_{dzi}^{(0)}$, $\lambda_{dyi}^{(0)}$, $T_{ij}^{(0)}$, $i, j \in S$, $j \neq i$, satisfying (4.10), (4.11), (4.12) and (4.22), and let $k := 0$.

2. Solve the following convex optimization problem for the decision variables $W_i$, $U_i$, $\lambda_{bi}$, $\lambda_{dzi}$, $\lambda_{dyi}$, $T_{ij}$, $i, j \in S$, $j \neq i$:

   \[
   \min \sum_{i=1}^{s} \text{trace} \left( W_i U_i^{(k)} + W_i^{(k)} U_i \right)
   \]

   subject to (4.10), (4.11), (4.12) and (4.22) for all $i \in S$.

3. Let $L_i^{(k)} := W_i$ and $R_i^{(k)} := U_i$ for all $i \in S$.

4. If

   \[
   \left| \sum_{i=1}^{s} \text{trace} \left( L_i^{(k)} U_i^{(k)} + W_i^{(k)} R_i^{(k)} \right) - 2 \sum_{i=1}^{s} \text{trace} \left( W_i U_i^{(k)} \right) \right| < \delta
   \]

then go to Step (7), otherwise go to Step (5).
(5) Compute $\theta^* \in [0, 1]$ by solving
\[
\min_{\theta \in [0, 1]} \sum_{i=1}^{s} \text{trace} \left( \left[ W_i^{(k)} + \theta \left( L_i^{(k)} - W_i^{(k)} \right) \right] \left[ U_i^{(k)} + \theta \left( R_i^{(k)} - U_i^{(k)} \right) \right] \right).
\]

(6) Let
\[
W_i^{(k+1)} := W_i^{(k)} + \theta^* \left( L_i^{(k)} - W_i^{(k)} \right), \quad U_i^{(k+1)} := U_i^{(k)} + \theta^* \left( R_i^{(k)} - U_i^{(k)} \right),
\]
for all $i \in S$, and $k := k + 1$. If $k < k_{\text{max}}$, then go to Step (2), otherwise go to Step (7).

(7) If $\sum_{i=1}^{s} \text{trace} \left( W_i^{(k)} U_i^{(k)} \right) = 3s + 2s(n + n_f)$, then a solution to (4.10)--(4.13) is found, and a desired fixed-order filter of form (4.3) can be obtained by solving (4.14), otherwise a solution cannot be found by this algorithm.

### 4.3 Numerical Examples

To illustrate the usefulness and flexibility of the theory developed in this chapter, we present a comparison with existing results [20, 21, 64] using a numerical example. Attention is focused on designing robust $H_\infty$ filters for Markovian jump linear systems with uncertainties in the mode transition rate matrix.

It is assumed that the system under consideration has two operation modes, and the uncertainties only exist in the mode transition rate matrix. The system data of (4.1) are as follows:

\[
A_1 = \begin{bmatrix}
-0.2 & 0 & -0.1 \\
0 & -0.1 & 0 \\
0.2 & 0 & -3
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
-1 & 0 & 0 \\
0.2 & -3 & -0.2 \\
0 & 0 & -4
\end{bmatrix},
\]
The nominal system of this uncertain system is robustly mean square stable. Suppose that a robust $H_{\infty}$ filter with level $\gamma_{H_{\infty}} \triangleq 0.4$ is desired over the mode transition rate uncertainties $\Delta \pi_{12} \in [-1.4, 1.4]$ and $\Delta \pi_{21} \in [-6, 6]$.

Filter Design Ignoring Uncertainties

Observer-Structured Filter

We can construct an observer-structured filter $\text{OF}_{d=0}$:

$$
\begin{align*}
\dot{x}_f(t) &= A(r(t))x_f(t) + K(r(t))\left[ y(t) - C_y(r(t))x_f(t) \right] \\
z_f(t) &= C_z(r(t))x_f(t)
\end{align*}
$$
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with

$$K_1 = \begin{bmatrix} 27.4126 & -16.7995 \\ -11.5524 & 13.9000 \\ 29.3859 & -7.9428 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 3.1135 & -0.0749 \\ 6.0657 & -3.8481 \\ 3.1646 & 7.3039 \end{bmatrix}$$

based on Theorem 3.1 and Corollary 3.1 of [20]. This filter is constructed from the solution

$$X_1 = \begin{bmatrix} 2.5965 & 1.9484 & -1.4655 \\ 1.9484 & 1.9958 & -1.0102 \\ -1.4655 & -1.0102 & 0.9670 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 5.7718 & 1.6895 & -3.3354 \\ 1.6895 & 1.5492 & -1.1940 \\ -3.3354 & -1.1940 & 2.2857 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 5.6034 & -4.8968 \\ 0.6692 & 3.0325 \\ -0.0850 & 2.8962 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 17.6625 & -31.2956 \\ 10.8783 & -14.8094 \\ -10.3943 & 21.5391 \end{bmatrix}$$

with $K_i = X_i^{-1}Y_i, i \in S$. The $H_\infty$ filtering level of this filter for the nominal system is $\gamma_{H_\infty}^* = 0.2318$. The shaded regions in the probability domain in Figure 4.1 illustrate different characteristics of the filter’s performance.

**Special Structured Full-Order Filter**

This method comes from Theorem 3.2 and Corollary 3.1 of [21] (also Theorem 1 of [64]). We can obtain a special structured full-order filter $\text{SFF}_{3,i=0}$ of the form

$$\begin{cases} \dot{x}_f(t) = A_f(r(t))x_f(t) + B_f(r(t))y(t) \\ z_f(t) = C_f(r(t))x_f(t) \end{cases}$$
This filter is constructed by $A_{fi} = Y_i^{-1}W_i$, $B_{fi} = Y_i^{-1}Z_i$, $i \in S$, from the solution

$$
X_1 = \begin{bmatrix}
0.1036 & 0.0027 & -0.0580 \\
0.0027 & 0.5074 & 0.0137 \\
-0.0580 & 0.0137 & 0.1468
\end{bmatrix}, \quad X_2 = \begin{bmatrix}
0.1027 & 0.0074 & -0.0803 \\
0.0074 & 0.4114 & 0.0049 \\
-0.0803 & 0.0049 & 0.1944
\end{bmatrix},
$$

$$
Y_1 = \begin{bmatrix}
1.8769 & 1.0983 & -1.0087 \\
1.0983 & 1.1581 & -0.4884 \\
-1.0087 & -0.4884 & 0.6298
\end{bmatrix}, \quad Y_2 = \begin{bmatrix}
3.9641 & 0.9278 & -1.9691 \\
0.9278 & 1.0741 & -0.5773 \\
-1.9691 & -0.5773 & 1.1978
\end{bmatrix},
$$

$$
W_1 = \begin{bmatrix}
-3.7881 & -1.2672 & 2.8052 \\
-0.5386 & -2.9700 & 1.2994 \\
-0.5895 & -1.3390 & -1.7119
\end{bmatrix}, \quad W_2 = \begin{bmatrix}
5.7107 & -2.6014 & -1.3153 \\
1.1704 & -2.2838 & -3.5378 \\
-4.5026 & 1.5910 & -0.0270
\end{bmatrix},
$$

$$
Z_1 = \begin{bmatrix}
3.2581 & -2.0642 \\
0.1670 & 2.6259 \\
0.9878 & 0.4994
\end{bmatrix}, \quad Z_2 = \begin{bmatrix}
9.0244 & -18.5017 \\
6.1887 & -8.2370 \\
-4.7772 & 11.1514
\end{bmatrix}.
$$

This filter is a special case of filter (4.3) with $n_f \equiv n$, $C_{fi} \equiv C_{z_i}$ and $D_{fi} \equiv 0$ for all $i \in S$. The $H_\infty$ filtering level of this filter for the nominal system is $\gamma_{H_\infty} = 0.2563$. The shaded regions in the probability domain in Figure 4.2 illustrate different characteristics of the filter’s performance.
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General Fixed-Order Filter

Based on Corollary 4.2 and Algorithm FRFP in this chapter, both full-order ($n_f = 3$) filters and reduced-order ($n_f = 2$) filters of form (4.3) can be obtained. To compute with Algorithm FRFP for this example, it is chosen that $\delta = 10^{-10}$, $k_{\text{max}} = 100$ and $\beta = 0.01$.

First, we can find a general full-order filter (GFF) of form (4.3) with

$$A_{f_1} = \begin{bmatrix} -369.9008 & 242.3131 & 241.7545 \\ 356.1902 & -240.0308 & -261.6523 \\ 313.2755 & -209.7969 & -260.0767 \end{bmatrix},$$

$$A_{f_2} = \begin{bmatrix} -758.5119 & 286.2321 & 250.9678 \\ 154.3482 & -69.0988 & -54.0923 \\ -209.8038 & 68.4000 & 66.0809 \end{bmatrix},$$

$$B_{f_1} = \begin{bmatrix} -11.5339 & 57.7992 \\ -17.0942 & -49.8951 \\ -47.6935 & -33.8549 \end{bmatrix},$$

$$B_{f_2} = \begin{bmatrix} -101.1364 & -51.8593 \\ 18.1410 & 15.2288 \\ -33.0899 & -6.0017 \end{bmatrix},$$

$$C_{f_1} = \begin{bmatrix} 3.8541 & -0.1083 & -1.3007 \\ 5.6026 & -3.4622 & -4.4919 \end{bmatrix},$$

$$C_{f_2} = \begin{bmatrix} -1.0122 & -1.2736 & 0.6262 \\ 0.4558 & -2.4948 & -1.5376 \end{bmatrix},$$

$$D_{f_1} = \begin{bmatrix} 0.0530 & 0.1765 \\ 0.1419 & 0.4126 \end{bmatrix},$$

$$D_{f_2} = \begin{bmatrix} 0.0216 & 0.0024 \\ -0.0473 & -0.0379 \end{bmatrix}.$$
This filter is denoted by $G_{F2,\Delta}$ in Table 4.1. The $H_{\infty}$ filtering level of this filter for the nominal system is $\gamma^*_{H_{\infty}} = 0.3171$. The shaded regions in the probability domain in Figure 4.3 illustrate different characteristics of the filter’s performance.

Also, a general reduced-order filter of form (4.3) can be found with

$$A_{f_1} = \begin{bmatrix} -8.5510 & -17.8723 \\ -4.7118 & -124.5348 \end{bmatrix}, \quad A_{f_2} = \begin{bmatrix} -33.9644 & -4.2556 \\ -31.7127 & -6.2280 \end{bmatrix},$$

$$B_{f_1} = \begin{bmatrix} -6.3671 & -9.2921 \\ -57.4942 & -53.2179 \end{bmatrix}, \quad B_{f_2} = \begin{bmatrix} -8.9711 & 18.8709 \\ -7.0880 & 15.0433 \end{bmatrix},$$

$$C_{f_1} = \begin{bmatrix} 2.9846 & 0.4146 \\ 0.3946 & -1.9282 \end{bmatrix}, \quad C_{f_2} = \begin{bmatrix} -3.5404 & -0.2118 \\ -2.1515 & -1.1681 \end{bmatrix},$$

$$D_{f_1} = \begin{bmatrix} -0.9365 & 0.9778 \\ -0.0512 & 0.2433 \end{bmatrix}, \quad D_{f_2} = \begin{bmatrix} 0.1114 & -0.0773 \\ 0.1981 & 0.0154 \end{bmatrix}.$$
We can verify that \( \|P_1X_1 - I\| = 6.9930 \times 10^{-12} \) and \( \|P_2X_2 - I\| = 5.5888 \times 10^{-12} \). The \( H_\infty \) filtering level of this filter for the nominal system is \( \gamma_{H_\infty} = 0.3740 \). The shaded regions in the probability domain in Figure 4.4 illustrate different characteristics of the filter’s performance.

**Filter Design Considering Uncertainties**

Based on Theorem 4.2 and Algorithm FRFP, general fixed-order filters of form (4.3) can be found which take into consideration of the uncertainties in the mode transition rate matrix.

A general full-order filter of form (4.3) is obtained with

\[
A_{f_1} = \begin{bmatrix}
-546.6673 & 67.4351 & 486.9535 \\
500.7348 & -448.6584 & -509.8440
\end{bmatrix},
\]

\[
A_{f_2} = \begin{bmatrix}
-2155.9862 & 1662.1406 & 1604.2583 \\
1456.7935 & -1165.6585 & -1126.1783 \\
2054.6194 & -1621.5165 & -1573.5652
\end{bmatrix}.
\]
\[ B_{f_1} = \begin{bmatrix} -157.1999 & 1.2619 \\ -1.2812 & -149.2214 \\ 83.9923 & -124.0781 \end{bmatrix}, \]
\[ B_{f_2} = \begin{bmatrix} -104.4875 & -30.1725 \\ 51.6677 & 53.8150 \\ 81.4388 & 63.1084 \end{bmatrix}, \]
\[ C_{f_1} = \begin{bmatrix} 6.4089 & -4.2098 & -3.5922 \\ -0.8566 & -3.3221 & 0.5734 \end{bmatrix}, \]
\[ C_{f_2} = \begin{bmatrix} -1.1112 & 0.5084 & -1.4766 \\ -1.7182 & -1.1175 & -0.1138 \end{bmatrix}, \]
\[ D_{f_1} = \begin{bmatrix} -0.1463 & 0.1400 \\ 0.0252 & 0.0417 \end{bmatrix}, \]
\[ D_{f_2} = \begin{bmatrix} 0.0489 & 0.0165 \\ 0.1040 & 0.0506 \end{bmatrix}. \]

This filter is denoted by GFF_{3.2=0} in Table 4.1. The performance of this filter for the nominal system is \( \gamma_{H_\infty} = 0.3359 \).

A general reduced-order filter of form (4.3) can also be constructed with

\[ A_{f_1} = \begin{bmatrix} -689.7027 & -564.3889 \\ -649.9022 & -539.5504 \end{bmatrix}, \quad A_{f_2} = \begin{bmatrix} -294.6021 & -373.9299 \\ -288.7422 & -371.0400 \end{bmatrix}, \]
\[ B_{f_1} = \begin{bmatrix} -42.4476 & 158.7354 \\ -43.0375 & 150.8787 \end{bmatrix}, \quad B_{f_2} = \begin{bmatrix} 34.8560 & -114.0897 \\ 32.5800 & -111.4928 \end{bmatrix}, \]
\[ C_{f_1} = \begin{bmatrix} 4.1136 & 0.4558 \\ 3.0435 & 2.1285 \end{bmatrix}, \quad C_{f_2} = \begin{bmatrix} 4.4366 & 6.4957 \\ 7.2700 & 7.4670 \end{bmatrix}, \]
\[ D_{f_1} = \begin{bmatrix} -1.1962 & -0.3154 \\ 1.0096 & 0.2990 \end{bmatrix}, \quad D_{f_2} = \begin{bmatrix} 0.4011 & 0.1030 \\ 0.8025 & 0.2184 \end{bmatrix}. \]
This filter is denoted by $GFF_{2,d=0}$ in Table 4.1. The performance of this filter for the nominal system is $\gamma^*_{H_{\infty}} = 0.3608$.

Table 4.1 gives a comparison of the performance of these filters on some points in the probability domain (including the no uncertainty case and the vertices of the probability domain). From this table, we can see that the filters designed by ignoring the uncertainties in the mode transition rate matrix do not always guarantee the desired $H_{\infty}$ filtering level, while the filters designed by considering the uncertainties can. Table 4.1 also shows that the uncertainties in the mode transition rate matrix can degrade the performance of filters, and even destabilize the filtering error system in some cases. Therefore, it is important and necessary to consider the effect of these uncertainties for Markovian jump linear systems when filters are designed. The developed theory in this chapter provides us with a powerful design procedure for such problems.

### 4.4 Summary

The fixed-order robust $H_{\infty}$ filter design problem has been investigated for a class of uncertain Markovian jump linear systems. The main result in this chapter is that if a system of coupled linear matrix inequalities with equality constraints is feasible, then a robust $H_{\infty}$ filter can be constructed by solving a set of linear matrix inequalities. The developed theory has also been illustrated by numerical comparisons with existing ones and presented powerful utility and flexibility.

Chapters 2–4 have provided a solution to Problem 1 (cf. page 10) in the context of robust stabilization, robust $H_2$ control, robust $H_{\infty}$ control and fixed-order robust $H_{\infty}$ filter design for uncertain Markovian jump linear systems. The attention has been mainly drawn on the handling of uncertainties in the mode transition rate matrix. It has been shown that such uncertainties can (a) destabilize the system in Chapter 2; (b) degrade the system performance in Chapter 3; (c) deteriorate the filtering level in Chapter 4. Moreover, new conditions and synthesis procedures have been proposed via
exploiting the probability constraints on the rows of the mode transition rate matrix. Both theoretical and numerical comparisons have been used to illustrate the developed theory as well.
Figure 4.1  Performance regions of the probability domain

Figure 4.2  Performance regions of the probability domain
Figure 4.3  Performance regions of the probability domain

Figure 4.4  Performance regions of the probability domain
Table 4.1  $H_\infty$ filtering performance of different filters

<table>
<thead>
<tr>
<th>$\Delta \pi_{12}$</th>
<th>$\Delta \pi_{21}$</th>
<th>$OF_{\Delta \pi_{12}=0}$</th>
<th>$SFF_{\Delta \pi_{12}=0}$</th>
<th>$GFF_{\Delta \pi_{12}=0}$</th>
<th>$GFF_{2, \pi_{12}=0}$</th>
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</table>

‡ Boldface figure indicates that the filter has violated the specified performance.
§ U indicates that the filtering error system is unstable.
Chapter 5

Stabilization of DMJLS via Delayed Information

Research Problem 2 (cf. page 14) is studied in this chapter. The system under consideration is a discrete-time Markovian jump linear system. Both the system state and the jumping parameter are accessible to the controller with time delays as shown in Figure 1.1 (cf. page 14). The delay in the system state is time-varying and bounded. The delay in the jumping parameter is constant, and manifested as a constant mismatch of the modes between the controller and the system. The objective of this chapter is to design a state-feedback control law, using the delayed mode signal and the delayed system state, such that the closed-loop system is stochastically stable.

The organization of this chapter is as follows. Section 5.1 formulates the problem to be solved. The main results are presented in Section 5.2. The closed-loop system is firstly shown to be a time-varying delayed Markovian jump linear system with extended operation modes. Then a sufficient condition is found to test the stochastic stability of the closed-loop system. Based on the analysis result, a stabilizing controller design method is provided. In Section 5.3, a numerical example is used to show the application of the developed theory. Section 5.4 is the summary of this chapter.
5.1 Problem Formulation

Consider the following discrete-time Markovian jump linear system defined on a complete probability space \((\Omega, \mathcal{F}, P)\):

\[
x(k + 1) = A(r(k))x(k) + B(r(k))u(k)
\]

where \(k \in \mathbb{Z}_+\), \(x(k) \in \mathbb{R}^n\) is the system state, and \(u(k) \in \mathbb{R}^m\) is the control input. \(\{r(k) : k \in \mathbb{Z}_+\}\) is a discrete-time homogeneous Markov chain. \(r(k)\) takes values in a finite state space \(S \triangleq \{1, 2, \ldots, s\}\) with transition probability matrix \(\Pi \triangleq (\pi_{ij}) \in \mathbb{R}^{s \times s}\), where

\[
\pi_{ij} \triangleq \Pr (r(k + 1) = j \mid r(k) = i) \geq 0
\]

for all \(i, j \in S\) and \(k \in \mathbb{Z}_+\), and

\[
\sum_{j=1}^{s} \pi_{ij} = 1
\]

for every \(i \in S\). Matrices \(A_i \triangleq A(r(k) = i)\) and \(B_i \triangleq B(r(k) = i)\), for all \(i \in S\), are constant matrices of appropriate dimensions.

Now consider a time-delayed, mode-dependent, state-feedback control law

\[
u(k) = K(r(k - \tau_r))x(k - \tau_s(k))
\]

where \(\tau_r \in \mathbb{N}\) is a constant delay occurring in mode signal \(r(k)\). \(\tau_s(k) \in \mathbb{N}\), the delay in system state \(x(k)\), may be time-varying, and satisfies \(\tau_{\min} \leq \tau_s(k) \leq \tau_{\max}\) with \(\tau_{\min}, \tau_{\max} \in \mathbb{N}\). Here, \(\tau_r\) is assumed to be constant and determines the structure of the closed-loop system. This point will be made clear in the following section. \(\phi(k) \in \mathbb{R}^n\), \(k = -\tau_{\max}, -\tau_{\max} + 1, \ldots, 0\), and \(\kappa(k) \in S\), \(k = -\tau_r, -\tau_r + 1, \ldots, 0\), are the initial condition of the system.
Applying control law (5.2) to open-loop system (5.1) results in the closed-loop system

\[ x(k + 1) = A(r(k))x(k) + B(r(k))K(r(k - \tau_r))x(k - \tau_r(k)). \]  
(5.3)

**Remark 5.1** Closed-loop system (5.3) is no longer a Markovian jump system with respect to \( r(k) \) because of the delayed mode signal \( r(k - \tau_r) \). However, if we extend the state space of the operation mode, it will be a Markovian jump system.

**Remark 5.2** One difficulty in the control problem considered in this chapter (compared with others [3, 7, 10, 54]) is that the state matrices of (5.3) are not affected by the introduction of control law (5.2). While in [3, 7, 10, 54], the state-feedback control law of the form \( u(k) = K(r(k))x(k) \), which changes the state matrices of the closed-loop system, is used to stabilize the system and also to tolerate the time-delayed term, for instance, \( A_d(r(k))x(k - d) \) in [54].

We have the following stochastic stability concept for system (5.3).

**Definition 5.1** [54] Let \( x(k; \phi(\cdot), \kappa(\cdot)) \) be the trajectory of the system state of closed-loop system (5.3). The closed-loop system in (5.3) is said to be stochastically stable if

\[ E\left( \sum_{k=0}^{\infty} \|x(k; \phi(\cdot), \kappa(\cdot))\|^2 \mid \phi(\cdot), \kappa(\cdot) \right) < \infty \]

for every initial condition \( \phi(k) \in \mathbb{R}^n, k = -\tau_{\max}, -\tau_{\max} + 1, \ldots, 0 \), and \( \kappa(k) \in \mathcal{S}, k = -\tau_r, -\tau_r + 1, \ldots, 0 \).

In this chapter, our attention is directed at designing a control law of form (5.2) such that the closed-loop system in (5.3) is stochastically stable, given a possibly unstable system (5.1).
5.2 Stabilization with Delayed Information

In this section, we first show that closed-loop system (5.3) is a time-varying delayed Markovian jump linear system with $s^{\tau_r+1}$ operation modes. Then a sufficient condition is established to test the stability of the system in terms of coupled linear matrix inequalities. Based on the analysis result, a controller design technique is proposed.

**Lemma 5.1** Closed-loop system (5.3) is a time-varying delayed Markovian jump linear system with $s^{\tau_r+1}$ modes.

**Proof:** Given $\tau_r \in \mathbb{N}$, we define two finite sets:

$$S^{\tau_r+1} = S \times S \times \cdots \times S_{\tau_r+1 \text{ times}},$$

$$S_{\tau_r+1} = \{1, 2, \ldots, s^{\tau_r+1}\},$$

and introduce a mapping $\psi : S^{\tau_r+1} \to S_{\tau_r+1}$ with

$$\psi(\theta) = i + (i_1 - 1) s + \cdots + (i_{-\tau_r+1} - 1) s^{\tau_r-1} + (i_{-\tau_r} - 1) s^{\tau_r}$$

where

$$\theta = \begin{bmatrix} i & i_{-1} & i_{-2} & \cdots & i_{-\tau_r} \end{bmatrix}^T \in S^{\tau_r+1}$$

and $i, i_{-1}, \ldots, i_{-\tau_r} \in S$.

For every element $\nu \in S_{\tau_r+1}$, an unique element $\theta \in S^{\tau_r+1}$, satisfying $\psi(\theta) = \nu$, can be determined by the following procedure:

1. Let $i_{-\tau_r} = \left\lceil \frac{\nu}{s^{\tau_r}} \right\rceil$ and $\nu_{-\tau_r+1} = \nu - (i_{-\tau_r} - 1) s^{\tau_r}$;
2. Let $i_{-\tau_r+1} = \left\lceil \frac{\nu_{-\tau_r+1}}{s^{\tau_r-1}} \right\rceil$ and $\nu_{-\tau_r+2} = \nu_{-\tau_r+1} - (i_{-\tau_r+1} - 1) s^{\tau_r-1}$;
3. Repeat Step (2) until $i_{-1} = \left\lceil \frac{\nu_{-1}}{s} \right\rceil$ and $\nu_0 = \nu_{-1} - (i_{-1} - 1) s$;
Let $i = \nu_0$.

Therefore, the mapping $\psi(\cdot)$ is a bijection from $S_{\tau+1}$ to $S_{\tau+1}$.

Now define a vector-valued random variable
\[
\tilde{r}(k) \triangleq \begin{bmatrix} r(k) & r(k-1) & \cdots & r(k-\tau_r) \end{bmatrix}^T.
\]

Then closed-loop system (5.3) can be written as
\[
x(k+1) = A(\tilde{r}(k))x(k) + B(\tilde{r}(k))K(\tilde{r}(k))x(k-\tau_s(k)) \tag{5.4}
\]
where $A(\tilde{r}(k)) = A(r(k))$, $B(\tilde{r}(k)) = B(r(k))$ and $K(\tilde{r}(k)) = K(r(k-\tau_r))$.

Note that the vector-valued stochastic process $\{\tilde{r}(k) : k \in \mathbb{Z}_+\}$, taking values in $S_{\tau+1}$, is a discrete-time, vector-valued Markov chain since $\{r(k) : k \in \mathbb{Z}_+\}$ is a Markov chain. Therefore, system (5.4) is a time-varying delayed Markovian jump linear system with $S_{\tau+1}$ operation modes, and so is closed-loop system (5.3). At time $k$, we say that jump system (5.3) is in mode $\nu = \psi(\tilde{r}(k)) \in S_{\tau+1}$.

In the following, we construct the extended transition probability matrix $\tilde{\Pi} = (\tilde{\pi}_{\nu\eta})$ for closed-loop system (5.3) according to matrix $\Pi$. For any two elements $\nu, \eta \in S_{\tau+1}$; because $\psi(\cdot)$ is bijective, we can uniquely obtain two vectors:

\[
\tilde{\nu} \triangleq \psi^{-1}(\nu) = \begin{bmatrix} i & i_{-1} & \cdots & i_{-\tau_r} \end{bmatrix}^T; \\
\tilde{\eta} \triangleq \psi^{-1}(\eta) = \begin{bmatrix} j & j_{-1} & \cdots & j_{-\tau_r} \end{bmatrix}^T.
\]

Then the mode transition probability from mode $\tilde{\nu}$ at time $k$ to mode $\tilde{\eta}$ at time $k+1$ is given by

\[
\tilde{\pi}_{\nu\eta} \triangleq \Pr (\tilde{r}(k+1) = \tilde{\eta} | \tilde{r}(k) = \tilde{\nu}) = \Pr (r(k+1) = j, r(k) = j_{-1}, \ldots, r(k-\tau_r + 1) = j_{-\tau_r} | r(k) = i, r(k-1) = i_{-1}, \ldots, r(k-\tau_r + 1) = i_{-\tau_r+1}, r(k-\tau_r) = i_{-\tau_r})
\]
\[ = \pi_{ij} \delta(i, j-1) \delta(i-1, j-2) \cdots \delta(i-\tau_r+1, j-\tau_r). \]

This completes the proof. \(\square\)

**Remark 5.3** The state-space matrices of closed-loop system (5.3) can be easily determined by the operation mode. Given any operation mode \(\nu \in S_{\tau_r+1}\) at time \(k\), we have

\[ \bar{r}(k) = \psi^{-1}(\nu) = \begin{bmatrix} i & i_{-1} & \cdots & i_{-\tau_r} \end{bmatrix}^T. \]

Then \(A(\bar{r}(k)) = A_i\), \(B(\bar{r}(k)) = B_i\) and \(K(\bar{r}(k)) = K_{i_{-\tau_r}}\). In other words, the values of \(A(\bar{r}(k))\) and \(B(\bar{r}(k))\) are determined by the first element of vector \(\bar{r}(k)\), and \(K(\bar{r}(k))\) is determined by the last element of \(\bar{r}(k)\). For example, if \(s = 2, \tau_r = 1\), we have

\[ S_{\tau_r+1} = \begin{Bmatrix} \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 2 \end{bmatrix}, \begin{bmatrix} 1 \end{bmatrix}, \begin{bmatrix} 2 \end{bmatrix} \end{Bmatrix}, \]

\[ S_{\tau_r+1} = \{1, 2, 3, 4\}. \]

That is, closed-loop system (5.3) has four operation modes and

\[ \bar{r}(k) = \begin{bmatrix} r(k) \\ r(k-1) \end{bmatrix} \in S_{\tau_r+1}. \]

According to the procedure described in the proof of Lemma 5.1,

- for mode \(\nu = 1\) at time \(k\), we first have \(i_{-1} = \left\lceil \frac{1}{2} \right\rceil = [0.5] = 1\), next \(i = \nu - (i_{-1} - 1) s = 1\). Thus \(\bar{r}(k) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\). Then the state-space matrices are given by \(A(\bar{r}(k)) = A_1, B(\bar{r}(k)) = B_1\) and \(K(\bar{r}(k)) = K_1\).

- for mode \(\nu = 2\) at time \(k\), we first have \(i_{-1} = \left\lceil \frac{2}{2} \right\rceil = [1] = 1\), next \(i = \nu - (i_{-1} - 1) s = 1\). Thus \(\bar{r}(k) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\). Then the state-space matrices are given by \(A(\bar{r}(k)) = A_1, B(\bar{r}(k)) = B_1\) and \(K(\bar{r}(k)) = K_1\).
(i_{-1} - 1) s = 2. Thus \( \bar{\tau}(k) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \). Then the state-space matrices are given by

\[
A(\bar{\tau}(k)) = A_2, \quad B(\bar{\tau}(k)) = B_2 \quad \text{and} \quad K(\bar{\tau}(k)) = K_1.
\]

- for mode \( \nu = 3 \) at time \( k \), we first have \( i_{-1} = \left\lceil \frac{3}{2} \right\rceil = \lceil 1.5 \rceil = 2 \), next \( i = \nu - (i_{-1} - 1) s = 1. \) Thus \( \bar{\tau}(k) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \). Then the state-space matrices are given by

\[
A(\bar{\tau}(k)) = A_1, \quad B(\bar{\tau}(k)) = B_1 \quad \text{and} \quad K(\bar{\tau}(k)) = K_2.
\]

- for mode \( \nu = 4 \) at time \( k \), we first have \( i_{-1} = \left\lceil \frac{4}{2} \right\rceil = \lceil 2 \rceil = 2 \), next \( i = \nu - (i_{-1} - 1) s = 2. \) Thus \( \bar{\tau}(k) = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \). Then the state-space matrices are given by

\[
A(\bar{\tau}(k)) = A_2, \quad B(\bar{\tau}(k)) = B_2 \quad \text{and} \quad K(\bar{\tau}(k)) = K_2.
\]

Eventually, the relationship among them is established (see Table 5.1). In fact, we only need to know the first and the last elements of vector \( \bar{\tau}(k) \) to determine the state-space matrices. However, the system mode is determined by the whole vector \( \bar{\tau}(k) \) and vice versa.

**Remark 5.4** When \( \tau_r = 0 \), we have \( \theta = i, \psi(\theta) = i \) and \( \bar{\tau}(k) = r(k) \). Thus \( \bar{\Pi} = \Pi \).
When $\tau_r = 1$, the extended mode transition probability matrix $\tilde{\Pi} \in \mathbb{R}^{s^2 \times s^2}$ is

$$
\tilde{\Pi} =
\begin{bmatrix}
\pi_{11} & \pi_{12} & \cdots & \pi_{1s} & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & \pi_{21} & \pi_{22} & \cdots & \pi_{2s} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\begin{bmatrix}
\pi_{s1} & \pi_{s2} & \cdots & \pi_{ss} \\
\end{bmatrix}
$$

For instance, if $s = 2$ and $\tau_r = 1$, we have

$$
\tilde{\Pi} =
\begin{bmatrix}
\pi_{11} & \pi_{12} & 0 & 0 \\
0 & 0 & \pi_{21} & \pi_{22} \\
\pi_{11} & \pi_{12} & 0 & 0 \\
0 & 0 & \pi_{21} & \pi_{22} \\
\end{bmatrix}
.$$
If \( s = 2 \) and \( \tau_r = 2 \), we have

\[
\tilde{\Pi} = \begin{bmatrix}
\pi_{11} & \pi_{12} & 0 & 0 & 0 & 0 \\
0 & 0 & \pi_{21} & \pi_{22} & 0 & 0 \\
0 & 0 & 0 & \pi_{11} & \pi_{12} & 0 \\
0 & 0 & 0 & 0 & \pi_{21} & \pi_{22} \\
0 & 0 & 0 & 0 & 0 & \pi_{11} & \pi_{12} \\
0 & 0 & 0 & 0 & 0 & \pi_{21} & \pi_{22}
\end{bmatrix}.
\]

**Remark 5.5** Different system mode delay \( \tau_r \) results in different closed-loop system (5.3) in the sense that closed-loop system (5.3) will have different number of operation modes and different mode transition probability matrix.

With Lemma 5.1 and Definition 5.1, we are now ready to analyze the stability and stabilization problems for discrete-time Markovian jump linear system (5.1) controlled by time-delayed state-feedback control law (5.2).

**Theorem 5.1** Closed-loop system (5.3) is stochastically stable, if there exist matrices \( P_\nu \in \mathbb{S}^{n \times n}, Q \in \mathbb{S}^{n \times n}, Z \in \mathbb{S}^{n \times n}, Y_{1v} \in \mathbb{R}^{n \times n}, Y_{2v} \in \mathbb{R}^{n \times n}, \nu \in S_{\tau_r+1} \), satisfying the coupled linear matrix inequalities

\[
\begin{bmatrix}
\Phi_{\nu 11} & \Phi_{\nu 12} & Y_{1v} \\
\Phi_{\nu 12}^T & \Phi_{\nu 22} & Y_{2v} \\
Y_{1v}^T & Y_{2v}^T & -\frac{1}{\tau_{\text{max}}} Z
\end{bmatrix} < 0
\]  \hspace{1cm} (5.5)

for all \( \nu \in S_{\tau_r+1} \), where

\[
\Phi_{\nu 11} = A_i^T \left( \sum_{j=1}^{s} \pi_{ij} P_{\mu j} \right) A_i - P_\nu + Y_{1v} + Y_{1v}^T + (\tau_{\text{max}} - \tau_{\text{min}} + 1) Q \\
+ \tau_{\text{max}} (A_i - I)^T Z (A_i - I)
\]
\[ \Phi_{v12} = A_i^T \left( \sum_{j=1}^{s} \pi_{ij} P_{\mu+j} \right) B_i K_{i,\tau} - Y_{1v} + Y_{2v}^T + \tau_{\text{max}} (A_i - I)^T Z B_i K_{i,\tau}, \]
\[ \Phi_{v22} = K_{i,\tau}^T B_i^T \left( \sum_{j=1}^{s} \pi_{ij} P_{\mu+j} \right) B_i K_{i,\tau} - Y_{2v} - Y_{2v}^T - Q + \tau_{\text{max}} K_{i,\tau}^T B_i^T Z B_i K_{i,\tau}, \]

and \[ i_{i-1} \cdots i_{i-\tau} \] = \psi^{-1}(\nu) and \( \mu = (i - 1) s + (i_{-1} - 1) s^2 + \cdots + (i_{-\tau+2} - 1) s^{\tau-1} + (i_{-\tau+1} - 1) s^{\tau}. \)

**Proof:** We define

\[ \zeta(k) \triangleq x(k+1) - x(k) \]
\[ x_k \triangleq \left[ x^T(k) \quad x^T(k-1) \quad \cdots \quad x^T(k-\tau_{\text{max}}) \right]^T \]

and adopt the Lyapunov functional

\[ V(x_k, \tilde{r}(k), k) \triangleq V_1(x_k, \tilde{r}(k), k) + V_2(x_k, \tilde{r}(k), k) + V_3(x_k, \tilde{r}(k), k) + V_4(x_k, \tilde{r}(k), k) \]

with

\[ V_1(x_k, \tilde{r}(k), k) \triangleq x^T(k) P(\tilde{r}(k)) x(k), \]
\[ V_2(x_k, \tilde{r}(k), k) \triangleq \sum_{l=k-\tau_{\text{min}}(k)}^{k-1} x^T(l) Q x(l), \]
\[ V_3(x_k, \tilde{r}(k), k) \triangleq \sum_{l=k-\tau_{\text{max}}+2}^{k-1} \sum_{l=k-1+h}^{k-1} x^T(l) Q x(l), \]
\[ V_4(x_k, \tilde{r}(k), k) \triangleq \sum_{l=k-\tau_{\text{max}}+1}^{k-1} \sum_{l=k-1+h}^{k-1} \zeta^T(l) Z \zeta(l). \]

For simplicity, suppose the mode at time \( k \) is \( \nu \in S_{\tau+1} \), that is,

\[ \nu = \psi(\tilde{r}(k)) = i + (i_{-1} - 1) s + \cdots + (i_{-\tau} - 1) s^{\tau}, \]

and matrices \( P(\tilde{r}(k)) = P_\nu, A(\tilde{r}(k)) = A_i, B(\tilde{r}(k)) = B_i, \) and \( K(\tilde{r}(k)) = K_{i,\tau} \). Then at
time \( k + 1 \), the system may jump to any mode \( \eta \in S_{\tau_r+1} \), that is,

\[
\eta = \psi(\tilde{r}(k + 1)) = j + (j_{-1} - 1)s + \cdots + (j_{-\tau_r} - 1)s^{\tau_r}.
\]

From the proof of Lemma 5.1, we have that the mode transition probability from mode \( \nu \) at time \( k \) to mode \( \eta \) at time \( k + 1 \) is given by

\[
\bar{\pi}_{\nu \eta} = \pi_{ij} \delta(i, j_{-1}) \delta(i_{-1}, j_{-2}) \cdots \delta(i_{-\tau_r}, j_{-\tau_r}).
\]

Now, comparing the expressions of \( \nu \) and \( \eta \), we have \( \bar{\pi}_{\nu \eta} = \pi_{ij} \) if \( \eta = \mu + j \); otherwise \( \bar{\pi}_{\nu \eta} = 0 \). Thus

\[
\sum_{\eta = 1}^{s^{\tau_r+1}} \bar{\pi}_{\nu \eta} P_{\eta} = \sum_{j = 1}^{s} \pi_{ij} P_{\mu + j}.
\]

Then

\[
E \left( V_1(x_{k+1}, \tilde{r}(k + 1), k + 1) \mid x_k, \tilde{r}(k), k \right) - V_1(x_k, \tilde{r}(k), k)
\]

\[
= E \left( x^T(k + 1)P(\tilde{r}(k + 1))x(k + 1) \mid x_k, \tilde{r}(k), k \right) - x^T(k)P(\tilde{r}(k))x(k)
\]

\[
= \left[ A_i x(k) + B_i K_{l_{\mu'}} x(k - \tau_s(k)) \right] \xi^T(k) \left[ \sum_{\eta = 1}^{s^{\tau_r+1}} \bar{\pi}_{\nu \eta} P_{\eta} \right] \left[ A_i x(k) + B_i K_{l_{\mu'}} x(k - \tau_s(k)) \right]
\]

\[
- x^T(k)P_{\nu} x(k)
\]

\[
= \xi^T(k) \left[ A^T_i \left( \sum_{j = 1}^{s} \pi_{ij} P_{\mu + j} \right) A_i - P_{\nu} + A^T_i \left( \sum_{j = 1}^{s} \pi_{ij} P_{\mu + j} \right) B_i K_{l_{\mu'}} \right] \xi(k)
\]

where

\[
\xi(k) = \begin{bmatrix} x(k) \\ x(k - \tau_s(k)) \end{bmatrix}.
\]

Furthermore,

\[
E \left( V_2(x_{k+1}, \tilde{r}(k + 1), k + 1) \mid x_k, \tilde{r}(k), k \right) - V_2(x_k, \tilde{r}(k), k)
\]
Moreover,
\[
E \left( V_3(x_{k+1}, \bar{r}(k + 1), k + 1) \mid x_k, \bar{r}(k), k \right) - V_3(x_k, \bar{r}(k), k) = \sum_{h=-\tau_{\text{max}}+2}^{\tau_{\text{max}}-\tau_{\text{min}}+1} \sum_{l=k+h}^{k} x^T(l) Q x(l) - \sum_{h=-\tau_{\text{max}}+2}^{\tau_{\text{max}}-\tau_{\text{min}}+1} \sum_{l=k-1+h}^{k-1} x^T(l) Q x(l)
\]
\[
= \sum_{h=-\tau_{\text{min}}+2}^{\tau_{\text{max}}-\tau_{\text{min}}+1} \left[ \sum_{l=k+h}^{k} x^T(l) Q x(l) - \sum_{l=k-1+h}^{k-1} x^T(l) Q x(l) \right]
\]
\[
= \sum_{h=-\tau_{\text{min}}+2}^{\tau_{\text{max}}-\tau_{\text{min}}+1} \left[ x^T(k) Q x(k) - x^T(k - 1 + h) Q x(k - 1 + h) \right]
\]
\[
= (\tau_{\text{max}} - \tau_{\text{min}}) x^T(k) Q x(k) - \sum_{h=-\tau_{\text{max}}+2}^{\tau_{\text{max}}-\tau_{\text{min}}+1} x^T(k - 1 + h) Q x(k - 1 + h)
\]
\[
= (\tau_{\text{max}} - \tau_{\text{min}}) x^T(k) Q x(k) - \sum_{l=k+1-\tau_{\text{max}}}^{k-\tau_{\text{min}}} x^T(l) Q x(l).
\]

Since \( \tau_{\text{min}} \leq \tau_s(k) \leq \tau_{\text{max}} \), that is, \( k - \tau_s(k) \leq k - \tau_{\text{min}} \), and \( k + 1 - \tau_{\text{max}} \leq k + 1 - \tau_s(k + 1) \), we have
\[
\sum_{l=k+1-\tau_s(k+1)}^{k-\tau_s(k)} x^T(l) Q x(l) - \sum_{l=k+1-\tau_{\text{max}}}^{k-\tau_{\text{min}}} x^T(l) Q x(l) \leq 0.
\]
Hence,

\[
E(V_2(x_{k+1}, \bar{r}(k+1), k+1) + V_3(x_{k+1}, \bar{r}(k+1), k+1) \mid x_k, \bar{r}(k), k) \\
- (V_2(x_k, \bar{r}(k), k) + V_3(x_k, \bar{r}(k), k)) \\
\leq \xi^T(k) \begin{bmatrix}
(\tau_{\text{max}} - \tau_{\text{min}} + 1) Q & 0 \\
0 & -Q
\end{bmatrix} \xi(k).
\]

Moreover, at time \( k \), we have

\[
\zeta(k) = (A_i - I) x(k) + B_i K_{i, r}, x(k - \tau_x(k)).
\]

So

\[
E(V_4(x_{k+1}, \bar{r}(k+1), k+1) \mid x_k, \bar{r}(k), k) - V_4(x_k, \bar{r}(k), k)
= E \left( \sum_{h=-\tau_{\text{max}}+1}^{0} \sum_{l=k+h}^{k} \xi^T(l) Z \xi(l) \mid x_k, \bar{r}(k), k \right) - \sum_{h=-\tau_{\text{max}}+1}^{0} \sum_{l=k-1+h}^{k-1} \xi^T(l) Z \xi(l)
= \sum_{h=-\tau_{\text{max}}+1}^{0} \left( \sum_{l=k+h}^{k} \xi^T(l) Z \xi(l) - \sum_{l=k-1+h}^{k-1} \xi^T(l) Z \xi(l) \right)
= \sum_{h=-\tau_{\text{max}}+1}^{0} \left[ \xi^T(k) Z \xi(k) - \xi^T(k-1+h) Z \xi(k-1+h) \right]
= \tau_{\text{max}} \xi^T(k) Z \xi(k) - \sum_{l=k-\tau_{\text{max}}}^{k-1} \xi^T(l) Z \xi(l)
= \tau_{\text{max}} \xi^T(k) Z \left[ (A_i - I) x(k) + B_i K_{i, r}, x(k - \tau_x(k)) \right]^T Z \left[ (A_i - I) x(k) + B_i K_{i, r}, x(k - \tau_x(k)) \right]
- \sum_{l=k-\tau_{\text{max}}}^{k-1} \xi^T(l) Z \xi(l)
= \tau_{\text{max}} \xi^T(k) \begin{bmatrix}
(A_i - I)^T Z (A_i - I) & (A_i - I)^T Z B_i K_{i, r} \\
K_{i, r}^T B_i^T Z (A_i - I) & K_{i, r}^T B_i^T Z B_i K_{i, r}
\end{bmatrix} \xi(k) - \sum_{l=k-\tau_{\text{max}}}^{k-1} \xi^T(l) Z \xi(l).
\]

Also note that for any matrices \( X_{11} = X_{11}^T \in \mathbb{R}^{n \times n}, X_{12} \in \mathbb{R}^{n \times n}, X_{22} = X_{22}^T \in \mathbb{R}^{n \times n} \).
and \( Y_{1\nu} \in \mathbb{R}^{n \times n}, Y_{2\nu} \in \mathbb{R}^{n \times n} \) satisfying

\[
\begin{bmatrix}
X_{\nu} & Y_{\nu} \\
Y_{\nu}^T & Z
\end{bmatrix} \geq 0
\]

where

\[
X_{\nu} = \begin{bmatrix}
X_{11\nu} & X_{12\nu} \\
X_{12\nu}^T & X_{22\nu}
\end{bmatrix},
\]

\[
Y_{\nu} = \begin{bmatrix}
Y_{1\nu} \\
Y_{2\nu}
\end{bmatrix}.
\]

We have the inequality

\[
0 \leq \sum_{l=k-\tau_{s}(k)}^{k-1} \left[ \xi(k) \begin{bmatrix} X_{\nu} & Y_{\nu} \\ Y_{\nu}^T & Z \end{bmatrix} \xi(l) \right] = \tau_{s}(k)\xi^T(k)X_{\nu}\xi(k) + 2\xi^T(k)Y_{\nu} \sum_{l=k-\tau_{s}(k)}^{k-1} \xi(l) + \sum_{l=k-\tau_{s}(k)}^{k-1} \xi^T(l)Z\xi(l)
\]

\[
= \tau_{s}(k)\xi^T(k)X_{\nu}\xi(k) + 2\xi^T(k)Y_{\nu} [x(k) - x(k - \tau_{s}(k))] + \sum_{l=k-\tau_{s}(k)}^{k-1} \xi^T(l)Z\xi(l)
\]

\[
\leq \xi^T(k) \begin{bmatrix}
Y_{1\nu} + Y_{1\nu}^T & -Y_{1\nu} + Y_{2\nu}^T \\
-Y_{1\nu}^T + Y_{2\nu} & -Y_{2\nu} - Y_{2\nu}^T + \tau_{\max}X_{\nu}
\end{bmatrix} \xi(k) + \sum_{l=k-\tau_{\max}}^{k-1} \xi^T(l)Z\xi(l)
\]

\[
\triangleq \Xi_{\nu},
\]

Therefore,

\[
E (V(x_{k+1}, \bar{r}(k + 1), k + 1) | x_{k}, \bar{r}(k), k) - V(x_{k}, \bar{r}(k), k)
\]

\[
\leq E (V(x_{k+1}, \bar{r}(k + 1), k + 1) | x_{k}, \bar{r}(k), k) - V(x_{k}, \bar{r}(k), k) + \Xi_{\nu}
\]

\[
\leq \xi^T(k) \left( \tilde{\Phi}_{\nu} + \tau_{\max}X_{\nu} \right) \xi(k)
\]
where

$$\hat{\Phi}_\nu = \begin{bmatrix} \Phi_{\nu11} & \Phi_{\nu12} \\ \Phi_{\nu12}^T & \Phi_{\nu22} \end{bmatrix}.$$ 

Hence, if

$$\hat{\Phi}_\nu + \tau_{\text{max}} X_\nu < 0$$

and

$$\begin{bmatrix} X_\nu & Y_\nu \\ Y^T_\nu & Z \end{bmatrix} \geq 0$$

then

$$\mathbb{E}(V(x_{k+1}, \tilde{r}(k+1), k+1) \mid x_k, \tilde{r}(k), k) - V(x_k, \tilde{r}(k), k) \leq -\delta \|x(k)\|^2 < 0$$

for any $x(k) \neq 0$, where

$$\delta = \inf_{\nu \in S_{\tau_{\text{max}}}} \left( \lambda_{\min}(\hat{\Phi}_\nu - \tau_{\text{max}} X_\nu) \right) > 0.$$ 

Following the similar line as in the proof of Theorem 1 in [3], it can be shown that

$$\lim_{N \to \infty} \mathbb{E} \left( \sum_{k=0}^{N} \|x(k; \phi(\cdot), \kappa(\cdot))\|^2 \mid \phi(\cdot), \kappa(\cdot) \right) < \infty,$$

that is, closed-loop system (5.3) is stochastically stable.

Note that there exist $X_\nu = X^T_\nu$ and $Y_\nu$ such that

$$\hat{\Phi}_\nu + \tau_{\text{max}} X_\nu < 0.$$
and

\[
\begin{bmatrix}
X_v & Y_v \\
Y_v^T & Z
\end{bmatrix} \geq 0
\]

hold if and only if there exists \(Y_v\) such that

\[
\hat{\Phi}_v + \tau_{\max} Y_v Z^{-1} Y_v^T < 0.
\]

Finally, this inequality is further equivalent to (5.5) in view of Schur complement equivalence. This completes the proof. \(\square\)

Theorem 5.1 can be used to check the stochastic stability of closed-loop system (5.3), as for the design of control law (5.2), we have the following theorem.

**Theorem 5.2** Consider the Markovian jump linear system given in (5.1), there exists a state-feedback control law (5.2) such that closed-loop system (5.3) is stochastically stable, if there exist matrices \(P_v \in \mathbb{S}^{n \times n}\), \(R_v \in \mathbb{S}^{n \times n}\), \(Q \in \mathbb{S}^{n \times n}\), \(Z \in \mathbb{S}^{n \times n}\), \(W \in \mathbb{S}^{n \times n}\), \(Y_1 \in \mathbb{R}^{n \times n}\), \(Y_2 \in \mathbb{R}^{n \times n}\), \(v \in S_{r+1}\), and \(K_i \in \mathbb{R}^{m \times n}\), \(i \in S\), satisfying the coupled linear matrix inequalities

\[
\begin{bmatrix}
\Theta_v & -Y_{1v} + Y_{2v}^T & Y_{1v} & A_i^T - I & \Gamma_{1i} \\
-Y_{1v}^T + Y_{2v} & -Y_{2v} - Y_{2v}^T - Q & Y_{2v} & K_i^T B_i^T & \Gamma_{2i} \\
Y_{1v}^T & Y_{2v}^T & -\frac{1}{\tau_{\max}} Z & 0 & 0 \\
A_i - I & B_i K_i - \tau_{\max} & 0 & -\frac{1}{\tau_{\max}} W & 0 \\
I_{1i}^T & I_{2i}^T & 0 & 0 & -A_v
\end{bmatrix} < 0 \quad (5.6)
\]

with equality constraints

\[
ZW = I, \quad P_v R_v = I \quad (5.7)
\]
for all $\nu \in S_{\tau_r+1}$, where

$$\Theta_\nu = -P_\nu + Y_{1\nu}^T + (\tau_{\text{max}} - \tau_{\text{min}} + 1) Q,$$

$$\Gamma_{1i} = \begin{bmatrix} \sqrt{\pi_{i_1}} A_{i_1}^T & \sqrt{\pi_{i_2}} A_{i_2}^T & \cdots & \sqrt{\pi_{i_s}} A_{i_s}^T \end{bmatrix},$$

$$\Gamma_{2i} = \begin{bmatrix} \sqrt{\pi_{i_1}} K_{i_{\tau_r}}^T B_{i_1}^T & \cdots & \sqrt{\pi_{i_s}} K_{i_{\tau_r}}^T B_{i_s}^T \end{bmatrix},$$

$$\Lambda_\nu = \text{diag}(R_{\mu+1}, R_{\mu+2}, \ldots, R_{\mu+s}),$$

and $i, i_{\tau_r}, \mu$ are given in Theorem 5.1.

**Proof:** Note that linear matrix inequality (5.5) is equivalent to $\dot{\Phi}_\nu + \tau_{\text{max}} Y_\nu Z^{-1} Y_\nu^T < 0$, which is equivalent to (5.6) and (5.7) by defining $W \triangleq Z^{-1}, R_\nu \triangleq P_\nu^{-1}$ and in view of Schur complement equivalence. This completes the proof. □

**Remark 5.6** Theorem 5.2 gives a delay-dependent condition for finding a time-delayed, mode-dependent controller such that the closed-loop system is stochastically stable.

**Remark 5.7** Although the solution set of (5.6)–(5.7) is not convex, a design procedure similar to Algorithm RSP could be used to facilitate the design of the controllers.

### 5.3 Numerical Example

In this section, we present numerical simulations to illustrate the application of the developed theory. Let us consider a Markovian jump system with two operation modes $s = 2$, system mode delay $\tau_r = 1$ and system state delay $1 \leq \tau_s(k) \leq 3$. The system data of (5.1) are as follows:

$$A_1 = \begin{bmatrix} 1.5 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.6 & 0 \\ 0.1 & 1.2 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$
Π = \begin{bmatrix} 0.1 & 0.9 \\ 0.5 & 0.5 \end{bmatrix},

and φ(k) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, k = -3, \ldots, 0, and κ(k) = 1, k = -1, 0, are the initial condition. This system with u(k) ≡ 0 is not stochastically stable (see Figure 5.1).

Applying Theorem 5.2 and an algorithm similar to Algorithm RSP, we obtain a controller

\[ K_1 = \begin{bmatrix} -0.0622 & -0.1477 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -0.0569 & -0.1391 \end{bmatrix}, \]

which stochastically stabilizes the system when time delays occur both in the system mode and in the system state (see Figure 5.2). One set of solutions is given by

\[ P_1 = \begin{bmatrix} 1.6035 & 0.8791 \\ 0.8791 & 2.2817 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.5251 & 0.3319 \\ 0.3319 & 3.4483 \end{bmatrix}, \]

\[ P_3 = \begin{bmatrix} 1.6054 & 0.9349 \\ 0.9349 & 1.8647 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 0.5611 & 0.3605 \\ 0.3605 & 3.3867 \end{bmatrix}, \]

\[ R_1 = \begin{bmatrix} 0.7907 & -0.3046 \\ -0.3046 & 0.5556 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 2.0276 & -0.1951 \\ -0.1951 & 0.3088 \end{bmatrix}, \]

\[ R_3 = \begin{bmatrix} 0.8797 & -0.4410 \\ -0.4410 & 0.7574 \end{bmatrix}, \quad R_4 = \begin{bmatrix} 1.9132 & -0.2037 \\ -0.2037 & 0.3170 \end{bmatrix}, \]

\[ Y_{11} = \begin{bmatrix} -0.1266 & -0.2757 \\ -0.2757 & -0.6910 \end{bmatrix}, \quad Y_{12} = \begin{bmatrix} -0.1266 & -0.2757 \\ -0.2757 & -0.6910 \end{bmatrix}, \]

\[ Y_{13} = \begin{bmatrix} -0.1266 & -0.2757 \\ -0.2757 & -0.6910 \end{bmatrix}, \quad Y_{14} = \begin{bmatrix} -0.1266 & -0.2757 \\ -0.2757 & -0.6910 \end{bmatrix}, \]

\[ Y_{21} = \begin{bmatrix} 0.1266 & 0.2757 \\ 0.2757 & 0.6910 \end{bmatrix}, \quad Y_{22} = \begin{bmatrix} 0.1266 & 0.2757 \\ 0.2757 & 0.6910 \end{bmatrix}. \]
CHAPTER 5. STABILIZATION OF DMJLS VIA DELAYED INFORMATION

\[
Y_{23} = \begin{bmatrix} 0.1266 & 0.2757 \\ 0.2757 & 0.6910 \end{bmatrix}, \quad Y_{24} = \begin{bmatrix} 0.1266 & 0.2757 \\ 0.2757 & 0.6910 \end{bmatrix},
\]

\[
Q = 10^{-7} \times \begin{bmatrix} 0.2730 & -0.1458 \\ -0.1458 & 0.5220 \end{bmatrix}, \quad Z = \begin{bmatrix} 0.3798 & 0.8270 \\ 0.8270 & 2.0730 \end{bmatrix},
\]

\[
W = \begin{bmatrix} 20.0389 & -7.9942 \\ -7.9942 & 3.6715 \end{bmatrix}.
\]

It can be verified that \(\|P_1R_1 - I\| = 2.5427 \times 10^{-12}, \|P_2R_2 - I\| = 2.5447 \times 10^{-12},\)

\(\|P_3R_3 - I\| = 2.5459 \times 10^{-12}, \|P_4R_4 - I\| = 2.4721 \times 10^{-12}, \text{ and } \|ZW - I\| = 2.5506 \times 10^{-12}\). Therefore, the equality constraints in (5.7) are satisfied.

5.4 Summary

Research Problem 2 (cf. page 14) has been studied in the context of stabilization of discrete-time Markovian jump linear systems. The accessibility of the system state and the system mode are time delayed. It has been shown that the delay in the system mode plays an essential role for the control of the system, and should be kept constant in applications. A sufficient condition for the design of the time-delayed, state-feedback control law has been established in terms of coupled linear matrix inequalities with equality constraints. A numerical example demonstrates the use of the developed theory.
Figure 5.1  Initial condition response of the open-loop system

Figure 5.2  Initial condition response of the closed-loop system
Stabilization of NCS with Packet Loss

Research Problem 3 (cf. page 14) will be investigated in this chapter and next chapter. The communication network exists not only between the sampler and the controller, but also between the controller and the actuator. Moreover, the network suffers from packet loss as well as time delay.

The main focus of this chapter is the packet-loss issue of networked control systems in the discrete-time domain. The chapter is organized as follows. Section 6.1 describes a general framework of networked control systems. Two models of packet-loss processes are considered. One is the arbitrary packet-loss process; the other is the Markovian packet-loss process. The stability of networked control systems is analyzed in Section 6.2, and Section 6.3 tackles the stabilization problem. A networked, state-feedback controller is to be designed so that the closed-loop system is stable. The results are also extended to the unit time case in Section 6.4. Numerical example and simulations are presented in Section 6.5. Finally, Section 6.6 concludes the chapter with a summary.
CHAPTER 6. STABILIZATION OF NCS WITH PACKET LOSS

6.1 Problem Formulation

The framework of the networked control system considered in this chapter is depicted in Figure 6.1. The process to be controlled is modeled by a linear discrete-time system

\[ x(k + 1) = Ax(k) + Bu(k) \]

(6.1)

where \( k \in \mathbb{Z}_+ \) is the time step, \( x(k) \in \mathbb{R}^n \) and \( u(k) \in \mathbb{R}^m \) are the system state and the control input, respectively. \( x_0 \triangleq x(0) \) is the initial state. \( A \) and \( B \) are two constant matrices of appropriate dimensions. Networks exist between the sampler and the controller and between the controller and the zero-order hold. The sampler is clock driven, the controller and zero-order hold are event driven. At each time step, the data are transmitted in a single packet.

Let \( I \triangleq \{ i_1, i_2, \ldots \} \), a subsequence of \( \{ 1, 2, 3, \ldots \} \), denote the sequence of the time points of the successful data transmissions from the sampler to the zero-order hold; \( s \triangleq \max_{i_k \in I} (i_{k+1} - i_k) \) be the maximum packet-loss upper bound. Then the following concepts and mathematical models can be used to capture the nature of the packet loss.

**Definition 6.1** Packet-loss process is defined as

\[ \{ \eta(i_k) \triangleq i_{k+1} - i_k : i_k \in I \} \]

(6.2)
which takes values in the finite state space $S \triangleq \{1, 2, \ldots, s\}$.

**Remark 6.1** $s = 1$ means that no packet is lost throughout the network transmission. The maximum allowable packet-loss rate may be defined as $\frac{s-1}{s} \times 100\%$.

**Definition 6.2** Packet-loss process (6.2) is said to be arbitrary if it takes values in $S$ arbitrarily.

**Remark 6.2** The model of the arbitrary packet-loss process is similar to the one considered in [69].

**Definition 6.3** Packet-loss process (6.2) is said to be Markovian if it is a discrete-time homogeneous Markov chain on a complete probability space $(\Omega, \mathcal{F}, P)$, and takes values in $S$ with known transition probability matrix $\Pi \triangleq (\pi_{ij}) \in \mathbb{R}^{s \times s}$, where

$$\pi_{ij} \triangleq \Pr (\eta(i_{k+1}) = j \mid \eta(i_k) = i) \geq 0$$

for all $i, j \in S$, and $\sum_{j=1}^{s} \pi_{ij} = 1$ for each $i \in S$.

**Remark 6.3** When $\Pi = \begin{bmatrix} p & 1 - p \\ p & 1 - p \end{bmatrix}$ with $0 \leq p \leq 1$, the two-state Markov chain is reduced to a Bernoulli process [50, 52].

The networked controller is a state-feedback controller

$$u = Kx$$

(6.3)

where $K \in \mathbb{R}^{m \times n}$ is to be designed.

From the viewpoint of the zero-order hold, the control input is

$$u(l) = u(i_k) = Kx(i_k)$$
for \( i_k \leq l \leq i_{k+1} - 1 \). The initial inputs are set to zero: \( u(l) = 0 \) for \( 0 \leq l \leq i_1 - 1 \). Hence the closed-loop system becomes

\[
x(l + 1) = Ax(l) + BKx(i_k)
\]

for \( i_k \leq l \leq i_{k+1} - 1, \ i_k \in I \).

The objective of this chapter is to construct controller (6.3) so that networked control system (6.4) is stable.

### 6.2 Stability of Networked Control Systems

In this section, we analyze the stability property of networked control systems. For networked control systems with arbitrary packet-loss process, a sufficient condition is derived by adopting a packet-loss dependent Lyapunov function approach. For networked control systems with Markovian packet-loss process, a necessary and sufficient condition is established by using the theory from Markovian jump systems. The conditions are given in terms of linear matrix inequalities.

#### 6.2.1 Arbitrary Packet-Loss Stability

**Definition 6.4** Let \( x(l; x_0) \) be the trajectory of networked control system (6.4) with initial state \( x_0 \). Then networked control system (6.4) with arbitrary packet-loss process (6.2) is said to be stable, if for any \( \varepsilon > 0 \) there exists a \( \delta \triangleq \delta(\varepsilon) > 0 \) such that \( \|x_0\| < \delta \) implies \( \|x(l; x_0)\| < \varepsilon \) for \( l \in \mathbb{Z}_+ \). Furthermore, (6.4) is said to be asymptotically stable if it is stable and \( \lim_{l \to \infty} \|x(l; x_0)\|^2 = 0 \) for any initial state \( x_0 \in \mathbb{R}^n \).

**Theorem 6.1** Networked control system (6.4) with arbitrary packet-loss process (6.2) is asymptotically stable, if there exist matrices \( P_i \in \mathbb{S}^{nxn}, \ i \in S \), such that

\[
\left( A^j + B_jK \right)^T P_j \left( A^j + B_jK \right) - P_i < 0
\]
holds for all $i, j \in S$, where $B_j = \sum_{r=0}^{j-1} A^r B$.

**Proof:** From system (6.4), we have

$$x(i_{k+1}) = \left[A^{\eta(i_k)} + \sum_{r=0}^{\eta(i_k)-1} A^r B K \right] x(i_k), \quad i_k \in I. \quad (6.6)$$

The initial state of system (6.6) is $x(i_1) = A^{i_1} x_0$. A packet-loss dependent Lyapunov function is taken as

$$V(l) \triangleq x^T(l) P_{(l-i_k)} x(l) \quad (6.7)$$

for $i_k + 1 \leq l \leq i_{k+1}, i_k \in I$. Now let $i = \eta(i_{k-1}), j = \eta(i_k)$, we have

$$V(i_k) = x^T(i_k) P_{(i_k-i_{k-1})} x(i_k) = x^T(i_k) P_i x(i_k),$$

$$V(i_{k+1}) = x^T(i_{k+1}) P_{(i_{k+1}-i_k)} x(i_{k+1}) = x^T(i_k) \left(A^j + B_j K\right)^T P_j \left(A^j + B_j K\right) x(i_k).$$

Therefore,

$$V(i_{k+1}) - V(i_k) = x^T(i_k) \left[\left(A^j + B_j K\right)^T P_j \left(A^j + B_j K\right) - P_i\right] x(i_k) < 0$$

for any $x(i_k) \neq 0$ if inequality (6.5) holds. Hence,

$$\lim_{i_k \to \infty} V(i_k) = 0.$$

Now consider system state $x(l)$ for $i_k + 1 \leq l \leq i_{k+1}$, we have

$$x(l) = \left(A^h + B_h K\right) x(i_k)$$

where $h = l - i_k \in S$ and $B_h = \sum_{r=0}^{h-1} A^r B$. Hence,

$$V(l) = x^T(l) P_{(l-i_k)} x(l) = x^T(i_k) \left[\left(A^h + B_h K\right)^T P_h \left(A^h + B_h K\right)\right] x(i_k)$$
and
\[ V(l) - V(i_k) = x^T(i_k) \left[ (A^h + B_hK)^T P_h (A^h + B_hK) - P_i \right] x(i_k) < 0 \]
for any \( x(i_k) \neq 0 \) in view of (6.5). That is, \( V(l) < V(i_k) \) for \( i_k + 1 < l < i_{k+1} \). Therefore, \( \lim_{l \to \infty} V(l) = 0 \) since \( \lim_{i_k \to \infty} V(i_k) = 0 \). As a result,
\[ \lim_{l \to \infty} \|x(l; x_0)\|^2 = 0. \]
That is, the sequence \( \{x(l) : l \in \mathbb{Z}_+\} \) converges to zero.

To prove networked control system (6.4) is stable, define
\[ \beta_1 \triangleq \max \left\{ \max_{h \in S} \|A^h\|^2, 1 \right\}, \]
\[ \beta_2 \triangleq \max_{h \in S} \|P_h\|, \]
\[ \beta_3 \triangleq \min_{h \in S} \|P^{-1}_h\|, \]
\[ \beta \triangleq \min \left\{ \sqrt{\frac{1}{\beta_1}}, \sqrt{\frac{\beta_3}{\beta_1 \beta_2}} \right\}. \]
Then given any \( \varepsilon > 0 \), we will prove that \( \|x_0\| < \beta \varepsilon \) implies \( \|x(l; x_0)\| < \varepsilon \) for all \( l \in \mathbb{Z}_+ \) in the following.

For \( 0 \leq l \leq i_1 \), since the initial inputs are zero, we have \( x(l) = A^l x_0 \), so
\[ \|x(l; x_0)\| \leq \sqrt{\beta_1} \|x_0\| < \sqrt{\beta_1} \beta \varepsilon \leq \varepsilon. \]
For $l > i_1$, we have

$$\beta_3 \|x(l; x_0)\|^2 < V(l)$$

according to the definition of the Lyapunov function. Moreover, we have

$$V(l) < V(i_1) \leq \beta_2 \|x(i_1)\|^2 \leq \beta_1 \beta_2 \|x_0\|^2$$

from the proof above. So

$$\|x(l; x_0)\| < \sqrt{\frac{\beta_1 \beta_2}{\beta_3}} \|x_0\| < \sqrt{\frac{\beta_1 \beta_2}{\beta_3} \beta \varepsilon} \leq \varepsilon.$$ 

Thus, we conclude that

$$\|x(l; x_0)\| < \varepsilon$$

for all $l \in \mathbb{Z}_+$ if

$$\|x_0\| < \delta \quad \text{with} \quad \delta = \beta \varepsilon.$$

According to Definition 6.4, networked control system (6.4) is asymptotically stable. □

**Remark 6.4** In the proof of Theorem 6.1, $V(l + 1) < V(l)$ is not necessarily true for all $l \in \mathbb{Z}_+$. 

### 6.2.2 Markovian Packet-Loss Stability

**Definition 6.5** Networked control system (6.4) with Markovian packet-loss process (6.2) is said to be mean square stable if

$$\lim_{l \to \infty} E(\|x(l; x_0)\|^2) = 0$$
for any initial state $x_0 \in \mathbb{R}^n$.

**Theorem 6.2** Networked control system (6.4) with Markovian packet-loss process (6.2) is mean square stable if, and only if, there exist matrices $P_i \in \mathbb{S}^{n \times n}$, $i \in \mathcal{S}$, such that

$$
\sum_{j=1}^{s} \left[ \pi_{ij} \left( A^j + B_j K \right)^T P_j \left( A^j + B_j K \right) \right] - P_i < 0
$$

(6.8)

holds for all $i \in \mathcal{S}$, where $B_j$ is given in Theorem 6.1.

**Proof:** (Sufficiency) Because $\{\eta(i_k) : i_k \in I\}$ is a discrete-time homogeneous Markov chain, system (6.6) is in fact a discrete-time Markovian jump linear system with $s$ operation modes [12, 35].

Consider the same Lyapunov function as in (6.7), we have

$$
E \left( V(i_{k+1}) \mid \eta(i_{k-1}) = \tilde{i} \right) = V(i_k)
$$

$$
= \left( x^T (i_{k+1}) P_{(i_{k+1}-i_k)} x(i_{k+1}) \mid \eta(i_{k-1}) = \tilde{i} \right) - x^T (i_k) P_{\eta(i_{k-1})} x(i_k)
$$

$$
= \left( x^T (i_k) \left( A^{\eta(i_k)} + B_{\eta(i_k)} K \right)^T P_{\eta(i_k)} \left( A^{\eta(i_k)} + B_{\eta(i_k)} K \right) x(i_k) \mid \eta(i_{k-1}) = \tilde{i} \right) - x^T (i_k) P_i x(i_k)
$$

$$
= x^T (i_k) \left( \sum_{j=1}^{s} \pi_{ij} \left( A^j + B_j K \right)^T P_j \left( A^j + B_j K \right) \right) x(i_k)
$$

$$
< 0
$$

for any $x(i_k) \neq 0$ if inequality (6.8) holds. Hence, $\lim_{i_k \to \infty} E \left( V(i_k) \right) = 0$ and

$$
\lim_{i_k \to \infty} E \left( ||x(i_k) - x(i_{1})||^2 \right) = 0.
$$

That is, system (6.6) is mean square stable.

Note that two consecutive successful control input packets arrive at the zero-order hold at $i_k$ and $i_{k+1}$, respectively. This means no new data arrive for $i_k + 1 \leq l \leq i_{k+1} - 1$. Therefore, we have

$$
V(l) = x^T (i_k) \left[ \left( A^h + B_h K \right)^T P_h \left( A^h + B_h K \right) \right] x(i_k)
$$
for $i_k + 1 \leq l \leq i_{k+1}$, where $h = l - i_k$.

Now define

$$\alpha_1 \triangleq \max_{h \in \mathcal{S}} \left\| (A^h + B_hK)^T P_h (A^h + B_hK) \right\|$$

and

$$\alpha \triangleq \frac{\alpha_1}{\beta_3} > 0$$

where $\beta_3$ was defined in the proof of Theorem 6.1. Then

$$V(l) \leq \alpha V(i_k)$$

for $i_k + 1 \leq l \leq i_{k+1}$. Therefore, $\lim_{l \to \infty} \mathbb{E}(V(l)) = 0$, so

$$\lim_{l \to \infty} \mathbb{E}(\| x(l; x_0) \|^2) = 0.$$

This completes the sufficient part of the proof.

(Necessity) Suppose that networked control system (6.4) is mean square stable, then system (6.6) must be mean square stable. Since system (6.6) is a Markovian jump system, according to the stability results from Markovian jump systems (see Proposition 2.1), there exist matrices $P_i \in \mathbb{S}^{n \times n}$, $i \in \mathcal{S}$, such that (6.8) holds. This completes the whole proof. \qed

**Remark 6.5** Inequality (6.8) can be written as

$$\sum_{j=1}^{s} \pi_{ij} \left[ (A^j + B_jK)^T P_j (A^j + B_jK) - P_i \right] < 0$$

because of $\sum_{j=1}^{s} \pi_{ij} = 1$. Therefore, as expected, arbitrary packet-loss stability implies Markovian packet-loss stability.
6.3 Stabilization of Networked Control Systems

With the stability results developed in Section 6.2, the controller design techniques are provided in this section.

**Theorem 6.3** Consider discrete-time system (6.1), there exists a state-feedback controller (6.3), for the network with arbitrary packet-loss process (6.2), such that networked control system (6.4) is asymptotically stable, if there exist matrices $X_i \in \mathbb{S}^{n \times n}$, $i \in S$, $G \in \mathbb{R}^{n \times n}$, and $Y \in \mathbb{R}^{m \times n}$, satisfying the coupled linear matrix inequalities

$$
\begin{bmatrix}
-G - G^T + X_i & (A^i G + B_j Y)^T \\
A^i G + B_j Y & -X_j
\end{bmatrix} < 0 \quad (6.9)
$$

for all $i, j \in S$, where $B_j$ is given in Theorem 6.1. In this case, the controller is given by $K = YG^{-1}$.

**Proof:** Pre- and post-multiplying (6.9) by

$$
\begin{bmatrix}
A^i + B_j K & 1
\end{bmatrix}
$$

and its transpose and noting that $Y = KG$, we have

$$
(A^i + B_j K)X_i(A^i + B_j K)^T - X_j < 0
$$

which is equivalent to inequality (6.5) with $P_i \triangleq X_i^{-1}$.

**Remark 6.6** Theorem 6.3 contains the quadratic stabilization result as a particular case. If we aggregate to the linear matrix inequalities in (6.9) the additional linear constraints $G = X \in \mathbb{S}^{n \times n}$ and $X_i = X, i \in S$, then Theorem 1 of [69] is recovered exactly.

The following theorem gives a sufficient mean square stabilization condition for discrete-time system (6.1) controlled by (6.3) over the network with Markovian packet-
loss process.

**Theorem 6.4** Consider discrete-time system (6.1), there exists a state-feedback controller (6.3), for the network with Markovian packet-loss process (6.2), such that networked control system (6.4) is mean square stable, if there exist matrices $X_i \in \mathbb{S}^{n \times n}$, $i \in S$, $G \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{m \times n}$, satisfying the coupled linear matrix inequalities

$$
\begin{bmatrix}
-G - G^T + X_i & \Gamma_{1i} \\
\Gamma_{1i}^T & -\Lambda
\end{bmatrix} < 0
$$

(6.10)

for all $i \in S$, where

$$
\Gamma_{1i} = \begin{bmatrix}
\sqrt{\pi_1} (A + BK)^T & \ldots & \sqrt{\pi_s} (A^*G + B_s Y)^T
\end{bmatrix},
\Lambda = \text{diag} (X_1, \ldots, X_s),
$$

and $B_j$ is given in Theorem 6.1. In this case, the controller is given by $K = YG^{-1}$.

**Proof:** Define

$$
\Xi_{1i} \triangleq \begin{bmatrix}
\sqrt{\pi_1} (A + BK) \\
\vdots \\
\sqrt{\pi_s} (A + BK)
\end{bmatrix}
$$

Pre- and post-multiplying inequality (6.10) by

$$
\begin{bmatrix}
\Xi_{1i} & I
\end{bmatrix}
$$

and its transpose, respectively, we have

$$
\Xi_{1i} X_i \Xi_{1i}^T - \Lambda < 0
$$
which is equivalent to
\[
\begin{bmatrix}
-X^{-1} & \Xi^T_P

\Xi & -\Lambda
\end{bmatrix} < 0.
\]

This inequality is equivalent to inequality (6.8) by defining \( P_i = X^{-1}_i \). This completes the proof. □

**Remark 6.7** By virtue of \( s \) being an integer, we can obtain the largest packet-loss upper bound \( s_{\text{max}} \) easily.

**Remark 6.8** If system (6.1) is discretized from a continuous-time system

\[
\dot{x}(t) = A_c x(t) + B_c u(t), \quad t \in [0, +\infty)
\]

with \( A = e^{A_c T_s} \) and \( B = \int_0^{T_s} e^{A_c \tau} B_c d\tau \), where \( T_s \) is the sampling period, then given the upper bound \( s \), a simple bisection method may be used to find out the maximal sampling period \( T_{s_{\text{max}}} \).

### 6.4 Network-Induced Delay

Network-induced time delay is another important issue to be dealt with, and can be modeled as an input delay. Hence we amend system (6.1) to

\[
x(k+1) = Ax(k) + Bu(k - \tau)
\]

(6.11)

where \( \tau \) is a constant time delay. The trajectory of (6.11) on time instants \( I \), under the control of (6.3), is generally a function of the previous ones. Specifically, \( x(i_{k+1}) \) is a function of \( x(i_k), x(i_{k-1}), \ldots, x(i_k-\tau) \). In contrast, \( x(i_{k+1}) \) is a function of \( x(i_k) \) when \( \tau = 0 \) as shown in (6.6). To simplify the analysis and synthesis, we consider the simplest case with \( \tau = 1 \). However, the principle used here remains valid for \( \tau > 1 \).
Let us consider system (6.11) with $\tau = 1$, the closed-loop system is

$$x(l + 1) = Ax(l) + BKx(i_k)$$  \hspace{1cm} (6.12)$$

for $i_k + 1 \leq l \leq i_{k+1}$, $i_k \in I$. Therefore, we have

$$x(i_{k+1}) = \left[ A^{\eta(i_k)} + \sum_{r=0}^{\eta(i_k)-2} A'^rBK \right] x(i_k) + A^{\eta(i_k)-1}BKx(i_{k-1})$$ \hspace{1cm} (6.13)$$

for $i_k \in I$.

**Remark 6.9** In the case when $\tau = 2$, the closed-loop system is

$$x(l + 1) = Ax(l) + BKx(i_k)$$

for $i_k + 2 \leq l \leq i_{k+1} + 1$, $i_k \in I$. Therefore, we have

$$x(i_{k+1}) = \left[ A^{\eta(i_k)} + \sum_{r=0}^{\eta(i_k)-3} A'^rBK \right] x(i_k)$$

$$+ \delta_{i_k-1}A^{\eta(i_k)-1}x(i_{k-1}) + (1 - \delta_{i_k-1})A^{\eta(i_k)-1}x(i_{k-2})$$

$$+ \delta_{i_k}A^{\eta(i_k)-2}BKx(i_{k-1})$$

for $i_k \in I$, where

$$\delta_k = \begin{cases} 
1, & \text{if } \eta(i_k) \geq 2 \\
0, & \text{if } \eta(i_k) = 1.
\end{cases}$$

The following theorem provides a stability result for the networked control system given in (6.13).

**Theorem 6.5** Networked control system (6.12) with arbitrary packet-loss process (6.2) is asymptotically stable, if there exist matrices $P_i \in S^{n \times n}$, $i \in S$, $Q \in S^{n \times n}$, $Z \in S^{n \times n}$,
$N_1 \in \mathbb{R}^{n \times n}$ and $N_2 \in \mathbb{R}^{n \times n}$ such that

$$
\begin{bmatrix}
\Phi_{ij1} & \Phi_{ij2} & N_1 \\
\Phi_{ij2}^T & \Phi_{ij3} & N_2 \\
N_1^T & N_2^T & -Z
\end{bmatrix} < 0
$$

(6.14)

holds for all $i, j \in S$, where

$$
\begin{align*}
\Phi_{ij1} &= \Gamma_{2j}^TP_j\Gamma_{2j} - P_i + Q + (\Gamma_{2j}^T - I)Z(\Gamma_{2j} - I) + N_1 + N_1^T, \\
\Phi_{ij2} &= \Gamma_{2j}^TP_jA^{j-1}BK + (\Gamma_{2j}^T - I)ZA^{j-1}BK - N_1 + N_1^T, \\
\Phi_{ij3} &= K^TB^T\left(A^{j-1}\right)^T\left(P_j + Z\right)A^{j-1}BK - Q - N_2 - N_2^T, \\
\Gamma_{2j} &= A^j + \sum_{r=0}^{j-2} A^rBK.
\end{align*}
$$

Proof: Consider system (6.13), define

$$
\begin{align*}
\zeta(i_k) &\triangleq x(i_{k+1}) - x(i_k), \\
\xi(i_k) &\triangleq \begin{bmatrix} x^T(i_k) & x^T(i_{k-1}) \end{bmatrix}^T,
\end{align*}
$$

and let $i = i_k - i_{k-1}$ and $j = i_{k+1} - i_k$, so

$$
\begin{align*}
x(i_{k+1}) &= \begin{bmatrix} \Gamma_{2j} & A^{j-1}BK \end{bmatrix} \xi(i_k), \\
\zeta(i_k) &= \begin{bmatrix} \Gamma_{2j} - I & A^{j-1}BK \end{bmatrix} \xi(i_k), \\
\zeta(i_{k-1}) &= \begin{bmatrix} I & -I \end{bmatrix} \xi(i_k).
\end{align*}
$$

Take the Lyapunov functional as

$$
V(l) \triangleq x^T(l)P_{(l-i_k)}x(l) + x^T(i_k)Qx(i_k) + \zeta^T(i_k)Z\zeta(i_k)
$$
for $i_k + 1 \leq l \leq i_{k+1}$, $i_k \in \mathcal{I}$. Then

$$V(i_{k+1}) - V(i_k) = x^T(i_{k+1}) P_{j} x(i_{k+1}) - x^T(i_{k}) P_{i} x(i_{k}) + x^T(i_{k}) Q x(i_{k})$$

$$- x^T(i_{k-1}) Q x(i_{k-1}) + \zeta^T(i_{k}) Z \zeta(i_{k}) - \zeta^T(i_{k-1}) Z \zeta(i_{k-1}).$$

Note that for any $Z \in \mathbb{S}^{n \times n}$ and $N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \in \mathbb{R}^{2n \times n}$, we have

$$-\zeta^T(i_{k-1}) Z \zeta(i_{k-1}) \leq \zeta^T(i_{k}) N Z - N^T \zeta(i_{k}) + 2 \zeta^T(i_{k}) N \zeta(i_{k-1}).$$

Therefore,

$$V(i_{k+1}) - V(i_k) \leq \zeta^T(i_k) \left( \Phi_{ij} + N Z^{-1} N^T \right) \zeta(i_k) < 0$$

for all $x(i_k) \neq 0$ if

$$\Phi_{ij} + N Z^{-1} N^T < 0$$

where $\Phi_{ij} = \begin{bmatrix} \Phi_{ij_1} & \Phi_{ij_2} \\ \Phi_{ij_2}^T & \Phi_{ij_3} \end{bmatrix}$. In view of Schur complement equivalence, this inequality is equivalent to (6.14). The rest of the proof can be carried out by following similar lines as in the proof of Theorem 6.1. The result then follows. □

**Theorem 6.6** Consider discrete-time system (6.11) with time delay $\tau = 1$, there exists a state-feedback controller (6.3), for the network with arbitrary packet-loss process (6.2), such that networked control system (6.12) is asymptotically stable, if there exist matrices $X_i \in \mathbb{S}^{n \times n}$, $i \in \mathcal{S}$, $\bar{Q} \in \mathbb{S}^{n \times n}$, $W \in \mathbb{S}^{n \times n}$, $G \in \mathbb{R}^{n \times n}$, $\bar{N}_1 \in \mathbb{R}^{n \times n}$, $\bar{N}_2 \in \mathbb{R}^{n \times n}$. 

and \( Y \in \mathbb{R}^{m \times n} \), satisfying the coupled linear matrix inequalities

\[
\begin{bmatrix}
\Theta_{1i} & \bullet & \bullet & \bullet & \bullet \\
-\tilde{N}_1^T + \tilde{N}_2 & \Theta_{2j} & \bullet & \bullet & \bullet \\
\tilde{N}_1^T & \tilde{N}_2^T & \Theta_3 & \bullet & \bullet \\
\Gamma_{3j} - G & A^{j-1}BY & 0 & -W & \bullet \\
\Gamma_{3j} & A^{j-1}BY & 0 & 0 & -X_{j1}
\end{bmatrix} < 0
\] (6.15)

for all \( i, j \in S \), where

\[
\begin{align*}
\Theta_{1i} &= X_i - G - G^T + \bar{Q} + \tilde{N}_1 + \tilde{N}_1^T, \\
\Theta_{2i} &= -\bar{Q} - \tilde{N}_2 - \tilde{N}_2^T, \\
\Theta_3 &= W - G - G^T, \\
\Gamma_{3j} &= A^jG + \sum_{r=0}^{j-2} A'^rBY.
\end{align*}
\]

In this case, the controller is given by \( K = YG^{-1} \).

**Proof:** Inequality (6.14) is equivalent to

\[
\begin{bmatrix}
\Xi_{2i} & \bullet & \bullet & \bullet & \bullet \\
-N_1^T + N_2 & \Xi_3 & \bullet & \bullet & \bullet \\
N_1^T & N_2^T & -Z & \bullet & \bullet \\
\Gamma_{2j} - I & A^{j-1}BK & 0 & -Z^{-1} & \bullet \\
\Gamma_{2j} & A^{j-1}BK & 0 & 0 & -P_j^{-1}
\end{bmatrix} < 0
\] (6.16)

where

\[
\begin{align*}
\Xi_{2i} &= -P_i + Q + N_1 + N_1^T, \\
\Xi_3 &= -Q - N_2 - N_2^T.
\end{align*}
\]
Note that
\[
\left( G^{-1} - P_i \right)^T P_i^{-1} \left( G^{-1} - P_i \right) \geq 0
\]
implies
\[
\Xi_4 \triangleq G^{-T} P_i^{-1} G^{-1} - G^{-1} - G^{-T} \geq -P_i
\]
for any nonsingular matrix \( G \in \mathbb{R}^{n \times n} \). Similarly,
\[
\Xi_5 \triangleq G^{-T} Z^{-1} G^{-1} - G^{-1} - G^{-T} \geq -Z.
\]
Hence inequality (6.16) holds if
\[
\begin{bmatrix}
\Xi_4 + Q + N_1 + N_1^T & \bullet & \bullet & \bullet & \bullet \\
-N_1^T + N_2 & \Xi_3 & \bullet & \bullet & \bullet \\
N_1^T & N_2^T & \Xi_5 & \bullet & \bullet \\
\Gamma_{2j} - I & A^{-1} BK & 0 & -Z^{-1} & \bullet \\
\Gamma_{2j} & A^{-1} BK & 0 & 0 & -P_j^{-1}
\end{bmatrix} < 0.
\]
Pre- and post-multiply this inequality by \( \text{diag} \left( G^T, G^T, G^T, I, I \right) \) and its transpose, and define \( X_i \triangleq P_i^{-1}, W \triangleq Z^{-1}, \tilde{Q} \triangleq G^T Q G, \tilde{N}_1 \triangleq G^T N_1 G, \tilde{N}_2 \triangleq G^T N_2 G, Y \triangleq KG \), we have inequality (6.15). \( \square \)

Remark 6.10 The above arguments are applicable to the case when packet-loss process (6.2) is Markovian as well.

6.5 Numerical Examples

In this section, a numerical example and simulations are used to illustrate the usefulness of the developed synthesis methods.
Let us consider the nominal continuous-time system borrowed from [70] with no disturbance input:

\[ \dot{x}(t) = \begin{bmatrix} -1 & 0 & -0.5 \\ 1 & -0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t). \]

When the plant is sampled with a sampling period \( T_s = 0.5s \), the discretized system is

\[ x(k + 1) = \begin{bmatrix} 0.6065 & 0 & -0.2258 \\ 0.3445 & 0.7788 & -0.0536 \\ 0 & 0 & 1.2840 \end{bmatrix} x(k) + \begin{bmatrix} -0.0582 \\ -0.0093 \\ 0.5681 \end{bmatrix} u(k). \]  

(6.17)

Since \( \text{eig}(A_c) = \{-0.5, -1, 0.5\} \) and \( \text{eig}(A) = \{0.7788, 0.6065, 1.2840\} \), both the continuous-time system and the discretized system are unstable. Furthermore, we assume that the packet-loss upper bound \( s = 5 \), which means that up to 80% of the packets can be lost during network transmissions.

Applying Theorem 6.3, a networked controller can be obtained as

\[ u = \begin{bmatrix} 0.0242 & 0.0117 & -0.8048 \end{bmatrix} x. \]

Our simulations are based on the framework in Figure 6.1 (cf. page 137), that is, the designed controller is used to control the continuous-time system over the network, not the discretized correspondence. To simulate, the initial state is taken as

\[ x_0 = \begin{bmatrix} -5 & 0 & 5 \end{bmatrix}^T. \]

Figure 6.2 depicts the trajectory of the system state when the packet-loss process is arbitrary (generated randomly from a uniform distribution). The time instants at which the zero-order hold updates its state are indicated with circles on the time axes. We can see that only 14 control input packets arrived at the zero-order hold during the first 25
seconds. This means that 72% of the packets were lost. Figure 6.3 shows an extreme case when no packet loss occurs.

In addition, from Remark 6.5, this controller will also be able to stabilize the system when the packet-lost process is governed by a Markov chain. Figure 6.4 illustrates this point with the transition probability matrix given by

\[
\Pi = \begin{bmatrix}
0.5 & 0.2 & 0.1 & 0.1 & 0.1 \\
0.2 & 0.5 & 0.3 & 0 & 0 \\
0 & 0.2 & 0.5 & 0.3 & 0 \\
0 & 0 & 0.2 & 0.5 & 0.3 \\
0.1 & 0.1 & 0.1 & 0.2 & 0.5
\end{bmatrix}
\]

Such a \(\Pi\) exhibits the bursty nature of packet loss. The bursty nature is modeled by \(\pi_{ii} > \pi_{ij}\), for all \(i, j \in S, j \neq i\), which indicates that the likelihood of losing a packet after a lost packet transmission is higher than after a successful packet transmission [49, 52].

It should be noted that our sampling period is larger than the one used in [70], where \(T_s = 0.2s\). Figure 6.5 shows the system state trajectory when the sampling period is \(T_s = 0.2s\) where the plant is controlled by the controller developed earlier. Of course, one could apply Theorem 6.3 again to obtain a new controller

\[
u = \begin{bmatrix}
0.0809 & 0.0233 & -1.5923
\end{bmatrix}x.
\]

The state trajectory of the plant controlled by this new controller is plotted in Figure 6.6. In fact, even with \(T_s = 1.5s\), Theorem 6.3 still gives a feasible solution. This demonstrates that larger sampling periods are allowable if both the communication channel and the sampling characteristics are considered in the controller design stage.

Finally, if we replace \(u(k)\) in (6.11) with \(u(k - 1)\), and suppose \(s = 2\), Theorem 6.6
successfully provides a controller

\[ u = \begin{bmatrix} 0.0077 & 0.0017 & -0.8633 \end{bmatrix} x. \]

Figure 6.7 is a simulation of the closed-loop system with unit time delay.

### 6.6 Summary

The stabilization problem of networked control systems with bounded packet loss has been studied in this chapter. Two types of networked controller design methods have been proposed via a packet-loss dependent Lyapunov approach. One ensures that the networked control system is asymptotically stable in the presence of arbitrary packet loss. The other ensures that the mean square stability of the system in the presence of Markovian packet loss. These results have also been extended to the unit time delay case. The powerful potential of the developed theory has been illustrated by a numerical example and simulations.
Figure 6.2  *Initial condition response of the system with arbitrary packet loss*

Figure 6.3  *Initial condition response of the system without packet loss*
Figure 6.4  Initial condition response of the system with Markovian packet loss

Figure 6.5  Initial condition response of the system with $T_s = 0.2s$
Figure 6.6  Initial condition response of the system with the new controller

Figure 6.7  Initial condition response of the system with unit time delay
Chapter 7

Stabilization of NCS with A Logic ZOH

This chapter continues to study Problem 3 (cf. page 22) by adopting a special zero-order hold. Specifically, the zero-order hold in this chapter is configured to be both time-driven and event-driven, and has the logical capability of choosing the newest control input packet to control the process. In addition, network-induced time delay and packet loss are considered at the same time. With such a framework, the continuous-time process can be discretized to a discrete-time system with input delay. A networked state-feedback controller is to be designed to stabilize the discretized system.

The organization of the chapter is as follows. Section 7.1 formulates the stabilization problem to be solved. Section 7.2 gives the main results of the chapter. A sufficient stability condition and several sufficient stabilization conditions are established. Numerical examples are used to illustrate the developed theory in Section 7.3. Section 7.4 summaries this chapter.

7.1 Problem Formulation

Figure 7.1 shows a general framework of networked control systems. The physical
process to be controlled is a linear continuous-time system. The system state is sampled periodically. Let \( t_k = kT_s : k \in \mathbb{Z}_+ \) be the sampling instants, where \( T_s > 0 \) is the sampling period. The sampler generates a new packet \( x(t_k) \) at time instant \( t_k \), and tries to send it into the network during time interval \( t_k \leq t < t_{k+1} \). The network is generally unreliable and problem prone, and may introduce time delays or even packet losses. However, a certain level of network performance is maintained such that the time delays and the consecutive packet losses are upper bounded. The networked controller is a time-invariant discrete-time system. It starts a new computation once it receives a system state data from the network, and sends a new control input packet (that is, the computed data) into the network when completing the computation. The logic zero-order hold can receive the control input packet from the network at any time, but accepts it only if the information of the control input packet is newest. Also, the zero-order hold is synchronized with the sampler by a clock (synchronization here means that the accepted new control input packet does not take effect immediately, but from the next sampling instant). The mechanism of the logic ZOH can be summarized by the following description.

**Logic ZOH** Given \( u(0) \), let \( i_1 := i_0 := 0 \) and \( k := 0 \).

1. At time \( t_k \), the ZOH changes its output to \( u(t) = u(i_k T_s) \) for \( t_k \leq t < t_{k+1} \).

2. During \( t_k < t \leq t_{k+1} \), if a packet \( u(jT_s) \) arrives and \( j > i_{k+1} \), then the ZOH stores \( u(jT_s) \) and let \( i_{k+1} := j \).
(3) Repeat Step (2) until time $t$ reaches $t_{k+1}$. Let $k := k + 1$, $i_{k+1} := i_k$, and go to Step (1).

Remark 7.1 In Step (1), the ZOH is time-driven and synchronized with the sampler and updates its output to the newest control information $u(i_k T_s)$. In Step (2), the ZOH is event-driven and only accepts the newest information. To complete this task, the ZOH has a logic to compare and a memory to store the arrived control input data. Here we have both $i_k \leq k$ and $i_k \leq i_{k+1} \leq k + 1$. Note that $i_{k+1} \geq i_k$ guarantees that ZOH always uses the newest control input data.

A ZOH working in this way enables us to model the overall closed-loop system as a discrete-time system with input delay. Let us define a virtual time delay $\tau(k) \triangleq k - i_k$ for $t_k \leq t < t_{k+1}$ so that the input $u(t)$ in Step (1) of Logic ZOH can be represented by $u(t) = u((k-\tau(k))T_s)$. As the ZOH guarantees that $i_{k+1} \geq i_k$, we have $\tau(k+1) \leq \tau(k) + 1$. Moreover, the characteristic of the networks ensures that $\tau(k)$ has an upper bound, that is, $0 \leq \tau(k) \leq \tau_{\max}$.

It is interesting to observe that the values of the virtual time delay $\tau(k)$ include all of the information to identify the packet losses and the actual time delays. Suppose $\tau(k) = c$ and $\tau(k + 1) = d$ (of course $d \leq c + 1$). Then there are three cases left: (a) if $d = c + 1$, then no new packet arrives during $t_k < t \leq t_{k+1}$, the ZOH continues to use the same control input data, so the virtual time delay just increases itself by one; (b) if $d \leq c$, then a new packet, which was time delayed by $d$, arrives during $t_k < t \leq t_{k+1}$, and $(c - d)$ consecutive packets prior to this packet have been lost during the transmission; (c) the ideal case happens when $d = 0$. In this case, $u(k + 1)$ arrives immediately, and the networks are perfect — neither time delay nor packet loss occurs. Figure 7.2 illustrates these cases.

Based upon the above analysis, the continuity of the process in Figure 7.1, together with the sampler and the logic ZOH, can be discretized as the following linear discrete-
**Figure 7.2** An example of the packets transmitted, lost and used

In this chapter, we are interested in a state-feedback controller of the form

$$ u = Kx $$

(7.2)

where $K$ is to be designed. The time step $k$ was omitted in (7.2) to signal that the controller is event-driven only. Then the resulting closed-loop system is a time-delay system

$$ x(k + 1) = Ax(k) + BKx(k - \tau(k)), \quad k \in \mathbb{Z}_+. $$

(7.3)

The objective of this chapter is to design networked controller (7.2) such that networked control system (7.3) is asymptotically stable.
Remark 7.2 Our proposed framework is similar to those in [67, 68, 70, 72]. In [70], the ZOH is event-driven only, and the newest control input packet takes effect immediately. In [72], the time delay $\tau(k)$ is further divided into two parts: one from the sampler to the controller, the other from the controller to the ZOH. In [67, 68], the time delay $\tau(k)$ is between two positive integers, and the ZOH has no logical decision and always uses the latest arrived control input packet. Specifically, $\tau(k + 1) \leq \tau(k) + 1$ is not guaranteed in [67, 68].

7.2 Stabilization of Networked Control Systems

In this section, we present a sufficient condition for the stability analysis and several sufficient conditions for the synthesis of networked control systems. A combined procedure is also suggested for the controller design.

The following theorem provides a sufficient condition of the asymptotic stability of networked control system (7.3), and plays an essential role in the controller design.

Theorem 7.1 Networked control system (7.3) is asymptotically stable, if there exist matrices $P \in \mathbb{S}^{n \times n}$, $Q \in \mathbb{S}^{n \times n}$, $Z \in \mathbb{S}^{n \times n}$, $T_1 \in \mathbb{R}^{n \times n}$ and $T_2 \in \mathbb{R}^{n \times n}$ satisfying

$$
\begin{bmatrix}
\Phi_{11} & \Phi_{12} & T_1 \\
\Phi_{12}^T & \Phi_{22} & T_2 \\
T_1^T & T_2^T & -\frac{1}{\tau_{\text{max}}}Z
\end{bmatrix} < 0
$$

(7.4)

where

$$
\Phi_{11} = A^TPA - P + (\tau_{\text{max}} + 2) Q + \tau_{\text{max}}(A^T - I)Z(A - I) + T_1 + T_1^T,
$$

$$
\Phi_{12} = A^TPBK + \tau_{\text{max}}(A^T - I)ZBK - T_1 + T_2^T,
$$

$$
\Phi_{22} = K^TB^TPBK - Q + \tau_{\text{max}}K^TB^TZBK - T_2 - T_2^T.
$$
Proof: To facilitate the proof, we define the following symbols:

\[ x_k \triangleq \begin{bmatrix} x^T(k) & x^T(k-1) & \cdots & x^T(k - \tau_{\text{max}}) \end{bmatrix}^T, \]
\[ \xi(k) \triangleq \begin{bmatrix} x^T(k) & x^T(k - \tau(k)) \end{bmatrix}^T, \]
\[ \zeta(k) \triangleq x(k + 1) - x(k). \]

From system (7.3), we have

\[ \zeta(k) = (A - I)x(k) + BKx(k - \tau(k)), \]
\[ \sum_{h=k-\tau(k)}^{k-1} \zeta(h) = x(k) - x(k - \tau(k)). \]

Now take the Lyapunov functional as

\[ V(x_k, k) = V_1(x_k, k) + V_2(x_k, k) + V_3(x_k, k) \]

where

\[ V_1(x_k, k) = x^T(k)Px(k), \]
\[ V_2(x_k, k) = \sum_{l=k-\tau(k)}^{k-1} x^T(l)Qx(l) + \sum_{l=-\tau_{\text{max}}+1}^{0} \sum_{h=k-1+l}^{k-1} x^T(h)Qx(h), \]
\[ V_3(x_k, k) = \sum_{l=-\tau_{\text{max}}+1}^{0} \sum_{h=k-1+l}^{k-1} \zeta^T(h)Z\zeta(h). \]

Then

\[ V_1(x_{k+1}, k + 1) - V_1(x_k, k) \]
\[ = x^T(k + 1)Px(k + 1) - x^T(k)Px(k) \]
\[ = x^T(k)\left( A^TPA - P \right)x(k) + 2x^T(k)A^TPBKx(k - \tau(k)) \]
\[ + x^T(k - \tau(k))K^TB^TPBKx(k - \tau(k)) \]
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Moreover,

\[ \xi^T(k) \begin{bmatrix} A^T PA - P & A^T PBK \\ K^T B^T PA & K^T B^T PBK \end{bmatrix} \xi(k). \]

Furthermore,

\[
V_2(x_{k+1}, k + 1) - V_2(x_k, k) \\
= \sum_{l=k+1-\tau(k+1)}^k x^T(l)Qx(l) - \sum_{l=\tau(k)}^{k-1} x^T(l)Qx(l) \\
+ \sum_{l=-\tau_{\text{max}}+1}^0 \left( \sum_{h=k+1}^l x^T(h)Qx(h) - \sum_{h=k-1+l}^{k-1} x^T(h)Qx(h) \right) \\
= x^T(k)Qx(k) + \sum_{l=k+1-\tau(k+1)}^{k-1} x^T(l)Qx(l) - \sum_{l=k-\tau_{\text{max}}}^{k-1} x^T(l)Qx(l) \\
+ \sum_{l=-\tau_{\text{max}}+1}^0 \left( x^T(k)Qx(k) - x^T(k - 1 + l)Qx(k - 1 + l) \right) \\
= (\tau_{\text{max}} + 1) x^T(k)Qx(k) - \sum_{l=k-\tau(k)}^{k-1} x^T(l)Qx(l) \\
+ \sum_{l=k+1-\tau(k+1)}^{k-1} x^T(l)Qx(l) - \sum_{l=k-\tau_{\text{max}}}^{k-1} x^T(l)Qx(l) \\
\leq (\tau_{\text{max}} + 1) x^T(k)Qx(k) - \sum_{l=k-\tau(k)}^{k-1} x^T(l)Qx(l) \\
\leq (\tau_{\text{max}} + 2) x^T(k)Qx(k) - x^T(k - \tau(k))Qx(k - \tau(k)).
\]

Moreover,

\[
V_3(x_{k+1}, k + 1) - V_3(x_k, k) \\
= \sum_{l=-\tau_{\text{max}}+1}^0 \left( \sum_{h=k+1}^l \xi^T(h)Z\xi(h) - \sum_{h=k-1+l}^{k-1} \xi^T(h)Z\xi(h) \right) \\
= \sum_{l=-\tau_{\text{max}}+1}^0 \left( \xi^T(k)Z\xi(k) - \xi^T(k - 1 + l)Z\xi(k - 1 + l) \right) \\
= \tau_{\text{max}}\xi^T(k)Z\xi(k) - \sum_{l=k-\tau_{\text{max}}}^{k-1} \xi^T(l)Z\xi(l).\]
Noting that for any matrix \( Z \in \mathbb{S}^{n \times n} \) and any matrix \( T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \in \mathbb{R}^{2n \times n} \), we have
\[
\begin{bmatrix} T \, Z^{-1}T^T \\ T^T \end{bmatrix} \geq 0
\]
and hence
\[
0 \leq \sum_{l=k-\tau(k)}^{k-1} \begin{bmatrix} \xi(k) \\ \zeta(l) \end{bmatrix}^T \begin{bmatrix} (T \, Z^{-1}T^T) \, T \\ T^T \, Z \end{bmatrix} \begin{bmatrix} \xi(k) \\ \zeta(l) \end{bmatrix} + \sum_{l=k-\tau(k)}^{k-1} \zeta(l)^T \, Z \, \zeta(l)
\]
for any \( \xi(k), 0 \) if linear matrix inequality (7.4) holds in view of Schur complement.
equivalence. This completes the proof. □

Remark 7.3 If $\tau(k) > 0$ for all $k$, then

$$
\Delta V_2(k) \triangleq V_2(x_{k+1}, k+1) - V_2(x_k, k)
\leq (\tau_{\max} + 1) x^T(k)Qx(k) - x^T(k - \tau(k))Qx(k - \tau(k))
$$

always holds. Therefore, $\tau_{\max} + 2$ in $\Phi_{11}$ of (7.4) can be replaced by $\tau_{\max} + 1$. However, in the ideal case with $\tau(k) = 0$ and $\tau(k + 1) = 1$, we have

$$
\Delta V_2(k) = (\tau_{\max} + 1) x^T(k)Qx(k) - \sum_{l=k-\tau_{\max}}^{k-1} x^T(l)Qx(l)
\leq (\tau_{\max} + 1) x^T(k)Qx(k) - x^T(k - \tau(k))Qx(k - \tau(k))
$$

holds only if

$$
x^T(k)Qx(k) \leq \sum_{l=k-\tau_{\max}}^{k-1} x^T(l)Qx(l).
$$

However, this condition is not always true. Thus we upper bound $\Delta V_2(k)$ by

$$
(\tau_{\max} + 2) x^T(k)Qx(k) - x^T(k - \tau(k))Qx(k - \tau(k)).
$$

Based upon Theorem 7.1, we are ready to present the corresponding procedures for controller design. They can be combined together to balance the computational complexity and the conservatism of the conditions.

Theorem 7.2 Consider system (7.1), there exists a networked controller (7.2) such that networked control system (7.3) is asymptotically stable, if there exist matrices $P \in \mathbb{S}^{n \times n}$, $X \in \mathbb{S}^{n \times n}$, $Q \in \mathbb{S}^{n \times n}$, $Z \in \mathbb{S}^{n \times n}$, $W \in \mathbb{S}^{n \times n}$, $T_1 \in \mathbb{R}^{n \times n}$, $T_2 \in \mathbb{R}^{n \times n}$, and $K \in \mathbb{R}^{m \times n}$
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satisfying the linear matrix inequality

\[
\begin{bmatrix}
\Theta_1 & \bullet & \bullet & \bullet & \bullet \\
-T_1^T + T_2 & -Q - T_2 - T_2^T & \bullet & \bullet & \bullet \\
T_1^T & T_2^T & -\frac{1}{\tau_{\text{max}}}Z & \bullet & \bullet \\
A - I & BK & 0 & -\frac{1}{\tau_{\text{max}}}W & \bullet \\
A & BK & 0 & 0 & -X
\end{bmatrix} < 0 \tag{7.5}
\]

with equality constraints

\[
ZW = I, \quad PX = I \tag{7.6}
\]

where

\[
\Theta_1 = -P + (\tau_{\text{max}} + 2)Q + T_1 + T_1^T.
\]

**Proof:** Note that (7.4) is equivalent to

\[
\begin{bmatrix}
\Theta_1 & \bullet & \bullet & \bullet & \bullet \\
-T_1^T + T_2 & -Q - T_2 - T_2^T & \bullet & \bullet & \bullet \\
T_1^T & T_2^T & -\frac{1}{\tau_{\text{max}}}Z & \bullet & \bullet \\
A - I & BK & 0 & -\frac{1}{\tau_{\text{max}}}Z^{-1} & \bullet \\
A & BK & 0 & 0 & -P^{-1}
\end{bmatrix} < 0. \tag{7.7}
\]

This inequality is further equivalent to (7.5)–(7.6) if we define \(W = Z^{-1}\) and \(X = P^{-1}\). \(\square\)

Although the CCL-type algorithms \[26, 38\] are efficient in solving such problems, in most cases, we may prefer LMI conditions since they can be solved more directly. Next we provide two LMI conditions by imposing some constraints on the matrix variables.

**Theorem 7.3** Consider system (7.1), there exists a networked controller (7.2) such
that networked control system (7.3) is asymptotically stable, if one of the following conditions holds:

(a) there exist matrices $X \in \mathbb{S}^{n \times n}$, $\tilde{Q} \in \mathbb{S}^{n \times n}$, $W \in \mathbb{S}^{n \times n}$, $\tilde{T}_1 \in \mathbb{R}^{n \times n}$, $\tilde{T}_2 \in \mathbb{R}^{n \times n}$, and $Y \in \mathbb{R}^{m \times n}$ satisfying

\[
\begin{bmatrix}
\Theta_2 & \bullet & \bullet & \bullet & \bullet \\
-\tilde{T}_1^T + \tilde{T}_2 & -\tilde{Q} - \tilde{T}_1 - \tilde{T}_2^T & \bullet & \bullet & \bullet \\
\tilde{T}_1^T & \tilde{T}_2^T & \frac{-1}{\tau_{\text{max}}} (2X - W) & \bullet & \bullet \\
(A - I) X & BY & 0 & \frac{-1}{\tau_{\text{max}}} W & \bullet \\
AX & BY & 0 & 0 & -X
\end{bmatrix} < 0 \tag{7.8}
\]

(b) there exist matrices $X \in \mathbb{S}^{n \times n}$, $\tilde{Q} \in \mathbb{S}^{n \times n}$, $\tilde{T}_1 \in \mathbb{R}^{n \times n}$ $\tilde{T}_2 \in \mathbb{R}^{n \times n}$, and $Y \in \mathbb{R}^{m \times n}$ such that

\[
\begin{bmatrix}
\Theta_2 & \bullet & \bullet & \bullet & \bullet \\
-\tilde{T}_1^T + \tilde{T}_2 & -\tilde{Q} - \tilde{T}_1 - \tilde{T}_2^T & \bullet & \bullet & \bullet \\
\tilde{T}_1^T & \tilde{T}_2^T & \frac{-1}{\tau_{\text{max}}} X & \bullet & \bullet \\
(A - I) X & BY & 0 & \frac{-\alpha}{\tau_{\text{max}}} X & \bullet \\
AX & BY & 0 & 0 & -X
\end{bmatrix} < 0 \tag{7.9}
\]

holds for some scalar $\alpha > 0$.

where

$$\Theta_2 = -X + (\tau_{\text{max}} + 2) \tilde{Q} + \tilde{T}_1 + \tilde{T}_1^T.$$ 

In this case, the controller is given by $K = YX^{-1}$.

**Proof:** Pre- and post-multiplying (7.7) by $\text{diag}(P^{-1}, P^{-1}, P^{-1}, I, I)$ and defining $X \triangleq
\[ P^{-1}, Y \triangleq KP^{-1}, W \triangleq Z^{-1}, \tilde{Q} \triangleq P^{-1}QP^{-1}, \tilde{T}_1 \triangleq P^{-1}T_1P^{-1}, \tilde{T}_2 \triangleq P^{-1}T_2P^{-1} \] yields
\[
\begin{bmatrix}
\Theta_2 & \bullet & \bullet & \bullet & \bullet \\
-\tilde{T}_1^T + \tilde{T}_2 - \tilde{Q} - \tilde{T}_2 - \tilde{T}_2^T & \bullet & \bullet & \bullet & \bullet \\
\tilde{T}_1^T & \tilde{T}_2^T & -\frac{1}{\tau_{\text{max}}}XW^{-1}X & \bullet & \bullet & < 0.
\end{bmatrix}
\]

Note that
\[
(X - W)W^{-1}(X - W) \geq 0
\]
implies
\[
2X - W \leq XW^{-1}X.
\]
Therefore, the solvability of linear matrix inequality (7.8) guarantees the solvability of inequality (7.10).

Furthermore, inequality (7.10) becomes linear matrix inequality (7.9) if we constrain \( W = \alpha X \) for some given scalar \( \alpha > 0 \). This completes the proof. \(\square\)

**Remark 7.4** \( W < 2X \) is necessary for inequality (7.8) to hold, while \( W \geq 2X \) is possible for inequality (7.9) if \( \alpha \geq 2 \). Therefore, linear matrix inequality (7.8) and linear matrix inequality (7.9) do not share the same feasible solution set. However, Theorem 7.3 is more stringent than Theorem 7.2.

**Remark 7.5** A design scheme can be devised by combining Theorem 7.2 and Theorem 7.3. First one should try to solve linear matrix inequality (7.8). If a solution cannot be found, try to solve linear matrix inequality (7.9) for some scalars \( \alpha \geq 2 \), then try to solve linear matrix inequality (7.5) with equality constraints (7.6).
7.3 Numerical Examples

In this section, we illustrate the developed theory via several numerical examples and simulations. The continuous-time systems are taken from the recently published papers [49, 59, 70]. Here, we first discretize the continuous-time systems to their corresponding discrete-time versions, then apply the controller design techniques developed in this chapter to construct the networked controllers. Finally we conduct numerical simulations based upon the framework in Figure 7.1 (cf. page 160). For the transmission characteristic of the networks, \(0 \leq \tau(k) \leq 5\) is assumed. The initial control inputs are chosen to zero.

Example 7.1 Consider an unstable batch reactor [59]:

\[
\dot{x}(t) = \begin{bmatrix} 1.38 & -0.2077 & 6.715 & -5.676 \\ -0.5814 & -4.29 & 0 & 0.675 \\ 1.067 & 4.273 & -6.654 & 5.893 \\ 0.048 & 4.273 & 1.343 & -2.104 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 5.679 \\ 1.136 \\ 1.136 \end{bmatrix} u(t).
\]

We discretize the system with \(T_s = 0.005s\), and obtain

\[
x(k + 1) = \begin{bmatrix} 1.0070 & -0.0010 & 0.0330 & -0.0278 \\ -0.0029 & 0.9788 & -0.0000 & 0.0034 \\ 0.0052 & 0.0211 & 0.9675 & 0.0288 \\ 0.0002 & 0.0211 & 0.0066 & 0.9897 \end{bmatrix} x(k) + \begin{bmatrix} 0.0000 & -0.0003 \\ 0.0281 & 0.0000 \\ 0.0060 & -0.0155 \\ 0.0060 & -0.0001 \end{bmatrix} u(k).
\]

Based upon part (a) of Theorem 7.3, a networked controller is

\[
u = \begin{bmatrix} 1.5735 & -3.3250 & 0.2113 & -2.8839 \\ 6.5701 & -0.3592 & 7.0343 & -2.6816 \end{bmatrix} x.
\]

The system state trajectory of the closed-loop system is plotted in Figure 7.3, and Figure 7.4 shows the distribution of the virtual time delay \(\tau(k)\) over the state space.
The initial system state is given by $x_0 = \begin{bmatrix} -5 & 0 & 5 \end{bmatrix}^T$. One solution set of part (a) of Theorem 7.3 is as follows:

$$
X = \begin{bmatrix}
5.2766 & 0.7569 & -2.6773 & 1.5957 \\
0.7569 & 7.3046 & -0.8524 & -1.1291 \\
-2.6773 & -0.8524 & 7.4645 & 0.4100 \\
1.5957 & -1.1291 & 0.4100 & 4.5807
\end{bmatrix},
$$

$$
Q = \begin{bmatrix}
0.0174 & 0.0013 & -0.0190 & 0.0086 \\
0.0013 & 0.0786 & -0.0060 & 0.0007 \\
-0.0190 & -0.0060 & 0.0878 & -0.0042 \\
0.0086 & 0.0007 & -0.0042 & 0.0159
\end{bmatrix},
$$

$$
W = \begin{bmatrix}
5.0728 & 0.9217 & -3.2400 & 1.9226 \\
0.9217 & 7.6936 & -1.1983 & -1.1494 \\
-3.2400 & -1.1983 & 8.4172 & 0.0739 \\
1.9226 & -1.1494 & 0.0739 & 4.4884
\end{bmatrix},
$$

$$
T_1 = \begin{bmatrix}
-0.4553 & -0.0178 & -0.0180 & -0.0538 \\
-0.0465 & -0.7310 & -0.0502 & -0.0244 \\
0.0244 & -0.0522 & -0.6505 & -0.0853 \\
-0.0557 & -0.0561 & -0.0650 & -0.4602
\end{bmatrix},
$$

$$
T_2 = \begin{bmatrix}
0.4681 & 0.0118 & 0.0114 & 0.0627 \\
0.0467 & 0.7297 & 0.0288 & -0.0041 \\
-0.0769 & 0.0330 & 0.6701 & 0.0683 \\
0.0709 & 0.0510 & 0.0570 & 0.4711
\end{bmatrix},
$$

$$
Y = \begin{bmatrix}
0.6185 & -20.0205 & -0.9843 & -6.8586 \\
11.2838 & -0.6187 & 34.1239 & 1.4900
\end{bmatrix}.
$$

**Example 7.2** Consider the linearized state-space model of the motion about the up-
ward unstable equilibrium position of a pendulum [49]:

\[
\dot{x}(t) = \begin{bmatrix}
0 & 1 & 0 & 0 \\
63.25 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
-33.31 & 0 & 0 & 0 \\
\end{bmatrix} x(t) + \begin{bmatrix}
0 \\
-520.72 \\
0 \\
804.13 \\
\end{bmatrix} u(t).
\]

The discretized system with \( T_s = 0.005 \) s is

\[
x(k + 1) = \begin{bmatrix}
1.0008 & 0.0050 & 0.0000 & 0.0000 \\
0.3163 & 1.0008 & 0.0000 & 0.0000 \\
-0.0004 & -0.0000 & 1.0000 & 0.0050 \\
-0.1666 & -0.0004 & 0.0000 & 1.0000 \\
\end{bmatrix} x(k) + \begin{bmatrix}
-0.0065 \\
-2.6043 \\
0.0101 \\
4.0210 \\
\end{bmatrix} u(k).
\]

Part (a) of Theorem 7.3 provides a networked controller

\[
u = \begin{bmatrix}
0.4373 \\
0.0649 \\
0.0055 \\
0.0105 \\
\end{bmatrix} x.
\]

A simulation of the networked system is given in Figure 7.5 and Figure 7.6, and the initial system state is \( x_0 = \begin{bmatrix}
-5 \\
0 \\
5 \\
0 \\
\end{bmatrix}^T \). One solution set of part (a) of Theorem 7.3 is as follows:

\[
X = \begin{bmatrix}
483.5732 & -2675.8968 & -221.9104 & -621.8891 \\
-2675.8968 & 19749.7143 & 587.7457 & -6365.8883 \\
-221.9104 & 587.7457 & 16905.5129 & -6420.8491 \\
-621.8891 & -6365.8883 & -6420.8491 & 34402.8108 \\
\end{bmatrix},
\]

\[
Q = \begin{bmatrix}
2.0442 & -13.2014 & 0.0864 & -0.6973 \\
-13.2014 & 120.9980 & 1.3495 & -62.4291 \\
0.0864 & 1.3495 & 5.1346 & -6.7088 \\
-0.6973 & -62.4291 & -6.7088 & 136.7783 \\
\end{bmatrix},
\]
Example 7.3 Consider the unstable dynamic system in [70]:

\[
\dot{x}(t) = \begin{bmatrix} -1 & 0 & -0.5 \\ 1 & -0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t).
\]

When the sampling period is chosen as \(T_s = 0.2\) s, the system is discretized as

\[
x(k+1) = \begin{bmatrix} 0.8187 & 0.0000 & -0.0955 \\ 0.1722 & 0.9048 & -0.0094 \\ 0.0000 & 0.0000 & 1.1052 \end{bmatrix} x(k) + \begin{bmatrix} -0.0097 \\ -0.0006 \\ 0.2103 \end{bmatrix} u(k).
\]

Theorem 7.2 gives a networked controller

\[
u = \begin{bmatrix} 0.0604 & 0.0283 & -0.8349 \end{bmatrix} x.
\]
A simulation is given in Figure 7.7 and Figure 7.8, and the initial system state is \( x_0 = \begin{bmatrix} -5 & 0 & 5 \end{bmatrix}^T \).

In the simulations, \( \tau(k+1) \) is first generated uniformly distributed over \{0, 1, \ldots, 5\}, then compared to \( \tau(k) + 1 \). If \( \tau(k+1) \geq \tau(k) + 1 \), we let \( \tau(k+1) := \tau(k) + 1 \) to force the ZOH to use the newest control input data. This is also the reason why the percentage of \( \tau(k) = 5 \) is very small. Thus, the ZOH acts as a filter which modifies the probability distribution of the network delays in the examples.

### 7.4 Summary

The stabilization problem of networked control systems has been studied in this chapter. A logic ZOH has been used so that the continuous-time process could be discretized to a discrete-time system with input delay. Several sufficient conditions for the stabilization of the discretized system have been addressed based upon the Lyapunov theory. Finally, numerical examples and simulations have been used to illustrate the developed theory.
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Figure 7.3 Initial condition response of the closed-loop system

Figure 7.4 Time delay probability distribution
Figure 7.5  Initial condition response of the closed-loop system

Figure 7.6  Time delay probability distribution
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Figure 7.7  Initial condition response of the closed-loop system

Figure 7.8  Time delay probability distribution
Conclusions and Future Work

This chapter draws conclusions on the thesis, and points out some possible research directions related to the work done in this thesis.

8.1 Conclusions

The focus of the thesis has been placed on various synthesis methods of dynamic systems with Markovian characteristics. Specifically, three research problems have been investigated in detail.

1. Robust control and robust filtering problems of uncertain Markovian jump linear systems have been studied. In this case, the uncertainties exist both in the system matrices and in the mode transition rate matrix. New matrix bounding techniques have been used to handle the uncertainties in the mode transition rate matrix. On the one hand, the thesis has demonstrated that the uncertainties in the mode transition rate matrix can (a) destabilize the system; (b) degrade the $H_2$ or $H_{\infty}$ performance of the closed-loop system; (c) deteriorate the $H_{\infty}$ performance of the filter. On the other hand, the probability constraints on the rows of the mode transition rate matrix have been utilized to reduce the conservatism of the conditions. Sufficient synthesis conditions have been derived based upon the stochastic Lyapunov theory for the robust stabilization, robust $H_2$ control,
robust $H_\infty$ control and fixed-order robust $H_\infty$ filter design of the uncertain system. These conditions are given in terms of coupled linear matrix inequalities with equality constraints, and can be solved by CCL-type algorithms effectively. Finally, comparisons have been used to illustrate the efficiency of the developed theory.

2. The problem of how to achieve efficient control without instantaneous system information has been studied. The system under consideration is a discrete-time Markovian jump linear system. The accessibility of the system state and the system mode are time delayed. The thesis has shown that the delay in the system mode plays an essential role for the control of the system, and should be constant in applications. A sufficient condition has been developed to design the state-feedback control law such that the closed-loop system is stochastically stable. Also, the efficiency of the developed theory has been demonstrated by a numerical example.

3. The stabilization of networked control systems has been studied. Two techniques are proposed to derive the stabilization conditions. The first is to find a new packet-loss dependent Lyapunov function. The network suffers from either arbitrary or Markovian packet loss. With this approach, the networked controller is designed to ensure the stability of the closed-loop system. The other is to configure a special zero-order hold in the system. Not only is the zero-order hold synchronized with the sampler, but also a logic is used by the zero-order hold to choose the newest control information to control the process. Several sufficient conditions for the stabilization of the system have been established based upon the Lyapunov theory. The potential application of the theory has been illustrated by numerical examples and simulations.
8.2 Future Work

Related topics for the future research work are listed below.

1. Although the techniques used to handle the element-wise uncertainties in the mode transition rate matrix are less conservative than existing ones, the obtained conditions are still sufficient conditions only. In the future, it is worth while and challenging either to seek new matrix upper bounding techniques to reduce the conservatism or to find necessary and sufficient conditions.

2. For the control of Markovian jump linear systems with delayed information, the technique developed in this thesis requires the delay in the system mode to be constant. The reason is that if the delay in the system mode is time-varying, the number of the operation modes and the transition probability matrix of the closed-loop system will be time-varying, too. This greatly complicates the analysis of the problem. Therefore, it is worth trying to look for other techniques to allow the system mode delay to be time-varying.

3. For the control of networked control systems, on the one hand, the advanced techniques developed for time-delay systems may be borrowed to reduce the conservatism of the results in this thesis as networked control systems with bounded packet loss and time delay can be discretized to a system with input delay. On the other hand, since the quantization of signals is inevitable in a networked control system design and the thesis considers only the packet-loss issue and time-delay issue, an interesting research direction is to incorporate the signal quantization issue into the framework developed in this thesis.
List of Publications

Most of the results presented in the thesis have been published during my PhD study except those in Chapter 7. Here is a list of publications.

- The main results of Chapter 2 have been published in a journal:

- The results of Section 3.2 has been published in a conference:
  J. Xiong, and J. Lam, “Robust $H_\infty$ control of Markovian jump linear systems with uncertain switching probabilities,” in *The 16th IFAC World Congress Proceedings*, 2005.

- Chapter 4 is based upon the journal paper:

- The results of Chapter 5 have been summarized in the journal paper:
• Chapter 6 is based upon the journal paper:
  J. Xiong, and J. Lam, “Stabilization of linear systems over networks with bounded

• The results of Chapter 7 have been summarized in a paper under review:
  J. Xiong, and J. Lam, “Stabilization of networked control systems with a logic
  ZOH,” 2007 (submitted for publication).
Bibliography


