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Automatica 42 (2006) 747–753
www.elsevier.com/locate/automatica

Brief paper

Stabilization of discrete-time Markovian jump linear systems via time-delayed controllers☆

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Received 16 September 2005; received in revised form 7 November 2005; accepted 11 December 2005
Available online 14 February 2006

Abstract

This paper is concerned with the stabilization problem for a class of discrete-time Markovian jump linear systems with time-delays both in the system state and in the mode signal. The delay in the system state may be time-varying. The delay in the mode signal is manifested as a constant mismatch of the modes between the controller and the system. We first show that the resulting closed-loop system is a time-varying delayed Markovian jump linear system with extended state space. Then a sufficient condition is proposed for the design of a controller such that the closed-loop system is stochastically stable. Finally, numerical simulation is used to illustrate the developed theory.

Keywords: Linear matrix inequalities; Markovian parameters; Stabilization; Time-delay

1. Introduction

Discrete-time Markovian jump linear system (DMJLS) may represent a large class of hybrid systems subject to abrupt changes in structures due to, for instance, random failures of the components, sudden disturbances and variations of the environment, changes of the subsystems interconnections. As a special class of hybrid systems, a DMJLS has finite operation modes, and evolves as a linear system within a fixed mode. The jumps between different modes are governed by a discrete Markov chain. The control issues of DMJLSs have attracted the attention of many researchers. For example, the stability and stabilization problems were investigated by Ji and Chizeck (1990), Costa (1993), and the equivalence of various second moment stability properties was established by Ji, Chizeck, Feng, and Loparo (1991). The linear quadratic control problem was studied by Ji and Chizeck (1990), Abou-Kandil, Freiling, and Jank (1995), and the filtering problem was considered by Costa and Guerra (2002), and Liu, Sun, and Sun (2004). The H_2 and H_∞ control problems were tackled by Costa and Marques (1998), and Seiler and Sengupta (2003). We refer the reader to Costa, Fragoso, and Marques (2005) for more information on DMJLSs.

On the other hand, time-delays are commonly encountered in many physical processes, and frequently a major source of instability and poor performance. The control problem of time-delayed DMJLSs has also received much attention recently. For example, the stochastic stabilization problem for DMJLSs with system state delays has been studied by Cao and Lam (1999), Shi, Boukas, and Agarwal (1999), where the results are delay-independent, and delay-dependent results have been developed by Boukas and Liu (2001), and Chen, Guan, and Yu (2004) as well. The basic idea of the control schemes in Cao and Lam (1999), Shi et al. (1999), Boukas and Liu (2001), and Chen et al. (2004) is to construct a control signal, according to current system mode and current system state, such that the unstable plant is stabilized without the delayed terms and remains stable in the presence of the delayed terms. However, if we consider a more realistic situation as shown in Fig. 1, where the system itself may not be time-delayed, time-delays exist in the channels from system to controller. Such delays often arise from the measurement and the network transmission of the signals. In this case, the overall

☆ This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor George Yin under the direction of Editor Ian Petersen. This work is partially supported by RGC HKU 7028/04P.
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0005-1098/$ - see front matter
doi:10.1016/j.automatica.2005.12.015

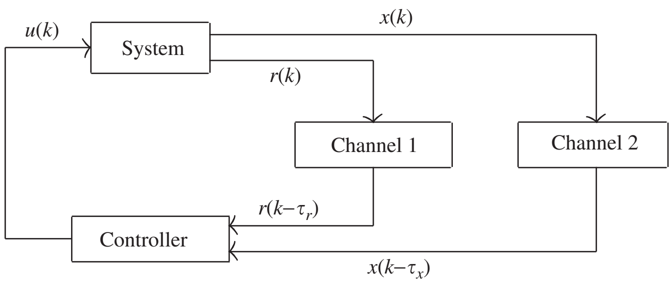

Fig. 1. Control of system with delayed system information.

closed-loop system is a time-delayed system, but the control techniques developed by Cao and Lam (1999), Shi et al. (1999), Boukas and Liu (2001), and Chen et al. (2004) are not applicable because current mode information and current system state are no longer available. Naturally, a new question is how to achieve effective control without instantaneous system information (Richard, 2003). As a result, new control techniques are needed to design the control signal based upon past system information (that is, delayed mode and delayed system state in our case) to stabilize an unstable plant. Such an observation motivates the current research.

This paper studies the stabilization problem for discrete-time Markovian jump linear systems. The objective is to design a state-feedback control law, using the delayed mode signal and the delayed system state, such that the closed-loop system is stochastically stable.

Notation: The notations in this paper are standard. $\mathbb{N}$ and $\mathbb{Z}_+$ denote the set of natural numbers and the set of nonnegative integer numbers, respectively. $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$ and $\mathbb{S}^+$ are, respectively, the n-dimensional Euclidean space, the set of $m \times n$ real matrices, and the set of $n \times n$ real symmetric positive definite matrices. Notation $X \geqslant Y$ (respectively, $X > Y$) where X and Y are real symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix of compatible dimensions. The superscript "T" denotes the transpose for vectors or matrices, and $\lambda_{\min}(\cdot)$ is the minimum of eigenvalues of symmetric matrices. $\|\cdot\|$ refers to the Euclidean norm for vectors and induced 2-norm for matrices. $\delta(\cdot,\cdot)$ stands for the Kronecker delta. For any $x \in \mathbb{R}$, $\lceil x \rceil$ means the integer $n \in \mathbb{N}$ such that $n-1 < x \leqslant n$. Moreover, let $(\Omega, \mathscr{F}, P)$ be a complete probability space. $E(\cdot)$ stands for the mathematical expectation operator.

2. Problem formulation

Consider the following discrete-time Markovian jump linear system defined on a complete probability space $(\Omega, \mathscr{F}, P)$:

$$x(k+1) = A(r(k))x(k) + B(r(k))u(k), \tag{1}$$

where $k \in \mathbb{Z}_+$, $x(k) \in \mathbb{R}^n$ is the system state, and $u(k) \in \mathbb{R}^m$ is the control input. $\{r(k) : k \in \mathbb{Z}_+\}$ is a discrete-time homogeneous Markov chain, takes values in a finite state space $\mathscr{S} \triangleq \{1, 2, \ldots, s\}$ with transition probability matrix $\Pi \triangleq (\pi_{ij})$, where $\pi_{ij} \triangleq \Pr(r(k+1) = j | r(k) = i) \geqslant 0$ for all $i, j \in \mathscr{S}$ and $k \in \mathbb{Z}_+$, and $\sum_{j=1}^{s} \pi_{ij} = 1$ for every $i \in \mathscr{S}$. The matrices $A_i \triangleq A(r(k) = i)$ and $B_i \triangleq B(r(k) = i)$, $i \in \mathscr{S}$, are constant matrices of appropriate dimensions.

Now consider a time-delayed, mode-dependent, state-feedback control law

$$u(k) = K(r(k-\tau_r))x(k-\tau_x(k)), \tag{2}$$

where $\tau_r \in \mathbb{N}$ is a constant delay occurring in the mode signal $r(k)$. $\tau_x(k) \in \mathbb{N}$, the delay in the system state $x(k)$, may be time-varying, and satisfies $\tau_{\min} \leqslant \tau_x(k) \leqslant \tau_{\max}$ with $\tau_{\min}, \tau_{\max} \in \mathbb{N}$. Here, τ_r is assumed to be constant and it determines the structure of the closed-loop system. This point will be made clear in the following section. $\phi(k) \in \mathbb{R}^n$, $k = -\tau_{\max}, -\tau_{\max}+1, \ldots, 0$, and $\kappa(k) \in \mathscr{S}$, $k = -\tau_r, -\tau_r+1, \ldots, 0$, are the initial conditions.

Applying controller (2) to the open-loop system (1) results in the closed-loop system

$$x(k+1) = A(r(k))x(k) + B(r(k))K(r(k-\tau_r))x(k-\tau_x(k)). \tag{3}$$

Remark 1. Closed-loop system (3) is no longer a Markovian jump system with respect to $r(k)$ because of the delayed mode signal $r(k-\tau_r)$. However, if we extend the state space, it will be a Markovian jump system.

Remark 2. One difficulty in the control problem considered in this paper compared with others (Boukas & Liu, 2001; Cao & Lam, 1999; Chen et al., 2004; Shi et al., 1999) is that the state matrices of (3) are not affected by the introduction of controller (2). While in Cao and Lam (1999), Shi et al. (1999), Boukas and Liu (2001), and Chen et al. (2004) the state-feedback controller of the form $u(k) = K(r(k))x(k)$, which changes the state matrices of the closed-loop system, is used to stabilize the system and also to tolerate the time-delayed term, for instance, $A_d(r(k))x(k-d)$ in Shi et al. (1999).

We have the following stochastic stability concept for system (3).

Definition 3 (*Shi et al., 1999*). Let $x(k; \phi(\cdot), \kappa(\cdot))$ be the trajectory of the state of closed-loop system (3). The closed-loop system in (3) is said to be stochastically stable if

$$E\left(\sum_{k=0}^{\infty} \|x(k; \phi(\cdot), \kappa(\cdot))\|^2 \mid \phi(\cdot), \kappa(\cdot)\right) < \infty$$

for every initial condition $\phi(k) \in \mathbb{R}^n$, $k = -\tau_{\max}, -\tau_{\max}+1, \ldots, 0$, and $\kappa(k) \in \mathscr{S}$, $k = -\tau_r, -\tau_r+1, \ldots, 0$.

In this paper, our attention is directed at designing a controller of form (2) such that the closed-loop system (3) is stochastically stable given a possibly unstable system (1).

3. Stabilization

In this section, we first show that closed-loop system (3) is a time-varying delayed Markovian jump linear system with s^{τ_r+1}

operation modes. Then a sufficient condition is established to verify the stability of such systems in terms of coupled LMIs. Based on the analysis result, a controller design technique is also proposed.

Lemma 4. *Closed-loop system* (3) *is a time-varying delayed Markovian jump linear system with* s^{τ_r+1} *modes.*

Proof. Given $\tau_r \in \mathbb{N}$, we define two finite sets $\mathscr{S}^{\tau_r+1} \triangleq \underbrace{\mathscr{S} \times \mathscr{S} \times \cdots \times \mathscr{S}}_{\tau_r+1 \text{ times}}$ and $\mathscr{S}_{\tau_r+1} \triangleq \{1, 2, \ldots, s^{\tau_r+1}\}$, and introduce a mapping $\psi : \mathscr{S}^{\tau_r+1} \to \mathscr{S}_{\tau_r+1}$ with

$$\psi(\theta) \triangleq i + (i_{-1} - 1)s + \cdots + (i_{-\tau_r+1} - 1)s^{\tau_r-1} + (i_{-\tau_r} - 1)s^{\tau_r},$$

where $\theta=[i \ i_{-1} \ i_{-2} \ \cdots \ i_{-\tau_r}]^{\mathrm{T}} \in \mathscr{S}^{\tau_r+1}$ and $i, i_{-1}, \ldots, i_{-\tau_r} \in \mathscr{S}$. For every element $v \in \mathscr{S}_{\tau_r+1}$, an unique element $\theta \in \mathscr{S}^{\tau_r+1}$, satisfying $\psi(\theta) = v$, can be determined by the following procedure: let $i_{-\tau_r} := \lceil v/s^{\tau_r} \rceil$ and $v_{-\tau_r+1} := v - (i_{-\tau_r} - 1)s^{\tau_r}$, next let $i_{-\tau_r+1} := \lceil v_{-\tau_r+1}/s^{\tau_r-1} \rceil$ and $v_{-\tau_r+2} := v_{-\tau_r+1} - (i_{-\tau_r+1} - 1)s^{\tau_r-1}, \ldots$, let $i_{-1} := \lceil v_{-1}/s \rceil$ and $v_0 := v_{-1} - (i_{-1} - 1)s$, finally, let $i := v_0$. Therefore, the mapping $\psi(\cdot)$ is a bijection from $\mathscr{S}^{\tau_r+1}$ to $\mathscr{S}_{\tau_r+1}$.

Now define a vector-valued random variable

$$\tilde{r}(k) \triangleq [r(k) \ \ r(k-1) \ \ \cdots \ \ r(k-\tau_r)]^{\mathrm{T}}.$$

Then closed-loop system (3) can be written as

$$x(k+1) = A(\tilde{r}(k))x(k) + B(\tilde{r}(k))K(\tilde{r}(k))x(k - \tau_x(k)), \quad (4)$$

where $A(\tilde{r}(k)) = A(r(k))$, $B(\tilde{r}(k)) = B(r(k))$ and $K(\tilde{r}(k)) = K(r(k - \tau_r))$. Note that the vector-valued stochastic process $\{\tilde{r}(k), k \in \mathbb{Z}_+\}$, taking values in $\mathscr{S}^{\tau_r+1}$, is a discrete-time vector-valued Markov chain since $\{r(k), k \in \mathbb{Z}_+\}$ is a Markov chain. Therefore, system (4) is a time-varying delayed Markovian jump linear system with s^{τ_r+1} operation modes, and so is closed-loop system (3). At time k, we say that jump system (3) is in mode $v = \psi(\tilde{r}(k)) \in \mathscr{S}_{\tau_r+1}$.

In the following, we construct the extended transition probability matrix $\tilde{\Pi} \triangleq (\tilde{\pi}_{v\eta})$ for closed-loop system (3) from the matrix Π. For any two elements $v, \eta \in \mathscr{S}_{\tau_r+1}$, because $\psi(\cdot)$ is bijective, we can uniquely obtain two vectors

$$\tilde{v} \triangleq \psi^{-1}(v) = [i \ \ i_{-1} \ \ \cdots \ \ i_{-\tau_r}]^{\mathrm{T}},$$

$$\tilde{\eta} \triangleq \psi^{-1}(\eta) = [j \ \ j_{-1} \ \ \cdots \ \ j_{-\tau_r}]^{\mathrm{T}}.$$

Then

$$\begin{aligned}\tilde{\pi}_{v\eta} &= \Pr(\tilde{r}(k+1) = \tilde{\eta} | \tilde{r}(k) = \tilde{v}) \\ &= \Pr(r(k+1) = j, r(k) = j_{-1}, \ldots, \\ &\qquad r(k-\tau_r+1) = j_{-\tau_r} | r(k) = i, r(k-1) = i_{-1}, \\ &\qquad \ldots, r(k-\tau_r+1) = i_{-\tau_r+1}, r(k-\tau_r) = i_{-\tau_r}) \\ &= \pi_{ij}\delta(i, j_{-1})\delta(i_{-1}, j_{-2}) \cdots \delta(i_{-\tau_r+1}, j_{-\tau_r}).\end{aligned}$$

This completes the proof. □

Remark 5. The state space matrices of closed-loop system (3) can be easily determined by the operation mode. Given any mode $v \in \mathscr{S}_{\tau_r+1}$ at time k, we have $\tilde{r}(k) = \psi^{-1}(v) = [i \ i_{-1} \ \cdots \ i_{-\tau_r}]^{\mathrm{T}}$. Then $A(\tilde{r}(k)) = A_i$, $B(\tilde{r}(k)) = B_i$ and $K(\tilde{r}(k)) = K_{i_{-\tau_r}}$. For example, if $s = 2$, $\tau_r = 1$, the relationship among them is established in Table 1.

Table 1
Relationship between matrices and modes

v	$\tilde{r}(k)$	$A(\tilde{r}(k))$	$B(\tilde{r}(k))$	$K(\tilde{r}(k))$
1	$\begin{bmatrix}1\\1\end{bmatrix}$	A_1	B_1	K_1
2	$\begin{bmatrix}2\\1\end{bmatrix}$	A_2	B_2	K_1
3	$\begin{bmatrix}1\\2\end{bmatrix}$	A_1	B_1	K_2
4	$\begin{bmatrix}2\\2\end{bmatrix}$	A_2	B_2	K_2

Remark 6. To give the reader an idea of the form of the extended transition probability matrix $\tilde{\Pi}$, consider the case where $s = 2$ and $\tau_r = 1$, we have

$$\tilde{\Pi} = \begin{bmatrix} \pi_{11} & \pi_{12} & 0 & 0 \\ 0 & 0 & \pi_{21} & \pi_{22} \\ \pi_{11} & \pi_{12} & 0 & 0 \\ 0 & 0 & \pi_{21} & \pi_{22} \end{bmatrix}.$$

If $s = 2$ and $\tau_r = 2$,

$$\tilde{\Pi} = \left[\begin{array}{cccc|cccc} \pi_{11} & \pi_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi_{21} & \pi_{22} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \pi_{11} & \pi_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pi_{21} & \pi_{22} \\ \hline \pi_{11} & \pi_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \pi_{21} & \pi_{22} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \pi_{11} & \pi_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \pi_{21} & \pi_{22} \end{array}\right].$$

Remark 7. Different delay τ_r results in different closed-loop system (3) in the sense that closed-loop system (3) will have different number of operation modes and different mode transition probability matrices.

With Lemma 4 and Definition 3, we are now ready to analyze the stability and stabilization problems for discrete-time Markovian jump linear system (1) controlled by a time-delayed state-feedback controller (2).

Theorem 8. *Closed-loop system* (3) *is stochastically stable if there exist matrices* $P_v \in \mathbb{S}^+$, $Q \in \mathbb{S}^+$, $Z \in \mathbb{S}^+$, $Y_{1v} \in \mathbb{R}^{n\times n}$, $Y_{2v} \in \mathbb{R}^{n\times n}$, $v \in \mathscr{S}_{\tau_r+1}$, *satisfying the coupled LMIs*

$$\begin{bmatrix} \Xi_{v11} & \Xi_{v12} & Y_{1v} \\ \Xi_{v12}^{\mathrm{T}} & \Xi_{v22} & Y_{2v} \\ Y_{1v}^{\mathrm{T}} & Y_{2v}^{\mathrm{T}} & -\frac{1}{\tau_{\max}} Z \end{bmatrix} < 0 \quad (5)$$

for all $v \in \mathscr{S}_{\tau_r+1}$, *where*

$$\Xi_{v11} = A_i^{\mathrm{T}}\left(\sum_{j=1}^{s}\pi_{ij}P_{\mu+j}\right)A_i - P_v + Y_{1v} + Y_{1v}^{\mathrm{T}} + (\tau_{\max}-\tau_{\min}+1)Q + \tau_{\max}(A_i - I)^{\mathrm{T}}Z(A_i - I),$$

$$\Xi_{v12} = A_i^{\mathrm{T}}\left(\sum_{j=1}^{s}\pi_{ij}P_{\mu+j}\right)B_iK_{i_{-\tau_r}} - Y_{1v} + Y_{2v}^{\mathrm{T}} + \tau_{\max}(A_i - I)^{\mathrm{T}}ZB_iK_{i_{-\tau_r}},$$

$$\Xi_{v22} = K_{i_{-\tau_r}}^{\mathrm{T}}B_i^{\mathrm{T}}\left(\sum_{j=1}^{s}\pi_{ij}P_{\mu+j}\right)B_iK_{i_{-\tau_r}} - Y_{2v} - Y_{2v}^{\mathrm{T}} - Q + \tau_{\max}K_{i_{-\tau_r}}^{\mathrm{T}}B_i^{\mathrm{T}}ZB_iK_{i_{-\tau_r}}$$

and $[i\ \ i_{-1}\ \ \cdots\ \ i_{-\tau_r}]^{\mathrm{T}} = \psi^{-1}(v)$ *and* $\mu = (i-1)s + (i_{-1}-1)s^2 + \cdots + (i_{-\tau_r+2}-1)s^{\tau_r-1} + (i_{-\tau_r+1}-1)s^{\tau_r}$.

Proof. We define

$$y(k) \triangleq x(k+1) - x(k)$$

$$x_k \triangleq [x^{\mathrm{T}}(k)\ \ x^{\mathrm{T}}(k-1)\ \ \cdots\ \ x^{\mathrm{T}}(k-\tau_{\max})]^{\mathrm{T}}$$

and adopt the Lyapunov functional

$$V(x_k, \tilde{r}(k), k) \triangleq V_1(x_k, \tilde{r}(k), k) + V_2(x_k, \tilde{r}(k), k) + V_3(x_k, \tilde{r}(k), k) + V_4(x_k, \tilde{r}(k), k)$$

with

$$V_1(x_k, \tilde{r}(k), k) = x^{\mathrm{T}}(k)P(\tilde{r}(k))x(k),$$

$$V_2(x_k, \tilde{r}(k), k) = \sum_{l=k-\tau_x(k)}^{k-1} x^{\mathrm{T}}(l)Qx(l),$$

$$V_3(x_k, \tilde{r}(k), k) = \sum_{h=-\tau_{\max}+2}^{-\tau_{\min}+1}\ \sum_{l=k-1+h}^{k-1} x^{\mathrm{T}}(l)Qx(l),$$

$$V_4(x_k, \tilde{r}(k), k) = \sum_{h=-\tau_{\max}+1}^{0}\ \sum_{l=k-1+h}^{k-1} y^{\mathrm{T}}(l)Zy(l).$$

For simplicity, let the mode at time k be $v \in \mathscr{S}_{\tau_r+1}$, that is,

$$v = \psi(\tilde{r}(k)) = i + (i_{-1}-1)s + \cdots + (i_{-\tau_r}-1)s^{\tau_r}$$

and $P(\tilde{r}(k)) = P_v$. Hence matrices $A(\tilde{r}(k)) = A_i$, $B(\tilde{r}(k)) = B_i$, $K(\tilde{r}(k)) = K_{i_{-\tau_r}}$. Then at time $k+1$, the system may jump to any mode $\eta \in \mathscr{S}_{\tau_r+1}$, that is,

$$\eta = \psi(\tilde{r}(k+1)) = j + (j_{-1}-1)s + \cdots + (j_{-\tau_r}-1)s^{\tau_r}.$$

From the proof of Lemma 4, we have

$$\tilde{\pi}_{v\eta} = \pi_{ij}\delta(i, j_{-1})\delta(i_{-1}, j_{-2})\cdots\delta(i_{-\tau_r+1}, j_{-\tau_r}).$$

Hence, $\tilde{\pi}_{v\eta} = \pi_{ij}$ if $\eta = \mu + j$, otherwise $\tilde{\pi}_{v\eta} = 0$. As a result, we have $\sum_{\eta=1}^{s^{\tau_r+1}}\tilde{\pi}_{v\eta}P_\eta = \sum_{j=1}^{s}\pi_{ij}P_{\mu+j}$.

Then,

$$\begin{aligned} &E(V_1(x_{k+1}, \tilde{r}(k+1), k+1) \mid x_k, \tilde{r}(k), k) - V_1(x_k, \tilde{r}(k), k) \\ &\quad = E(x^{\mathrm{T}}(k+1)P(\tilde{r}(k+1))x(k+1) \mid x_k, \tilde{r}(k), k) - x^{\mathrm{T}}(k)P(\tilde{r}(k))x(k) \\ &\quad = \xi^{\mathrm{T}}(k)\begin{bmatrix} A_i^{\mathrm{T}}\left(\sum_{j=1}^{s}\pi_{ij}P_{\mu+j}\right)A_i - P_v & A_i^{\mathrm{T}}\left(\sum_{j=1}^{s}\pi_{ij}P_{\mu+j}\right)B_iK_{i_{-\tau_r}} \\ K_{i_{-\tau_r}}^{\mathrm{T}}B_i^{\mathrm{T}}\left(\sum_{j=1}^{s}\pi_{ij}P_{\mu+j}\right)A_i & K_{i_{-\tau_r}}^{\mathrm{T}}B_i^{\mathrm{T}}\left(\sum_{j=1}^{s}\pi_{ij}P_{\mu+j}\right)B_iK_{i_{-\tau_r}} \end{bmatrix}\xi(k), \end{aligned}$$

where $\xi(k) = [x^{\mathrm{T}}(k)\ \ x^{\mathrm{T}}(k-\tau_x(k))]^{\mathrm{T}}$, and

$$\begin{aligned} &E(V_2(x_{k+1}, \tilde{r}(k+1), k+1) \mid x_k, \tilde{r}(k), k) - V_2(x_k, \tilde{r}(k), k) \\ &\quad = \sum_{l=k+1-\tau_x(k+1)}^{k} x^{\mathrm{T}}(l)Qx(l) - \sum_{l=k-\tau_x(k)}^{k-1} x^{\mathrm{T}}(l)Qx(l) \\ &\quad = x^{\mathrm{T}}(k)Qx(k) - x^{\mathrm{T}}(k-\tau_x(k))Qx(k-\tau_x(k)) \\ &\qquad + \sum_{l=k+1-\tau_x(k+1)}^{k-\tau_x(k)} x^{\mathrm{T}}(l)Qx(l) \end{aligned}$$

and

$$\begin{aligned} &E(V_3(x_{k+1}, \tilde{r}(k+1), k+1) \mid x_k, \tilde{r}(k), k) - V_3(x_k, \tilde{r}(k), k) \\ &\quad = \sum_{h=-\tau_{\max}+2}^{-\tau_{\min}+1}\left[\sum_{l=k+h}^{k} x^{\mathrm{T}}(l)Qx(l) - \sum_{l=k-1+h}^{k-1} x^{\mathrm{T}}(l)Qx(l)\right] \\ &\quad = (\tau_{\max}-\tau_{\min})x^{\mathrm{T}}(k)Qx(k) - \sum_{l=k+1-\tau_{\max}}^{k-\tau_{\min}} x^{\mathrm{T}}(l)Qx(l). \end{aligned}$$

Since $\tau_{\min} \leqslant \tau_x(k) \leqslant \tau_{\max}$, we have

$$\sum_{l=k+1-\tau_x(k+1)}^{k-\tau_x(k)} x^{\mathrm{T}}(l)Qx(l) - \sum_{l=k+1-\tau_{\max}}^{k-\tau_{\min}} x^{\mathrm{T}}(l)Qx(l) \leqslant 0.$$

Hence,

$$\begin{aligned} &E(V_2(x_{k+1}, \tilde{r}(k+1), k+1) + V_3(x_{k+1}, \tilde{r}(k+1), k+1) \mid \\ &\quad x_k, \tilde{r}(k), k) - (V_2(x_k, \tilde{r}(k), k) + V_3(x_k, \tilde{r}(k), k)) \\ &\qquad \leqslant \xi^{\mathrm{T}}(k)\begin{bmatrix} (\tau_{\max}-\tau_{\min}+1)Q & 0 \\ 0 & -Q \end{bmatrix}\xi(k). \end{aligned}$$

Moreover, at time k, we have $y(k)=(A_i-I)x(k)+B_iK_{i_{-\tau_r}}x(k-\tau_x(k))$, so

$$E(V_4(x_{k+1},\tilde{r}(k+1),k+1)\mid x_k,\tilde{r}(k),k)-V_4(x_k,\tilde{r}(k),k)=\sum_{h=-\tau_{\max}+1}^{0}\left(\sum_{l=k+h}^{k}y^{\mathrm T}(l)Zy(l)-\sum_{l=k-1+h}^{k-1}y^{\mathrm T}(l)Zy(l)\right)$$

$$=\sum_{h=-\tau_{\max}+1}^{0}\left[y^{\mathrm T}(k)Zy(k)-y^{\mathrm T}(k-1+h)Zy(k-1+h)\right]=\tau_{\max}y^{\mathrm T}(k)Zy(k)-\sum_{l=k-\tau_{\max}}^{k-1}y^{\mathrm T}(l)Zy(l)$$

$$=\tau_{\max}\xi^{\mathrm T}(k)\begin{bmatrix}(A_i-I)^{\mathrm T}Z(A_i-I) & (A_i-I)^{\mathrm T}ZB_iK_{i_{-\tau_r}}\\ K_{i_{-\tau_r}}^{\mathrm T}B_i^{\mathrm T}Z(A_i-I) & K_{i_{-\tau_r}}^{\mathrm T}B_i^{\mathrm T}ZB_iK_{i_{-\tau_r}}\end{bmatrix}\xi(k)-\sum_{l=k-\tau_{\max}}^{k-1}y^{\mathrm T}(l)Zy(l).$$

Also note that for any matrices $X_{11v}=X_{11v}^{\mathrm T}\in\mathbb{R}^{n\times n}$, $X_{12v}\in\mathbb{R}^{n\times n}$, $X_{22v}=X_{22v}^{\mathrm T}\in\mathbb{R}^{n\times n}$ and $Y_{1v}\in\mathbb{R}^{n\times n}$, $Y_{2v}\in\mathbb{R}^{n\times n}$ satisfying

$$\begin{bmatrix}X_v & Y_v\\ Y_v^{\mathrm T} & Z\end{bmatrix}\geqslant 0,$$

where

$$X_v=\begin{bmatrix}X_{11v} & X_{12v}\\ X_{12v}^{\mathrm T} & X_{22v}\end{bmatrix}\quad\text{and}\quad Y_v=\begin{bmatrix}Y_{1v}\\ Y_{2v}\end{bmatrix},$$

we have the inequality

$$0\leqslant\sum_{l=k-\tau_x(k)}^{k-1}\begin{bmatrix}\xi(k)\\ y(l)\end{bmatrix}^{\mathrm T}\begin{bmatrix}X_v & Y_v\\ Y_v^{\mathrm T} & Z\end{bmatrix}\begin{bmatrix}\xi(k)\\ y(l)\end{bmatrix}$$
$$=\tau_x(k)\xi^{\mathrm T}(k)X_v\xi(k)+2\xi^{\mathrm T}(k)Y_v\sum_{l=k-\tau_x(k)}^{k-1}y(l)$$
$$+\sum_{l=k-\tau_x(k)}^{k-1}y^{\mathrm T}(l)Zy(l)$$
$$\leqslant\xi^{\mathrm T}(k)\left(\begin{bmatrix}Y_{1v}+Y_{1v}^{\mathrm T} & -Y_{1v}+Y_{2v}^{\mathrm T}\\ -Y_{1v}^{\mathrm T}+Y_{2v} & -Y_{2v}-Y_{2v}^{\mathrm T}\end{bmatrix}+\tau_{\max}X_v\right)\xi(k)$$
$$+\sum_{l=k-\tau_{\max}}^{k-1}y^{\mathrm T}(l)Zy(l)$$
$$\triangleq\Gamma_v.$$

Therefore,

$$E(V(x_{k+1},\tilde{r}(k+1),k+1)\mid x_k,\tilde{r}(k),k)-V(x_k,\tilde{r}(k),k)$$
$$\leqslant E(V(x_{k+1},\tilde{r}(k+1),k+1)\mid x_k,\tilde{r}(k),k)$$
$$-V(x_k,\tilde{r}(k),k)+\Gamma_v$$
$$\leqslant\xi^{\mathrm T}(k)(\hat{\Xi}_v+\tau_{\max}X_v)\xi(k),$$

where

$$\hat{\Xi}_v=\begin{bmatrix}\Xi_{v11} & \Xi_{v12}\\ \Xi_{v12}^{\mathrm T} & \Xi_{v22}\end{bmatrix}.$$

Hence, if $\hat{\Xi}_v+\tau_{\max}X_v<0$ and $\begin{bmatrix}X_v & Y_v\\ Y_v^{\mathrm T} & Z\end{bmatrix}\geqslant 0$, then

$$E\,(V(x_{k+1},\tilde{r}(k+1),k+1)\mid x_k,\tilde{r}(k),k)$$
$$-V(x_k,\tilde{r}(k),k)\leqslant-\delta\|x(k)\|^2<0$$

for all $x(k)\neq 0$, where

$$\delta=\inf_{v\in\mathscr{S}_{\tau_r+1}}\left(\lambda_{\min}(-\hat{\Xi}_v-\tau_{\max}X_v)\right)>0.$$

Following a similar line as in the proof of Theorem 1 in Boukas and Liu (2001), it can be shown that

$$\lim_{N\to\infty}E\left(\sum_{k=0}^{N}\|x(k;\phi(\cdot),\kappa(\cdot))\|^2\mid\phi(\cdot),\kappa(\cdot)\right)<\infty$$

that is, closed-loop system (3) is stochastically stable.

Finally, note that there exist $X_v=X_v^{\mathrm T}$ and Y_v such that $\hat{\Xi}_v+\tau_{\max}X_v<0$ and

$$\begin{bmatrix}X_v & Y_v\\ Y_v^{\mathrm T} & Z\end{bmatrix}\geqslant 0$$

if and only if there exists Y_v such that $\hat{\Xi}_v+\tau_{\max}Y_vZ^{-1}Y_v^{\mathrm T}<0$. Moreover, the last inequality is further equivalent to (5) in view of Schur complement equivalence. This completes the proof. □

Theorem 8 can be used to check the stochastic stability of closed-loop system (3), as for the design of controller (2), we have the following theorem.

Theorem 9. *Consider Markovian jump linear system* (1), *there exists a state-feedback control law* (2) *such that closed-loop system* (3) *is stochastically stable if there exist matrices* $P_v\in\mathbb{S}^+$, $R_v\in\mathbb{S}^+$, $Q\in\mathbb{S}^+$, $Z\in\mathbb{S}^+$, $W\in\mathbb{S}^+$, $Y_{1v}\in\mathbb{R}^{n\times n}$, $Y_{2v}\in\mathbb{R}^{n\times n}$, $v\in\mathscr{S}_{\tau_r+1}$, *and* $K_{i_{-\tau_r}}\in\mathbb{R}^{m\times n}$, $i_{-\tau_r}\in\mathscr{S}$, *satisfying the coupled LMIs*

$$\begin{bmatrix}\Theta_{v11} & -Y_{1v}+Y_{2v}^{\mathrm T} & Y_{1v} & A_i^{\mathrm T}-I & M_{1i}\\ -Y_{1v}^{\mathrm T}+Y_{2v} & -Y_{2v}-Y_{2v}^{\mathrm T}-Q & Y_{2v} & K_{i_{-\tau_r}}^{\mathrm T}B_i^{\mathrm T} & M_{2i}\\ Y_{1v}^{\mathrm T} & Y_{2v}^{\mathrm T} & -\dfrac{1}{\tau_{\max}}Z & 0 & 0\\ A_i-I & B_iK_{i_{-\tau_r}} & 0 & -\dfrac{1}{\tau_{\max}}W & 0\\ M_{1i}^{\mathrm T} & M_{2i}^{\mathrm T} & 0 & 0 & -\Lambda_v\end{bmatrix}<0 \tag{6}$$

with equality constraints

$$ZW=I,\qquad P_vR_v=I \tag{7}$$

for all $v \in \mathscr{S}_{\tau_r+1}$, *where*

$$\Theta_{v11} = -P_v + Y_{1v} + Y_{1v}^{\mathrm{T}} + (\tau_{\max} - \tau_{\min} + 1)Q,$$

$$M_{1i} = [\sqrt{\pi_{i1}}A_i^{\mathrm{T}} \quad \sqrt{\pi_{i2}}A_i^{\mathrm{T}} \quad \cdots \quad \sqrt{\pi_{is}}A_i^{\mathrm{T}}],$$

$$M_{2i} = [\sqrt{\pi_{i1}}K_{i_{-\tau_r}}^{\mathrm{T}}B_i^{\mathrm{T}} \quad \cdots \quad \sqrt{\pi_{is}}K_{i_{-\tau_r}}^{\mathrm{T}}B_i^{\mathrm{T}}],$$

$$\Lambda_v = \mathrm{diag}(R_{\mu+1}, R_{\mu+2}, \ldots, R_{\mu+s})$$

and $i, i_{-\tau_r}, \mu$ *are as defined as in Theorem* 8.

Proof. Note that LMIs (5) is equivalent to $\hat{\Xi}_v + \tau_{\max} Y_v Z^{-1} Y_v^{\mathrm{T}} < 0$, which is equivalent to (6) and (7) by defining $W \triangleq Z^{-1}$, $R_v \triangleq P_v^{-1}$ and in view of Schur complement equivalence. This completes the proof. □

Remark 10. Theorem 9 gives a delay-dependent condition for finding a time-delayed mode-dependent controller such that the closed-loop system is stochastically stable.

Remark 11. Although the solution set of (6) and (7) is not convex, the cone complementarity linearization type (CCL-type) algorithms (Ghaoui, Oustry, & Rami, 1997; Leibfritz, 2001) can be employed to solve such problems effectively. For an application of these algorithms to the controller design techniques of Markovian jump systems, we refer readers to Xiong, Lam, Gao, and Ho (2005) for details, where an algorithm is presented explicitly, and applies to our case similarly.

4. Numerical example

In this section, we present a numerical simulation to show the application of the developed theory. Let us consider a Markovian jump system with two operation modes, system mode delay $\tau_r = 1$ and system state delay $1 \leqslant \tau_x(k) \leqslant 3$. The system data of (1) are as follows:

$$A_1 = \begin{bmatrix} 1.5 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.6 & 0 \\ 0.1 & 1.2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 0.1 & 0.9 \\ 0.5 & 0.5 \end{bmatrix}$$

and $\phi(k) = [\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}]$, $k = -3, \ldots, 0$, and $\kappa(k) = 1$, $k = -1, 0$, are the initial conditions. This system with $u(k) \equiv 0$ is not stochastically stable (see Fig. 2).

Applying Theorem 9 and an algorithm similar to those in Leibfritz (2001) and Xiong et al. (2005), we obtain a controller

$$K_1 = [-0.0630 \quad -0.1481], \quad K_2 = [-0.0574 \quad -0.1393]$$

which stochastically stabilizes the system where time-delays occur both in the system mode and in the system state (see Fig. 3).

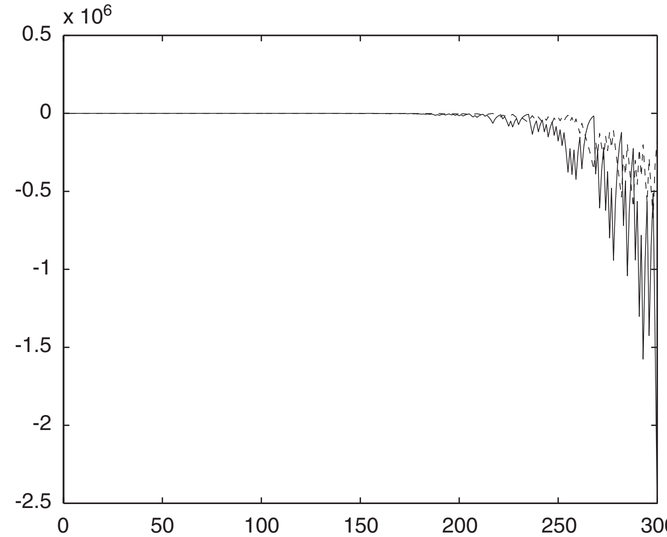

Fig. 2. Open-loop system (x_1(solid), x_2(dashed)).

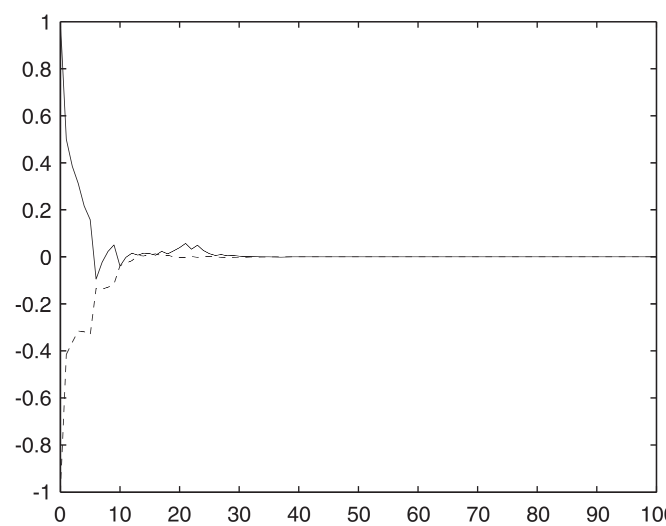

Fig. 3. Closed-loop system (x_1(solid), x_2(dashed)).

5. Conclusions

This paper studied the stabilization problem for discrete-time Markovian jump linear systems by a time-delayed state-feedback controller. A sufficient condition for the design of the controller is given in terms of coupled linear matrix inequalities with equality constraints. Such a non-convex problem can be solved by existing optimization algorithms effectively. A numerical example demonstrated the developed theory.

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