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Brief paper

Stabilization of linear systems over networks with bounded packet loss☆

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Abstract

This paper is concerned with the stabilization problem of networked control systems where the main focus is the packet-loss issue. Two types of packet-loss processes are considered. One is the arbitrary packet-loss process, the other is the Markovian packet-loss process. The stability conditions of networked control systems with both arbitrary and Markovian packet losses are established via a packet-loss dependent Lyapunov approach. The corresponding stabilizing controller design techniques are also given based upon the stability conditions. These results are also extended to the unit time delay case. Finally, the numerical example and simulations have demonstrated the usefulness of the developed theory.

Keywords: Linear systems; Networked control systems; Packet losses; Stabilization; Time delays

1. Introduction

Networked control systems (NCSs) are feedback control systems with control loops closed via digital communication channels. Compared with the traditional point-to-point wiring, the use of the communication channels can reduce the costs of cables and power, simplify the installation and maintenance of the whole system, and increase the reliability. NCSs have many industrial applications in automobiles, manufacturing plants, aircrafts, and HVAC systems (Walsh & Ye, 2001). However, the insertion of the communication channels creates discrepancies between the data records to be transmitted and their associated remotely transmitted images, and hence raises new interesting and challenging problems such as quantization, time delays, and packet losses. As a result, conventional control theories must be re-evaluated before applying to NCSs. Recently, NCSs have been a hot research topic and a wealth of literature have appeared. For example, the discussions of the signal quantization (Brockett & Liberzon, 2000; Liberzon, 2003), time delays (Nilsson, Bernhardsson, & Wittenmark, 1998; Yue, Han, & Lam, 2005; Zhang, Branicky, & Phillips, 2001; Zhang, Shi, Chen, & Huang, 2005), packet losses (Azimi-Sadjadi, 2003; Seiler & Sengupta, 2005; Yu, Wang, Chu, & Xie, 2004; Zhang et al., 2001), and limited information issue (Shi & Murray, 2005; Tatikonda & Mitter, 2004), scheduling (Walsh & Ye, 2001) were presented and some useful results were reported. In general, there are two major approaches to accommodate these issues in an NCS design. One way is that one first designs the control system without regard to the networks, and then determines a performance level that the networks should satisfy (for example, maximum allowable transfer interval) so that the closed-loop system maintains its performance (for example, stability) when some control and sensor signals are transmitted via the networks (Nešić & Teel, 2004; Zhang et al., 2001). The other approach is to treat the network protocol and traffic as given conditions and design the control strategies that explicitly take the network-induced issues into account (Azimi-Sadjadi, 2003; Seiler & Sengupta, 2005; Yu et al., 2004).

Packet loss is one of the most important and special issues of NCSs. Some results have been available. Under the assumption that the network is modeled as a switch governed by a Bernoulli process, Zhang et al. (2001) proposed a criterion to check whether the NCS is stable at a certain rate of packet losses, and searched for the maximum packet-loss rate under

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which the overall system remains stable. The method they used derives from the stability analysis for asynchronous dynamic systems. With packet-loss rate known and constant, Seiler and Sengupta (2005) formulated the NCS as a Markovian jump system with two operation modes, then applied the techniques developed for Markovian jump systems. A dynamic output feedback controller design method was proposed such that the NCS is mean square stable and has H_∞ gain below certain value in terms of linear matrix inequalities (LMIs). Moreover, Yu et al. (2004) modeled the packet-loss process as an arbitrary but finite switching signal. This enables them to apply the theory from switched systems to stabilize the NCS. However, in the framework considered in the references mentioned above, the controller is directly connected to the actuator. That means no packets are dropped in control signals. A general framework was considered in Azimi-Sadjadi (2003), where both sampling signals and control commands are transmitted through the network and may be dropped during the transmissions. The linear quadratic Gaussian control problem was studied based on dynamic programming approach.

In this paper, we consider the stabilization problem of NCSs under a general framework. The packet-loss process, defined as the sequence of the time intervals between consecutively successfully transmitted data, is categorized into two types. One type is called the arbitrary packet-loss process which takes values in a finite set arbitrarily. This model is similar to the one considered in Yu et al. (2004). However, there are at least two differences between this paper and (Yu et al., 2004). The first is that our framework is more general. Our framework accommodates both-side networks (one between sampler and controller, and the other between controller and actuator), while Yu et al. (2004) allows the network from sampler to controller only. The second is that our results can be reduced to the results in Yu et al. (2004). The packet-loss dependent Lyapunov function is used here to establish the stabilization conditions, while Yu et al. (2004) adopted a common Lyapunov function and the results are quadratic. The other type is called the Markovian packet-loss process which is modeled as a discrete-time Markov chain with known transition probability matrix. This model takes the Bernoulli model (Seiler & Sengupta, 2005) as a special case, and enables us to take full advantage of the well-developed theory for Markovian jump systems to tackle our problem.

The organization of the paper is as follows. Section 2 describes the NCS framework and two models of the packet-loss process. The stability of NCSs is analyzed in Section 3, and Section 4 tackles the stabilization problem. We extend the results to the case when NCSs suffer from both packet losses and unit time delays in Section 5. Numerical example and simulations are presented in Section 6. Finally, Section 7 concludes the paper.

Notation: $\mathbb{Z}_+$ denotes the set of nonnegative integers. $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$ and $\mathbb{S}^+$ denote, respectively, the n-dimensional Euclidean space, the set of $m \times n$ real matrices, and the set of $n\times n$ real symmetric positive definite matrices. Notation $X \geqslant Y$ (respectively, $X > Y$) where X and Y are real symmetric matrices, means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix of compatible dimensions. The superscript "T" denotes the transpose for vectors or matrices. $\|\cdot\|$ refers to the Euclidean norm for vectors and induced 2-norm for matrices. $\mathrm{E}(\cdot)$ stands for the mathematical expectation operator. "$*$" is used to complete symmetric matrices as in

$$\begin{bmatrix} M_1 & * \\ M_2^{\mathrm{T}} & M_3 \end{bmatrix} = \begin{bmatrix} M_1 & M_2 \\ M_2^{\mathrm{T}} & M_3 \end{bmatrix}.$$

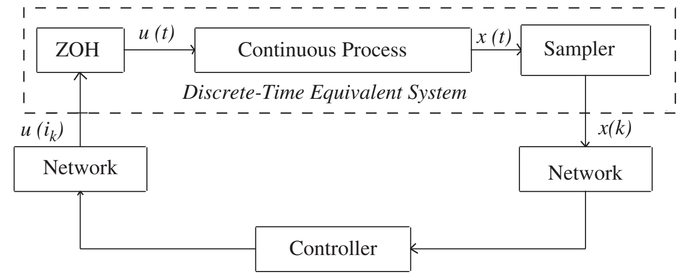

Fig. 1. Networked control systems with packet-losses.

2. Problem formulation

The framework of NCSs considered in the paper is depicted in Fig. 1. The process to be controlled is modeled by a linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k), \tag{1}$$

where $k \in \mathbb{Z}_+$ is the time step, $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$ are the system state and control input, respectively. $x_0 \triangleq x(0)$ is the initial state. A and B are two constant matrices of appropriate dimensions. Networks exist between sampler and controller, and between controller and zero-order hold. The sampler is clock driven, the controller and zero-order hold are event driven and the data are transmitted in a single packet at each time step.

Let $\mathscr{I} \triangleq \{i_1, i_2, \ldots\}$, a subsequence of $\{1, 2, 3, \ldots\}$, denote the sequence of time points of successful data transmissions from the sampler to the zero-order hold, and $s \triangleq \max_{i_k \in \mathscr{I}} (i_{k+1} - i_k)$ be the maximum packet-loss upper bound. Then the following concept and mathematical models are introduced to capture the nature of packet losses.

Definition 1. Packet-loss process is defined as

$$\{\eta(i_k) \triangleq i_{k+1} - i_k : i_k \in \mathscr{I}\} \tag{2}$$

which takes values in the finite state space $\mathscr{S} \triangleq \{1, 2, \ldots, s\}$.

Definition 2. Packet-loss process (2) is said to be arbitrary if it takes values in $\mathscr{S}$ arbitrarily.

Definition 3. Packet-loss process (2) is said to be Markovian if it is a discrete-time homogeneous Markov chain on a complete probability space $(\Omega, \mathscr{F}, \mathrm{P})$, and takes values in $\mathscr{S}$ with known transition probability matrix $\Pi \triangleq (\pi_{ij}) \in \mathbb{R}^{s\times s}$, where

$$\pi_{ij} \triangleq \Pr(\eta(i_{k+1}) = j \mid \eta(i_k) = i) \geqslant 0$$

for all $i, j \in \mathscr{S}$, and $\sum_{j=1}^{s} \pi_{ij} = 1$ for each $i \in \mathscr{S}$.

Remark 4. When $\Pi = \begin{bmatrix} p & 1-p \\ p & 1-p \end{bmatrix}$ with $0 \leqslant p \leqslant 1$, the two-state Markov process is reduced to a Bernoulli process (Seiler & Sengupta, 2005).

The networked controller is a state-feedback controller

$$u = Kx, \tag{3}$$

where $K \in \mathbb{R}^{m\times n}$ is to be designed. From the viewpoint of the zero-order hold, the control input is

$$u(l) = u(i_k) = Kx(i_k)$$

for $i_k \leqslant l \leqslant i_{k+1} - 1$. The initial inputs are set to zeros: $u(l) = 0$, $0 \leqslant l \leqslant i_1 - 1$. Hence the closed-loop system becomes

$$x(l+1) = Ax(l) + BKx(i_k) \tag{4}$$

for $i_k \leqslant l \leqslant i_{k+1} - 1$, $i_k \in \mathscr{I}$. The objective of this paper is to construct controller (3) so that NCS (4) is stable.

3. Stability of NCS

In this section, we analyze the stability property of NCSs. For NCSs with the arbitrary packet-loss process, a sufficient condition is derived by adopting a packet-loss dependent Lyapunov function approach. For NCSs with the Markovian packet-loss process, a necessary and sufficient condition is established by using the theory from Markovian jump systems. The conditions are given in terms of LMIs.

3.1. Arbitrary packet-loss stability

Definition 5. Let $x(l; x_0)$ be the trajectory of NCS (4) with initial state x_0. Then NCS (4) with arbitrary packet-loss process (2) is said to be stable if for any $\varepsilon > 0$ there exists a $\delta \triangleq \delta(\varepsilon) > 0$ such that $\|x_0\| < \delta$ implies $\|x(l; x_0)\| < \varepsilon$ for $l \in \mathbb{Z}_+$. Furthermore, it is said to be asymptotically stable if it is stable and $\lim_{l\to\infty} \|x(l; x_0)\|^2 = 0$ for any initial state $x_0 \in \mathbb{R}^n$.

Theorem 6. *NCS* (4) *with arbitrary packet-loss process* (2) *is asymptotically stable if there exist matrices $P_i \in \mathbb{S}^+$, $i \in \mathscr{S}$, such that*

$$(A^j + B_jK)^{\mathrm{T}} P_j (A^j + B_jK) - P_i < 0 \tag{5}$$

holds for all $i, j \in \mathscr{S}$, where $B_j = \sum_{r=0}^{j-1} A^r B$.

Proof. From system (4), we have

$$x(i_{k+1}) = \left[A^{\eta(i_k)} + \sum_{r=0}^{\eta(i_k)-1} A^r BK \right] x(i_k), \quad i_k \in \mathscr{I}. \tag{6}$$

The initial state of system (6) is $x(i_1) = A^{i_1} x_0$. Now take the packet-loss dependent Lyapunov function as

$$V(l) \triangleq x^{\mathrm{T}}(l) P_{(l-i_k)} x(l) \tag{7}$$

for $i_k + 1 \leqslant l \leqslant i_{k+1}$, $i_k \in \mathscr{I}$, and let $i \triangleq \eta(i_{k-1})$, $j \triangleq \eta(i_k)$, we have

$$V(i_k) = x^{\mathrm{T}}(i_k) P_{(i_k - i_{k-1})} x(i_k) = x^{\mathrm{T}}(i_k) P_i x(i_k),$$

$$V(i_{k+1}) = x^{\mathrm{T}}(i_k)(A^j + B_jK)^{\mathrm{T}} P_j (A^j + B_jK) x(i_k).$$

Therefore, $V(i_{k+1}) - V(i_k) < 0$ for any $x(i_k) \neq 0$ if inequality (5) holds. Hence, $\lim_{i_k\to\infty} V(i_k) = 0$.

Now consider the system state $x(l)$ for $i_k + 1 \leqslant l \leqslant i_{k+1}$, we have

$$x(l) = (A^h + B_hK)x(i_k)$$

where $h = l - i_k \in \mathscr{S}$ and $B_h = \sum_{r=0}^{h-1} A^r B$. Hence $V(l) - V(i_k) = x^{\mathrm{T}}(i_k)[(A^h + B_hK)^{\mathrm{T}} P_h (A^h + B_hK) - P_i] x(i_k) < 0$ for any $x(i_k) \neq 0$ in view of (5). That is, $V(l) < V(i_k)$ for $i_k + 1 \leqslant l \leqslant i_{k+1}$. Therefore, $\lim_{l\to\infty} V(l) = 0$ since $\lim_{i_k\to\infty} V(i_k) = 0$, and $\lim_{l\to\infty} \|x(l; x_0)\|^2 = 0$. That is, the sequence $\{x(l) : l \in \mathbb{Z}_+\}$ converges to zero.

To prove NCS (4) is stable, let $\beta_1 \triangleq \max\{\max_{h\in\mathscr{S}} \|A^h\|^2, 1\}$, $\beta_2 \triangleq \max_{h\in\mathscr{S}} \|P_h\|$, $\beta_3 \triangleq \min_{h\in\mathscr{S}} \{1/\|P_h^{-1}\|\}$, and $\beta \triangleq \min\{\sqrt{1/\beta_1}, \sqrt{\beta_3/(\beta_1\beta_2)}\}$. Then given any $\varepsilon > 0$, we prove that $\|x_0\| < \beta\varepsilon$ implies $\|x(l; x_0)\| < \varepsilon$ for $l \in \mathbb{Z}_+$ in the following. For $0 \leqslant l \leqslant i_1$, since the initial inputs are zeros, we have $x(l) = A^l x_0$, so $\|x(l; x_0)\| \leqslant \sqrt{\beta_1}\|x_0\| < \sqrt{\beta_1}\beta\varepsilon \leqslant \varepsilon$. For $l > i_1$, we have $\beta_3 \|x(l; x_0)\|^2 < V(l)$ according to the definition of Lyapunov function, and $V(l) < V(i_1) \leqslant \beta_2 \|x(i_1)\|^2 \leqslant \beta_1\beta_2\|x_0\|^2$ from the proof above. So $\|x(l; x_0)\| < \sqrt{(\beta_1\beta_2)/\beta_3}\|x_0\| < \sqrt{(\beta_1\beta_2)/\beta_3}\beta\varepsilon \leqslant \varepsilon$. Thus, we conclude that $\|x(l; x_0)\| < \varepsilon$ for all $l \in \mathbb{Z}_+$ if $\|x_0\| < \delta$ with $\delta = \beta\varepsilon$. According to Definition 5, NCS (4) is asymptotically stable. □

Remark 7. In the proof of Theorem 6, $V(l+1) < V(l)$ is not necessarily true for all $l \in \mathbb{Z}_+$.

3.2. Markovian packet-loss stability

Definition 8. NCS (4) with Markovian packet-loss process (2) is said to be mean square stable if $\lim_{l\to\infty} \mathrm{E}(\|x(l; x_0)\|^2) = 0$ for any initial state $x_0 \in \mathbb{R}^n$.

Theorem 9. *NCS* (4) *with Markovian packet-loss process* (2) *is mean square stable if, and only if, there exist matrices $P_i \in \mathbb{S}^+$, $i \in \mathscr{S}$, such that*

$$\sum_{j=1}^{s} [\pi_{ij}(A^j + B_jK)^{\mathrm{T}} P_j (A^j + B_jK)] - P_i < 0 \tag{8}$$

holds for all $i \in \mathscr{S}$, where B_j is defined as in Theorem 6.

Proof. (*Sufficiency*) Because $\{\eta(i_k) : i_k \in \mathscr{I}\}$ is a discrete-time homogeneous Markov chain, system (6) is in fact a discrete-time Markovian jump linear system with s operation modes (Costa, 1993; Ji, Chizeck, Feng, & Loparo, 1991). Consider the same Lyapunov function as in (7), we have

$$\begin{aligned}&\mathrm{E}(V(i_{k+1})\mid \eta(i_{k-1})=i)-V(i_k)\\&\quad=x^{\mathrm{T}}(i_k)\left(\sum_{j=1}^{s}[\pi_{ij}(A^j+B_jK)^{\mathrm{T}}P_j(A^j+B_jK)]-P_i\right)\\&\qquad\times x(i_k)<0\end{aligned}$$

for any $x(i_k)\neq 0$ if inequality (8) holds. Hence, $\lim_{i_k\to\infty}\mathrm{E}(V(i_k))=0$ and $\lim_{i_k\to\infty}\mathrm{E}(\|x(i_k;x(i_1))\|^2)=0$. That is, system (6) is mean square stable.

Note that two consecutive successful control inputs arrive at the zero-order hold at i_k and i_{k+1}, respectively. This means no new data is available for $i_k+1\leqslant l\leqslant i_{k+1}-1$. Therefore, we have

$$V(l)=x^{\mathrm{T}}(i_k)[(A^h+B_hK)^{\mathrm{T}}P_h(A^h+B_hK)]x(i_k)$$

for $i_k+1\leqslant l\leqslant i_{k+1}$, where $h=l-i_k$. Now let $\alpha_1\triangleq\max_{h\in\mathscr{S}}\|(A^h+B_hK)^{\mathrm{T}}P_h(A^h+B_hK)\|$ and $\alpha\triangleq\alpha_1/\beta_3>0$, where β_3 was defined in the proof of Theorem 6. Then $V(l)\leqslant\alpha V(i_k)$ for $i_k+1\leqslant l\leqslant i_{k+1}$. Therefore, $\lim_{l\to\infty}\mathrm{E}(V(l))=0$, so $\lim_{l\to\infty}\mathrm{E}(\|x(l;x_0)\|^2)=0$. This completes the sufficient part of the proof.

(*Necessity*) Suppose that NCS (4) is mean square stable, then system (6) must be mean square stable. Because system (6) is a Markovian jump system, according to the stability results from Markovian jump systems, there exist matrices $P_i\in\mathbb{S}^+$, $i\in\mathscr{S}$, such that (8) holds. This completes the whole proof. □

Remark 10. Inequality (8) can be written as $\sum_{j=1}^{s}\pi_{ij}[(A^j+B_jK)^{\mathrm{T}}P_j(A^j+B_jK)-P_i]<0$ because of $\sum_{j=1}^{s}\pi_{ij}=1$. Therefore, as expected, the arbitrary packet-loss stability implies the Markovian packet-loss stability.

4. Stabilization of NCS

With the stability results developed in Section 3, the controller design techniques are provided in this section.

Theorem 11. *Consider discrete-time system* (1), *there exists a state-feedback controller* (3), *over network with arbitrary packet-loss process* (2), *such that NCS* (4) *is asymptotically stable if there exist matrices $X_i\in\mathbb{S}^+$, $i\in\mathscr{S}$, $G\in\mathbb{R}^{n\times n}$, and $Y\in\mathbb{R}^{m\times n}$, satisfying the coupled LMIs*

$$\begin{bmatrix}-G-G^{\mathrm{T}}+X_i & (A^jG+B_jY)^{\mathrm{T}}\\ A^jG+B_jY & -X_j\end{bmatrix}<0 \tag{9}$$

for all $i,j\in\mathscr{S}$, where B_j is defined as in Theorem 6. *In this case, the controller is given by $K=YG^{-1}$.*

Proof. Pre- and post-multiplying (9) by $[A^j+B_jK, I]$ and its transpose and noting that $Y=KG$, we have

$$(A^j+B_jK)X_i(A^j+B_jK)^{\mathrm{T}}-X_j<0$$

which is equivalent to inequality (5) with $P_i=X_i^{-1}$. □

Remark 12. Theorem 11 contains the quadratic stabilization result as a particular case. If we aggregate to LMIs (9) the additional linear constraints $G=X\in\mathbb{S}^+$ and $X_i=X$, $i\in\mathscr{S}$, then we recover Theorem 1 of Yu et al. (2004) exactly.

The following theorem gives us a sufficient mean square stabilization condition for discrete-time system (1) controlled by (3) over network with Markovian packet-loss process. The proof is similar to that of Theorem 11 and hence omitted.

Theorem 13. *Consider discrete-time system* (1), *there exists a state-feedback controller* (3), *over network with Markovian packet-loss process* (2), *such that NCS* (4) *is mean square stable if there exist matrices $X_i\in\mathbb{S}^+$, $i\in\mathscr{S}$, $G\in\mathbb{R}^{n\times n}$ and $Y\in\mathbb{R}^{m\times n}$, satisfying the coupled LMIs*

$$\begin{bmatrix}-G-G^{\mathrm{T}}+X_i & M_i\\ M_i^{\mathrm{T}} & -\Lambda\end{bmatrix}<0 \tag{10}$$

for all $i\in\mathscr{S}$, where

$$M_i=[\sqrt{\pi_{i1}}(AG+BY)^{\mathrm{T}}\ \cdots\ \sqrt{\pi_{is}}(A^sG+B_sY)^{\mathrm{T}}],$$
$$\Lambda=\mathrm{diag}(X_1,\ldots,X_s)$$

and B_j is defined as in Theorem 6. *In this case, the controller is given by $K=YG^{-1}$.*

Remark 14. By virtue of s being an integer, we can obtain the largest packet-loss upper bound $s_{\max}$ easily.

Remark 15. If system (1) is discretized from a continuous-time system

$$\dot{x}(t)=\Phi x(t)+\Psi u(t),\quad t\in[0,+\infty)$$

with $A=\mathrm{e}^{\Phi T_s}$ and $B=\int_0^{T_s}\mathrm{e}^{\Phi\tau}\Psi\,\mathrm{d}\tau$, where T_s is the sampling period, then given the upper bound s, a simple bisection method may be used to find out the maximal sampling period $T_{s\,\max}$.

5. Network-induced delay

Network-induced time delay is another important issue to be dealt with, and can be modeled as an input delay. So we amend system (1) to

$$x(k+1)=Ax(k)+Bu(k-\tau), \tag{11}$$

where τ is a constant time delay. The trajectory of (11) on time instants $\mathscr{I}$, under the control of (3), is generally a function of the previous ones. Specifically, $x(i_{k+1})$ is a function of $x(i_k), x(i_{k-1}),\ldots,x(i_{k-\tau})$. In contrast, $x(i_{k+1})$ is a function of $x(i_k)$ when $\tau=0$ as shown in (6). To simplify the analysis and synthesis, we consider only the simplest case with $\tau=1$. However, the principle used here remains valid for $\tau>1$.

Let us consider system (11) with $\tau=1$, the closed-loop system is

$$x(l+1)=Ax(l)+BKx(i_k) \tag{12}$$

for $i_k+1\leqslant l\leqslant i_{k+1}$, $i_k\in\mathscr{I}$. So we have

$$x(i_{k+1})=\left[A^{\eta(i_k)}+\sum_{r=0}^{\eta(i_k)-2}A^rBK\right]x(i_k)+A^{\eta(i_k)-1}BKx(i_{k-1}),\quad i_k\in\mathscr{I}. \tag{13}$$

The following theorem provides us a stability result.

Theorem 16. *NCS* (12) *with arbitrary packet-loss process* (2) *is asymptotically stable if there exist matrices* $P_i\in\mathbb{S}^+$, $i\in\mathscr{S}$, $Q\in\mathbb{S}^+$, $Z\in\mathbb{S}^+$, $N_1\in\mathbb{R}^{n\times n}$ *and* $N_2\in\mathbb{R}^{n\times n}$ *such that*

$$\begin{bmatrix}\Xi_{ij1} & \Xi_{ij2} & N_1\\ \Xi_{ij2}^{\mathrm T} & \Xi_{ij3} & N_2\\ N_1^{\mathrm T} & N_2^{\mathrm T} & -Z\end{bmatrix}<0 \tag{14}$$

hold for all $i,j\in\mathscr{S}$, *where*

$$\Xi_{ij1}=R_j^{\mathrm T}P_jR_j-P_i+Q+(R_j^{\mathrm T}-I)Z(R_j-I)+N_1+N_1^{\mathrm T},$$

$$\Xi_{ij2}=R_j^{\mathrm T}P_jA^{j-1}BK+(R_j^{\mathrm T}-I)ZA^{j-1}BK-N_1+N_2^{\mathrm T},$$

$$\Xi_{ij3}=K^{\mathrm T}B^{\mathrm T}(A^{j-1})^{\mathrm T}(P_j+Z)A^{j-1}BK-Q-N_2-N_2^{\mathrm T},$$

$$R_j=A^j+\sum_{r=0}^{j-2}A^rBK.$$

Proof. Consider system (13), we define

$$\zeta(i_k)\triangleq x(i_{k+1})-x(i_k),$$

$$\xi(i_k)\triangleq[x^{\mathrm T}(i_k)\ \ x^{\mathrm T}(i_{k-1})]^{\mathrm T}$$

and let $i\triangleq i_k-i_{k-1}$ and $j\triangleq i_{k+1}-i_k$, so

$$x(i_{k+1})=[R_j\ \ A^{j-1}BK]\xi(i_k).$$

$$\zeta(i_k)=[R_j-I\ \ A^{j-1}BK]\xi(i_k),$$

$$\zeta(i_{k-1})=[I\ \ -I]\xi(i_k).$$

Take the Lyapunov functional as

$$V(l)\triangleq x^{\mathrm T}(l)P_{(l-i_k)}x(l)+x^{\mathrm T}(i_k)Qx(i_k)+\zeta^{\mathrm T}(i_k)Z\zeta(i_k)$$

for $i_k+1\leqslant l\leqslant i_{k+1}$, $i_k\in\mathscr{I}$. Then

$$\begin{aligned}V(i_{k+1})-V(i_k) &=x^{\mathrm T}(i_{k+1})P_jx(i_{k+1})-x^{\mathrm T}(i_k)P_ix(i_k)+x^{\mathrm T}(i_k)Qx(i_k)\\ &\quad -x^{\mathrm T}(i_{k-1})Qx(i_{k-1})+\zeta^{\mathrm T}(i_k)Z\zeta(i_k)\\ &\quad -\zeta^{\mathrm T}(i_{k-1})Z\zeta(i_{k-1}).\end{aligned}$$

Note that for any $Z>0$ and $N=\begin{bmatrix}N_1\\ N_2\end{bmatrix}\in\mathbb{R}^{2n\times n}$, we have $-\zeta^{\mathrm T}(i_{k-1})Z\zeta(i_{k-1})\leqslant\xi^{\mathrm T}(i_k)NZ^{-1}N^{\mathrm T}\xi(i_k)+2\xi^{\mathrm T}(i_k)N\zeta(i_{k-1})$. Therefore,

$$V(i_{k+1})-V(i_k)\leqslant\xi^{\mathrm T}(i_k)(\Xi_{ij}+NZ^{-1}N^{\mathrm T})\xi(i_k)<0$$

for all $x(i_k)\neq 0$ if $\Xi_{ij}+NZ^{-1}N^{\mathrm T}<0$, where $\Xi_{ij}=\begin{bmatrix}\Xi_{ij1} & \Xi_{ij2}\\ \Xi_{ij2}^{\mathrm T} & \Xi_{ij3}\end{bmatrix}$. In view of Schur complement equivalence, this inequality is equivalent to (14). The rest of the proof can be carried out by following similar lines as in the proof of Theorem 6, and hence omitted. □

Theorem 17. *Consider discrete-time system* (11) *with time delay* $\tau=1$, *there exists a state-feedback controller* (3), *over network with arbitrary packet-loss process* (2), *such that NCS* (12) *is asymptotically stable if there exist matrices* $X_i\in\mathbb{S}^+$, $i\in\mathscr{S}$, $\bar Q\in\mathbb{S}^+$, $W\in\mathbb{S}^+$, $G\in\mathbb{R}^{n\times n}$, $\bar N_1\in\mathbb{R}^{n\times n}$, $\bar N_2\in\mathbb{R}^{n\times n}$, *and* $Y\in\mathbb{R}^{m\times n}$, *satisfying the coupled LMIs*

$$\begin{bmatrix}\Upsilon_{1i} & * & * & * & *\\ -\bar N_1^{\mathrm T}+\bar N_2 & \Upsilon_{2i} & * & * & *\\ \bar N_1^{\mathrm T} & \bar N_2^{\mathrm T} & \Upsilon_3 & * & *\\ \Gamma_j-G & A^{j-1}BY & 0 & -W & *\\ \Gamma_j & A^{j-1}BY & 0 & 0 & -X_j\end{bmatrix}<0 \tag{15}$$

for all $i,j\in\mathscr{S}$, *where*

$$\Upsilon_{1i}=X_i-G-G^{\mathrm T}+\bar Q+\bar N_1+\bar N_1^{\mathrm T},$$

$$\Upsilon_{2i}=-\bar Q-\bar N_2-\bar N_2^{\mathrm T},$$

$$\Upsilon_3=W-G-G^{\mathrm T},$$

$$\Gamma_j=A^jG+\sum_{r=0}^{j-2}A^rBY.$$

In this case, the controller is given by $K=YG^{-1}$.

Proof. Inequality (14) is equivalent to

$$\begin{bmatrix}T_{2i} & * & * & * & *\\ -N_1^{\mathrm T}+N_2 & T_3 & * & * & *\\ N_1^{\mathrm T} & N_2^{\mathrm T} & -Z & * & *\\ R_j-I & A^{j-1}BK & 0 & -Z^{-1} & *\\ R_j & A^{j-1}BK & 0 & 0 & -P_j^{-1}\end{bmatrix}<0,$$

where $T_{2i}=-P_i+Q+N_1+N_1^{\mathrm T}$ and $T_3=-Q-N_2-N_2^{\mathrm T}$. Note that $(G^{-1}-P_i)^{\mathrm T}P_i^{-1}(G^{-1}-P_i)\geqslant 0$ implies that $-P_i\leqslant G^{-\mathrm T}P_i^{-1}G^{-1}-G^{-1}-G^{-\mathrm T}\triangleq T_{4i}$ for any nonsingular matrix $G\in\mathbb{R}^{n\times n}$. Similarly, $-Z\leqslant G^{-\mathrm T}Z^{-1}G^{-1}-G^{-1}-G^{-\mathrm T}\triangleq T_5$. Hence the above inequality holds if

$$\begin{bmatrix}T_{4i}+Q+N_1+N_1^{\mathrm T} & * & * & * & *\\ -N_1^{\mathrm T}+N_2 & T_3 & * & * & *\\ N_1^{\mathrm T} & N_2^{\mathrm T} & T_5 & * & *\\ R_j-I & A^{j-1}BK & 0 & -Z^{-1} & *\\ R_j & A^{j-1}BK & 0 & 0 & -P_j^{-1}\end{bmatrix}<0.$$

Pre- and post-multiply this inequality by $\mathrm{diag}(G^{\mathrm T},G^{\mathrm T},G^{\mathrm T},I,I)$ and its transpose, and define $X_i\triangleq P_i^{-1}$, $W\triangleq Z^{-1}$, $\bar Q\triangleq G^{\mathrm T}QG$, $\bar N_1\triangleq G^{\mathrm T}N_1G$, $\bar N_2\triangleq G^{\mathrm T}N_2G$, $Y\triangleq KG$, we have inequality (15). □

Remark 18. The above arguments are applicable to the case when packet-loss process (2) is Markovian as well.

6. Numerical examples

In this section, a numerical example and simulations are used to illustrate the usefulness of the developed synthesis methods. Let us consider the nominal continuous-time system borrowed from Yue et al. (2005) with no disturbance input:

$$\dot{x}(t)=\begin{bmatrix}-1 & 0 & -0.5\\ 1 & -0.5 & 0\\ 0 & 0 & 0.5\end{bmatrix}x(t)+\begin{bmatrix}0\\0\\1\end{bmatrix}u(t).$$

When the plant is sampled with a sampling period $T_s=0.5\,\mathrm{s}$, the discretized system is system (1) with

$$A=\begin{bmatrix}0.6065 & 0 & -0.2258\\ 0.3445 & 0.7788 & -0.0536\\ 0 & 0 & 1.2840\end{bmatrix},$$
$$B=\begin{bmatrix}-0.0582\\ -0.0093\\ 0.5681\end{bmatrix}. \quad (16)$$

Both the continuous-time system and the discretized system are unstable because of $\mathrm{eig}(\Phi)=-0.5,-1,0.5$ and $\mathrm{eig}(A)=0.7788, 0.6065, 1.2840$. Furthermore, we assume that the packet-loss upper bound $s=5$, which means that up to 80% of the packets can be lost during the network transmissions. Applying Theorem 11, we obtain a networked controller $u=[0.0399\ \ 0.0217\ \ -0.8172]x$.

Our simulations are based on the framework in Fig. 1, that is, the designed controller is used to control the continuous-time system over the network, not the discretized correspondence. To simulate, we take the initial state as $x_0=[-5\ \ 0\ \ 5]^{\mathrm{T}}$. Fig. 2 depicts the trajectory of the system state when the packet-loss process is arbitrary. The time instants when the zero-order hold updates its state are indicated with circles on the time axes. We can see that only 18 control inputs arrived at the zero-order hold during the first 25 s, which means that 64% of the packets were lost. Fig. 3 shows an extreme case when no packet loss occurs.

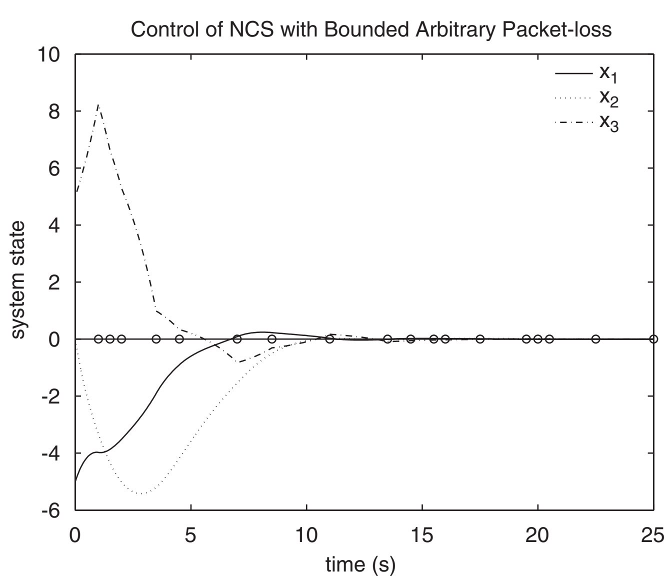

Fig. 2. State response (arbitrary packet losses).

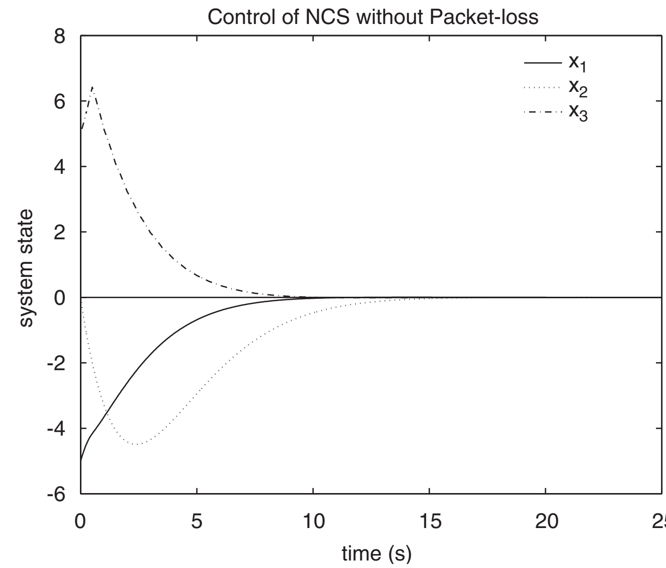

Fig. 3. State response (no packet losses).

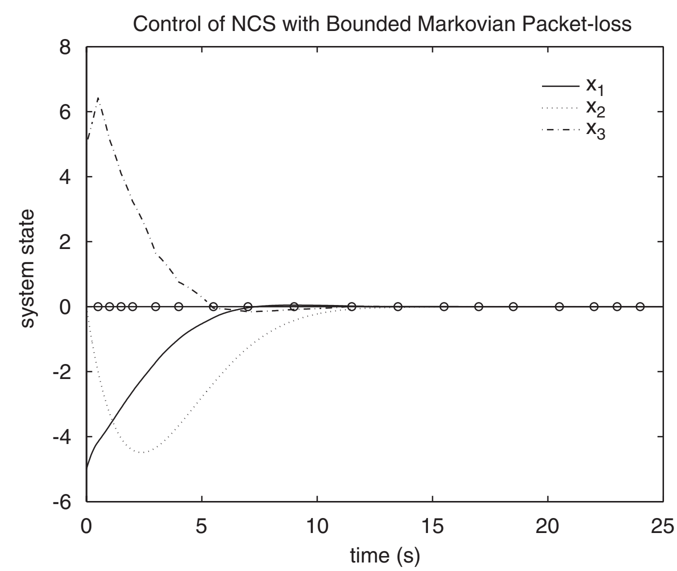

Fig. 4. State response (Markovian packet losses).

In addition, from Remark 10, this controller will also able to stabilize the system when the packet-loss process is governed by a Markov chain. Fig. 4 illustrates this point with the transition probability matrix given by

$$\Pi=\begin{bmatrix}0.5 & 0.2 & 0.1 & 0.1 & 0.1\\ 0.2 & 0.5 & 0.3 & 0 & 0\\ 0 & 0.2 & 0.5 & 0.3 & 0\\ 0 & 0 & 0.2 & 0.5 & 0.3\\ 0.1 & 0.1 & 0.1 & 0.2 & 0.5\end{bmatrix}.$$

Such a Π exhibits the bursty nature of packet losses. The bursty nature is modeled with $\pi_{ii}>\pi_{ij}$, for all $i,j\in\mathscr{S}, j\neq i$, which says that the likelihood of losing a packet after a lost packet transmission is higher than after a successful packet

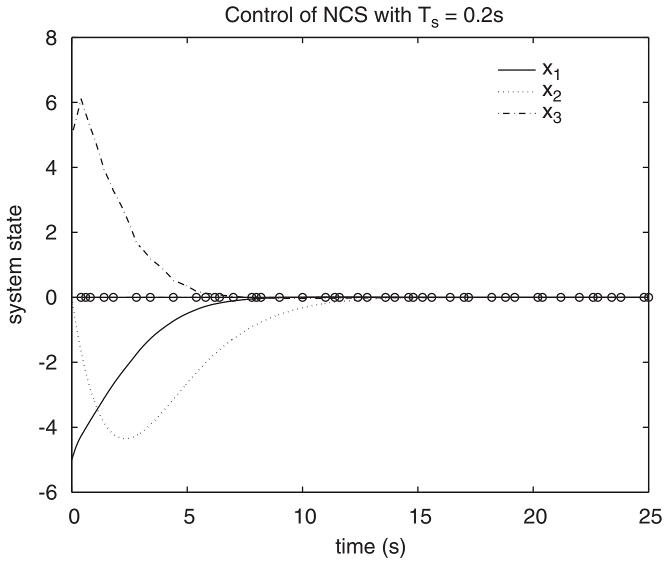

Fig. 5. State response ($T_s = 0.2\,\mathrm{s}$).

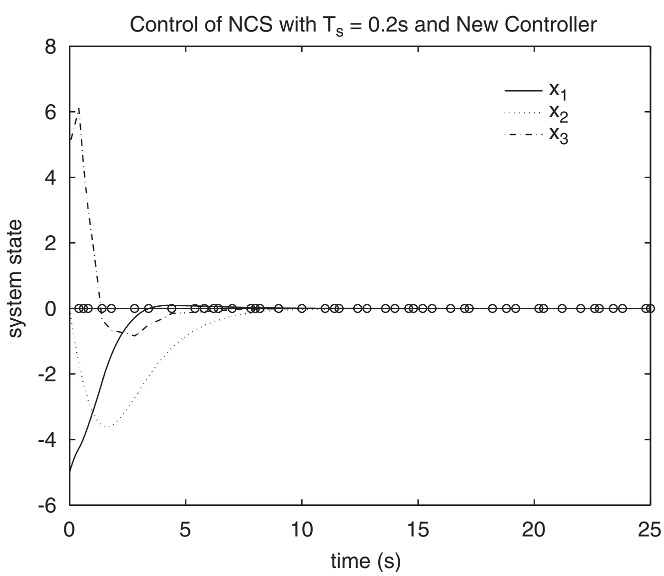

Fig. 6. State response ($T_s = 0.2\,\mathrm{s}$ and new controller).

transmission (Ploplys, Kawka, & Alleyne, 2004; Seiler & Sengupta, 2005).

It should be noticed that our sampling period is larger than the choice in Yue et al. (2005), where $T_s = 0.2\,\mathrm{s}$. Fig. 5 shows the system state trajectory when the sampling period is $T_s = 0.2\,\mathrm{s}$ where the plant is controlled by the controller above. Of course, we could apply Theorem 11 again to get a new controller $u = [0.2701 \;\; 0.1427 \;\; -1.6422]x$. The state trajectory of the plant controlled by this new controller is plotted in Fig. 6. In fact, even with $T_s = 1.5\,\mathrm{s}$, Theorem 11 still gives us a feasible solution. This demonstrates that larger sampling periods are allowable if both the communication channel and the sampling characteristics are considered in the controller design stage.

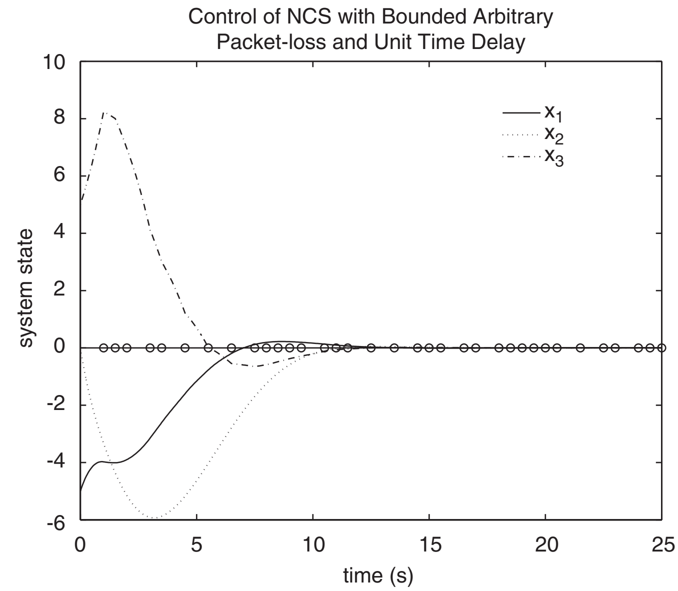

Fig. 7. State response (unit time delay).

Finally, if we consider system (11) with $\tau = 1$ and the coefficient matrices in (16), and suppose $s = 2$, Theorem 17 successfully provides us a controller $u = [-0.0109 \;\; -0.0074 \;\; -0.7176]x$. Fig. 7 is a simulation of the closed-loop system with unit time delay.

7. Conclusions

The stabilization problem of NCSs with bounded packet losses was studied in this paper. Two types of networked controller design methods were proposed. One ensures the networked control system is asymptotically stable in the presence of the arbitrary packet losses. The other ensures the mean square stability in the presence of the Markovian packet losses. The powerful potential of the developed theory was illustrated by a numerical example and simulations.

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