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Brief paper

On lossless negative imaginary systems[☆]

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ABSTRACT

The paper is concerned with the notion of lossless negative imaginary systems and their stabilization using strictly negative imaginary controllers through positive feedback. Firstly, the concept of lossless negative imaginary transfer functions is introduced and some properties of such transfer functions are studied. Secondly, a Lossless Negative Imaginary Lemma is given which establishes conditions on matrices appearing in a minimal state-space realization that are necessary and sufficient for a transfer function to be lossless negative imaginary. Thirdly, a necessary and sufficient condition is provided for the stabilization of a lossless negative imaginary system by a strictly negative imaginary controller. Finally, a flexible structure example is presented to illustrate the theory.

1. Introduction

Positive real systems are systems which have positive real transfer functions. Such systems could model many practical systems which dissipate energy, and hence have many uses in practice. For instance, the use of velocity sensors and force actuators in mechanical systems often leads to positive real transfer functions and could be used to implement a control system with a guarantee of closed loop stability (Brogliato, Lozano, Maschke, & Egeland, 2007). Moreover, positive real systems can be realized with an electrical circuit using only resistors, inductors and capacitors (Anderson & Vongpanitlerd, 1973). One major limitation of positive real systems is that their relative degree must be zero or one (Brogliato et al., 2007). For example, a lightly damped flexible structure with collocated velocity sensors and force actuators can typically be modeled by a sum of second-order transfer functions as $F(s) = \sum_{i=0}^{\infty} \frac{\psi_i \psi_i^T s}{s^2+2\zeta_i\omega_i s+\omega_i^2}$, where ω_i is the mode frequency, $\zeta_i > 0$ is the damping coefficient associated with the ith mode, and ψ_i is a column vector determined by the boundary conditions on the partial differential equation. However, in some cases (for example, when using piezoelectric sensors), the sensor output is proportional to position rather than velocity. So the transfer function $F(s)$ given above is the transfer function from the actuator input to the derivative of the sensor output. In the case of a lightly damped flexible structure with collocated position sensors and force actuators, the transfer function will be of the form $R(s) = \sum_{i=0}^{\infty} \frac{\psi_i \psi_i^T}{s^2+2\zeta_i\omega_i s+\omega_i^2}$. It can be seen that the relative degree of $R(s)$ is more than unity. Hence, the standard positive real theory will not be helpful in establishing closed loop stability. However, such a transfer function satisfies the following negative imaginary condition: $j[R(j\omega) - R^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$. Such systems are called *negative imaginary systems* in Lanzon and Petersen (2008), Petersen and Lanzon (2010) and Xiong, Petersen, and Lanzon (2010). Examples of negative imaginary systems can also be found in positive position feedback control of active structures (Fanson & Caughey, 1990; Moheimani, Vautier, & Bhikkaji, 2006).

A state-space characterization of negative imaginary transfer functions has been successfully established in Lanzon and Petersen (2008) and Xiong et al. (2010). A necessary and sufficient condition, expressed as the product of the DC gains (that is, the gains at zero frequency) being less than unity, has also been derived to guarantee the internal stability of a positive feedback interconnection of linear time-invariant multiple-input multiple-output negative imaginary systems. The stability result in Lanzon and Petersen (2008) and Xiong et al. (2010) has also been extended to the case of a string of arbitrarily many coupled strictly negative imaginary systems (Cai & Hagen, 2010), where the stability condition is given in terms of a continued fraction of the subsystem

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DC gains. Moreover, the controller synthesis problem for negative imaginary systems has been explored in Song, Lanzon, Patra, and Petersen (2010) by reformulating the closed loop systems into closed loop systems that have bounded gain.

A special and important class of negative imaginary systems will be further studied in this paper. Such systems are referred to as *lossless negative imaginary systems* since their transfer functions satisfy the lossless negative imaginary condition: $j[R(j\omega) - R^*(j\omega)] = 0$ for all $\omega \in (0, \infty)$ except values of ω where $j\omega$ is a pole. The study of lossless negative imaginary systems is significant because many applications can be found. For example, an m-port electrical network consisting of lossless circuit elements such as capacitors, inductors and transformers is lossless (Anderson & Vongpanitlerd, 1973). The transfer function of a lossless m-port network can be lossless negative imaginary. In addition, undamped flexible structures with collocated position sensors and force actuators are lossless negative imaginary systems since their transfer functions are of the form $R(s) = \sum_{i=0}^{\infty} \frac{\psi_i \psi_i^T}{s^2+\omega_i^2}$, which is lossless negative imaginary.

The novelties of the paper lie in the introduction of the concept of lossless negative imaginary systems and the establishment of a lossless negative imaginary lemma. Some properties of lossless negative imaginary transfer functions are also developed and the relationship between lossless negative imaginary transfer function matrices and lossless positive real transfer function matrices is derived. The results of the paper draw a nice parallel to the well understood results on lossless positive real systems.

The organization of the paper is as follows. The notion of lossless negative imaginary transfer functions is introduced in Section 2. In Section 3, a Lossless Negative Imaginary Lemma is established when the transfer functions satisfy the lossless negative imaginary condition. This lemma can be considered as a modification of the Negative Imaginary Lemma in Xiong et al. (2010) to the lossless negative imaginary case. Section 4 studies the stabilization of a lossless negative imaginary system by a strictly negative imaginary controller through a positive feedback interconnection. A necessary and sufficient condition for stabilization is proposed in terms of the DC loop gain of the systems. This result is a special case of the main result in Xiong et al. (2010) with the system being lossless negative imaginary. An undamped flexible structure example is presented in Section 5 to illustrate the theory and Section 6 concludes the paper.

Notation: Let $\mathbb{R}^{m\times n}$ and $\mathcal{R}^{m\times n}$ denote the set of $m \times n$ real matrices and real-rational proper transfer function matrices, respectively. $\lambda_{\max}(A)$ denotes the maximum eigenvalue for a square complex matrix A that has only real eigenvalues. $\bar{A}$, A^T and A^* denotes the complex conjugate, the transpose and the complex conjugate transpose of a complex matrix A, respectively. $R^{\sim}(s)$ presents the adjoint of transfer function matrix $R(s)$ given by $R^T(-s)$.

2. Lossless negative imaginary transfer functions

The concept of lossless negative imaginary transfer functions is introduced in this section and some properties of such functions are studied.

Definition 1. A real-rational proper transfer function matrix $R(s) \in \mathcal{R}^{m\times m}$ is *lossless negative imaginary* if

(1) $R(s)$ is negative imaginary;
(2) $j[R(j\omega) - R^*(j\omega)] = 0$ for all $\omega \in (0, \infty)$ except values of ω where $j\omega$ is a pole of $R(s)$.

Remark 1. The lossless negative imaginary property of a transfer function is simply defined by replacing the "$\geq$" sign with the "=" sign in the definition of negative imaginary transfer function matrices; see Definition 1 in Xiong et al. (2010).

Definition 2 (*Anderson & Vongpanitlerd, 1973*)**.** A real-rational proper transfer function matrix $F(s) \in \mathcal{R}^{m\times m}$ is lossless positive real if

(1) $F(s)$ is positive real;
(2) $F(j\omega) + F^*(j\omega) = 0$ for all real ω except values of ω where $j\omega$ is a pole of $F(s)$.

The following lemma provides a relationship between lossless negative imaginary transfer functions and lossless positive real transfer functions.

Lemma 1. *Given a real-rational strictly proper transfer function matrix $R(s) \in \mathcal{R}^{m\times m}$. Then $R(s)$ is lossless negative imaginary if and only if*

(1) *$R(s)$ has no poles at the origin;*
(2) *$F(s) = sR(s)$ is lossless positive real.*

Proof. (Necessity) Suppose $R(s)$ is lossless negative imaginary. Condition 1 of Definition 1 implies that $R(s)$ is also a negative imaginary transfer function. In view of Lemma 3 in Xiong et al. (2010), we have that $R(s)$ has no poles at the origin and $F(s)$ is positive real.

Now we prove that condition 2 of Definition 2 holds. Note that $F(s)$ and $R(s)$ have the same set of poles. For any $\omega \geq 0$ where $j\omega$ is not a pole of $R(s)$, we have $F(j\omega) + F^*(j\omega) = j\omega[R(j\omega) - R^*(j\omega)] = 0$. Also, by taking the complex conjugate we have $\overline{F(j\omega) + F^*(j\omega)} = 0$ for $\omega \geq 0$, which is equivalent to $F(-j\omega) + F^*(-j\omega) = 0$ for $\omega \geq 0$. So $F(j\omega) + F^*(j\omega) = 0$ for $\omega \leq 0$. Finally, we have $F(j\omega) + F^*(j\omega) = 0$ for all real ω with $j\omega$ not a pole. According to Definition 2, $F(s)$ is a lossless positive real transfer function.

(Sufficiency) Suppose $F(s)$ is lossless positive real, and $R(s)$ has no poles at the origin. Then $F(s)$ is also positive real. In view of Lemma 3 in Xiong et al. (2010), we have that $R(s)$ is negative imaginary.

Moreover, for any $\omega > 0$ where $j\omega$ is not a pole of $R(s)$, condition 2 of Definition 2 implies condition 2 of Definition 1 in view of $F(s) = sR(s)$. Therefore, $R(s)$ is lossless negative imaginary. □

Remark 2. Although the relationships stated in Lemma 1 exist between (lossless) negative imaginary and (lossless) positive real systems, similar relationships do not exist between strictly negative imaginary systems and strictly positive real systems. For example, $R(s) = \frac{1}{s+1}$ is strictly negative imaginary, while $F(s) = sR(s) = \frac{s}{s+1}$ is not (weakly) strictly positive real because the (weakly) strictly positive real frequency condition does not hold at zero frequency.

Lemma 2. *A real-rational strictly proper transfer function matrix $R(s) \in \mathcal{R}^{m\times m}$ is lossless negative imaginary if and only if*

(1) *$R(s)$ has no poles at the origin;*
(2) *All poles of elements of $R(s)$ are simple poles and purely imaginary, and the residue matrix of $jR(s)$ at any pole, $K_0 = \lim_{s\to j\omega_0}(s - j\omega_0)jR(s)$, is positive semidefinite Hermitian;*
(3) *$R(s) = R^{\sim}(s)$ for all s such that s is not a pole of $R(s)$.*

Proof. (Necessity) Suppose $R(s)$ is lossless negative imaginary. Then Lemma 1 implies $F(s) = sR(s)$ is lossless positive real and condition 1 holds. In view of Anderson and Vongpanitlerd (1973, p. 57), we have $F(s) + F^{\sim}(s) = 0$ for all s such that s is not a pole of $F(s)$. So $sR(s) - sR^{\sim}(s) = 0$ for all s such that s is not a pole of $R(s)$. Hence, we have $R(s) = R^{\sim}(s)$ for all s such that s is not a pole of $R(s)$. Here $R(0) = R^{\sim}(0)$ is due to the continuity of $R(s)$. Thus condition 3 holds.

Suppose that a complex number s_0 is a pole of $R(s)$. Then it follows from $R(s) = R^{\sim}(s) = R^T(-s)$ that $-s_0$ is also a pole of $R(s)$. On the other hand, $R(s)$ has no poles in the open right half plane according to the definition. Therefore, all poles of $R(s)$ must be purely imaginary. Moreover, condition 3 of Definition 1 in Xiong et al. (2010) implies that the poles are simple poles and the residue matrix of $jR(s)$ at any pole is positive semidefinite Hermitian. Thus condition 2 holds.

(Sufficiency) Suppose conditions 1–3 hold. First, condition 1 together with condition 2 implies condition 1 and condition 3 of Definition 1 in Xiong et al. (2010). Secondly, from condition 3, we have $R(j\omega) = R^{\sim}(j\omega) = R^{*}(j\omega)$, so condition 2 of Definition 1 is also true. Therefore, $R(s)$ is lossless negative imaginary. □

Example 1. As an application of Lemma 1, we can say that $R(s) = \frac{1}{s^2+1}$ is lossless negative imaginary if and only if $F(s) = \frac{s}{s^2+1}$ is lossless positive real. This can be actually verified by directly using Definitions 1 and 2 and Theorem 2.7.2 in Anderson and Vongpanitlerd (1973). We could also conclude that $R(s)$ is a lossless negative imaginary transfer function using Lemma 2.

The following lemma characterizes the properties of a sum of negative imaginary transfer functions.

Lemma 3. *Given two lossless negative imaginary transfer functions $R_1(s)$ and $R_2(s)$, and a negative imaginary transfer function $R(s)$. Then*

(1) *$R_1(s) + R_2(s)$ is lossless negative imaginary;*
(2) *$R_1(s) + R(s)$ is negative imaginary.*

Proof. The proof follows along similar lines to that of Lemma 6 in Xiong et al. (2010), and is hence omitted. □

3. Lossless negative imaginary lemma

The Lossless Negative Imaginary Lemma proposed in this section is a modification of the Negative Imaginary Lemma in Xiong et al. (2010) applied to the case where the transfer functions being considered are lossless negative imaginary. This Lossless Negative Imaginary Lemma, which is the main result of this paper, is analogous to the Lossless Positive Real Lemma (Anderson & Vongpanitlerd, 1973, pp. 221–222).

Theorem 1 (*Lossless Negative Imaginary Lemma*). *Let (A, B, C, D) be a minimal state-space realization of a real-rational proper transfer function $R(s) \in \mathcal{R}^{m\times m}$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$. Then $R(s)$ is lossless negative imaginary if and only if*

(1) $\det(A) \neq 0$, $D = D^T$;
(2) *there exists a matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, such that*

$$AY + YA^T = 0, \quad \text{and} \quad B + AYC^T = 0.$$

Proof. (Necessity) Note that a lossless negative imaginary transfer function $R(s)$ is also negative imaginary. Hence, according to Lemma 7 in Xiong et al. (2010), condition 1 holds. Furthermore, it follows from Corollary 1 in Xiong et al. (2010) that there exists a transfer function matrix

$$M(s) \sim \left[\begin{array}{c|c} A & B \\ \hline LY^{-1}A^{-1} & 0 \end{array}\right],$$

where $Y = Y^T > 0$ and L are the solutions of $L^TL = -AY - YA^T$ and $B + AYC^T = 0$. Moreover, $M(s)$ satisfies the spectral-factor property

$$j[R(j\omega) - R^{*}(j\omega)] = \omega M^{*}(j\omega)M(j\omega)$$

whenever $j\omega$ is not a pole of $R(s)$.

Since $R(s)$ is lossless, according to Lemma 2, we have $R(j\omega) = R^{\sim}(j\omega) = R^{*}(j\omega)$ for any real ω with $j\omega$ not a pole of $R(s)$. That is, $j[R(j\omega) - R^{*}(j\omega)] = 0$. Thus $M^{*}(j\omega)M(j\omega) = 0$, and hence $M(j\omega) = 0$ for all real ω with $j\omega$ not a pole of $M(s)$. Moreover, $M(0) = 0$ due to the continuity of $M(s)$. Therefore, we have $M(s) = LY^{-1}A^{-1}(sI - A)^{-1}B = 0$. By the controllability of the pair (A, B), we conclude that $LY^{-1}A^{-1} = 0$. That is, $L = 0$. Therefore, condition 2 holds. This completes the necessity part of the proof.

(Sufficiency) Suppose conditions 1–2 hold. We know that $R(s)$ is negative imaginary in view of Lemma 7 in Xiong et al. (2010), and $M(s) = 0$ in view of Corollary 1 in Xiong et al. (2010). Therefore, $j[R(j\omega) - R^{*}(j\omega)] = \omega M^{*}(j\omega)M(j\omega) = 0$ whenever $j\omega$ is not a pole of $R(s)$. It follows from the definition that $R(s)$ is lossless negative imaginary. □

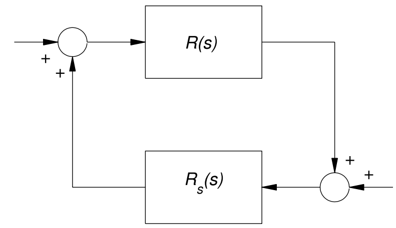

Fig. 1. Positive feedback interconnection.

Remark 3. The lossless negative imaginary lemma is different from the negative imaginary lemma in Xiong et al. (2010). Here, condition (2) contains two equalities while the negative imaginary lemma in Xiong et al. (2010) has one inequality and one equality. As a result, lossless negative imaginary systems are actually negative imaginary systems with all poles on the imaginary axis. It is also noticed that the definition of negative imaginary systems given in Lanzon and Petersen (2008) requires that all the system poles lie in the open left half plane, and the same concept was redefined in Xiong et al. (2010) so that the systems are allowed to have poles on the imaginary axis. Hence, all the results developed in Xiong et al. (2010) are valid for lossless negative imaginary systems, and all the results in Lanzon and Petersen (2008) are not valid for lossless negative imaginary systems.

4. Stabilization of lossless negative imaginary systems

In this section, we consider the stabilization of lossless negative imaginary systems via positive feedback as shown in Fig. 1. The positive feedback interconnection is denoted by $[R(s), R_s(s)]$. The results in this section are in fact special cases of the corresponding results in Xiong et al. (2010) with one system being lossless negative imaginary, and the proofs are the same as those in Xiong et al. (2010).

A necessary and sufficient condition is provided for the stability of the interconnected system given in Fig. 1 in terms of the DC loop gain.

Theorem 2. *Given a lossless negative imaginary transfer function $R(s)$ and a strictly negative imaginary transfer function $R_s(s)$ that satisfy $R(\infty)R_s(\infty) = 0$ and $R_s(\infty) \geq 0$. Then the positive feedback interconnection $[R(s), R_s(s)]$ is internally stable if and only if $\lambda_{\max}(R(0)R_s(0)) < 1$.*

The following corollaries are a restatement of the above theorem, written in the same form as the small gain theorem.

Corollary 1. *Given $\gamma > 0$ and a strictly negative imaginary transfer function $R(s)$ with $R(\infty) \geq 0$. Then, the positive feedback interconnection $[\Delta(s), R(s)]$ is internally stable for all lossless negative imaginary transfer functions $\Delta(s)$ satisfying $\Delta(\infty)R(\infty) = 0$ and $\lambda_{\max}(\Delta(0)) < \gamma$ (respectively $\leq \gamma$) if and only if $\lambda_{\max}(R(0)) \leq \frac{1}{\gamma}$ (respectively $< \frac{1}{\gamma}$).*

Corollary 2. *Given $\gamma > 0$ and a lossless negative imaginary transfer function $R(s)$. Then the positive feedback interconnection $[\Delta(s), R(s)]$ is internally stable for all strictly negative imaginary transfer functions $\Delta(s)$ satisfying $\Delta(\infty)R(\infty) = 0$, $\Delta(\infty) \geq 0$ and $\lambda_{\max}(\Delta(0)) < \gamma$ (respectively $\leq \gamma$) if and only if $\lambda_{\max}(R(0)) \leq \frac{1}{\gamma}$ (respectively $< \frac{1}{\gamma}$).*

Remark 4. Theorem 2 cannot be obtained from the stability results for positive real systems (for example, Lemma 3.37 of Brogliato et al. (2007)). The reason is that (weakly) strictly positive real systems need to satisfy the strictly positive real frequency domain conditions on the whole imaginary axis, while strictly negative imaginary systems satisfy the strictly negative imaginary frequency conditions on a punctured imaginary axis excluding the origin. Actually, Theorem 2 has a mix of both gain conditions (at zero and infinite frequencies) and phase conditions (at finite positive frequencies), while Lemma 3.37 of Brogliato et al. (2007) has pure phase conditions (at all frequencies). The readers are referred to Lanzon and Petersen (2008), Lanzon, Song, Patra, and Petersen (2011) and Xiong et al. (2010) for more details about strictly negative imaginary transfer functions.

Remark 5. The advantages of adopting the negative imaginary approach, compared to the positive real approach, are twofold: (1) the relative degrees of negative imaginary systems are between zero and two while positive real systems must have a relative degree of zero or one; and (2) the strictly negative imaginary frequency conditions need not hold at all frequencies while strictly positive real frequency conditions do.

Remark 6. Theorem 2 could be considered as a generalization of the positive position control results in Moheimani et al. (2006) and Fanson and Caughey (1990). In Moheimani et al. (2006) and Fanson and Caughey (1990), both the plants and the controllers are described by second-order differential equations, and in fact strictly negative imaginary systems. For a lossless negative imaginary system, a strictly negative imaginary controller can be used to stabilize the system as long as the conditions stated in Theorem 2 are satisfied. The design of such a strictly negative imaginary controller will be illustrated in the next section.

5. Illustrative example

To illustrate the main results of this paper, we consider a robust active vibration control problem. The flexible structure to be stabilized is an MIMO undamped mass–spring plant as depicted in Fig. 2. The displacement outputs and the force inputs are collocated, and denoted by q_1, q_2, f_1 and f_2, respectively. The parameters $m_1 > 0, m_2 > 0, k_1 > 0$ and $k_2 > 0$ are assumed to be unknown but belong to known intervals: $m_1 \in [\underline{m}_1, \overline{m}_1]$, $m_2 \in [\underline{m}_2, \overline{m}_2]$, $k_1 \in [\underline{k}_1, \overline{k}_1]$, $k_2 \in [\underline{k}_2, \overline{k}_2]$. Any uncertain parameters in those intervals are called admissible uncertainties. A robust stabilizing dynamic controller is to be designed to reduce the vibration exponentially when the flexible structure suffers from external impulse disturbances or nonzero initial conditions.

By letting

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \qquad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \qquad x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix},$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \qquad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix},$$

the ordinary differential equations describing the motion of the uncertain flexible structure in Fig. 2 are given by

$$M\ddot{q} + Kq = f.$$

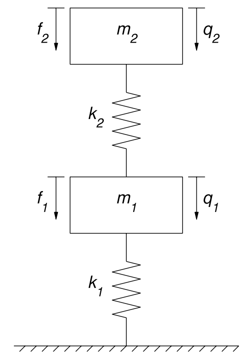

Fig. 2. An MIMO undamped flexible structure.

These equations represent the general case of $M\ddot{q} + E\dot{q} + Kq = f$ when the damping matrix E is negligible. Note that the above equations can model a large class of practical systems, such as large space structures (Fanson & Caughey, 1990) and active structures (Moheimani et al., 2006).

The corresponding state-space equation is

$$\begin{cases} \dot{x} = Ax + Bf, \\ q = Cx + Df, \end{cases}$$

where x is the system state, f is the control force input, and q is the measured displacement output. The system matrices are given by

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix},$$

$$C = \begin{bmatrix} I & 0 \end{bmatrix}, \qquad D = 0.$$

Note that (A, B, C, D) is a minimal state-space realization, and $M > 0, K > 0$.

To illustrate the Lossless Negative Imaginary Lemma, we note firstly that $\det(A) \neq 0$, and $D = D^T$. That is, condition 1 in Theorem 1 holds. Secondly, the real positive definite matrix

$$Y = \begin{bmatrix} K^{-1} & 0 \\ 0 & M^{-1} \end{bmatrix}$$

is found to satisfy $AY + YA^T = 0$ and $B + AYC^T = 0$. That is, condition 2 in Theorem 1 is satisfied. Applying Theorem 1, we can conclude that $P(s) \triangleq C(sI - A)^{-1}B + D$ is a lossless negative imaginary transfer function. In other words, the undamped mass–spring flexible structure is a lossless negative imaginary system.

In view of Theorem 2, the uncertain flexible structure $P(s)$ can be robustly stabilized by any controller $C(s)$ which is strictly negative imaginary and satisfies $C(\infty) \geq 0$ and $\lambda_{\max}(C(0)) < \frac{1}{\lambda_{\max}(P(0))}$ for all admissible parameter uncertainties. Note that the maximum eigenvalue of $P(0)$ is found to be $\lambda_{\max}(P(0)) = \frac{k_1+2k_2+\sqrt{k_1^2+4k_2^2}}{2k_1k_2}$, which is upper bounded by $\frac{\overline{k}_1+2\overline{k}_2+\sqrt{\overline{k}_1^2+4\overline{k}_2^2}}{2\underline{k}_1\underline{k}_2}$.

To illustrate the use of Theorem 2 in this example, we now present several simulations. The lower bounds on the uncertain parameters in the flexible structure are assumed to be $\underline{m}_1 = \underline{m}_2 = \underline{k}_1 = \underline{k}_2 = 0.9$ and the upper bounds are assumed to be $\overline{m}_1 = \overline{m}_2 = \overline{k}_1 = \overline{k}_2 = 1.1$. A strictly negative imaginary controller is simply chosen as $C(s) = \frac{\epsilon}{s+\alpha}I$, where $\alpha = 1$ and $\epsilon = 0.2$. It can be verified that $\lambda_{\max}(C(0)) < \frac{2\underline{k}_1\underline{k}_2}{\overline{k}_1+2\overline{k}_2+\sqrt{\overline{k}_1^2+4\overline{k}_2^2}} < \frac{1}{\lambda_{\max}(P(0))}$ for all admissible parameter uncertainties. The structure parameters are assumed to be $k_1 = m_2 = 0.9$ and $k_2 = m_1 = 1.1$. The initial condition of the structure is given by $q_1(0) = 2, \dot{q}_1(0) = 0$, $q_2(0) = -2$ and $\dot{q}_2(0) = 0$, and the controller is assumed to have zero initial condition. The initial condition responses of the flexible

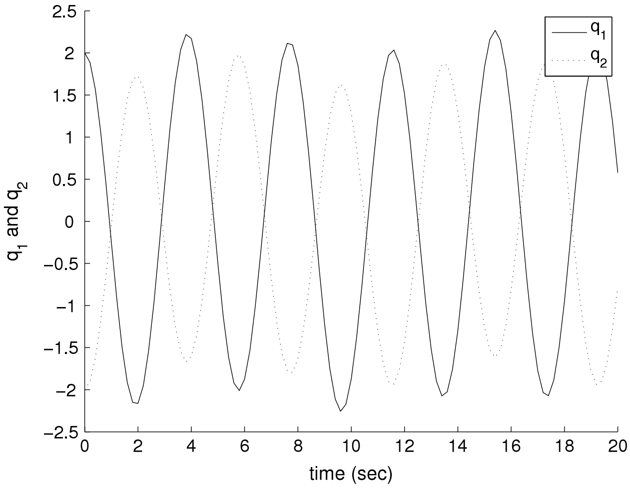

Fig. 3. Initial condition response of the flexible structure without control.

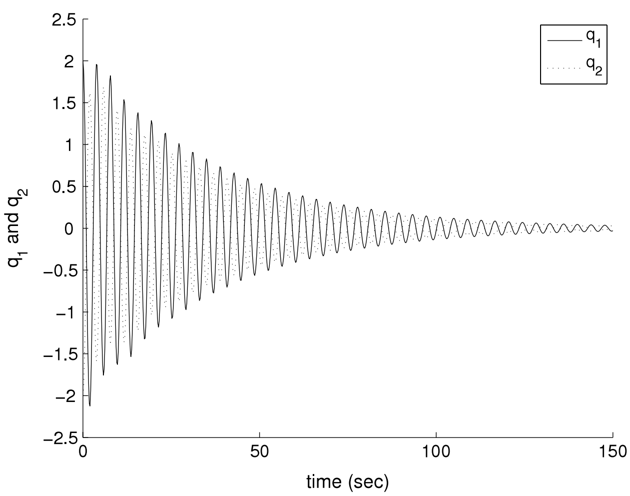

Fig. 4. Initial condition response of the flexible structure with control.

structure without control and with control are shown in Figs. 3 and 4, respectively. It can be seen from the simulations that the structure vibrations were reduced exponentially by the designed controller.

6. Conclusions

This paper has studied the lossless negative imaginary properties of square real-rational proper transfer functions. Dynamical systems with lossless negative imaginary transfer functions have applications in the control of lossless electrical circuits and positive position feedback control of undamped flexible structures. The Lossless Negative Imaginary Lemma was derived for transfer functions that are lossless negative imaginary. Moreover, a necessary and sufficient condition was established for the stabilization of a lossless negative imaginary system using a strictly negative imaginary system. Finally, the developed theory in the paper was illustrated by an uncertain flexible structure example.

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