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H_∞ and H_2 filtering for linear systems with uncertain Markov transitions[☆]

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ABSTRACT

This paper is concerned with H_∞ and H_2 filtering for Markovian jump linear systems with uncertain transition probabilities. Motivated by the fact that the existing results either impose severe restrictions on some key matrices or introduce some unnecessary matrix variables, this paper is focused on developing a new approach to systematically relax these restrictions for filter design. By applying a novel technique to eliminate the product terms between the Lyapunov matrices and the filter parameters, an improved condition is first obtained for analyzing the H_∞ performance of the filtering error system. Then sufficient conditions in terms of linear matrix inequalities are presented for designing filters with a guaranteed H_∞ filtering performance level. The proposed method is further extended to H_2 filtering. Theoretical analyses followed by a few numerical examples show that the proposed filter design method outperforms some existing results with respect to reduction of conservatism or variables needed for computation. The filter design problems for both continuous-time and discrete-time Markovian jump linear systems are addressed in a unified framework.

1. Introduction

As an important class of stochastic systems, Markovian jump systems can be used to effectively model practical plants of the multi-mode nature, that is, plants have different working situations and may continuously switch between them in a random way. Due to the usefulness in describing complicated time-varying dynamics, Markov jump systems have been extensively investigated and the control theory for this type of systems has undergone great development in the past decade (Boukas, 2005; Costa, Fragoso, & Marques, 2005; Feng, Lam, & Shu, 2010; Shu, Lam, & Xiong, 2010). An active area in recent years regarding Markovian jump systems is those with uncertain transition probabilities, which are motivated by the fact that it is sometimes difficult to obtain the accurate information of transition probabilities of practical plants. In de Souza, Trofino, and Barbosa (2006), Karan, Shi, and Kaya (2006) and Xiong, Lam, Gao, and Ho (2005), the uncertain transition probabilities are assumed to be of the norm-bounded or polytope-bounded type which is commonly used in robust control theory so that some robust analysis methodologies can be applied; recent research attention is mostly focused on the case that the transition probabilities are partly known (He,Zhang, Wu & She, 2011; Li, Lam, Gao, & Li, 2014; Zhang & Boukas, 2009a,b; Zhang, Boukas, & Lam, 2008; Zhang, He, Wu, & Zhang, 2011; Zhang & Lam, 2010; Zuo, Li, Liu, & Wang, 2012), which bridges the arbitrarily switching case and the completely known case; a new modeling method is considered in Gonçalves, Fioravanti, and Geromel (2011), Morais, Braga, Oliveira, and Peres (2013) and Morais, Braga, Lacerda, Oliveira, and Peres (2014), which models each row of the transition probability matrix as an independent polytopic domain and covers the method in Zhang and Boukas (2009b) and Zhang and Lam (2010) as a special case.

As a fundamental issue in systems and control theory, state estimation or filtering is to estimate signals that are of particular

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interest but not available due to technical difficulties. On the other hand, it is known that H_∞ technique is an effective approach to deal with non-statistic disturbances. Some recent results on H_∞ technique applied to dynamic systems can be found in, for instance, Hernandez-Gonzaleza and Basin (2014, 2015) and You, Gao, and Basin (2013). Parallel to H_∞ robust control theory, one of the prevailing filtering methods is H_∞ filtering, which does not need to know the statistics of noises and can lead to more robust filters. Many results on H_∞ filtering have been reported; see Basin, Shi, and Calderon-Alvarez (2009, 2010), He, Liu, Rees, and Wu (2009), Shi (1997, 1998), Shi, Luan, and Liu (2012) and Wang, Lam, and Liu (2004). Design of H_∞ filters for Markov jump systems also has drawn considerable attention (de Souza et al., 2006; Gonçalves, Fioravanti, & Geromel, 2009; Gonçalves et al., 2011; Liu, Ho, & Sun, 2008; Morais et al., 2014; Shu, Lam, & Hu, 2009; Xiong & Lam, 2006; Zhang & Boukas, 2009a). For Markov jump linear systems (MJLSs) with uncertain transition probabilities, sufficient conditions in terms of linear matrix inequalities (LMIs) for the existence of H_∞ filters were derived in de Souza et al. (2006), Dong, Wang, and Gao (2012); Dong, Wang, Ho, and Gao (2011), Gonçalves et al. (2009, 2011), Morais et al. (2014) and Zhang and Boukas (2009a). Especially, the methods in de Souza et al. (2006), Gonçalves et al. (2009, 2011), Liu et al. (2008) and Morais et al. (2014) can be used to design partly-mode-dependent or mode-independent H_∞ filters when the mode information of MJLSs is partly known or unknown. Besides H_∞ filtering, H_2 filtering also has been extensively studied for both continuous- and discrete-time MJLSs (Fioravanti, Gonçalves, & Geromel, 2008; Liu, Zhang, & Chen, 2012; Morais, Braga, Lacerda, Oliveira, & Peres, 2015).

It is worth pointing out that, when dealing with uncertain transition probabilities or designing mode-independent filters, a crucial procedure that is frequently used in Gonçalves et al. (2011), Liu et al. (2008), Morais et al. (2014) and Zhang and Boukas (2009a) is, by introducing some slack variables in the bounded real lemma for MJLSs, to eliminate the product terms between the Lyapunov matrices and the filter parameters. Although this procedure could relax the sufficient conditions for filter design, two important aspects on this procedure have been overlooked in these existing results. On one hand, the methods in Liu et al. (2008) and Zhang and Boukas (2009a) have restricted the slack matrices so much that the resulting design conditions can be further improved. On the other hand, although the latest conditions in Morais et al. (2014, 2015) outperform the previous ones with respect to conservatism reduction, it will be shown that there are too many unnecessary slack variables in these conditions (see Section 3.3.2). The above two aspects make it necessary to systematically develop new filter design methods to mitigate the drawback of each of the existing methods, which motivates the work in this paper.

In this paper, we will study the H_∞ and H_2 filter design problems of MJLSs with uncertain transition probabilities and in particular focus on dealing with the mentioned drawbacks of the existing design methods. First, for characterizing the H_∞ performance of the filtering error system, a new version of the bounded real lemma for MJLSs will be obtained through applying a novel technique to decouple the product terms between the Lyapunov matrices and the filter parameters. Based on the new bounded real lemma, sufficient conditions in terms of LMIs are then derived for the existence of full-order filters that guarantee an H_∞ disturbance attenuation level for the filtering error system. In addition, we will further extend the proposed method to the H_2 filtering problem of MJLSs. Due to the use of the new technique to eliminate the undesired product terms, the proposed filter design method is either less conservative or less computationally demanding than some existing methods. The advantages of the proposed method will be clearly shown by theoretical analyses and numerical results. The contributions of the paper are summarized as follows: (1) a new technique is proposed for introducing extra matrix variables so as to decouple the product terms between Lyapunov matrices and the filter parameters; and (2) the proposed filter design method has systematically improved the existing results from both conservatism reduction and variable reduction points of view.

Notation. *The superscripts "−1" and "T" stand for matrix inverse and matrix transpose, respectively.* $\mathbb{R}$ *denotes the set of real numbers and* $\mathbb{R}^{m\times n}$ *is the set of all* $m \times n$ *real matrices.* $\mathbb{N}$ *represents the set of nonnegative integers.* $|\cdot|$ *represents the Euclidean norm of a vector, and* L_2 *and* l_2 *are the space of square Lebesgue integrable functions and summable infinite sequences, respectively. For* $w \in L_2$ *or* l_2*, its 2-norm is denoted as* $\|w\|_2 \triangleq \sqrt{\int_0^\infty |w_k|^2 \,\mathrm{d}k}$ *and* $\|w\|_2 \triangleq \sqrt{\sum_{k=0}^\infty |w_k|^2}$*, respectively. The triplet notation* $(\Omega, \mathscr{F}, \mathscr{P})$ *refers to a probability space, where* Ω *is the sample space,* $\mathscr{F}$ *the* σ*-algebra of subsets of the sample space and* $\mathscr{P}$ *the probability measure on* $\mathscr{F}$*, respectively.* $\mathbf{E}[\cdot]$ *denotes the mathematical expectation. The notation* $P > 0$ (≥ 0) *means that matrix* P *is real symmetric and positive definite (semi-definite).* I *denotes an identity matrix with appropriate dimension.* $\mathrm{diag}\{A_1, A_2, \ldots, A_n\}$ *stands for a block diagonal matrix with* $A_1, A_2, \ldots, A_n$ *on the diagonal. Throughout the paper, a symbol with superscript "ct" or "dt" stand for the continuous- or discrete-time case, respectively. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.*

2. Problem statement

On a given probability space $(\Omega, \mathscr{F}, \mathscr{P})$, consider an MJLS represented by the following differential or difference equations:

$$\begin{aligned} \delta[x_k] &= A(r_k)x_k + B(r_k)w_k \\ y_k &= C(r_k)x_k + D(r_k)w_k \\ z_k &= E(r_k)x_k + F(r_k)w_k \end{aligned} \tag{1}$$

where $x_k \in \mathbb{R}^{n_x}$ is the state vector, $w_k \in \mathbb{R}^{n_w}$ is the external disturbance, $y_k \in \mathbb{R}^{n_y}$ is the measurement output and $z_k \in \mathbb{R}^{n_z}$ is the target output to be estimated. The symbol $\delta[x_k]$ denotes $\dot{x}_k$ $(k \in [0, +\infty))$ for the continuous-time case and x_{k+1} $(k \in \mathbb{N})$ for the discrete-time case, respectively. For the two cases, the disturbance signal w_k is assumed to belong to $L_2[0, +\infty)$ and $l_2(\mathbb{N})$, respectively. The scalar r_k takes values from a finite set $\mathbb{M} \triangleq \{1, 2, \ldots, M\}$, which is a switching signal determining which mode of the system is activated. For each mode $r_k = i \in \mathbb{M}$, the real matrices $A(r_k)$, $B(r_k)$, $C(r_k)$, $D(r_k)$, $E(r_k)$ and $F(r_k)$, denoted by A_i, B_i, C_i, D_i, E_i and F_i, respectively, are known and appropriately dimensioned.

For the continuous-time MJLS, $\{r_k\}$ is a continuous-time, discrete-state homogeneous Markov process with mode transition rates:

$$\Pr(r_{k+d} = j | r_k = i) = \begin{cases} \pi_{ij} d + o(d), & \text{for } j \neq i \\ 1 + \pi_{ii} d + o(d), & \text{for } j = i \end{cases}$$

where $d > 0$, $\lim_{d\to 0} \frac{o(d)}{d} = 0$, and $\pi_{ij} \geq 0$ for $i, j \in \mathbb{M}, j \neq i$ and $\pi_{ii} = -\sum_{j\in\mathbb{M}, j\neq i} \pi_{ij} \leq 0$. For $j \neq i$, π_{ij} denotes the switching rate from the ith mode at time k to the jth model at time $k + d$. For the discrete-time MJLS, $\{r_k\}$ is a discrete-time Markov chain with mode transition probabilities:

$$\Pr(r_{k+1} = j | r_k = i) = \pi_{ij}$$

where $\pi_{ij} \geq 0$ for $i, j \in \mathbb{M}$ and $\sum_{j\in\mathbb{M}} \pi_{ij} = 1$. Define π_i as the ith row of the probability matrix $[\pi_{ij}] \triangleq \Pi$, that is,

$$\pi_i \triangleq \begin{bmatrix} \pi_{i1} & \pi_{i2} & \cdots & \pi_{iM} \end{bmatrix}.$$

Due to the difficulty in obtaining the exact value of the transition probabilities in practice, we assume that π_i is unknown but belongs to a convex set. As in Gonçalves et al. (2011), we describe the uncertain probability vector π_i for $i \in \mathbb{M}$ in the following way:

$$\pi_i(\lambda_i) \in \left\{ \sum_{s=1}^{S_i} \lambda_{is} \pi_i^{(s)} \mid \lambda_i \in \Lambda_i \right\} \tag{2}$$

where $\lambda_i \triangleq \begin{bmatrix} \lambda_{i1} & \lambda_{i2} & \cdots & \lambda_{iS_i} \end{bmatrix}$ is the uncertain parameter vector, Λ_i is a unit simplex defined as

$$\Lambda_i \triangleq \left\{ \lambda_i \in \mathbb{R}^{S_i} \mid \sum_{s=1}^{S_i} \lambda_{is} = 1, \lambda_{is} \geq 0 \right\}$$

with S_i being the number of vertices of the polytope, and $\pi_i^{(s)} \triangleq \begin{bmatrix} \pi_{i1}^{(s)} & \pi_{i2}^{(s)} & \cdots & \pi_{iM}^{(s)} \end{bmatrix}$ is the known value of $\pi_i(\lambda_i)$ at the sth vertex of the polytope. For later use, define

$$\lambda \triangleq \lambda_1 \times \lambda_2 \times \cdots \times \lambda_M, \qquad \Lambda \triangleq \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_M,$$
$$\mathbb{S}_i \triangleq \{1, 2, \ldots, S_i\}.$$

We will make use of the measurement y_k to estimate the output z_k for the MJLS in (1). To this end, our attention is focused on designing a filter in the following form:

$$\begin{aligned} \delta\left[\hat{x}_k\right] &= \hat{A}(r_k)\hat{x}_k + \hat{B}(r_k)y_k \\ \hat{z}_k &= \hat{C}(r_k)\hat{x}_k + \hat{D}(r_k)y_k \end{aligned} \tag{3}$$

where $\hat{x}_k \in \mathbb{R}^{n_x}$ is the state vector of the filter, and $\hat{A}(r_k)$, $\hat{B}(r_k)$, $\hat{C}(r_k)$ and $\hat{D}(r_k)$ are real matrices with compatible dimensions. For each $i \in \mathbb{M}$, $\hat{A}(r_k = i)$, $\hat{B}(r_k = i)$, $\hat{C}(r_k = i)$ and $\hat{D}(r_k = i)$, denoted by $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$ and $\hat{D}_i$, respectively, are the filter parameters to be determined. If we restrict $\hat{A}_i = \hat{A}$, $\hat{B}_i = \hat{B}$, $\hat{C}_i = \hat{B}$ and $\hat{D}_i = \hat{D}$ for all $i \in \mathbb{M}$, the filter in (3) reduces to a mode-independent one that does not need the information of the switching signal r_k, which also can be handled by the proposed method.

Define the augmented state vector $\bar{x}_k \triangleq \begin{bmatrix} x_k^{\mathrm{T}} & \hat{x}_k^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$ and the filtering error $e_k \triangleq z_k - \hat{z}_k$, respectively. Combining the MJLS in (1) and the filter in (3) leads to the following filtering error system:

$$\begin{aligned} \delta\left[\bar{x}_k\right] &= \bar{A}(r_k)\bar{x}_k + \bar{B}(r_k)w_k \\ e_k &= \bar{C}(r_k)\bar{x}_k + \bar{D}(r_k)w_k \end{aligned} \tag{4}$$

where

$$\bar{A}(r_k) = \begin{bmatrix} A(r_k) & 0 \\ \hat{B}(r_k)C(r_k) & \hat{A}(r_k) \end{bmatrix}, \bar{B}(r_k) = \begin{bmatrix} B(r_k) \\ \hat{B}(r_k)D(r_k) \end{bmatrix}$$
$$\bar{C}(r_k) = \begin{bmatrix} E(r_k) - \hat{D}(r_k)C(r_k) & -\hat{C}(r_k) \end{bmatrix}$$
$$\bar{D}(r_k) = F(r_k) - \hat{D}(r_k)D(r_k).$$

To state the filtering objectives, the definitions of stability, H_∞ performance (Boukas, 2005; Costa et al., 2005) and H_2 performance (Costa, do Val, & Geromel, 1997; de Farias, Geromel, do Val, & Costa, 2000) are introduced for the filtering error system in (4).

Definition 1. The system in (4) with $w_k \equiv 0$ is said to be stochastically stable if

$$\mathbf{E}\left[\|\bar{x}\|_2 \mid \bar{x}_0, r_0\right] < \infty$$

for every initial condition $\bar{x}_0 \in \mathbb{R}^{2n_x}$ and $r_0 \in \mathbb{M}$.

Definition 2. Assume that the system in (4) is stochastically stable. Given a scalar $\gamma > 0$, the system in (4) is said to have an H_∞ performance level γ if it satisfies

$$\mathbf{E}\left[\|e\|_2^2\right] < \gamma^2 \|w\|_2^2$$

for all nonzero $w_k \in L_2[0, +\infty)$ for the continuous-time case (respectively, $w_k \in l_2(\mathbb{N})$ for the discrete-time case) under zero initial conditions $\bar{x}_0 = 0$.

Definition 3. Assume that the system in (4) is stochastically stable. Given a scalar $\rho > 0$, the system in (4) (with $\bar{D}(r_k) = 0$ for the continuous-time case) is said to have an H_2 performance level ρ if it satisfies

$$\sum_{s=1}^{n_w} \sum_{i=1}^{M} \mu_i \mathbf{E}\left[\left\|g_{s,i}\right\|_2^2\right] < \rho^2$$

where $\mu_i = \Pr(r_0 = i)$ and $g_{s,i}(k)$ is the output e_k under zero initial conditions $\bar{x}_0 = 0, r_0 = i$ and the input $w_k = \boldsymbol{e}_s \sigma_k$ with $\boldsymbol{e}_s$ being the sth column of the identity matrix and σ_k being the unitary impulse.

The H_∞ (respectively, H_2) filtering problem for MJLSs with uncertain transition probabilities to be considered in the paper is stated as follows: *Find a filter in (3) for the MJLS in (1) such that the filtering error system in (4) is stochastically stable with an H_∞ (respectively, H_2) performance level γ (respectively, ρ) for all $\lambda \in \Lambda$.*

3. H_∞ filtering

In this section, we will first provide a new condition for H_∞ filtering analysis and then propose a filter design method for MJLSs with uncertain transition probabilities.

3.1. H_∞ filtering analysis

We first introduce the following lemma that is useful for dealing with the formulated H_∞ filtering problem, which is known as the bounded real lemma for continuous-time MJLSs (de Souza et al., 2006) and discrete-time MJLSs (Seiler & Sengupta, 2003), respectively.

Lemma 1. *Given transition probabilities π_{ij} for $i, j \in \mathbb{M}$ and a scalar γ, the system in (4) is stochastically stable with an H_∞ performance level γ if and only if there exist symmetric matrices $P_i > 0$, $\forall i \in \mathbb{M}$ such that for the continuous-time case,*

$$\begin{bmatrix} \bar{A}_i^{\mathrm{T}} P_i + P_i \bar{A}_i + \boldsymbol{P}_i & P_i \bar{B}_i & \bar{C}_i^{\mathrm{T}} \\ * & -\gamma^2 I & \bar{D}_i^{\mathrm{T}} \\ * & * & -I \end{bmatrix} < 0, \quad \forall i \in \mathbb{M} \tag{5}$$

and for the discrete-time case,

$$\begin{bmatrix} \bar{A}_i & \bar{B}_i \\ \bar{C}_i & \bar{D}_i \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{P}_i & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}_i & \bar{B}_i \\ \bar{C}_i & \bar{D}_i \end{bmatrix} - \begin{bmatrix} P_i & 0 \\ 0 & \gamma^2 I \end{bmatrix} < 0, \quad \forall i \in \mathbb{M} \tag{6}$$

where $\boldsymbol{P}_i \triangleq \sum_{j=1}^{M} \pi_{ij} P_j$.

Given the MJLS in (1) and the filter in (3), Lemma 1 provides an LMI condition for analyzing the H_∞ performance of the filtering error system in (4). However, the conditions in (5) and (6) are not tractable for designing filters subject to uncertain transition probabilities, because the filter parameters are coupled with the Lyapunov matrices P_i in multiple inequalities. To handle this difficulty, we give the following theorem, a new condition for analyzing the H_∞ filtering performance for MJLSs. For convenience, define a symbol Φ_i as

$$\Phi_i^{\mathrm{ct}} \triangleq \begin{bmatrix} 0 & P_i & 0 & 0 \\ P_i & \boldsymbol{P}_i & 0 & \bar{C}_i^{\mathrm{T}} \\ 0 & 0 & -\gamma^2 I & \bar{D}_i^{\mathrm{T}} \\ 0 & \bar{C}_i & \bar{D}_i & -I \end{bmatrix}$$
$$\Phi_i^{\mathrm{dt}} \triangleq \begin{bmatrix} \boldsymbol{P}_i & 0 & 0 & 0 \\ 0 & -P_i & 0 & \bar{C}_i^{\mathrm{T}} \\ 0 & 0 & -\gamma^2 I & \bar{D}_i^{\mathrm{T}} \\ 0 & \bar{C}_i & \bar{D}_i & -I \end{bmatrix}. \tag{7}$$

Theorem 1. *Consider the system in (1) with known transition probabilities π_{ij} for $i, j \in \mathbb{M}$ and the filter in (3). Given a scalar $\gamma > 0$, the filtering error system in (4) is stochastically stable with an H_∞ performance level γ if and only if there exist matrices $P_i > 0$ and K_i for $i \in \mathbb{M}$ such that*

$$\Psi_i^{\mathrm{T}} \Phi_i \Psi_i + G_i^{\mathrm{T}} K_i H_i + H_i^{\mathrm{T}} K_i^{\mathrm{T}} G_i < 0, \quad \forall i \in \mathbb{M} \tag{8}$$

where Φ_i is defined in (7) and

$$\Psi_i = \left[\begin{array}{ccccc} 0 & A_i & 0 & B_i & 0 \\ I & 0 & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & 0 & I \end{array}\right]$$

$$G_i = \begin{bmatrix} 0 & A_i + \varepsilon_1 I & 0 & B_i & 0 \\ I & 0 & \varepsilon_2 I & 0 & 0 \end{bmatrix}$$

$$H_i = \begin{bmatrix} -I & \hat{B}_i C_i & \hat{A}_i & \hat{B}_i D_i & 0 \end{bmatrix}.$$

In H_i, scalars ε_1 and ε_2 take $\varepsilon_1 = \varepsilon_2 \gg 0$ for the continuous-time case and take zero for the discrete-time case, respectively.

Proof. According to Lemma 1, we need to prove the equivalent solvability of the conditions in (5) (respectively, (6)) and those in (8). To this end, we re-write the conditions in (5) and (6) in the following unified form:

$$W_i^{\mathrm{T}} \Phi_i W_i < 0 \tag{9}$$

where Φ_i is defined in (7) and

$$W_i = \begin{bmatrix} \bar{A}_i & \bar{B}_i & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

For the continuous-time case, it can be directly verified that the conditions in (5) are those in (9). For the discrete-time case, the conditions in (6) can be expressed as

$$\begin{bmatrix} \bar{A}_i & \bar{B}_i \\ I & 0 \\ 0 & I \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{P}_i & 0 & 0 \\ * & -P_i & 0 \\ * & * & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \bar{A}_i & \bar{B}_i \\ I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} 0 & \bar{C}_i & \bar{D}_i \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} 0 & \bar{C}_i & \bar{D}_i \end{bmatrix} < 0.$$

By applying Schur Complement Equivalence, the above inequality is equivalent to

$$\begin{bmatrix} \begin{bmatrix} \bar{A}_i & \bar{B}_i \\ I & 0 \\ 0 & I \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{P}_i & 0 & 0 \\ * & -P_i & 0 \\ * & * & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \bar{A}_i & \bar{B}_i \\ I & 0 \\ 0 & I \end{bmatrix} & \begin{bmatrix} 0 \\ \bar{C}_i^{\mathrm{T}} \\ \bar{D}_i^{\mathrm{T}} \end{bmatrix} \\ \begin{bmatrix} 0 & \bar{C}_i & \bar{D}_i \end{bmatrix} & -I \end{bmatrix} < 0$$

which is those in (9) for the discrete-time case. Now, we shall prove that the conditions in (9) and those in (8) have equivalent solvability (that is, the existence of $P_i > 0$ and K_i satisfying (8) implies the existence of $P_i > 0$ satisfying (9) and vice versa).

(8) $\Rightarrow$ (9): Suppose that the conditions in (8) are satisfied. Define a symbol T_i for $i \in \mathbb{M}$ as

$$T_i \triangleq \left[\begin{array}{cccc} \hat{B}_i C_i & \hat{A}_i & \hat{B}_i D_i & 0 \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & I \end{array}\right]$$

for which the row dimension is such that $H_i T_i = 0$ with H_i in (8). With $\bar{A}_i$ and $\bar{B}_i$ in W_i substituted by the expressions in (4), it can be shown that $\Psi_i T_i = W_i$. Hence, pre- and post-multiplying (8) by T_i^{T} and T_i, respectively, and noting that T_i has full column rank, we have

$$W_i^{\mathrm{T}} \Phi_i W_i = T_i^{\mathrm{T}} \left(\Psi_i^{\mathrm{T}} \Phi_i \Psi_i + G_i^{\mathrm{T}} K_i H_i + H_i^{\mathrm{T}} K_i^{\mathrm{T}} G_i \right) T_i < 0$$

that is, the conditions in (8) imply those in (9).

(9) $\Rightarrow$ (8): Define a symbol Γ_i as

$$\Gamma_i \triangleq \left[\begin{array}{cccc|c} \hat{B}_i C_i & \hat{A}_i & \hat{B}_i D_i & 0 & I \\ \hline I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & I & 0 \end{array}\right]$$

which satisfies the following relations:

$$\Psi_i \Gamma_i = \left[\begin{array}{cccc|c} A_i & 0 & B_i & 0 & 0 \\ \hat{B}_i C_i & \hat{A}_i & \hat{B}_i D_i & 0 & I \\ \hline I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ \hline 0 & 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & I & 0 \end{array}\right] = \begin{bmatrix} W_i & V_i \end{bmatrix} \tag{10}$$

$$G_i \Gamma_i = \left[\begin{array}{cccc|c} A_i + \varepsilon_1 I & 0 & B_i & 0 & 0 \\ \hat{B}_i C_i & \hat{A}_i + \varepsilon_2 I & \hat{B}_i D_i & 0 & I \end{array}\right] = J^{\mathrm{T}} \begin{bmatrix} W_i & V_i \end{bmatrix} \tag{11}$$

$$H_i \Gamma_i = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & -I \end{array}\right] = \begin{bmatrix} 0 & -I \end{bmatrix} \tag{12}$$

where

$$V_i = \left[\begin{array}{c} 0 \\ I \\ \hline 0 \\ 0 \\ \hline 0 \\ \hline 0 \end{array}\right], \qquad J = \left[\begin{array}{cc} I & 0 \\ 0 & I \\ \hline \varepsilon_1 I & 0 \\ 0 & \varepsilon_2 I \\ \hline 0 & 0 \\ \hline 0 & 0 \end{array}\right].$$

Combining (10)–(12), we have

$$\begin{aligned} &\Gamma_i^{\mathrm{T}} \left(\Psi_i^{\mathrm{T}} \Phi_i \Psi_i + G_i^{\mathrm{T}} K_i H_i + H_i^{\mathrm{T}} K_i^{\mathrm{T}} G_i \right) \Gamma_i \\ &= \begin{bmatrix} W_i^{\mathrm{T}} \\ V_i^{\mathrm{T}} \end{bmatrix} \Phi_i \begin{bmatrix} W_i & V_i \end{bmatrix} + \begin{bmatrix} W_i^{\mathrm{T}} \\ V_i^{\mathrm{T}} \end{bmatrix} J K_i \begin{bmatrix} 0 & -I \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ -I \end{bmatrix} K_i^{\mathrm{T}} J^{\mathrm{T}} \begin{bmatrix} W_i & V_i \end{bmatrix} \\ &= \begin{bmatrix} W_i^{\mathrm{T}} \Phi_i W_i & W_i^{\mathrm{T}} (\Phi_i V_i - J K_i) \\ * & V_i^{\mathrm{T}} \Phi_i V_i - V_i^{\mathrm{T}} J K_i - K_i^{\mathrm{T}} J^{\mathrm{T}} V_i \end{bmatrix}. \end{aligned}$$

Since Γ_i is invertible, it follows from the previous equation that the inequalities in (8) are satisfied if

$$\begin{bmatrix} W_i^{\mathrm{T}} \Phi_i W_i & W_i^{\mathrm{T}} (\Phi_i V_i - J K_i) \\ * & V_i^{\mathrm{T}} \Phi_i V_i - V_i^{\mathrm{T}} J K_i - K_i^{\mathrm{T}} J^{\mathrm{T}} V_i \end{bmatrix} < 0. \tag{13}$$

Therefore, we only need to show (9) $\Rightarrow$ (13) in the sequel.

For the continuous-time case, (13) can be explicitly expressed as

$$\begin{bmatrix} W_i^{\mathrm{T}} \Phi_i W_i & W_i^{\mathrm{T}} L_i^{\mathrm{T}} \\ * & -K_{i2} - K_{i2}^{\mathrm{T}} \end{bmatrix} < 0 \tag{14}$$

where

$$L_i = \left[\begin{array}{cc|cc|c|c} -K_{i1}^{\mathrm{T}} & -K_{i2}^{\mathrm{T}} & P_{i12}^{\mathrm{T}} - \varepsilon_1 K_{i1}^{\mathrm{T}} & P_{i22}^{\mathrm{T}} - \varepsilon_2 K_{i2}^{\mathrm{T}} & 0 & 0 \end{array}\right]$$

and K_{i1} and K_{i2} are the upper and lower n_x rows of K_i, respectively, and P_{i12} and P_{i22} are the upper and lower right $n_x \times n_x$ blocks of P_i, respectively. Suppose that the inequalities in (9) hold for some

$P_i > 0$ for all $i \in \mathbb{M}$. Note that $P_{i22} > 0$ is implied by $P_i > 0$. Let δ_1 and δ_2 be two scalars such that $W_i^{\mathrm{T}}\Phi_i W_i < -\delta_1\mathbf{I} < 0$ and $W_i^{\mathrm{T}}N_i^{\mathrm{T}}P_{i22}^{-1}N_iW_i < \delta_2\mathbf{I}$, where

$$N_i = \begin{bmatrix} -P_{i12}^{\mathrm{T}} & -P_{i22}^{\mathrm{T}} & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Further let ϵ be a scalar satisfying $0 \le \frac{\epsilon}{2} \le \frac{\delta_1}{\delta_2}$, then we have

$$W_i^{\mathrm{T}}\Phi_i W_i < -\delta_1\mathbf{I} \le -\frac{\delta_1}{\delta_2}W_i^{\mathrm{T}}N_i^{\mathrm{T}}P_{i22}^{-1}N_iW_i \le -\frac{\epsilon}{2}W_i^{\mathrm{T}}N_i^{\mathrm{T}}P_{i22}^{-1}N_iW_i$$

which implies

$$W_i^{\mathrm{T}}\Phi_i W_i + \frac{1}{2}\epsilon W_i^{\mathrm{T}}N_i^{\mathrm{T}}P_{i22}^{-1}N_iW_i < 0. \tag{15}$$

By applying Schur Complement Equivalence, (15) is equivalent to

$$\begin{bmatrix} W_i^{\mathrm{T}}\Phi_i W_i & \epsilon W_i^{\mathrm{T}}N_i^{\mathrm{T}} \\ * & -2\epsilon P_{i22} \end{bmatrix} < 0$$

which are the inequalities in (14) with

$$K_{i1} = \varepsilon_1^{-1}P_{i12}, \qquad K_{i2} = \varepsilon_2^{-1}P_{i22}, \quad \varepsilon_1 = \varepsilon_2 = \epsilon^{-1}.$$

Consequently, we have the implication (9) ⇒ (15) ⇒ (14) ⇔ (13). To complete the proof of the implication (9) ⇒ (8) for the continuous-time case, finally note that ε_1 and ε_2 can be fixed as a sufficient large positive scalar without any loss of generality.

For the discrete-time case, (14) should be re-written as

$$\begin{bmatrix} W_i^{\mathrm{T}}\Phi_i W_i & W_i^{\mathrm{T}}L_i^{\mathrm{T}} \\ * & \boldsymbol{P}_{i22} - K_{i2} - K_{i2}^{\mathrm{T}} \end{bmatrix} < 0 \tag{16}$$

where L_i is re-defined as

$$L_i = \left[\begin{array}{cc|c|c|c} \boldsymbol{P}_{i12}^{\mathrm{T}} - K_{i1}^{\mathrm{T}} & \boldsymbol{P}_{i22}^{\mathrm{T}} - K_{i2}^{\mathrm{T}} & 0 \quad 0 & 0 & 0 \end{array}\right]$$

and $\boldsymbol{P}_{i12}$ and $\boldsymbol{P}_{i22}$ are the upper and lower right $n_x \times n_x$ blocks of $\boldsymbol{P}_i$, respectively. Suppose that the inequalities in (9) hold for some $P_i > 0$ for all $i \in \mathbb{M}$. Since $\boldsymbol{P}_{i22} > 0$ is guaranteed by $P_i > 0$ for all $i \in \mathbb{M}$, we can augment the inequalities in (9) as

$$\begin{bmatrix} W_i^{\mathrm{T}}\Phi_i W_i & 0 \\ 0 & -\boldsymbol{P}_{i22} \end{bmatrix} < 0 \tag{17}$$

which are the inequalities in (16) with

$$K_{i1} = \boldsymbol{P}_{i12}, \qquad K_{i2} = \boldsymbol{P}_{i22}.$$

Thus, we establish the implication (9) ⇒ (17) ⇒ (16) ⇔ (13) ⇒ (8) for the discrete-time case.

Consequently, we have proven (9) ⇔ (8) and using Lemma 1, the proof can be completed. ■

Compared with Lemma 1, the benefit of the conditions in (8) is that no product term between the Lyapunov matrices P_i and the filter parameters $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$ and $\hat{D}_i$ exists in these conditions. It is known that such a feature can be made use of to relax the restrictions imposed on the Lyapunov matrices P_i in the filter design problem with uncertain transition probabilities and/or the mode-independent filter design problem (see de Souza et al., 2006, Liu et al., 2008 for continuous-time MJLSs and de Souza, 2003, Gonçalves et al., 2011, Morais et al., 2014 and Zhang & Boukas, 2009a for discrete-time MJLSs).

It should be emphasized that the method for decoupling the mentioned product terms is *different* from those in the literature. To see the differences, we specifically compare Theorem 1 in this paper with Liu et al. (2008, Theorem 3.1) and with de Souza (2003, Theorem 3.1):

- Besides the Lyapunov matrices, Liu et al. (2008, Theorem 3.1) still includes $8n_x^2$ extra scalar variables (in terms of G_i and Z_i therein) for each $i \in \mathbb{M}$, while Theorem 1 includes only $2n_x^2$ extra scalar variables (in terms of K_i here), much fewer than the former.
- Similarly, de Souza (2003, Theorem 3.1) includes $4n_x^2$ extra scalar variables (in terms of G_i therein) for each $i \in \mathbb{M}$, which is still more than the number of the extra scalar variables in Theorem 1 in this paper.

The above differences show that, although the existing conditions, Liu et al. (2008, Theorem 3.1) and de Souza (2003, Theorem 3.1), and Theorem 1 in this paper all are necessary and sufficient for H_∞ filtering analysis for MJLSs with exactly known transition probabilities, Theorem 1 in this paper has fewer variables for H_∞ filtering analysis, thanks to which, a straightforward consequence is that our filter method in the sequel is *less computationally demanding* than some of the existing ones in the literature. Besides, it will be shown later that our filter design method has an advantage with respect to conservatism reduction when compared with some existing ones.

3.2. H_∞ filter design

The following theorem presents a sufficient condition for the existence of filters that guarantee the stochastic stability and an H_∞ filtering performance level for MJLSs with uncertain transition probabilities. It can be obtained by further imposing a constraint on the extra variables K_i in Theorem 1.

Theorem 2. *Consider the system in (1) with uncertain transition probabilities $\pi_{ij}(\lambda_i)$ for $\lambda_i \in \Lambda_i$ and $i, j \in \mathbb{M}$. Given scalars $\gamma > 0$, ε_1 and ε_2, a filter in (3) exists such that the filtering error system in (4) is stochastically stable with an H_∞ performance level γ if there exist matrices $P_i(\lambda) > 0$, $\mathscr{K}_i$, $\mathscr{A}_i$, $\mathscr{B}_i$, $\mathscr{C}_i$ and $\mathscr{D}_i$ for $i \in \mathbb{M}$ such that*

$$\Psi_i^{\mathrm{T}}\Phi_i(\lambda)\Psi_i + \mathscr{G}_i^{\mathrm{T}}\mathscr{H}_i + \mathscr{H}_i^{\mathrm{T}}\mathscr{G}_i < 0, \quad \forall (i, \lambda) \in \mathbb{M} \times \Lambda \tag{18}$$

where $\Phi_i(\lambda)$ is Φ_i in (7) with π_i, P_i, $\hat{C}_i$ and $\hat{D}_i$ replaced by $\pi_i(\lambda_i)$, $P_i(\lambda)$, $\mathscr{C}_i$ and $\mathscr{D}_i$, respectively, Ψ_i is in (8) and

$$\mathscr{G}_i = \begin{bmatrix} I & A_i + \varepsilon_1 I & \varepsilon_2 I & B_i & 0 \end{bmatrix}$$
$$\mathscr{H}_i = \begin{bmatrix} -\mathscr{K}_i & \mathscr{B}_iC_i & \mathscr{A}_i & \mathscr{B}_iD_i & 0 \end{bmatrix}.$$

Moreover, if the conditions in (18) are feasible, the filter parameters can be given by

$$\hat{A}_i = \mathscr{K}_i^{-1}\mathscr{A}_i, \qquad \hat{B}_i = \mathscr{K}_i^{-1}\mathscr{B}_i, \qquad \hat{C}_i = \mathscr{C}_i, \hat{D}_i = \mathscr{D}_i. \tag{19}$$

Proof. Suppose that the conditions in (18) are feasible for some matrices $P_i(\lambda) > 0$ and $\mathscr{K}_i$, $\mathscr{A}_i$, $\mathscr{B}_i$, $\mathscr{C}_i$, $\mathscr{D}_i$ ($i \in \mathbb{M}$). Construct a candidate filter in (3) with the parameters given by (19), then we have

$$\begin{aligned} &\Psi_i^{\mathrm{T}}\Phi_i(\lambda)\Psi_i + G_i^{\mathrm{T}}K_iH_i + H_i^{\mathrm{T}}K_i^{\mathrm{T}}G_i \\ &= \Psi_i^{\mathrm{T}}\Phi_i(\lambda)\Psi_i + \mathscr{G}_i^{\mathrm{T}}\mathscr{H}_i + \mathscr{H}_i^{\mathrm{T}}\mathscr{G}_i < 0, \forall (i, \lambda) \in \mathbb{M} \times \Lambda \end{aligned} \tag{20}$$

where G_i and H_i are defined as in (8) and K_i is specified as

$$K_i = \begin{bmatrix} \mathscr{K}_i \\ \mathscr{K}_i \end{bmatrix}. \tag{21}$$

According to Theorem 1, the conditions in (20) guarantee that, for the system in (1) and the filter parameters given by (19), the filtering error system in (4) is stochastically stable with an H_∞ performance level γ for all $\lambda \in \mathbb{M}$. The proof is completed. ■

The conditions in (18) are linear with respect to $P_i(\lambda)$, $\mathscr{K}_i$, $\mathscr{A}_i$, $\mathscr{B}_i$, $\mathscr{C}_i$ and $\mathscr{D}_i$. Due to the dependence on the uncertain parameter λ, there are an infinite number of LMIs in (18) needing to be checked. To make these conditions numerically tractable, we in the next employ a relaxation method similar to Morais et al. (2014, 2013) to obtain an alternative set of a finite number of LMIs such that those in (18) are guaranteed for all $\lambda \in \Lambda$. Consider the parameter-dependent Lyapunov matrix $P_i(\lambda)$ given by

$$P_i(\lambda) = \sum_{k \in \mathscr{I}(g)} \lambda^k P_{i,k} \tag{22}$$

where $g = (g_1, g_2, \ldots, g_M) \in \mathbb{N}^M$ is the degree vector, $k = (k_1, k_2, \ldots, k_M)$ with $k_i = (k_{i1}, k_{i2}, \ldots, k_{iS_i}) \in \mathbb{N}^{S_i}$ is the exponent vector, $\lambda^k = \lambda_1^{k_1}\lambda_2^{k_2}\ldots\lambda_M^{k_M}$ with $\lambda_i^{k_i} = \lambda_{i1}^{k_{i1}}\lambda_{i2}^{k_{i2}}\ldots\lambda_{iS_i}^{k_{iS_i}}$ is a homogeneous monomial in λ and $P_{i,k}$ is the corresponding coefficient matrix. The set $\mathscr{I}(g)$ is defined as $\mathscr{I}(g) \triangleq \mathscr{I}_1(g_1) \times \mathscr{I}_2(g_2) \times \cdots \times \mathscr{I}_M(g_M)$ with $\mathscr{I}_i(g_i) \subset \mathbb{N}^{S_i}$ being the set of all possible S_i-tuple vectors such that for any $k \in \mathscr{I}(g)$, $\sum_{s=1}^{S_i} k_{is} = g_i$, $i \in \mathbb{M}$. In other words, $P_i(\lambda)$ given by (22) represents a homogeneous polynomial matrix with g_r as the partial degree of variable λ_r. For two exponent vectors $k, l \in \mathscr{I}(g)$, operation $k \geq l$ or $k - l \geq 0$ indicates that no element of $k - l$ is smaller than zero. For later use, denote by $\mathbf{1}$ a vector of ones, and for a vector k_i with one non-zero element, denote by $\upsilon(k_i)$ the position of the non-zero element. In addition, define a function $\phi(h)$ as $\phi(h) = h_1!h_2!\ldots h_n!$ for $h = (h_1, h_2, \ldots, h_n) \in \mathbb{N}^n$, and further define a symbol $\Xi_{i,k,l,h}$ as

$$\Xi_{i,k,l,h}^{\mathrm{ct}} \triangleq \begin{bmatrix} 0 & P_{i,k-l-h} & 0 & 0 \\ P_{i,k-l-h} & \sum_{j=1}^{M} \pi_{ij}^{(\upsilon(h_i))} P_{j,k-l-h} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Xi_{i,k,l,h}^{\mathrm{dt}} \triangleq \mathrm{diag}\left\{ \sum_{j=1}^{M} \pi_{ij}^{(\upsilon(h_i))} P_{j,k-l-h}, -P_{i,k-l-h}, 0, 0 \right\}.$$

With the above preparation, we can obtain the following theorem which provides a sufficient condition consisting of a finite number of LMIs for the existence of a filter with a guaranteed H_∞ filtering performance.

Theorem 3. *Consider the system in (1) with uncertain transition probabilities $\pi_{ij}(\lambda_i)$ for $\lambda_i \in \Lambda_i$ and $i, j \in \mathbb{M}$. Given scalars $\gamma > 0$, ε_1, ε_2 and vectors $g, d, b \in \mathbb{N}^M$, a filter in (3) exists such that the filtering error system in (4) is stochastically stable with an H_∞ performance level γ if there exist matrices $P_{i,k}$, $k \in \mathscr{I}(g)$, $\mathscr{K}_i$, $\mathscr{A}_i$, $\mathscr{B}_i$, $\mathscr{C}_i$ and $\mathscr{D}_i$ for $i \in \mathbb{M}$ such that*

$$\mathscr{P}_{i,k} > 0, \quad \forall (i,k) \in \mathbb{M} \times \mathscr{I}(g+d) \tag{23}$$

$$\mathscr{Q}_{i,k} < 0, \quad \forall (i,k) \in \mathbb{M} \times \mathscr{I}(g+b+\mathbf{1}) \tag{24}$$

where

$$\mathscr{P}_{i,k} = \sum_{l \in \mathscr{I}(d), l \leq k} \frac{\phi(d)}{\phi(l)} P_{i,k-l}$$

$$\mathscr{Q}_{i,k} = \sum_{\substack{l \in \mathscr{I}(b) \\ l \leq k}} \sum_{\substack{h \in \mathscr{I}(\mathbf{1}) \\ h \leq k-l}} \frac{\phi(b)}{\phi(l)} \Psi_i^{\mathrm{T}} \Xi_{i,k,l,h} \Psi_i + \frac{\phi(g+b+\mathbf{1})}{\phi(k)} \left(\Psi_i^{\mathrm{T}} \Upsilon_i \Psi_i + \mathscr{G}_i^{\mathrm{T}} \mathscr{H}_i + \mathscr{H}_i^{\mathrm{T}} \mathscr{G}_i \right)$$

$$\Upsilon_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{C}_i^{\mathrm{T}} \\ 0 & 0 & -\gamma^2 I & \bar{D}_i^{\mathrm{T}} \\ 0 & \bar{C}_i & \bar{D}_i & -I \end{bmatrix} \tag{25}$$

and $\hat{C}_i$, $\hat{D}_i$ in $\bar{C}_i$ and $\bar{D}_i$ are replaced by $\mathscr{C}_i$ and $\mathscr{D}_i$, respectively, and Ψ_i, $\mathscr{G}_i$ and $\mathscr{H}_i$ are as in (18).

Moreover, if the conditions in (28) are feasible, the filter parameters can be given by (19).

Proof. Using the homogeneous polynomial representation of $P_i(\lambda)$ in (22) and noting the fact that

$$\prod_{r=1}^{M} \left(\sum_{s=1}^{S_r} \lambda_{rs} \right)^{d_r} = \left(\sum_{l \in \mathscr{I}(d)} \frac{\phi(d)}{\phi(l)} \lambda^l \right) = 1$$

for any degree $d \in \mathbb{N}^M$, we have

$$\begin{aligned} P_i(\lambda) &= \prod_{r=1}^{M} \left(\sum_{s=1}^{S_r} \lambda_{rs} \right)^{d_r} P_i(\lambda) \\ &= \left(\sum_{l \in \mathscr{I}(d)} \frac{\phi(d)}{\phi(l)} \lambda^l \right) \left(\sum_{k \in \mathscr{I}(g)} \lambda^k P_{i,k} \right) \\ &= \sum_{k \in \mathscr{I}(g+d)} \lambda^k \left(\sum_{l \in \mathscr{I}(d), l \leq k} \frac{\phi(d)}{\phi(l)} P_{i,k-l} \right) \\ &= \sum_{k \in \mathscr{I}(g+d)} \lambda^k \mathscr{P}_{i,k}. \end{aligned}$$

Thus, the inequalities in (23) guarantee $P_i(\lambda) > 0$ for all $i \in \mathbb{M}$ and $\lambda \in \Lambda$. Similarly, for any $b \in \mathbb{N}^M$, it follows that

$$\begin{aligned} &P_i(\lambda) \\ &= \prod_{r=1}^{M} \left(\sum_{s=1}^{S_r} \lambda_{rs} \right)^{b_r+1} P_i(\lambda) \\ &= \sum_{k \in \mathscr{I}(g+b+\mathbf{1})} \lambda^k \left(\sum_{l \in \mathscr{I}(b+\mathbf{1}), l \leq k} \frac{\phi(b+\mathbf{1})}{\phi(l)} P_{i,k-l} \right) \\ &= \sum_{k \in \mathscr{I}(g+b+\mathbf{1})} \lambda^k \left(\sum_{\substack{l \in \mathscr{I}(b) \\ l \leq k}} \sum_{\substack{h \in \mathscr{I}(\mathbf{1}) \\ h \leq k-l}} \frac{\phi(b)}{\phi(l)} P_{i,k-l-h} \right) \end{aligned} \tag{26}$$

$$\begin{aligned} &\sum_{j=1}^{M} \pi_{ij}(\lambda_i) P_j(\lambda) \\ &= \prod_{r=1}^{M} \left(\sum_{s=1}^{S_r} \lambda_{rs} \right)^{b_r} \prod_{\substack{t=1 \\ t \neq i}}^{M} \left(\sum_{s=1}^{S_t} \lambda_{ts} \right) \left(\sum_{j=1}^{M} \pi_{ij}(\lambda_i) P_j(\lambda) \right) \\ &= \sum_{k \in \mathscr{I}(g+b+\mathbf{1})} \lambda^k \left(\sum_{\substack{l \in \mathscr{I}(b) \\ l \leq k}} \sum_{\substack{h \in \mathscr{I}(\mathbf{1}) \\ h \leq k-l}} \frac{\phi(b)}{\phi(l)} \sum_{j=1}^{M} \pi_{ij}^{(\upsilon(h_i))} P_{j,k-l-h} \right). \end{aligned} \tag{27}$$

Decompose the matrix $\Phi_i(\lambda)$ in (18) as $\Phi_i(\lambda) = \Xi_i(\lambda) + \Upsilon_i$, where Υ_i is defined in (25) and $\Xi_i(\lambda)$ consists of the remaining terms in $\Phi_i(\lambda)$. By combining (26) and (27), it can be verified that

$$\begin{aligned} &\Psi_i^{\mathrm{T}} \Phi_i(\lambda) \Psi_i + \mathscr{G}_i^{\mathrm{T}} \mathscr{H}_i + \mathscr{H}_i^{\mathrm{T}} \mathscr{G}_i \\ &= \Psi_i^{\mathrm{T}} \Xi_i(\lambda) \Psi_i + \left(\Psi_i^{\mathrm{T}} \Upsilon_i \Psi_i + \mathscr{G}_i^{\mathrm{T}} \mathscr{H}_i + \mathscr{H}_i^{\mathrm{T}} \mathscr{G}_i \right) \\ &= \sum_{k \in \mathscr{I}(g+b+\mathbf{1})} \lambda^k \left(\sum_{\substack{l \in \mathscr{I}(b) \\ l \leq k}} \sum_{\substack{h \in \mathscr{I}(\mathbf{1}) \\ h \leq k-l}} \frac{\phi(b)}{\phi(l)} \Psi_i^{\mathrm{T}} \Xi_{i,k,l,h} \Psi_i \right) \\ &\quad + \prod_{r=1}^{M} \left(\sum_{s=1}^{S_r} \lambda_{rs} \right)^{g_r+b_r+1} \left(\Psi_i^{\mathrm{T}} \Upsilon_i \Psi_i + \mathscr{G}_i^{\mathrm{T}} \mathscr{H}_i + \mathscr{H}_i^{\mathrm{T}} \mathscr{G}_i \right) \\ &= \sum_{k \in \mathscr{I}(g+b+\mathbf{1})} \lambda^k \mathscr{Q}_{i,k}. \end{aligned}$$

Thus the inequalities in (24) guarantee the satisfaction of those in (18) for all $i \in \mathbb{M}$ and $\lambda \in \Lambda$. Finally, by virtue of Theorem 2, the proof can be completed. ■

The conditions in (23) and (24) include three degree vectors g, d and b, of which g is the degree of the Lyapunov matrices as in (22) while d and b are the levels of Pólya's relaxation (see Morais et al., 2015, 2013). By increasing the degrees, Theorem 3 becomes less conservative but requires more computational demand.

In order to highlight the contribution of this paper, we here are also interested in the case that the Lyapunov matrices $P_i(\lambda)$ are fixed as an uncertainty-independent form, P_i, for each $i \in \mathbb{M}$. Under this setting, the conditions in (18) are simplified to a finite number of LMIs as the following corollary shows.

Corollary 1. *Consider the system in (1) with uncertain transition probabilities $\pi_{ij}(\lambda_i)$ for $\lambda_i \in \Lambda_i$ and $i, j \in \mathbb{M}$. Given scalars $\gamma > 0$, ε_1 and ε_2, a filter in (3) exists such that the filtering error system in (4) is stochastically stable with an H_∞ performance level γ if there exist matrices $P_i > 0$, $\mathscr{K}_i$, $\mathscr{A}_i$, $\mathscr{B}_i$, $\mathscr{C}_i$ and $\mathscr{D}_i$ for $i \in \mathbb{M}$ such that*

$$\Psi_i^{\mathrm{T}}\Phi_i^{(s)}\Psi_i + \mathscr{G}_i^{\mathrm{T}}\mathscr{H}_i + \mathscr{H}_i^{\mathrm{T}}\mathscr{G}_i < 0, \quad \forall (i, s) \in \mathbb{M} \times \mathbb{S}_i \tag{28}$$

where $\Phi_i^{(s)}$ is Φ_i in (7) with π_i, $\hat{C}_i$, $\hat{D}_i$ replaced by $\pi_i^{(s)}$, $\mathscr{C}_i$ and $\mathscr{D}_i$, respectively, and Ψ_i, $\mathscr{G}_i$ and $\mathscr{H}_i$ are as in (18).

Moreover, if the conditions in (28) are feasible, the filter parameters can be given by (19).

Proof. Constraining $P_i(\lambda)$ as $P_i(\lambda) = P_i$ for all $\lambda \in \Lambda$ and noting $\sum_{s=1}^{S_i} \lambda_i^{(s)} = 1$, we have

$$\begin{aligned}&\Psi_i^{\mathrm{T}}\Phi_i(\lambda)\Psi_i + \mathscr{G}_i^{\mathrm{T}}\mathscr{H}_i + \mathscr{H}_i^{\mathrm{T}}\mathscr{G}_i \\ &\quad = \sum_{s=1}^{S_i} \lambda_i^{(s)} \left(\Psi_i^{\mathrm{T}}\Phi_i^{(s)}\Psi_i + \mathscr{G}_i^{\mathrm{T}}\mathscr{H}_i + \mathscr{H}_i^{\mathrm{T}}\mathscr{G}_i\right)\end{aligned}$$

that is, the inequalities in (18) for each $i \in \mathbb{M}$ are affine with respect to λ_i. Therefore, under the setting $P_i(\lambda) = P_i$, the conditions in (18) are equivalent to those in (28). Then applying Theorem 2 can complete the proof. ■

The proposed method can be modified for designing partly-mode-dependent (or mode-cluster-dependent) filters (Gonçalves et al., 2009) or mode-independent filters (de Souza et al., 2006). To show the two cases, suppose that the modes of the original MJLS are grouped into N clusters and the mode indices in the nth cluster are denoted by a set $\mathbb{M}_n$ that satisfies $\mathbb{M} = \bigcup_{n=1}^{N} \mathbb{M}_n$ and $\mathbb{M}_i \bigcap \mathbb{M}_j = \emptyset$ for $i \neq j$. A mode-cluster-dependent filter is concerned with a filter that, for the nth cluster, satisfies $\hat{A}_i = \tilde{A}_n$, $\hat{B}_i = \tilde{B}_n$, $\hat{C}_i = \tilde{C}_n$ and $\hat{D}_i = \tilde{D}_n$ for $i \in \mathbb{M}_n$, which means that the filter has the same parameters, $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$ and $\tilde{D}_n$ for the switching signal $r_k \in \mathbb{M}_n$. To apply the proposed method, for instance, Corollary 1, we only need to replace $\mathscr{K}_i$, $\mathscr{A}_i$, $\mathscr{B}_i$, $\mathscr{C}_i$ and $\mathscr{D}_i$ by the new matrix variables $\tilde{\mathscr{K}}_n$, $\tilde{\mathscr{A}}_n$, $\tilde{\mathscr{B}}_n$, $\tilde{\mathscr{C}}_n$ and $\tilde{\mathscr{D}}_n$, respectively, for all $n \in \{1, 2, \ldots, N\}$ and $i \in \mathbb{M}_n$. A mode-independent filter is a special case of a mode-cluster-dependent one with $N = 1$ and $\mathbb{M}_1 = \mathbb{M}$. Let $\mathscr{K}_i = \mathscr{K}$, $\mathscr{A}_i = \mathscr{A}$, $\mathscr{B}_i = \mathscr{B}$, $\mathscr{C}_i = \mathscr{C}$ and $\mathscr{D}_i = \mathscr{D}$ for all $i \in \mathbb{M}$, then the proposed method can be applied to design mode-independent filters.

Remark 1. It has been shown that, for filtering performance analysis with known transition probabilities, appropriately fixing the scalars ε_1 and ε_2 in Theorem 1 will not introduce conservatism. However, this is not the case for Theorem 2 and Corollary 1 because they only give sufficient conditions for the existence of a required filter. In other words, given different values of ε_1 and ε_2, Corollary 1 usually leads to different design results. Thus, one may tune the scalars for Corollary 1 so as to further improve the guaranteed filter performance, that is, reduce the H_∞ filtering performance level γ. According to the existing numerical results in the literature, the settings $\varepsilon_1 = \varepsilon_2 = 1$ and $\varepsilon_1 = \varepsilon_2 = 0$ are usually chosen for the continuous- and discrete-time case, respectively, which will also be adopted for the numerical examples in this paper. A straightforward way to achieve better results is trial and error (for instance, check some gridding points of a given $(\varepsilon_1, \varepsilon_2)$ domain); another approach is to utilize the function `fminsearch` in Matlab with $(\varepsilon_1, \varepsilon_2)$ as the argument and the minimization of γ subject to the LMIs in (23) and (24) or (28) as the objective.

3.3. Comparisons with some existing results

In this section, we theoretically compare the proposed filter design method with some existing representatives in order to show that our method is either less conservative than or equivalent to but less computationally demanding than them.

3.3.1. Comparison with Liu et al. (2008)

In Liu et al. (2008), the design problem of mode-independent filters for continuous-time MJLSs was considered. Instead of directly comparing the LMI approaches to filter design, that is, Corollary 1 in this paper and Theorem 3.2 in the reference, we resort to the H_∞ filtering analysis conditions for the sake of statement convenience. Keep in mind that the filter discussed in this subsection is strictly proper, mode-independent and the MJLS has exactly known transition probabilities.

On one hand, note that Theorem 3.2 in Liu et al. (2008) was obtained by applying a congruence transformation to the H_∞ filtering analysis condition, Theorem 3.1 therein, and further restricting $G_i = Z_i = G$ for all $i \in \mathbb{M}$. Hence, it is not difficult to show that the optimal H_∞ filtering performance obtained by Theorem 3.2 in the reference is equivalent to

$$\gamma_1^* = \min_{\gamma, P_i > 0, G, \hat{A}, \hat{B}, \hat{C}} \gamma \; : \; \text{s.t. (29)}$$

where

$$\Theta_i = \begin{bmatrix} \bar{A}_i^{\mathrm{T}}G + G^{\mathrm{T}}\bar{A}_i + \boldsymbol{P}_i & \bar{A}_i^{\mathrm{T}}G + P_i - G^{\mathrm{T}} & G\bar{B}_i & \bar{C}_i^{\mathrm{T}} \\ * & -G - G^{\mathrm{T}} & G\bar{B}_i & 0 \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix} < 0, \; i \in \mathbb{M}. \tag{29}$$

For convenience, the above condition has been re-written with the notation in this paper. Introduce a matrix U_i and partition the matrix G as

$$U_i = \left[\begin{array}{ccccc} 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ \hline 0 & A_i & 0 & B_i & 0 \\ I & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & 0 & I \end{array}\right], \qquad G = \begin{bmatrix} \bullet & \bullet \\ \mathscr{K}_1^{\mathrm{T}} & \mathscr{K}_2^{\mathrm{T}} \end{bmatrix} \tag{30}$$

where $\mathscr{K}_1$ and $\mathscr{K}_2$ are $n_x \times n_x$ matrix blocks and "$\bullet$" denotes matrices that do not matter with the following derivations. Then multiplying Θ_i on the left and right by U_i^{T} and U_i, respectively, we can show

$$U_i^{\mathrm{T}}\Theta_i U_i = \Psi_i^{\mathrm{T}}\Phi_i\Psi_i + G_i^{\mathrm{T}}K_iH_i + H_i^{\mathrm{T}}K_i^{\mathrm{T}}G_i$$

where $\hat{A}_i = \hat{A}$, $\hat{B}_i = \hat{B}$, $\hat{C}_i = \hat{C}$ in Φ_i and H_i, $K_i = [\mathscr{K}_1^{\mathrm{T}} \quad \mathscr{K}_2^{\mathrm{T}}]^{\mathrm{T}}$ and $\varepsilon_1 = \varepsilon_2 = 1$ in G_i (see (8) for symbols Φ_i, Ψ_i, G_i, H_i and K_i). The above relation implies that the solvability of the inequalities in (29) guarantee that of those in (8) with $K_i = [\mathscr{K}_1^{\mathrm{T}} \quad \mathscr{K}_2^{\mathrm{T}}]^{\mathrm{T}}$ and $\varepsilon_1 = \varepsilon_2 = 1$. However, the converse is *not* always true, because

U_i is of full column rank only. In other words, we obtain $\gamma_1^* \geq \gamma_2^*$, where

$$\gamma_2^* = \min_{\gamma, P_i>0, K_i, \hat{A}, \hat{B}, \hat{C}} \gamma \;:\; \text{s.t. (8) with } \begin{cases} K_i = [\mathscr{K}_1^{\mathrm{T}} \quad \mathscr{K}_2^{\mathrm{T}}]^{\mathrm{T}} \\ \text{and } \varepsilon_1 = \varepsilon_2 = 1. \end{cases} \tag{31}$$

On the other hand, by way of Theorem 2, it is seen that Corollary 1 in this paper for mode-independent filter design is derived from Theorem 1 with K_i specified as $K_i = [\mathscr{K}^{\mathrm{T}} \quad \mathscr{K}^{\mathrm{T}}]^{\mathrm{T}}$. Thus, the optimal H_∞ filtering performance level guaranteed by Corollary 1 under $\varepsilon_1 = \varepsilon_2 = 1$ is equal to

$$\gamma^* = \min_{\gamma, P_i>0, K_i, \hat{A}, \hat{B}, \hat{C}} \gamma \;:\; \text{s.t. (8) with } \begin{cases} K_i = [\mathscr{K}^{\mathrm{T}} \quad \mathscr{K}^{\mathrm{T}}]^{\mathrm{T}} \\ \text{and } \varepsilon_1 = \varepsilon_2 = 1. \end{cases} \tag{32}$$

We give the following result on the relation between γ^* and γ_2^*.

Proposition 1. *Consider the system in (1) with known transition probabilities π_{ij} for $i, j \in \mathbb{M}$ and define scalars γ_2^* and γ^* in (31) and (32), respectively. Then $\gamma^* = \gamma_2^*$.*

Proof. The case $\gamma^* \geq \gamma_2^*$ is obviously true. We show $\gamma^* \leq \gamma_2^*$ in the next. To this end, suppose that there exist γ, $P_i > 0$, $K_i = [\mathscr{K}_1^{\mathrm{T}} \quad \mathscr{K}_2^{\mathrm{T}}]^{\mathrm{T}}$, $\hat{A}$, $\hat{B}$ and $\hat{C}$ such that (8) under $\varepsilon_1 = \varepsilon_2 = 1$ holds. Define a matrix

$$X = \mathrm{diag}\{\mathscr{K}_2^{-\mathrm{T}}\mathscr{K}_1^{\mathrm{T}}, I, \mathscr{K}_2^{-\mathrm{T}}\mathscr{K}_1^{\mathrm{T}}, I, I\}. \tag{33}$$

Then performing a congruence transformation to the inequalities in (8) by X, we have

$$X^{\mathrm{T}}\left(\Psi_i^{\mathrm{T}}\Phi_i\Psi_i + G_i^{\mathrm{T}}K_iH_i + H_i^{\mathrm{T}}K_i^{\mathrm{T}}G_i\right)X < 0.$$

Note that the above inequality can be re-formulated as (8) with the following change of variables:

$$\mathscr{K}_1^{-\mathrm{T}}\mathscr{K}_2^{\mathrm{T}}\hat{A}\mathscr{K}_2^{-\mathrm{T}}\mathscr{K}_1^{\mathrm{T}} \to \hat{A}, \quad \mathscr{K}_1^{-\mathrm{T}}\mathscr{K}_2^{\mathrm{T}}\hat{B} \to \hat{B}, \quad \hat{C}\mathscr{K}_2^{-\mathrm{T}}\mathscr{K}_1^{\mathrm{T}} \to \hat{C}$$

$$\begin{bmatrix} \mathscr{K}_1\mathscr{K}_2^{-\mathrm{T}}\mathscr{K}_1^{\mathrm{T}} \\ \mathscr{K}_1\mathscr{K}_2^{-\mathrm{T}}\mathscr{K}_1^{\mathrm{T}} \end{bmatrix} \to \begin{bmatrix} \mathscr{K} \\ \mathscr{K} \end{bmatrix} = K_i$$

$$\mathrm{diag}\{I, \mathscr{K}_1\mathscr{K}_2^{-1}\}P_i\mathrm{diag}\{I, \mathscr{K}_2^{-\mathrm{T}}\mathscr{K}_1^{\mathrm{T}}\} \to P_i.$$

Therefore, there must exist $P_i > 0$, $K_i = [\mathscr{K}^{\mathrm{T}} \quad \mathscr{K}^{\mathrm{T}}]^{\mathrm{T}}$, $\hat{A}$, $\hat{B}$ and $\hat{C}$ and the same γ such that (8) also holds, implying that $\gamma^* \leq \gamma_2^*$. The proof is completed. ■

According to the above discussions, one sees that $\gamma^* = \gamma_2^* \leq \gamma_1^*$, showing that for mode-independent filter design, our method, Corollary 1, is less conservative than the one in Liu et al. (2008). Moreover, as aforementioned, Corollary 1 is also computationally advantageous over the one in Liu et al. (2008) because fewer variables are needed by our method for computing the filter parameters.

Remark 2. Specifically, the system considered in Liu et al. (2008) has no transition probability uncertainty. However, the method therein can also be extended to the case with transition probability uncertainty described by (2). In a general setting, this can be achieved by relaxing matrices W_{11i}, W_{21i} and W_{22i} in the reference to be λ-dependent, which is the same treatment used in deriving Theorem 2 in this paper. It is worth pointing out that, even compared with this adaptation, Theorem 2 in this paper still has advantages in both conservatism reduction and variable reduction. To see this, it is needed to relax the matrix P_i in (8) and (29) as $P_i(\lambda)$ and to note that all the derivations in this subsection still hold because the transformation matrices U_i in (30) and X in (33) are independent of the uncertain parameter λ. In Section 5, the simplest adaptation, $P_i(\lambda) = P_i$, for the uncertain probability case is specifically used for numerical illustration, which follows the same treatment of Corollary 1. As can be seen from Table 1, the advantages of Corollary 1 are obvious.

Table 1
γ^* and NoV for different methods in Example 1 (H_∞ filtering).

Method	Strictly proper		Proper	
	γ^*	NoV	γ^*	NoV
(Known Π:)				
de Souza et al. (2006)	0.7404	85	Inapplicable	
Liu et al. (2008)	0.3028	85	Inapplicable	
Morais et al. (2015)	0.2019	151	0.1880	152
Corollary 1	0.2019	67	0.1880	68
(Uncertain Π:)				
de Souza et al. (2006)	0.9264	85	Inapplicable	
Liu et al. (2008)	0.3111[a]	85	Inapplicable	
Morais et al. (2015) ($g = 0, h = 0$)	0.2076	151	0.1925	152
Theorem 3 ($g = 0$)	0.2076	67	0.1925	68
Morais et al. (2015) ($g = \mathbf{1}, h = 0$)	0.2059	277	0.1914	278
Morais et al. (2015) ($g = \mathbf{1}, h = \mathbf{1}$)	0.2059	529	0.1914	530
Theorem 3 ($g = \mathbf{1}$)	0.2059	193	0.1914	194

[a] The reference Liu et al. (2008) does not address this case directly, and the result is obtained from an adaptation according to Remark 2.

3.3.2. Comparison with Morais et al. (2014)

In Morais et al. (2014), the discrete-time case of a similar problem to the one in the paper was considered based on the multi-simplex representation method. As we mentioned, Theorem 2 in this paper combined with the multi-simplex representation method can produce a series of LMI conditions for filter design, leading to Theorem 3. Regardless of the degrees of the polynomials, the general formulation of the condition in Morais et al. (2014) used for computing the filtering performance level and the filter parameters is given by (15) therein, which with the notation in this paper is re-written as (34) which is given in Box I, where

$$Q_i(\lambda) = \begin{bmatrix} \bullet & \mathscr{K}_i \\ \bullet & \mathscr{K}_i \end{bmatrix}, \qquad X_i(\lambda) = \begin{bmatrix} \bullet & \varepsilon_1\mathscr{K}_i \\ \bullet & \varepsilon_2\mathscr{K}_i \end{bmatrix}$$
$$Y_i(\lambda) = \begin{bmatrix} \bullet & 0 \end{bmatrix}, \qquad Z_i(\lambda) = \begin{bmatrix} \bullet & 0 \end{bmatrix} \tag{35}$$

with "•" denoting matrices that do not matter with the following discussion. Multiplying $\Sigma_i(\lambda)$ on the left and right by U_i^{T} and U_i (see (30)), respectively, we have

$$U_i^{\mathrm{T}}\Sigma_i(\lambda)U_i = \Psi_i^{\mathrm{T}}\Phi_i(\lambda)\Psi_i + G_i^{\mathrm{T}}K_iH_i + H_i^{\mathrm{T}}K_i^{\mathrm{T}}G_i \tag{36}$$

where $K_i = [\mathscr{K}_i^{\mathrm{T}} \quad \mathscr{K}_i^{\mathrm{T}}]^{\mathrm{T}}$ (see (8) for other symbols). The above relation implies that the solvability of the inequalities in (34) for $P_i(\lambda)$, $Q_i(\lambda)$, $X_i(\lambda)$, $Y_i(\lambda)$, $Z_i(\lambda)$, $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$ and $\hat{D}_i$ guarantees that of those in (8) for some $P_i(\lambda)$, K_i, $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$ and $\hat{D}_i$ with $K_i = [\mathscr{K}_i^{\mathrm{T}} \quad \mathscr{K}_i^{\mathrm{T}}]^{\mathrm{T}}$.

On one hand, according to the proof of Theorem 2, one sees that the best filtering performance level achieved by Theorem 2 is equivalent to minimizing γ subject to (8) for $P_i(\lambda)$, K_i, $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$ and $\hat{D}_i$ with $K_i = [\mathscr{K}_i^{\mathrm{T}} \quad \mathscr{K}_i^{\mathrm{T}}]^{\mathrm{T}}$, which together with (36) shows that the best H_∞ filtering performance level achieved by Theorem 2 is not worse than that by the method in Morais et al. (2014). On the other hand, noting that the columns of U_i form the nullspace of $\begin{bmatrix} A_i & 0 & -I & 0 & B_i & 0 \end{bmatrix}$ and using Finsler's Lemma (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994), one can introduce extra slack matrix variables (that is, those denoted by "•") so as to recover (34) from (8) with $K_i = [\mathscr{K}_i^{\mathrm{T}} \quad \mathscr{K}_i^{\mathrm{T}}]^{\mathrm{T}}$. These discussions show that for the discrete-time case, Theorem 2 and the method in Morais et al. (2014) can attain the same best H_∞ filtering performance level, but obviously the former involves much *fewer* variables for computing the filter parameters.

For the polynomial relaxation conditions of a specific degree, that is, Theorem 1 in Morais et al. (2014) and Theorem 3 in this paper, a similar argument also follows. Actually, multiplying (8) of Morais et al. (2014) by U_i^{T} on the left and U_i on the right, respectively, one sees that the resulting term $U_i^{\mathrm{T}}\Psi_kU_i$ can be

$$\Sigma_i(\lambda) = \begin{bmatrix} \bar{A}_i^{\mathrm{T}}X_i^{\mathrm{T}}(\lambda) + X_i(\lambda)\bar{A}_i - P_i(\lambda) & \bar{A}_i^{\mathrm{T}}Q_i^{\mathrm{T}}(\lambda) - X_i(\lambda) & \Sigma_{31i}(\lambda) = X_i(\lambda)\bar{B}_i + \bar{A}_i^{\mathrm{T}}Y_i^{\mathrm{T}}(\lambda) & \bar{A}_i^{\mathrm{T}}Z_i^{\mathrm{T}}(\lambda) + \bar{C}_i^{\mathrm{T}} \\ * & \boldsymbol{P}_i(\lambda) - Q_i(\lambda) - Q_i^{\mathrm{T}}(\lambda) & \Sigma_{32i}(\lambda) = Q_i(\lambda)\bar{B}_i - Y_i^{\mathrm{T}}(\lambda) & -Z_i^{\mathrm{T}}(\lambda) \\ * & * & \Sigma_{33i}(\lambda) = \bar{B}_i^{\mathrm{T}}Y_i^{\mathrm{T}}(\lambda) + Y_i(\lambda)\bar{B}_i - \gamma^2 I & \bar{B}_i^{\mathrm{T}}Z_i^{\mathrm{T}}(\lambda) + \bar{D}_i^{\mathrm{T}} \\ * & * & * & -I \end{bmatrix} < 0 \quad (34)$$

Box I.

eliminated and the remaining two additive terms $U_i^{\mathrm{T}}(\Theta_k + \Phi_k)U_i$ are finally reduced to (24) in this paper with the relaxation level $b = w - g - \mathbf{1}$.[2] Furthermore, the continuous-time case of a similar problem has been studied in Morais et al. (2015) most recently. However, when system matrices are known, it is found that the slack matrices $G_{1_{i_k}}$, $G_{2_{i_k}}$, $K_{1_{i_k}}$, $K_{2_{i_k}}$, $Q_{1_{i_k}}$ and $F_{1_{i_k}}$ in the reference correspond to those in (35) denoted by "•" and thus can be eliminated using the same method. Consequently, Theorem 3 is less computationally demanding than the conditions in Morais et al. (2014, 2015) for filter design of MJLSs with known system matrices at each mode.

Remark 3. It should be clarified that the presented comparisons in this subsection are valid only when the matrix U_i in (30) are invariant for all $\lambda \in \Lambda$, which implies that system matrices A_i and B_i are exactly known, as is previously assumed. If system matrices also contain uncertainty, it is known that introducing extra slack matrices are useful to reduce conservatism (de Oliveira, Bernussou, & Geromel, 1999). In this paper, we focus on coping with uncertainties in the transition matrix only, and aim to exploit more efficient conditions for filter design for this specific case. If the system matrices at each mode in (1) are also uncertain, it is still suggested to apply the results, for instance, in Morais et al. (2015) to solve the corresponding filter design problem.

4. H_2 filtering

In this section, we consider the H_2 filtering counterpart for MJLSs with uncertain transition probabilities. New LMI conditions will be presented for H_2 filtering performance analysis and filter design, respectively.

4.1. H_2 filtering analysis

To deal with the H_2 performance of the filtering error system, we first introduce the following lemma, which for the continuous- and discrete-time cases can be found in, e.g., Costa et al. (1997) and de Farias et al. (2000), respectively.

Lemma 2. *Given initial mode probabilities μ_i, transition probabilities π_{ij} for $i, j \in \mathbb{M}$ and a scalar ρ, the system in (4) (suppose $\bar{D}(r_k) = 0$ for the continuous-time case) is stochastically stable with an H_2 performance level ρ if and only if there exist symmetric matrices $P_i > 0$, $\forall i \in \mathbb{M}$ such that for the continuous-time case,*

$$\begin{aligned} &\sum_{i=1}^{M} \mu_i \mathrm{Tr}\left(\bar{B}_i^{\mathrm{T}} P_i \bar{B}_i\right) < \rho^2, \\ &\bar{A}_i^{\mathrm{T}} P_i + P_i \bar{A}_i + \boldsymbol{P}_i + \bar{C}_i^{\mathrm{T}} \bar{C}_i < 0, \quad \forall i \in \mathbb{M} \end{aligned} \tag{37}$$

and for the discrete-time case,

$$\begin{aligned} &\sum_{i=1}^{M} \mu_i \mathrm{Tr}\left(\bar{B}_i^{\mathrm{T}} \boldsymbol{P}_i \bar{B}_i + \bar{D}_i^{\mathrm{T}} \bar{D}_i\right) < \rho^2, \\ &\bar{A}_i^{\mathrm{T}} \boldsymbol{P}_i \bar{A}_i - P_i + \bar{C}_i^{\mathrm{T}} \bar{C}_i < 0, \quad \forall i \in \mathbb{M} \end{aligned} \tag{38}$$

where $\boldsymbol{P}_i \triangleq \sum_{j=1}^{M} \pi_{ij} P_j$.

According to Lemma 2, the task of computing the H_2 performance level of the filtering error system in (4) is cast into the solvability of the inequalities in (37) or (38) for some matrices $P_i > 0$. By introducing an auxiliary matrix R_i such that $\bar{B}_i^{\mathrm{T}} P_i \bar{B}_i < R_i$ and applying Schur Complement Equivalence, the inequalities in (37) can be re-written as $\sum_{i=1}^{M} \mu_i \mathrm{Tr}\,(R_i) - \rho^2 < 0$ and

$$\bar{B}_i^{\mathrm{T}} P_i \bar{B}_i - R_i < 0, \begin{bmatrix} \bar{A}_i^{\mathrm{T}} P_i + P_i \bar{A}_i + \boldsymbol{P}_i & \bar{C}_i^{\mathrm{T}} \\ \bar{C}_i & -I \end{bmatrix} < 0. \tag{39}$$

Similarly, by introducing the matrix R_i such that $\bar{B}_i^{\mathrm{T}} \boldsymbol{P}_i \bar{B}_i + \bar{D}_i^{\mathrm{T}} \bar{D}_i < R_i$, the inequalities in (37) are converted into $\sum_{i=1}^{M} \mu_i \mathrm{Tr}\,(R_i) - \rho^2 < 0$ and

$$\begin{bmatrix} \bar{B}_i^{\mathrm{T}} \boldsymbol{P}_i \bar{B}_i - R_i & \bar{D}_i^{\mathrm{T}} \\ \bar{D}_i & -I \end{bmatrix} < 0, \qquad \begin{bmatrix} \bar{A}_i^{\mathrm{T}} \boldsymbol{P}_i \bar{A}_i - P_i & \bar{C}_i^{\mathrm{T}} \\ \bar{C}_i & -I \end{bmatrix} < 0. \tag{40}$$

Define symbols $\tilde{\Phi}_i$ and $\breve{\Phi}_i$ as

$$\begin{aligned} &\tilde{\Phi}_i^{\mathrm{ct}} \triangleq \begin{bmatrix} P_i & 0 \\ 0 & -R_i \end{bmatrix}, \qquad \tilde{\Phi}_i^{\mathrm{dt}} \triangleq \begin{bmatrix} \boldsymbol{P}_i & 0 & 0 \\ 0 & -R_i & \bar{D}_i^{\mathrm{T}} \\ 0 & \bar{D}_i & -I \end{bmatrix} \\ &\breve{\Phi}_i^{\mathrm{ct}} \triangleq \begin{bmatrix} 0 & P_i & 0 \\ P_i & \boldsymbol{P}_i & \bar{C}_i^{\mathrm{T}} \\ 0 & \bar{C}_i & -I \end{bmatrix}, \qquad \breve{\Phi}_i^{\mathrm{dt}} \triangleq \begin{bmatrix} \boldsymbol{P}_i & 0 & 0 \\ 0 & -P_i & \bar{C}_i^{\mathrm{T}} \\ 0 & \bar{C}_i & -I \end{bmatrix}. \end{aligned} \tag{41}$$

Note that the inequalities in (39) and (40) are not easy-to-handle if the transition probabilities are uncertain. To cope with this difficulty, following a similar technique that is used to derive Theorem 1, we can obtain the following new condition for analyzing the H_2 performance of the filtering error system in (4).

Theorem 4. *Consider the system in (1) and the filter in (3) with known initial mode probabilities μ_i and transition probabilities π_{ij} for $i, j \in \mathbb{M}$. Given a scalar $\rho > 0$, the filtering error system in (4) (suppose $\bar{D}(r_k) = 0$ for the continuous-time case) is stochastically stable with an H_2 performance level ρ if and only if there exist symmetric matrices $P_i > 0$, R_i and matrices K_i for $i \in \mathbb{M}$ such that*

$$\sum_{i=1}^{M} \mu_i \mathrm{Tr}\,(R_i) - \rho^2 < 0 \tag{42}$$

$$\tilde{\Psi}_i^{\mathrm{T}} \tilde{\Phi}_i \tilde{\Psi}_i + \tilde{G}_i^{\mathrm{T}} K_i \tilde{H}_i + \tilde{H}_i^{\mathrm{T}} K_i^{\mathrm{T}} \tilde{G}_i < 0, \quad \forall i \in \mathbb{M} \tag{43}$$

$$\breve{\Psi}_i^{\mathrm{T}} \breve{\Phi}_i \breve{\Psi}_i + \breve{G}_i^{\mathrm{T}} K_i \breve{H}_i + \breve{H}_i^{\mathrm{T}} K_i^{\mathrm{T}} \breve{G}_i < 0, \quad \forall i \in \mathbb{M} \tag{44}$$

where $\tilde{\Phi}_i$ and $\breve{\Phi}_i$ are defined in (41) and

$$\breve{\Psi}_i = \left[\begin{array}{cccc} 0 & A_i & 0 & 0 \\ I & 0 & 0 & 0 \\ \hline 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ \hline 0 & 0 & 0 & I \end{array}\right]$$

$$\tilde{\Psi}_i^{\mathrm{ct}} = \left[\begin{array}{cc} 0 & B_i \\ I & 0 \\ \hline 0 & I \end{array}\right], \qquad \tilde{\Psi}_i^{\mathrm{dt}} = \left[\begin{array}{ccc} 0 & B_i & 0 \\ I & 0 & 0 \\ \hline 0 & I & 0 \\ \hline 0 & 0 & I \end{array}\right]$$

[2] The meaning of Ψ_k, Φ_k, Θ_k and w is the same as those in Morais et al. (2014).

$$\tilde{G}_i^{\rm ct} = \begin{bmatrix} 0 & \varepsilon_3 B_i \\ \varepsilon_4 I & 0 \end{bmatrix}, \qquad \tilde{G}_i^{\rm dt} = \begin{bmatrix} 0 & \varepsilon_3 B_i & 0 \\ \varepsilon_4 I & 0 & 0 \end{bmatrix}$$

$$\tilde{H}_i^{\rm ct} = \begin{bmatrix} -I & \hat{B}_i D_i \end{bmatrix}, \qquad \tilde{H}_i^{\rm dt} = \begin{bmatrix} -I & \hat{B}_i D_i & 0 \end{bmatrix}$$

$$\breve{G}_i = \begin{bmatrix} 0 & A_i + \varepsilon_1 I & 0 & 0 \\ I & 0 & \varepsilon_2 I & 0 \end{bmatrix} \quad \breve{H}_i = \begin{bmatrix} -I & \hat{B}_i C_i & \hat{A}_i & 0 \end{bmatrix} \tag{45}$$

with $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 > 0$ sufficiently large for the continuous-time case and $\varepsilon_1 = \varepsilon_2 = 0$, $\varepsilon_3 = \varepsilon_4 = 1$ for the discrete-time case, respectively.

Proof. We can re-write the conditions in (39) or (40) in the following unified form:

$$\tilde{W}_i^{\rm T} \tilde{\Phi}_i \tilde{W}_i < 0, \qquad \breve{W}_i^{\rm T} \breve{\Phi}_i \breve{W}_i < 0 \tag{46}$$

where $\tilde{\Phi}_i$ and $\breve{\Phi}_i$ are defined in (41) and

$$\tilde{W}_i^{\rm ct} = \begin{bmatrix} \bar{B}_i \\ I \end{bmatrix}, \qquad \tilde{W}_i^{\rm dt} = \begin{bmatrix} \bar{B}_i & 0 \\ I & 0 \\ 0 & I \end{bmatrix}, \qquad \breve{W}_i = \begin{bmatrix} \bar{A}_i & 0 \\ I & 0 \\ 0 & I \end{bmatrix}.$$

(43) and (44) $\Rightarrow$ (46): Suppose that the conditions in (43) and (44) are satisfied. Define two symbols $\tilde{T}_i$ and $\breve{T}_i$ for $i \in \mathbb{M}$ as

$$\tilde{T}_i^{\rm ct} \triangleq \begin{bmatrix} \hat{B}_i D_i \\ I \end{bmatrix}, \quad \tilde{T}_i^{\rm dt} \triangleq \begin{bmatrix} \hat{B}_i D_i & 0 \\ I & 0 \\ 0 & I \end{bmatrix}, \quad \breve{T}_i \triangleq \left[\begin{array}{ccc} \hat{B}_i C_i & \hat{A}_i & 0 \\ \hline I & 0 & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & I \end{array}\right]$$

for which the row dimensions are such that $\tilde{H}_i \tilde{T}_i = 0$ and $\breve{H}_i \breve{T}_i = 0$ with $\tilde{H}_i$ and $\breve{H}_i$ in (45). With $\bar{A}_i$ and $\bar{B}_i$ in $\breve{W}_i$ substituted by the expressions in (4), it can be shown that $\tilde{\Psi}_i \tilde{T}_i = \tilde{W}_i$ and $\breve{\Psi}_i \breve{T}_i = \breve{W}_i$. Noting that $\tilde{T}_i$ and $\breve{T}_i$ have full column rank, we have

$$\tilde{W}_i^{\rm T} \tilde{\Phi}_i \tilde{W}_i = \tilde{T}_i^{\rm T} \left(\tilde{\Psi}_i^{\rm T} \tilde{\Phi}_i \tilde{\Psi}_i + \tilde{G}_i^{\rm T} K_i \tilde{H}_i + \tilde{H}_i^{\rm T} K_i^{\rm T} \tilde{G}_i \right) \tilde{T}_i < 0$$

$$\breve{W}_i^{\rm T} \breve{\Phi}_i \breve{W}_i = \breve{T}_i^{\rm T} \left(\breve{\Psi}_i^{\rm T} \breve{\Phi}_i \breve{\Psi}_i + \breve{G}_i^{\rm T} K_i \breve{H}_i + \breve{H}_i^{\rm T} K_i^{\rm T} \breve{G}_i \right) \breve{T}_i < 0$$

that is, the conditions in (43) and (44) imply those in (46).

(46) $\Rightarrow$ (43) and (44): Define symbols $\tilde{\Gamma}_i$ and $\breve{\Gamma}_i$, respectively, as

$$\tilde{\Gamma}_i^{\rm ct} \triangleq \left[\begin{array}{c|c} \hat{B}_i D_i & I \\ I & 0 \end{array}\right], \qquad \tilde{\Gamma}_i^{\rm dt} \triangleq \left[\begin{array}{cc|c} \hat{B}_i D_i & 0 & I \\ \hline I & 0 & 0 \\ 0 & I & 0 \end{array}\right]$$

$$\breve{\Gamma}_i \triangleq \left[\begin{array}{ccc|c} \hat{B}_i C_i & \hat{A}_i & 0 & I \\ \hline I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & I & 0 \end{array}\right]$$

which satisfy the following relations:

$$\tilde{\Psi}_i \tilde{\Gamma}_i = \begin{bmatrix} \tilde{W}_i & \tilde{V}_i \end{bmatrix}, \quad \tilde{G}_i \tilde{\Gamma}_i = \tilde{J}^{\rm T} \begin{bmatrix} \tilde{W}_i & \tilde{V}_i \end{bmatrix}, \quad \tilde{H}_i \tilde{\Gamma}_i = \begin{bmatrix} 0 & -I \end{bmatrix}$$

$$\breve{\Psi}_i \breve{\Gamma}_i = \begin{bmatrix} \breve{W}_i & \breve{V}_i \end{bmatrix}, \quad \breve{G}_i \breve{\Gamma}_i = \breve{J}^{\rm T} \begin{bmatrix} \breve{W}_i & \breve{V}_i \end{bmatrix}, \quad \breve{H}_i \breve{\Gamma}_i = \begin{bmatrix} 0 & -I \end{bmatrix} \tag{47}$$

where

$$\tilde{V}_i^{\rm ct} = \left[\begin{array}{c} 0 \\ I \\ \hline 0 \end{array}\right], \qquad \tilde{V}_i^{\rm dt} = \left[\begin{array}{c} 0 \\ I \\ \hline 0 \\ 0 \end{array}\right], \qquad \tilde{J}^{\rm ct} = \left[\begin{array}{cc} \varepsilon_3 I & 0 \\ 0 & \varepsilon_4 I \\ \hline 0 & 0 \end{array}\right]$$

$$\tilde{J}^{\rm dt} = \left[\begin{array}{cc} \varepsilon_3 I & 0 \\ 0 & \varepsilon_4 I \\ \hline 0 & 0 \\ \hline 0 & 0 \end{array}\right], \qquad \breve{V}_i = \left[\begin{array}{c} 0 \\ I \\ \hline 0 \\ 0 \\ \hline 0 \end{array}\right], \qquad \breve{J} = \left[\begin{array}{cc} I & 0 \\ 0 & I \\ \hline \varepsilon_1 I & 0 \\ 0 & \varepsilon_2 I \\ \hline 0 & 0 \end{array}\right].$$

Combining these relations, we have

$$\tilde{\Gamma}_i^{\rm T} \left(\tilde{\Psi}_i^{\rm T} \tilde{\Phi}_i \tilde{\Psi}_i + \tilde{G}_i^{\rm T} K_i \tilde{H}_i + \tilde{H}_i^{\rm T} K_i^{\rm T} \tilde{G}_i \right) \tilde{\Gamma}_i = \begin{bmatrix} \tilde{W}_i^{\rm T} \tilde{\Phi}_i \tilde{W}_i & \tilde{W}_i^{\rm T} (\tilde{\Phi}_i \tilde{V}_i - \tilde{J} K_i) \\ * & \tilde{V}_i^{\rm T} \tilde{\Phi}_i \tilde{V}_i - \tilde{V}_i^{\rm T} \tilde{J} K_i - K_i^{\rm T} \tilde{J}^{\rm T} \tilde{V}_i \end{bmatrix}$$

$$\breve{\Gamma}_i^{\rm T} \left(\breve{\Psi}_i^{\rm T} \breve{\Phi}_i \breve{\Psi}_i + \breve{G}_i^{\rm T} K_i \breve{H}_i + \breve{H}_i^{\rm T} K_i^{\rm T} \breve{G}_i \right) \breve{\Gamma}_i = \begin{bmatrix} \breve{W}_i^{\rm T} \breve{\Phi}_i \breve{W}_i & \breve{W}_i^{\rm T} (\breve{\Phi}_i \breve{V}_i - \breve{J} K_i) \\ * & \breve{V}_i^{\rm T} \breve{\Phi}_i \breve{V}_i - \breve{V}_i^{\rm T} \breve{J} K_i - K_i^{\rm T} \breve{J}^{\rm T} \breve{V}_i \end{bmatrix}.$$

Since $\tilde{\Gamma}_i$ and $\breve{\Gamma}_i$ are invertible, it follows from the above equations that the inequalities in (43) and (44) are satisfied if

$$\begin{bmatrix} \tilde{W}_i^{\rm T} \tilde{\Phi}_i \tilde{W}_i & \tilde{W}_i^{\rm T} (\tilde{\Phi}_i \tilde{V}_i - \tilde{J} K_i) \\ * & \tilde{V}_i^{\rm T} \tilde{\Phi}_i \tilde{V}_i - \tilde{V}_i^{\rm T} \tilde{J} K_i - K_i^{\rm T} \tilde{J}^{\rm T} \tilde{V}_i \end{bmatrix} < 0 \tag{48}$$

$$\begin{bmatrix} \breve{W}_i^{\rm T} \breve{\Phi}_i \breve{W}_i & \breve{W}_i^{\rm T} (\breve{\Phi}_i \breve{V}_i - \breve{J} K_i) \\ * & \breve{V}_i^{\rm T} \breve{\Phi}_i \breve{V}_i - \breve{V}_i^{\rm T} \breve{J} K_i - K_i^{\rm T} \breve{J}^{\rm T} \breve{V}_i \end{bmatrix} < 0. \tag{49}$$

For the continuous-time case, (48) and (49) can be explicitly expressed, respectively, as

$$\begin{bmatrix} \tilde{W}_i^{\rm T} \tilde{\Phi}_i \tilde{W}_i & \tilde{W}_i^{\rm T} \tilde{L}_i^{\rm T} \\ * & P_{i22} - \varepsilon_4 K_{i2} - \varepsilon_4 K_{i2}^{\rm T} \end{bmatrix} < 0 \tag{50}$$

$$\begin{bmatrix} \breve{W}_i^{\rm T} \breve{\Phi}_i \breve{W}_i & \breve{W}_i^{\rm T} \breve{L}_i^{\rm T} \\ * & -K_{i2} - K_{i2}^{\rm T} \end{bmatrix} < 0 \tag{51}$$

where

$$\tilde{L}_i = \left[\begin{array}{cc|c} P_{i12}^{\rm T} - \varepsilon_3 K_{i1}^{\rm T} & P_{i22}^{\rm T} - \varepsilon_4 K_{i2}^{\rm T} & 0 \end{array}\right]$$

$$\breve{L}_i = \left[\begin{array}{cc|cc|c} -K_{i1}^{\rm T} & -K_{i2}^{\rm T} & P_{i12}^{\rm T} - \varepsilon_1 K_{i1}^{\rm T} & P_{i22}^{\rm T} - \varepsilon_2 K_{i2}^{\rm T} & 0 \end{array}\right]$$

and K_{i1} and K_{i2} are the upper and lower n_x rows of K_i, respectively, and P_{i12} and P_{i22} are the upper and lower right $n_x \times n_x$ blocks of P_i, respectively. Using similar arguments as in the proof of Theorem 1, it can be verified that if the two inequalities in (46) hold for some $P_i > 0$, those in (50) and (51) are also satisfied for the following assignment:

$$K_{i1} = \varepsilon_1^{-1} P_{i12}, \qquad K_{i2} = \varepsilon_2^{-1} P_{i22}, \qquad \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \epsilon^{-1}$$

where $\epsilon > 0$ is a sufficiently small scalar. Consequently, the inequalities in (43) and (44) can be obtained.

For the discrete-time case where $\varepsilon_1 = \varepsilon_2 = 0$ and $\varepsilon_3 = \varepsilon_4 = 1$, (48) and (49) can be explicitly expressed, respectively, as

$$\begin{bmatrix} \tilde{W}_i^{\rm T} \tilde{\Phi}_i \tilde{W}_i & \tilde{W}_i^{\rm T} \tilde{L}_i^{\rm T} \\ * & \boldsymbol{P}_{i22} - K_{i2} - K_{i2}^{\rm T} \end{bmatrix} < 0$$

$$\begin{bmatrix} \breve{W}_i^{\rm T} \breve{\Phi}_i \breve{W}_i & \breve{W}_i^{\rm T} \breve{L}_i^{\rm T} \\ * & \boldsymbol{P}_{i22} - K_{i2} - K_{i2}^{\rm T} \end{bmatrix} < 0$$

where

$$\tilde{L}_i = \left[\begin{array}{cc|c|c} \boldsymbol{P}_{i12}^{\rm T} - K_{i1}^{\rm T} & \boldsymbol{P}_{i22}^{\rm T} - K_{i2}^{\rm T} & 0 & 0 \end{array}\right]$$

$$\breve{L}_i = \left[\begin{array}{cc|cc|c} \boldsymbol{P}_{i12}^{\rm T} - K_{i1}^{\rm T} & \boldsymbol{P}_{i22}^{\rm T} - K_{i2}^{\rm T} & 0 & 0 & 0 \end{array}\right]$$

and $\boldsymbol{P}_{i12}$ and $\boldsymbol{P}_{i22}$ are the upper and lower right $n_x \times n_x$ blocks of $\boldsymbol{P}_i$, respectively. The above two inequalities can be directly obtained from those in (46) with $K_{i1} = \boldsymbol{P}_{i12}$, $K_{i2} = \boldsymbol{P}_{i22}$.

We have established the equivalence between the inequalities in (46) and those in (43) and (44), and further applying Lemma 2, can conclude the proof. ■

4.2. H_2 filter design

Based on Theorem 4, we can obtain the following result which provides a new parameter-dependent LMI condition for the existence of a filter with a guaranteed H_2 filtering performance for all transition uncertainties. The theorem can be proved by applying Theorem 4 and specifying K_i as in (21). The proof follows similar lines as that of Theorem 3 and is thus omitted for saving space.

Theorem 5. *Consider the system in (1) with known initial mode probabilities μ_i and uncertain transition probabilities $\pi_{ij}(\lambda_i)$ for $\lambda_i \in \Lambda_i$ and $i, j \in \mathbb{M}$. Given scalars $\rho > 0$, ε_1, ε_2, ε_3 and ε_4, a filter in (3) exists such that the filtering error system in (4) (suppose $\bar{D}(r_k) = 0$ for the continuous-time case) is stochastically stable with an H_2 performance level ρ if there exist symmetric matrices $P_i(\lambda) > 0$, $R_i(\lambda)$ and matrices $\mathscr{K}_i$, $\mathscr{A}_i$, $\mathscr{B}_i$, $\mathscr{C}_i$, $\mathscr{D}_i$ for $i \in \mathbb{M}$ such that*

$$\sum_{i=1}^{M} \mu_i \mathrm{Tr}\left(R_i(\lambda)\right) - \rho^2 < 0, \quad \forall \lambda \in \Lambda \tag{52}$$

$$\tilde{\Psi}_i^{\mathrm{T}} \tilde{\Phi}_i(\lambda) \tilde{\Psi}_i + \tilde{\mathscr{G}}_i^{\mathrm{T}} \tilde{\mathscr{H}}_i + \tilde{\mathscr{H}}_i^{\mathrm{T}} \tilde{\mathscr{G}}_i < 0, \quad \forall (i, \lambda) \in \mathbb{M} \times \Lambda \tag{53}$$

$$\breve{\Psi}_i^{\mathrm{T}} \breve{\Phi}_i(\lambda) \breve{\Psi}_i + \breve{\mathscr{G}}_i^{\mathrm{T}} \breve{\mathscr{H}}_i + \breve{\mathscr{H}}_i^{\mathrm{T}} \breve{\mathscr{G}}_i < 0, \quad \forall (i, \lambda) \in \mathbb{M} \times \Lambda \tag{54}$$

where $\tilde{\Phi}_i(\lambda)$ and $\breve{\Phi}_i(\lambda)$ are $\tilde{\Phi}_i$ and $\breve{\Phi}_i$ in (41) with π_i, P_i, R_i, $\hat{C}_i$ and $\hat{D}_i$ replaced by $\pi_i(\lambda_i)$, $P_i(\lambda)$, $R_i(\lambda)$, $\mathscr{C}_i$ and $\mathscr{D}_i$, respectively, $\tilde{\Psi}_i$ and $\breve{\Psi}_i$ are defined in (45) and

$$\tilde{\mathscr{G}}_i^{\mathrm{ct}} = \begin{bmatrix} \varepsilon_4 I & \varepsilon_3 B_i \end{bmatrix}, \qquad \tilde{\mathscr{H}}_i^{\mathrm{ct}} = \begin{bmatrix} -\mathscr{K}_i & \mathscr{B}_i D_i \end{bmatrix}$$

$$\tilde{\mathscr{G}}_i^{\mathrm{dt}} = \begin{bmatrix} \varepsilon_4 I & \varepsilon_3 B_i & 0 \end{bmatrix}, \qquad \tilde{\mathscr{H}}_i^{\mathrm{dt}} = \begin{bmatrix} -\mathscr{K}_i & \mathscr{B}_i D_i & 0 \end{bmatrix}$$

$$\breve{\mathscr{G}}_i = \begin{bmatrix} I & A_i + \varepsilon_1 I & \varepsilon_2 I & 0 \end{bmatrix}, \quad \breve{\mathscr{H}}_i = \begin{bmatrix} -\mathscr{K}_i & \mathscr{B}_i C_i & \mathscr{A}_i & 0 \end{bmatrix}.$$

Moreover, if the conditions in (52)–(54) are feasible, the filter parameters can be given by (18).

To ensure the conditions in (52)–(54) by a finite number of LMIs, consider the parameter-dependent matrices $P_i(\lambda)$ and $R_i(\lambda)$ in the following homogeneous polynomial form:

$$P_i(\lambda) = \sum_{k \in \mathscr{I}(g)} \lambda^k P_{i,k}, \qquad R_i(\lambda) = \sum_{k \in \mathscr{I}(g)} \lambda^k R_{i,k}.$$

The meaning of the symbol $\mathscr{I}(g)$ is the same as that in the previous section. Define some symbols as follows:

$$\tilde{\Xi}_{i,k,l}^{\mathrm{ct}} \triangleq \mathrm{diag}\left\{P_{i,k-l}, -R_{i,k-l}\right\}$$

$$\tilde{\Xi}_{i,k,l,h}^{\mathrm{dt}} \triangleq \mathrm{diag}\left\{\sum_{j=1}^{M} \pi_{ij}^{(v(h_i))} P_{j,k-l-h}, -R_{i,k-l-h}, 0\right\}$$

$$\breve{\Xi}_{i,k,l,h}^{\mathrm{ct}} \triangleq \begin{bmatrix} 0 & P_{i,k-l-h} & 0 \\ P_{i,k-l-h} & \sum_{j=1}^{M} \pi_{ij}^{(v(h_i))} P_{j,k-l-h} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\breve{\Xi}_{i,k,l,h}^{\mathrm{dt}} \triangleq \mathrm{diag}\left\{\sum_{j=1}^{M} \pi_{ij}^{(v(h_i))} P_{j,k-l-h}, -P_{i,k-l-h}, 0\right\}.$$

Using the same polynomial relaxation method in deriving the previous H_∞ filtering condition, we can obtain the following result for H_2 filter design. The proof can be completed by following similar lines as that of Theorem 3.

Theorem 6. *Consider the system in (1) with known initial mode probabilities μ_i and uncertain transition probabilities $\pi_{ij}(\lambda_i)$ for $\lambda_i \in \Lambda_i$ and $i, j \in \mathbb{M}$. Given scalars $\rho > 0$, ε_1, ε_2, ε_3, ε_4 and vectors $g, d, b \in \mathbb{N}^M$, a filter in (3) exists such that the filtering error system in (4) (suppose $\bar{D}(r_k) = 0$ for the continuous-time case) is stochastically stable with an H_2 performance level ρ if there exist symmetric matrices $P_{i,k}$, $R_{i,k}$, $k \in \mathscr{I}(g)$ and matrices $\mathscr{K}_i$, $\mathscr{A}_i$, $\mathscr{B}_i$, $\mathscr{C}_i$ and $\mathscr{D}_i$ for $i \in \mathbb{M}$ such that the LMIs in (23) and (55)–(57) hold,*

$$\hat{\mathscr{Q}}_k < 0, \quad \forall k \in \mathscr{I}(g+d) \tag{55}$$

$$\tilde{\mathscr{Q}}_{i,k}^{\mathrm{ct}} < 0, \quad \forall (i, k) \in \mathbb{M} \times \mathscr{I}(g+b) \tag{56-ct}$$

$$\tilde{\mathscr{Q}}_{i,k}^{\mathrm{dt}} < 0, \quad \forall (i, k) \in \mathbb{M} \times \mathscr{I}(g+b+\mathbf{1}) \tag{56-dt}$$

$$\breve{\mathscr{Q}}_{i,k} < 0, \quad \forall (i, k) \in \mathbb{M} \times \mathscr{I}(g+b+\mathbf{1}) \tag{57}$$

where

$$\hat{\mathscr{Q}}_k = \sum_{l \in \mathscr{I}(d), l \preceq k} \frac{\phi(d)}{\phi(l)} \sum_{i=1}^{M} \mu_i \mathrm{Tr}\left(R_{i,k-l}\right) - \frac{\phi(g+d)}{\phi(k)} \rho^2$$

$$\tilde{\mathscr{Q}}_{i,k}^{\mathrm{ct}} = \sum_{l \in \mathscr{I}(b), l \preceq k} \frac{\phi(b)}{\phi(l)} \left(\tilde{\Psi}_i^{\mathrm{ct}}\right)^{\mathrm{T}} \tilde{\Xi}_{i,k,l}^{\mathrm{ct}} \tilde{\Psi}_i^{\mathrm{ct}} + \frac{\phi(g+b)}{\phi(k)} \left(\left(\tilde{\mathscr{G}}_i^{\mathrm{ct}}\right)^{\mathrm{T}} \tilde{\mathscr{H}}_i^{\mathrm{ct}} + \left(\tilde{\mathscr{H}}_i^{\mathrm{ct}}\right)^{\mathrm{T}} \tilde{\mathscr{G}}_i^{\mathrm{ct}}\right)$$

$$\tilde{\mathscr{Q}}_{i,k}^{\mathrm{dt}} = \sum_{\substack{l \in \mathscr{I}(b) \\ l \preceq k}} \sum_{\substack{h \in \mathscr{I}(\mathbf{1}) \\ h \preceq k-l}} \frac{\phi(b)}{\phi(l)} \left(\tilde{\Psi}_i^{\mathrm{dt}}\right)^{\mathrm{T}} \tilde{\Xi}_{i,k,l,h}^{\mathrm{dt}} \tilde{\Psi}_i^{\mathrm{dt}} + \frac{\phi(g+b+\mathbf{1})}{\phi(k)} \times \left(\left(\tilde{\Psi}_i^{\mathrm{dt}}\right)^{\mathrm{T}} \tilde{\Upsilon}_i^{\mathrm{dt}} \tilde{\Psi}_i^{\mathrm{dt}} + \left(\tilde{\mathscr{G}}_i^{\mathrm{dt}}\right)^{\mathrm{T}} \tilde{\mathscr{H}}_i^{\mathrm{dt}} + \left(\tilde{\mathscr{H}}_i^{\mathrm{dt}}\right)^{\mathrm{T}} \tilde{\mathscr{G}}_i^{\mathrm{dt}}\right)$$

$$\breve{\mathscr{Q}}_{i,k} = \sum_{\substack{l \in \mathscr{I}(b) \\ l \preceq k}} \sum_{\substack{h \in \mathscr{I}(\mathbf{1}) \\ h \preceq k-l}} \frac{\phi(b)}{\phi(l)} \breve{\Psi}_i^{\mathrm{T}} \breve{\Xi}_{i,k,l,h} \breve{\Psi}_i + \frac{\phi(g+b+\mathbf{1})}{\phi(k)} \left(\breve{\Psi}_i^{\mathrm{T}} \breve{\Upsilon}_i \breve{\Psi}_i + \breve{\mathscr{G}}_i^{\mathrm{T}} \breve{\mathscr{H}}_i + \breve{\mathscr{H}}_i^{\mathrm{T}} \breve{\mathscr{G}}_i\right)$$

$$\tilde{\Upsilon}_i^{\mathrm{dt}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{D}_i^{\mathrm{T}} \\ 0 & \bar{D}_i & -I \end{bmatrix}, \qquad \breve{\Upsilon}_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \bar{C}_i^{\mathrm{T}} \\ 0 & \bar{C}_i & -I \end{bmatrix}$$

and $\hat{C}_i$, $\hat{D}_i$ in $\bar{C}_i$ and $\bar{D}_i$ are replaced by $\mathscr{C}_i$ and $\mathscr{D}_i$, respectively, and $\tilde{\Psi}_i$, $\breve{\Psi}_i$, $\tilde{\mathscr{G}}_i$, $\breve{\mathscr{G}}_i$, $\tilde{\mathscr{H}}_i$ and $\breve{\mathscr{H}}_i$ are as in (53) and (54).

Moreover, if the conditions in (23) and (55)–(57) are feasible, the filter parameters can be given by (19).

Parallel to Corollary 1 for H_∞ filter design, when the matrices $P_i(\lambda)$ and $R_i(\lambda)$ are parameter-independent, we also obtain the following corollary for H_2 filter design.

Corollary 2. *Consider the system in (1) with known initial mode probabilities μ_i and uncertain transition probabilities $\pi_{ij}(\lambda_i)$ for $\lambda_i \in \Lambda_i$ and $i, j \in \mathbb{M}$. Given scalars $\rho > 0$, ε_1, ε_2, ε_3 and ε_4, a filter in (3) exists such that the filtering error system in (4) (suppose $\bar{D}(r_k) = 0$ for the continuous-time case) is stochastically stable with an H_2 performance level ρ if there exist symmetric matrices $P_i > 0$, R_i and matrices $\mathscr{K}_i$, $\mathscr{A}_i$, $\mathscr{B}_i$, $\mathscr{C}_i$ and $\mathscr{D}_i$ for $i \in \mathbb{M}$ such that*

$$\sum_{i=1}^{M} \mu_i \mathrm{Tr}\left(R_i\right) - \rho^2 < 0 \tag{58}$$

$$\tilde{\Psi}_i^{\mathrm{T}} \tilde{\Phi}_i^{(s)} \tilde{\Psi}_i + \tilde{\mathscr{G}}_i^{\mathrm{T}} \tilde{\mathscr{H}}_i + \tilde{\mathscr{H}}_i^{\mathrm{T}} \tilde{\mathscr{G}}_i < 0, \quad \forall (i, s) \in \mathbb{M} \times \mathbb{S}_i \tag{59}$$

$$\breve{\Psi}_i^{\mathrm{T}} \breve{\Phi}_i^{(s)} \breve{\Psi}_i + \breve{\mathscr{G}}_i^{\mathrm{T}} \breve{\mathscr{H}}_i + \breve{\mathscr{H}}_i^{\mathrm{T}} \breve{\mathscr{G}}_i < 0, \quad \forall (i, s) \in \mathbb{M} \times \mathbb{S}_i \tag{60}$$

where $\tilde{\Phi}_i^{(s)}$ and $\breve{\Phi}_i^{(s)}$ are $\tilde{\Phi}_i$ and $\breve{\Phi}_i$ in (41) with π_i, $\hat{C}_i$ and $\hat{D}_i$ replaced by $\pi_i^{(s)}$, $\mathscr{C}_i$ and $\mathscr{D}_i$, respectively, and $\tilde{\Psi}_i$, $\breve{\Psi}_i$, $\tilde{\mathscr{G}}_i$, $\breve{\mathscr{G}}_i$, $\tilde{\mathscr{H}}_i$ and $\breve{\mathscr{H}}_i$ are as in (53) and (54).

Moreover, if the conditions in (58)–(60) are feasible, the filter parameters can be given by (19).

Remark 4. For the continuous-time case, when P_i and R_i are parameter-independent, the LMIs in (53) are also parameter-independent. Hence, the index "(s)" of the LMIs in (59) can be removed for the continuous-time case.

Remark 5. For H_2 filtering in the continuous-time case, the auxiliary matrix R_i in (39) stems from the relation $\bar{B}_i^{\mathrm{T}} P_i \bar{B}_i < R_i$, which is introduced to linearize the synthesis condition. Instead, we can employ an alternative relation $\mu_i \bar{B}_i^{\mathrm{T}} P_i \bar{B}_i < R_i$ for the same purpose. In particular, Theorem 1 in Morais et al. (2015) is based on this choice. Furthermore, we have the following two comments. *First*, the filter design conditions resulting from the two relations are not equivalent to each other. This is because the obtained design conditions are sufficient only. Consequently, we cannot specifically relate Theorem 6 in this paper with Theorem 1 in Morais et al. (2015). *Second*, both conditions can be easily modified such that the modified ones are based on the same relation. For instance, under the relation $\bar{B}_i^{\mathrm{T}} P_i \bar{B}_i < R_i$, the matrix Θ_{Tk} in (15) of Morais et al. (2015) will not be multiplied by the coefficient μ_i, but the matrix $W_{j_{k-k'}}$ in (14) therein should be multiplied by μ_j instead. Note that these modifications will not change the sufficiency of the design conditions. Then, as in Section 3.3.2 for H_∞ filtering, we can exactly compare the modified design conditions and draw similar conclusions in principle. For numerical results on H_2 filtering in the sequel, the method in Morais et al. (2015) specifically indicates the modified version of Theorem 1 therein.

Remark 6. For H_2 filtering in the discrete-time case, the methods in Fioravanti et al. (2008) and Liu et al. (2012) do not address the uncertain situation described by (2). However, following the technique to obtain Corollary 2, it is easy to adapt the methods in Fioravanti et al. (2008) and Liu et al. (2012) to the situation considered in this paper. Moreover, using similar arguments as in Section 3.3.1 for H_∞ filtering, one can see that, compared with the method in Liu et al. (2012), Corollary 2 in the paper is not only less conservative but also includes fewer variables.

5. Numerical examples

In the section, we present three numerical examples to show the merits of the proposed filter design method.

Example 1. Consider a two-mode continuous-time example in (1) with state-space matrices from de Souza et al. (2006) and also given as follows:

$$A_1 = \begin{bmatrix} -3 & 1 & 0 \\ 0.3 & -2.5 & 1 \\ -0.1 & 0.3 & -3.8 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} -2.5 & 0.5 & -0.1 \\ 0.1 & -3.5 & 0.3 \\ -0.1 & 1 & -2 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -0.6 \\ 0.5 \\ 0 \end{bmatrix}, \qquad D_1 = 0.2, D_2 = 0.5$$

$$C_1 = \begin{bmatrix} 0.8 & 0.3 & 0 \end{bmatrix}, \qquad C_2 = \begin{bmatrix} -0.5 & 0.2 & 0.3 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 0.5 & -0.1 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0.6 \end{bmatrix}, \quad F_1 = F_2 = 0.$$

For the case that the transition rates are exactly known, the matrix Π is given by

$$\Pi = \begin{bmatrix} -0.5 & 0.5 \\ 0.3 & -0.3 \end{bmatrix}.$$

For the case that the transition rates are uncertain, the vertices $\pi_i^{(s)}$ of the rows of the matrix Π is given by

$$\pi_1^{(1)} = \begin{bmatrix} -0.35 & 0.35 \end{bmatrix}, \qquad \pi_1^{(2)} = \begin{bmatrix} -0.65 & 0.65 \end{bmatrix},$$
$$\pi_2^{(1)} = \begin{bmatrix} 0.2 & -0.2 \end{bmatrix}, \qquad \pi_2^{(2)} = \begin{bmatrix} 0.4 & -0.4 \end{bmatrix}.$$

H_∞ **filtering**. Since the methods in de Souza et al. (2006) and Liu et al. (2008) can design mode-independent filters only, we consider the design of mode-independent filters for this example. Using Corollary 1 and Theorem 3 in this paper and the design methods in de Souza et al. (2006), Liu et al. (2008) and Morais et al. (2015), the results on the minimum H_∞ filtering performance level γ^* achieved by these methods are listed in Table 1, which also includes the information on the number of variables (NoV) of the LMI conditions for each method. The parameters in Corollary 1 and Theorem 3 are set as $\varepsilon_1 = \varepsilon_2 = 1$, $d = b = 0$, while those in the method in Morais et al. (2015) are set as $\lambda_1 = \lambda_2 = 1$, $d = 0$. It is seen from the table that, compared with de Souza et al. (2006) and Liu et al. (2008), Corollary 1 or Theorem 3 with $g = 0$ not only gives rise to better results on the minimum H_∞ disturbance attenuation level, but also involves fewer variables for computation. Compared with Morais et al. (2015), the conditions in this paper are still less computationally demanding. For the case of the uncertain Π, the parameters of the mode-independent proper filter designed by Corollary 1 are given by

$$\left[\begin{array}{cc} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{array}\right] = \left[\begin{array}{ccc|c} -4.2382 & -2.3638 & 3.7667 & -0.4365 \\ -0.0305 & -2.6680 & 0.3480 & -0.6369 \\ -1.2147 & -2.5653 & -0.4671 & -0.9001 \\ \hline 0.0921 & -1.0228 & -0.3247 & 0.2122 \end{array}\right].$$

For the design case of mode-independent and strictly proper filters, Theorem 3 ($d = b = 0, g = \mathbf{1}$) gives rise to $\gamma^* = 0.1918$ for parameters $(\varepsilon_1, \varepsilon_2) = (4.001, 7.1570)$ (obtained by directly applying the `fminsearch` function in Matlab). This H_∞ filtering performance level is smaller than $\gamma^* = 0.2059$ for $\varepsilon_1 = \varepsilon_2 = 1$ (see Table 1), showing that it is possible to improve a filter by adjusting the two scalars $\varepsilon_1, \varepsilon_2$.

H_2 **filtering**. Let the initial probabilities $\mu = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix}$ and via this example, we compare Corollary 2 and Theorem 6 in this paper with Theorem 1 in Morais et al. (2015). Since matrices D_1 and D_2 are non-zero, filters to be computed should be strictly proper. The parameters of these methods, besides $\varepsilon_3 = \varepsilon_4 = 1$ and others to be indicated, are set the same as those for H_∞ filtering. Table 2 lists the calculated results. It is clear that, under the same degree g, the methods in this paper and Morais et al. (2015) give rise to the same H_2 filtering performance level ρ^* but those in this paper consist of much fewer variables.

Table 2
ρ^* and NoV for different methods in Example 1 (H_2 filtering).

	Method	ρ^*	NoV
(Known Π:)	Morais et al. (2015)[a]	0.1054	243
	Corollary 1	0.1054	93
(Uncertain Π:)	Morais et al. (2015) ($g = 0, h = 0$)	0.1152	243
	Theorem 3 ($g = 0$)	0.1152	93
	Morais et al. (2015) ($g = \mathbf{1}, h = 0$)	0.1151	375
	Morais et al. (2015) ($g = \mathbf{1}, h = \mathbf{1}$)	0.1151	825
	Theorem 3 ($g = \mathbf{1}$)	0.1151	225

[a] Theorem 1 in Morais et al. (2015) is modified as pointed out in Remark 5.

Example 2. Consider a four-mode discrete-time example in (1) from Zhang and Boukas (2009a) with matrices given by

$$A_1 = \begin{bmatrix} 0 & -0.405 \\ 0.81 & 0.81 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & -0.2673 \\ 0.81 & 1.134 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & -0.81 \\ 0.81 & 0.972 \end{bmatrix}, \qquad A_4 = \begin{bmatrix} 0 & -0.1863 \\ 0.81 & 0.891 \end{bmatrix}$$

$$B_i = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}, \qquad C_i = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad D_i = E_i = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$F_i = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad i \in \{1, 2, 3, 4\}.$$

Table 3
γ^* and NoV for different methods in Example 2 (H_∞ filtering).

Method	Case Π_1		Case Π_2	
	γ^*	NoV	γ^*	NoV
(Mode-dependent filters:)				
Zhang and Boukas (2009a)	3.8215	125	4.2793	125
Morais et al. (2014)	1.6422	181	1.6436	181
Corollary 1	1.6422	93	1.6436	93
(Mode-independent filters:)				
Morais et al. (2014) ($g = 0, h = 0$)	1.7361	142	1.7625	142
Theorem 3 ($g = 0$)	1.7361	54	1.7625	54
Morais et al. (2014) ($g = \mathbf{1}, h = 0$)	1.7336	342	1.7425	582
Morais et al. (2014) ($g = \mathbf{1}, h = \mathbf{1}$)	1.7336	782	1.7425	1550
Theorem 3 ($g = \mathbf{1}$)	1.7336	254	1.7425	494

Table 4
ρ^* and NoV for different methods in Example 2 (H_2 filtering).

Method	Case Π_1		Case Π_2	
	ρ^*	NoV	ρ^*	NoV
(Mode-dependent filters:)				
Fioravanti et al. (2008)[a]	0.7092	73	0.7112	73
Corollary 2 or Theorem 6 ($g = 0$)	0.6876	105	0.6893	105
Theorem 6 ($g = \mathbf{1}$)	0.6726	365	0.6734	677
(Mode-independent filters:)				
Fioravanti et al. (2008)	0.7410	37	0.7410	37
Liu et al. (2012)	0.7271	74	0.7315	74
Corollary 2 or Theorem 6 ($g = 0$)	0.7134	66	0.7199	66
Theorem 6 ($g = \mathbf{1}$)	0.6982	326	0.7004	638

[a] The methods in Fioravanti et al. (2008) and Liu et al. (2012) are adapted according to Remark 6.

The following two cases are considered for the uncertain transition probabilities:

$$\Pi_1 = \begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.4 \\ ? & ? & 0.3 & 0.2 \\ 0.1 & 0.1 & 0.5 & 0.3 \\ 0.2 & ? & ? & ? \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.4 \\ ? & ? & 0.3 & 0.2 \\ ? & 0.1 & ? & 0.3 \\ 0.2 & ? & ? & ? \end{bmatrix}$$

where "?" denotes the unknown transition probabilities. The two cases correspond to Case I and Case II of the example in Zhang and Boukas (2009a).

H_∞ **filtering**. For this example, we first design mode-dependent filters by Corollary 1 in this paper, and compare the achieved minimum H_∞ filtering performance level with that obtained by the methods in Morais et al. (2014) and Zhang and Boukas (2009a). Set $\varepsilon_1 = \varepsilon_2 = 0$ for Corollary 1 and set $\lambda_1 = \lambda_2 = 0, d = g = h = 0$ for the method in Morais et al. (2014). The first part of Table 3 summarizes the results on the achieved minimum H_∞ filtering performance level as well as the data on NoV of each method. On one hand, it is shown that with fewer variables to be optimized, Corollary 1 still obtains better H_∞ filtering performance levels than those by Zhang and Boukas (2009a). On the other hand, although Corollary 1 and the method in Morais et al. (2014) give rise to the same H_∞ filtering performance level, Corollary 1 merits an obvious computational advantage, because the NoV for Corollary 1 is almost only a half of that for the method in Morais et al. (2014). This fact well verifies the discussions in Section 3.3.2 regarding the theoretical comparison between the two methods.

We further consider the design case of mode-independent filters. Set $\varepsilon_1 = \varepsilon_2 = 0, d = b = 0$ for Theorem 3 and $\lambda_1 = \lambda_2 = 0$, $d = 0$ for the method in Morais et al. (2014). The calculated results by different methods are presented in the second part of Table 3, which show that, using the same degree of polynomially parameter-dependent Lyapunov matrices, Theorem 3 and the method in Morais et al. (2014) can produce filters with the same guaranteed H_∞ filtering performance level, but obviously the former needs much fewer variables.

H_2 **filtering**. Suppose that the initial probabilities $\mu = \begin{bmatrix} 0.22 & 0.18 & 0.22 & 0.38 \end{bmatrix}$. We compute the minimum H_2 filtering performance level by Corollary 2 and Theorem 6 in this paper and the methods in Fioravanti et al. (2008); Liu et al. (2012). For Theorem 6, parameters $\varepsilon_i, i = 1, \ldots, 4$, d and b are set as $\varepsilon_1 = \varepsilon_2 = 0, \varepsilon_3 = \varepsilon_4 = 1$ and $d = b = 0$. Table 4 shows the computed results. It is easy to see that Corollary 2 gives smaller H_2 performance levels than those by the methods in Fioravanti et al. (2008) and Liu et al. (2012). Moreover, increasing the degree of polynomial Lyapunov matrices can further reduce the H_2 filtering error level.

Example 3. This two-mode discrete-time example is borrowed from de Souza and Fragoso (2003) with matrices given by

$$A_1 = \begin{bmatrix} 1 & 5.2529 \times 10^{-2} \\ 1.5146 \times 10^{-3} & 1.1022 \end{bmatrix}, \qquad B_i = \begin{bmatrix} 0.5 & 0 \\ 0 & 0 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0.9955 & 4.9660 \times 10^{-2} \\ -0.2669 & 0.8075 \end{bmatrix}, \qquad C_i = \begin{bmatrix} -1 & 1 \end{bmatrix}$$
$$D_i = E_i = \begin{bmatrix} 0 & 1 \end{bmatrix}, \qquad F_i = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad i \in \{1, 2\}.$$

The transition probabilities are uncertain with the vertices $\pi_i^{(s)}$ of the rows of the matrix Π given by

$$\pi_1^{(1)} = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix}, \qquad \pi_1^{(2)} = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix},$$
$$\pi_2^{(1)} = \begin{bmatrix} 0.2 & 0.8 \end{bmatrix}, \qquad \pi_2^{(2)} = \begin{bmatrix} 0.3 & 0.7 \end{bmatrix}.$$

Mode-independent filters for this example are designed by Corollary 1 in this paper and the method in Gonçalves et al. (2011). The scalars ε_1 and ε_2 for Corollary 1 are specified as $\varepsilon_1 = \varepsilon_2 = 0$. The design results are shown in Table 5, where the results for both strictly proper and proper filters are calculated. Apparently the proposed method, Corollary 1, generates filters with improved guaranteed H_∞ filtering performance bounds than that obtained by the method in Gonçalves et al. (2011), although the former needs to solve an optimization problem with more variables. The proper filter obtained by Corollary 1 has the following state-space realization:

$$\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} = \left[\begin{array}{cc|c} 0.1710 & 1.0022 & 0.9204 \\ 0.2544 & 0.4892 & -0.4422 \\ \hline -0.3744 & -0.6256 & 0.3744 \end{array}\right].$$

To further show the effectiveness of the obtained filter, we do some time-domain simulations under the following disturbance

$$w(k) = \begin{bmatrix} e^{-0.5k} \sin(0.1k) \\ e^{-0.7k} \sin(0.3k) \end{bmatrix}.$$

The initial states of the system are set as $x_0 = \begin{bmatrix} 0.2 & 0.4 \end{bmatrix}^T$ and that of the filter are set to zero. The system output $z(k)$ and its estimation $\hat{z}(k)$ are plotted in Fig. 1, where the switching signals $r(k)$ are randomly generated according to the transition probabilities at the four vertices of the uncertain transition probability domain. It is shown that the filter output can effectively track the target signal $z(k)$ for all the four cases of the transition probabilities.

6. Conclusion

The H_∞ and H_2 filtering problems of MJLSs with uncertain transition probabilities have been investigated in the paper and new approaches have been systematically proposed for designing

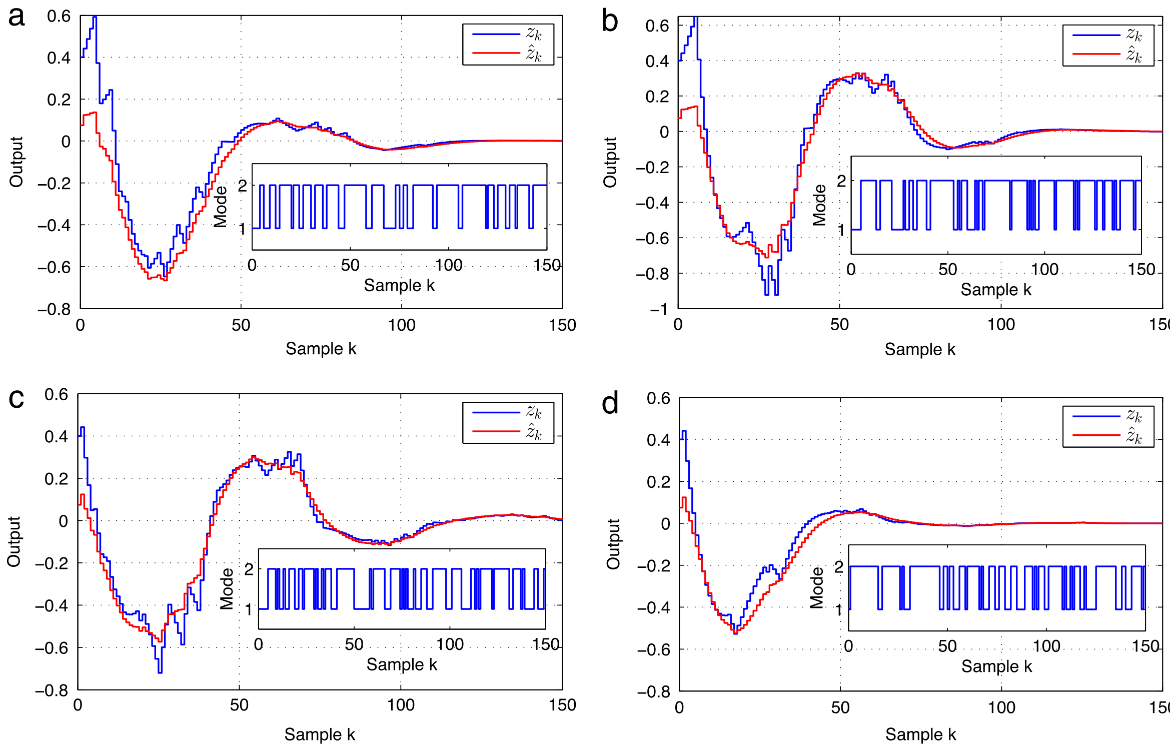

Fig. 1. Time-domain response of the system and filter in Example 3. (a): $\Pi = [0.6, 0.4; 0.2, 0.8]$; (b): $\Pi = [0.5, 0.5; 0.2, 0.8]$; (c): $\Pi = [0.6, 0.4; 0.3, 0.7]$; (d) $\Pi = [0.5, 0.5; 0.3, 0.7]$.

Table 5
γ^* and NoV for different methods in Example 3.

Method	Strictly proper		Proper	
	γ^*	NoV	γ^*	NoV
Gonçalves et al. (2011)	9.8077	25	9.7234	26
Corollary 1	7.7995	33	7.7946	34

H_∞ or H_2 filters for both continuous- and discrete-time MJLSs. To overcome the drawbacks of the existing results, a new technique has been applied to decouple the product terms between the Lyapunov matrices and the filter parameters, leading to improved conditions for H_∞ and H_2 filtering analysis. LMI conditions have been proposed for checking the existence of filters that satisfy a guaranteed H_∞ or H_2 filtering performance, so that the filter design problems are cast into convex optimization problems that can be effectively solved. Theoretical comparisons between the proposed method and some existing ones have been presented, showing that the proposed one has advantages in conservatism reduction or variables reduction. Finally, numerical examples have been provided to further illustrate the effectiveness and improvements of the proposed method. In the future, the method developed in the paper may be used to study control of networked control systems (Wang, Gao, & Qiu, 2015) or control of switching electrical devices (Napoles et al., 2013).

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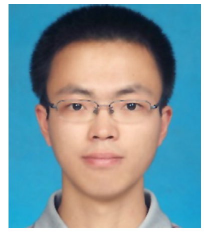

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