$H_\infty$ and $H_2$ filtering for linear systems with uncertain Markov transitions

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Abstract

This paper is concerned with $H_\infty$ and $H_2$ filtering for Markovian jump linear systems with uncertain transition probabilities. Motivated by the fact that the existing results either impose severe restrictions on some key matrices or introduce some unnecessary matrix variables, this paper is focused on developing a new approach to systematically relax these restrictions for filter design. By applying a novel technique to eliminate the product terms between the Lyapunov matrices and the filter parameters, an improved condition is first obtained for analyzing the $H_\infty$ performance of the filtering error system. Then sufficient conditions in terms of linear matrix inequalities are presented for designing filters with a guaranteed $H_\infty$ filtering performance level. The proposed method is further extended to $H_2$ filtering. Theoretical analyses followed by a few numerical examples show that the proposed filter design method outperforms some existing results with respect to reduction of conservatism or variables needed for computation. The filter design problems for both continuous-time and discrete-time Markovian jump linear systems are addressed in a unified framework.

Keywords: $H_\infty$ filtering, $H_2$ filtering, Markovian jump linear system, Uncertain transition probability, Linear matrix inequality

1. Introduction

As an important class of stochastic systems, Markovian jump systems can be used to effectively model practical plants of the multi-mode nature, that is, plants have different working situations and may continuously switch between them in a random way. Due to the usefulness in describing complicated time-varying dynamics, Markov jump systems have been extensively investigated and the control theory for this type of systems has undergone great development in the past decade (Boukas, 2005; Costa, Fragoso, & Marques, 2005; Feng, Lam, & Shu, 2010; Shu, Lam, & Xiong, 2010). An active area in recent years regarding Markovian jump systems is those with uncertain transition probabilities, which are motivated by the fact that it is sometimes difficult to obtain the accurate information of transition probabilities of practical plants. In de Souza, Trofino, and Barbosa (2006), Karan, Shi, and Kaya (2006) and Xiong, Lam, Gao, and Ho (2005), the uncertain transition probabilities are assumed to be of the norm-bounded or polytope-bounded type which is commonly used in robust control theory so that some robust analysis methodologies can be applied; recent research attention is mostly focused on the case that the transition probabilities are partly known (He, Zhang, Wu, & She, 2011; Li, Lam, Gao, & Li, 2014; Zhang & Boukas, 2009a, b; Zhang, Boukas, & Lam, 2008; Zhang, He, Wu, & Zhang, 2011; Zhang & Lam, 2010; Zuo, Li, Liu, & Wang, 2012), which bridges the arbitrarily switching case and the completely known case: a new modeling method is considered in Gonçalves, Fioravanti, and Geromel (2011), Morais, Braga, Oliveira, and Peres (2013) and Morais, Braga, Lacerda, Oliveira, and Peres (2014), which models each row of the transition probability matrix as an independent polytopic domain and covers the method in Zhang and Boukas (2009b) and Zhang and Lam (2010) as a special case.

As a fundamental issue in systems and control theory, state estimation or filtering is to estimate signals that are of particular...
interest but not available due to technical difficulties. On the other hand, it is known that $H_\infty$ technique is an effective approach to deal with non-statistic disturbances. Some recent results on $H_\infty$ technique applied to dynamic systems can be found in, for instance, Hernandez-Gonzalez and Basin (2014, 2015) and You, Gao, and Basin (2013). Parallel to $H_\infty$ robust control theory, one of the prevailing filtering methods is $H_\infty$ filtering, which does not need to know the statistics of noises and can lead to more robust filters. Many results on $H_\infty$ filtering have been reported; see Basin, Shi, and Calderon-Alvarez (2009, 2010), He, Liu, Rees, and Wu (2009), Shi (1997, 1998), Shi, Luan, and Liu (2012) and Wang, Lam, and Luan (2004). Design of $H_\infty$ filters for Markov jump systems also has drawn considerable attention (de Souza et al., 2006; Goncalves, Fioravanti, & Geromel, 2009; Goncalves et al., 2011; Liu, Ho, & Sun, 2008; Morais et al., 2014; Shu, Lam, & Hu, 2009; Xiong & Lam, 2006; Zhang & Boukas, 2009a). For Markov jump linear systems (MJLS) with uncertain transition probabilities, sufficient conditions in terms of linear matrix inequalities (LMIs) for the existence of $H_\infty$ filters were derived in de Souza et al. (2006), Dong, Wang, and Gao (2012); Dong, Wang, Ho, and Gao (2011), Goncalves et al. (2009, 2011), Morais et al. (2014) and Zhang and Boukas (2009a). Especially, the methods in de Souza et al. (2006), Goncalves et al. (2009, 2011), Liu et al. (2008) and Morais et al. (2014) can be used to design partly-mode-dependent or mode-dependent $H_\infty$ filters when the mode information of MJLS is partly known or unknown. Besides $H_\infty$ filtering, $H_2$ filtering also has been extensively studied for both continuous- and discrete-time MJLS (Fioravanti, Goncalves, & Geromel, 2008; Liu, Zhang, & Chen, 2012; Morais, Braga, Lacerda, Oliveira, & Peres, 2015). It is worth pointing out that, when dealing with uncertain transition probabilities or designing mode-independent filters, a crucial procedure that is frequently used in Goncalves et al. (2011), Liu et al. (2008), Morais et al. (2014) and Zhang and Boukas (2009a) is, by introducing some slack variables in the bounded real lemma for MJLS, to eliminate the product terms between the Lyapunov matrices and the filter parameters. Although this procedure could relax the sufficient conditions for filter design, two important aspects on this procedure have been overlooked in these existing results. On one hand, the methods in Liu et al. (2008) and Zhang and Boukas (2009a) have restricted the slack matrices so much that the resulting design conditions can be further improved. On the other hand, although the latest conditions in Morais et al. (2014, 2015) outperform the previous ones with respect to conservatism reduction, it will be shown that there are too many unnecessary slack variables in these conditions (see Section 3.3.2). The above two aspects make it necessary to systematically develop new filter design methods to mitigate the drawback of each of the existing methods, which motivates the work in this paper.

In this paper, we will study the $H_\infty$ and $H_2$ filter design problems of MJLS with uncertain transition probabilities and in particular focus on dealing with the mentioned drawbacks of the existing design methods. First, for characterizing the $H_\infty$ performance of the filtering error system, a new version of the bounded real lemma for MJLS will be obtained through applying a novel technique to decouple the product terms between the Lyapunov matrices and the filter parameters. Based on the new bounded real lemma, sufficient conditions in terms of LMIs are then derived for the existence of full-order filters that guarantee an $H_\infty$ disturbance attenuation level for the filtering error system. In addition, we will further extend the proposed method to the $H_2$ filtering problem of MJLS. Due to the use of the new technique to eliminate the undesired product terms, the proposed filter design method is either less conservative or less computationally demanding than some existing methods. The advantages of the proposed method will be clearly shown by theoretical analyses and numerical results. The contributions of the paper are summarized as follows:

(1) a new technique is proposed for introducing extra matrix variables so as to decouple the product terms between Lyapunov matrices and the filter parameters; and (2) the proposed filter design method has systematically improved the existing results from both conservatism reduction and variable reduction points of view.

**Notation.** The superscripts “$-1$” and “$T$” stand for matrix inverse and matrix transpose, respectively. $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^{m \times n}$ is the set of all $m \times n$ real matrices. $\mathbb{N}$ represents the set of nonnegative integers. $|| \cdot ||$ represents the Euclidean norm of a vector, and $L_2$ and $L_\infty$ are the space of square Lebesgue integrable functions and summable infinite sequences, respectively. For $w \in L_2$ or $L_\infty$, its $2$-norm is denoted as $\| w \|_2 \triangleq \sqrt{\int_0^\infty |w_k|^2 \, dk}$ and $\| w \|_\infty \triangleq \sqrt{\sum_{k=0}^\infty |w_k|^2}$, respectively. The triplet notation $(\Omega, \mathcal{F}, \mathbb{P})$ refers to a probability space, where $\Omega$ is the sample space, the $\sigma$-algebra of subsets of the sample space and $\mathbb{P}$ the probability measure on $\mathcal{F}$, respectively. $\mathbb{E} \{ \cdot \}$ denotes the mathematical expectation. The notation $P > 0$ (or $P \succeq 0$) means that matrix $P$ is real symmetric and positive definite (semi-definite), $I$ denotes an identity matrix with appropriate dimension, $\text{diag}(A_1, A_2, \ldots, A_n)$ stands for a block diagonal matrix with $A_1, A_2, \ldots, A_n$ on the diagonal. Throughout the paper, a symbol with superscript “ct” or “dt” stands for the continuous- or discrete-time case, respectively. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

### 2. Problem statement

On a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider an MJLS represented by the following differential or difference equations:

$$
\delta \begin{bmatrix} x_k \\ y_k \\ z_k 
\end{bmatrix} = \begin{bmatrix} A(r_k) x_k + B(r_k) w_k \\ C(r_k) x_k + D(r_k) u_k \\ E(r_k) x_k + F(r_k) v_k 
\end{bmatrix}
$$

where $x_k \in \mathbb{R}^n$ is the state vector, $w_k \in \mathbb{R}^m$ is the external disturbance, $y_k \in \mathbb{R}^p$ is the measurement output and $z_k \in \mathbb{R}^q$ is the target output to be estimated. The symbol $\delta \begin{bmatrix} x_k \\ y_k \\ z_k 
\end{bmatrix}$ denotes $x_k (k \in \{0, +\infty\})$ for the continuous-time case and $x_{k+1} (k \in \mathbb{N})$ for the discrete-time case, respectively. For the two cases, the disturbance signal $w_k$ is assumed to belong to $L_2(0, +\infty)$ and $L_\infty(\mathbb{N})$, respectively. The scalar $r_k$ takes values from a finite set $M = \{1, 2, \ldots, M\}$, which is a switching signal determining which mode of the system is activated. For each mode $r_k = i \in M$, the real matrices $A(r_k), B(r_k), C(r_k), D(r_k), E(r_k)$ and $F(r_k)$, denoted by $A_i, B_i, C_i, D_i, E_i$ and $F_i$, respectively, are known and appropriately dimensioned.

For the continuous-time MJLS, $[r_k]$ is a continuous-time, discrete-state homogeneous Markov process with mode transition rates:

$$
\Pr(r_{k+1} = j | r_k = i) = \begin{cases} \pi_{i,j} d + o(d), & \text{for } j \neq i \\ 1 + \pi_i d + o(d), & \text{for } j = i \end{cases}
$$

where $d > 0$, $\lim_{d \to 0} \frac{o(d)}{d} = 0$, and $\pi_{ij} \geq 0$ for $i, j \in M, j \neq i$ and $\pi_i = -\sum_{j \in M, j \neq i} \pi_{ij} \leq 0$. For $j \neq i, \pi_{ij}$ denotes the switching rate from the $i$th mode at time $k$ to the $j$th model at time $k + d$. For the discrete-time MJLS, $[r_k]$ is a discrete-time Markov chain with mode transition probabilities:

$$
\Pr(r_{k+1} = j | r_k = i) = \pi_{ij}
$$

where $\pi_{ij} \geq 0$ for $i, j \in M$ and $\sum_{j \in M} \pi_{ij} = 1$. Define $\pi_i$ as the $i$th row of the probability matrix $[\pi_{ij}] \triangleq \Pi$, that is,

$$
\pi_i \triangleq [\pi_{i1} \pi_{i2} \cdots \pi_{iM}].
$$

Due to the difficulty in obtaining the exact value of the transition probabilities in practice, we assume that $\pi_i$ is unknown but belongs
to a convex set. As in Gonçalves et al. (2011), we describe the uncertain probability vector \( \pi_i \) for \( i \in M \) in the following way:

\[
\pi_i(\lambda_i) = \left\{ \sum_{j=1}^S \lambda_{ij} \pi_{ij}^{(s)} \mid \lambda_i \in A_i \right\}
\]

(2)

where \( \lambda_i \equiv \left[ \lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{iS} \right] \) is the uncertain parameter vector, \( A_i \) is a unit simplex defined as

\[
A_i \equiv \left\{ \lambda_i \in \mathbb{R}^S \mid \sum_{j=1}^S \lambda_{ij} = 1, \lambda_{ij} \geq 0 \right\}
\]

with \( S \) being the number of vertices of the polytope, and \( \pi_{ij}^{(s)} \equiv \left[ \pi_{ij1}^{(s)}, \pi_{ij2}^{(s)}, \ldots, \pi_{ijM}^{(s)} \right] \) is the known value of \( \pi_i(\lambda_i) \) at the \( s \)th vertex of the polytope. For later use, define

\[
\lambda \equiv \lambda_1 \times \lambda_2 \times \cdots \times \lambda_M, \quad A \equiv A_1 \times A_2 \times \cdots \times A_M,
\]

\( S \equiv \{1, 2, \ldots, S\} \).

We will make use of the measurement \( y_k \) to estimate the output \( z_k \) for the MJLS in (1). To this end, our attention is focused on designing a filter in the following form:

\[
\delta \hat{z}_k = \hat{A}(r_k) \hat{x}_k + \hat{B}(r_k) y_k
\]

\[
\hat{z}_k = \hat{C}(r_k) \hat{x}_k + \hat{D}(r_k) y_k
\]

(3)

where \( \hat{x}_k \in \mathbb{R}^{n_x} \) is the state vector of the filter, and \( \hat{A}(r_k), \hat{B}(r_k), \hat{C}(r_k), \) and \( \hat{D}(r_k) \) are real matrices with compatible dimensions. For each \( i \in M \), \( \hat{A}(r_k = i), \hat{B}(r_k = i), \hat{C}(r_k = i), \) and \( \hat{D}(r_k = i) \), denoted by \( \hat{A}_i, \hat{B}_i, \hat{C}_i, \) and \( \hat{D}_i \), respectively, are the filter parameters to be determined. If we restrict \( \hat{A}_i = \hat{A}, \hat{B}_i = \hat{B}, \hat{C}_i = \hat{C}, \) and \( \hat{D}_i = \hat{D} \) for all \( i \in M \), the filter in (3) reduces to a mode-independent one that does not need the information of the switching signal \( r_k \), which also can be handled by the proposed method.

Define the augmented state vector \( \hat{x}_k \equiv \left[ \hat{x}_k^T, \hat{x}_k \right]^T \) and the filtering error \( e_k = \hat{x}_k - x_k \), respectively. Combining the MJLS in (1) and the filter in [3] leads to the following filtering error system:

\[
\delta \hat{z}_k = \hat{A}(r_k) \hat{x}_k + \hat{B}(r_k) w_k
\]

\[
e_k = \hat{C}(r_k) \hat{x}_k + \hat{D}(r_k) w_k
\]

(4)

\[
\hat{A}(r_k) = \left[ \begin{array}{c} \hat{A}(r_k) \hat{C}(r_k) \\ 0 \hat{A}(r_k) \end{array} \right], \quad \hat{B}(r_k) = \left[ \begin{array}{c} \hat{B}(r_k) \\ \hat{B}(r_k) \hat{D}(r_k) \end{array} \right],
\]

\[
\hat{C}(r_k) = \left[ \begin{array}{c} E(r_k) - \hat{D}(r_k) \hat{C}(r_k) \\ -\hat{C}(r_k) \end{array} \right],
\]

\[
\hat{D}(r_k) = F(r_k) - \hat{D}(r_k) \hat{D}(r_k),
\]

To state the filtering objectives, the definitions of stability, \( H_{\infty} \) performance (Boukas, 2005; Costa et al., 2005) and \( H_2 \) performance (Costa, do Val, & Geromel, 1997; de Farias, Geromel, do Val, & Costa, 2000) are introduced for the filtering error system in (4).

**Definition 1.** The system in (4) with \( w_k \equiv 0 \) is said to be stochastically stable if

\[
\mathbb{E} \left[ \| \hat{x}_0 \|_2^2 | \hat{x}_0, r_0 \right] < \infty
\]

for every initial condition \( \hat{x}_0 \in \mathbb{R}^{2n_x} \) and \( r_0 \in M \).

**Definition 2.** Assume that the system in (4) is stochastically stable. Given a scalar \( \gamma > 0 \), the system in (4) is said to have an \( H_{\infty} \) performance level \( \gamma \) if it satisfies

\[
\mathbb{E} \left[ \| e_k \|_2^2 \right] < \gamma^2 \mathbb{E} \left[ \| w_k \|_2^2 \right]
\]

for all nonzero \( w_k \in L_2[0, +\infty) \) for the continuous-time case (respectively, \( w_k \in L_2(\mathbb{N}) \) for the discrete-time case) under zero initial conditions \( \hat{x}_0 = 0 \).

**Definition 3.** Assume that the system in (4) is stochastically stable. Given a scalar \( \rho > 0 \), the system in (4) (with \( D(r_k) = 0 \) for the continuous-time case) is said to have an \( H_2 \) performance level \( \rho \) if it satisfies

\[
\sum_{i=1}^{n_w} \sum_{j=1}^{M} \mu_i \mathbb{E} \left[ \| g_{ij}(k) \|_2^2 \right] < \rho^2
\]

where \( \mu_i = \Pr(r_0 = i) \) and \( g_{ij}(k) \) is the output \( e_k \) under zero initial conditions \( \hat{x}_0 = 0, r_0 = i \) and the input \( w_k = e_{i,k} \sigma_k \) with \( e_i \) being the \( i \)th column of the identity matrix and \( \sigma_k \) being the unitary impulse.

The \( H_{\infty} \) (respectively, \( H_2 \)) filtering problem for MJLSs with uncertain transition probabilities to be considered in the paper is stated as follows: Find a filter in (3) for the MJLS in (1) such that the filtering error system in (4) is stochastically stable with an \( H_{\infty} \) (respectively, \( H_2 \)) performance level \( \gamma \) (respectively, \( \rho \)) for all \( \lambda \in \Lambda \).

### 3. \( H_{\infty} \) filtering

In this section, we will first provide a new condition for \( H_{\infty} \) filtering analysis and then propose a filter design method for MJLSs with uncertain transition probabilities.

#### 3.1. \( H_{\infty} \) filtering analysis

We first introduce the following lemma that is useful for dealing with the formulated \( H_{\infty} \) filtering problem, which is known as the bounded real lemma for continuous-time MJLSs (de Souza et al., 2006) and discrete-time MJLSs (Seiler & Sengupta, 2003), respectively.

**Lemma 1.** Given transition probabilities \( \pi_{ij} \) for \( i, j \in M \) and a scalar \( \gamma \), the system in (4) is stochastically stable with an \( H_{\infty} \) performance level \( \gamma \) if and only if there exist symmetric matrices \( P_i > 0, \forall i \in M \) such that for the continuous-time case,

\[
\left[ \begin{array}{c} \hat{A}_i \hat{P}_i + \hat{P}_i \hat{A}_i^T + \hat{P}_i \hat{B}_i \hat{C}_i^T \\ \hat{C}_i \hat{P}_i + \hat{P}_i \hat{C}_i^T \\ \hat{D}_i \hat{P}_i + \hat{P}_i \hat{D}_i \end{array} \right] < 0, \quad \forall i \in M
\]

(5)

and for the discrete-time case,

\[
\left[ \begin{array}{c} \hat{A}_i \hat{D}_i^T + \hat{D}_i \hat{A}_i^T + \hat{D}_i \hat{B}_i \hat{C}_i^T \\ \hat{C}_i \hat{D}_i^T + \hat{D}_i \hat{C}_i^T \\ \hat{D}_i \hat{D}_i^T + \hat{D}_i \hat{D}_i^T \end{array} \right] < 0, \quad \forall i \in M
\]

(6)

where \( P_i \equiv \sum_{j=1}^{M} \pi_{ij} P_j \).

Given the MJLS in (1) and the filter in (3), Lemma 1 provides an LMI condition for analyzing the \( H_{\infty} \) performance of the filtering error system in (4). However, the conditions in (5) and (6) are not tractable for designing filters subject to uncertain transition probabilities, because the filter parameters are coupled with the Lyapunov matrices \( P_i \) in multiple inequalities. To handle this difficulty, we give the following theorem, a new condition for analyzing the \( H_{\infty} \) filtering performance for MJLSs. For convenience, define a symbol \( \Phi_i \) as

\[
\Phi_i^{ct} \equiv \left[ \begin{array}{c} 0 \hat{P}_i \hat{C}_i^T \\ \hat{D}_i \hat{P}_i \end{array} \right], \quad \Phi_i^{dt} \equiv \left[ \begin{array}{c} -\hat{P}_i \hat{C}_i \end{array} \right]
\]

(7)
Theorem 1. Consider the system in (1) with known transition probabilities \( \pi_{ij} \) for \( i, j \in \mathbb{M} \) and the filter in (3). Given a scalar \( \gamma > 0 \), the filtering error system in (4) is stochastically stable with an \( H_\infty \) performance level \( \gamma \) if and only if there exist matrices \( P_i > 0 \) and \( K_i \) for \( i \in \mathbb{M} \) such that

\[
\Psi_i^T \Phi_i \Psi_i + G_i^T K_i H_i + H_i^T K_i^T G_i < 0, \quad \forall i \in \mathbb{M}
\]  

(8)

where \( \Phi_i \) is defined in (7) and

\[
\psi_i = \begin{bmatrix} 0 & A_i & 0 & B_i & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}
\]

\( G_i = \begin{bmatrix} 0 & A_i + \varepsilon_1 I & 0 & B_i & 0 \\ I & 0 & \varepsilon_2 I & 0 & 0 \end{bmatrix} \)

\( H_i = \begin{bmatrix} -1 & \hat{B}_i C_i & \hat{A}_i & \hat{B}_i D_i & 0 \end{bmatrix} \).

In \( H_i \), scalars \( \varepsilon_1 \) and \( \varepsilon_2 \) take \( \varepsilon_1 = \varepsilon_2 \gg 0 \) for the continuous-time case and take zero for the discrete-time case, respectively.

Proof. According to Lemma 1, we need to prove the equivalent solvability of the conditions in (5) (respectively, (6)) and those in (8). To this end, we re-write the conditions in (5) and (6) in the following unified form:

\[
W_i^T \Phi_i W_i < 0
\]

(9)

where \( \Phi_i \) is defined in (7) and

\[
W_i = \begin{bmatrix} \tilde{A}_i & \tilde{B}_i & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}
\]

For the continuous-time case, it can be directly verified that the conditions in (5) are those in (9). For the discrete-time case, the conditions in (6) can be expressed as

\[
\begin{bmatrix} \tilde{A}_i & \tilde{B}_i^T \end{bmatrix}^T \begin{bmatrix} P_i & 0 & 0 \\ 0 & -P_i & 0 \\ 0 & 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \tilde{A}_i & \tilde{B}_i \\ 0 & \tilde{C}_i & \tilde{D}_i \end{bmatrix} < 0
\]

By applying Schur Complement Equivalence, the above inequality is equivalent to

\[
\begin{bmatrix} \tilde{A}_i & \tilde{B}_i^T \end{bmatrix}^T \begin{bmatrix} P_i & 0 & 0 \\ 0 & -P_i & 0 \\ 0 & 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} \tilde{A}_i & \tilde{B}_i \\ 0 & \tilde{C}_i & \tilde{D}_i \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \\ -I \end{bmatrix} < 0
\]

which is those in (9) for the discrete-time case. Now, we shall prove that the conditions in (9) and those in (8) have equivalent solvability (that is, the existence of \( P_i > 0 \) and \( K_i \) satisfying (8) implies the existence of \( P_i > 0 \) satisfying (9) and vice versa).

(8) \( \Rightarrow \) (9): Suppose that the conditions in (8) are satisfied. Define a symbol \( T_i \) for \( i \in \mathbb{M} \) as

\[
T_i \triangleq \begin{bmatrix} \hat{B}_i C_i & \hat{A}_i & \hat{B}_i D_i & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}
\]

for which the row dimension is such that \( H_i T_i = 0 \) with \( H_i \) in (8). With \( \hat{A}_i \) and \( \hat{B}_i \) in \( W_i \) substituted by the expressions in (4), it can be shown that \( \psi_i T_i = W_i \). Hence, pre- and post-multiplying (8) by \( T_i^T \) and \( T_i \), respectively, and noting that \( T_i \) has full column rank, we have

\[
W_i^T \Phi_i W_i = T_i^T (\psi_i^T \Phi_i \psi_i + G_i^T K_i H_i + H_i^T K_i^T G_i) T_i < 0
\]

that is, the conditions in (8) imply those in (9).

(9) \( \Rightarrow \) (8): Define a symbol \( I_i \) as

\[
I_i \triangleq \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}
\]

which satisfies the following relations:

\[
\psi_i I_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}
\]

(10)

\[
K_i I_i = \begin{bmatrix} A_i + \varepsilon_1 I & 0 & B_i & 0 \\ \hat{B}_i C_i & \hat{A}_i & \hat{B}_i D_i & 0 \end{bmatrix}
\]

(11)

\[
H_i I_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

(12)

where

\[
V_i = \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix}, \quad J = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}
\]

Combining (10)–(12), we have

\[
I_i^T (W_i^T \Phi_i W_i + G_i^T K_i H_i + H_i^T K_i^T G_i) I_i
\]

\[
= W_i^T \Phi_i [W_i \ V_i] + \begin{bmatrix} W_i^T \ & V_i^T \end{bmatrix} J K_i [0 \ -I]
\]

\[
+ \begin{bmatrix} 0 & -I \end{bmatrix} K_i J^T [W_i \ V_i]
\]

\[
= W_i^T \Phi_i W_i \ + W_i^T \Phi_i \ V_i \ [J \ K_i - K_i^T J^T V_i].
\]

Since \( I_i \) is invertible, it follows from the previous equation that the inequalities in (8) are satisfied if

\[
W_i^T \Phi_i W_i \ 
W_i^T \Phi_i [V_i \ J K_i - K_i^T J^T V_i] < 0.
\]

(13)

Therefore, we only need to show (9) \( \Rightarrow \) (13) in the sequel.

For the continuous-time case, (13) can be explicitly expressed as

\[
W_i^T \Phi_i W_i \ W_i^T (\Phi_i V_i - J K_i) \ * \ -K_i - K_i^T < 0
\]

(14)

where

\[
L_i = \begin{bmatrix} -K_{11}^T & -K_{12} \end{bmatrix} \begin{bmatrix} P_{11} & -\varepsilon_1 K_{11}^T \ P_{12} & -\varepsilon_2 K_{12} \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}
\]

and \( K_{11} \) and \( K_{12} \) are the upper and lower \( n \times n \) blocks of \( P_i \), respectively. Suppose that the inequalities in (9) hold for some
$P_i > 0$ for all $i \in \mathbb{M}$. Note that $P_{22} > 0$ is implied by $P_i > 0$. Let $\delta_1$ and $\delta_2$ be two scalars such that $W_i^T \Phi_i W_i < -\delta_1 I < 0$ and $W_i^T N_i^T P_{22}^{-1} N_i W_i < \delta_2 I$, where

$$N_i = \begin{bmatrix} -I & -P_{122} & 0 & 0 & 0 \end{bmatrix}.$$

Further let $\varepsilon$ be a scalar satisfying $0 \leq \varepsilon \leq \frac{\delta_1}{\delta_2}$, then we have

$$W_i^T \Phi_i W_i < -\delta_1 I < W_i^T N_i^T P_{22}^{-1} N_i W_i < -\frac{\varepsilon}{2} W_i^T N_i^T P_{22}^{-1} N_i W_i$$

which implies

$$W_i^T \Phi_i W_i + \frac{1}{2} W_i^T N_i^T P_{22}^{-1} N_i W_i < 0. \quad (15)$$

By applying Schur Complement Equivalence, (15) is equivalent to

$$[W_i^T \Phi_i W_i \hfill \epsilon W_i^T N_i^T \hfill -2\varepsilon P_{222}] < 0$$

which are the inequalities in (14) with

$$K_1 = \varepsilon_1^{-1} P_{112}, \quad K_2 = \varepsilon_2^{-1} P_{222}, \quad \varepsilon_1 = \varepsilon_2 = \varepsilon^{-1}.$$

Consequently, we have the implication (9) $\Rightarrow$ (15) $\Rightarrow$ (14) $\Rightarrow$ (13). To complete the proof of the implication (9) $\Rightarrow$ (8) for the continuous-time case, finally note that $\varepsilon_1$ and $\varepsilon_2$ can be fixed as a sufficient large positive scalar without any loss of generality.

For the discrete-time case, (14) should be re-written as

$$[W_i^T \Phi_i W_i \hfill \epsilon W_i^T N_i^T \hfill -2\varepsilon P_{222}] < 0 \quad (16)$$

where $L_i$ is re-defined as

$$L_i = \begin{bmatrix} P_{112}^T - K_1^T & P_{222} - K_2^T \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix}$$

and $P_{112}$ and $P_{222}$ are the upper and lower right $n_x \times n_x$ blocks of $P_i$, respectively. Suppose that the inequalities in (9) hold for some $P_i > 0$ for all $i \in \mathbb{M}$. Since $P_{222} > 0$ is guaranteed by $P_i > 0$ for all $i \in \mathbb{M}$, we can augment the inequalities in (9) as

$$[W_i^T \Phi_i W_i \hfill 0 \hfill -P_{222}] < 0 \quad (17)$$

which are the inequalities in (16) with

$$K_1 = P_{112}, \quad K_2 = P_{222}.$$

Thus, we establish the implication (9) $\Rightarrow$ (17) $\Rightarrow$ (16) $\Rightarrow$ (13) $\Rightarrow$ (8) for the discrete-time case.

Consequently, we have proven (9) $\Leftrightarrow$ (8) and using Lemma 1, the proof can be completed.

Compared with Lemma 1, the benefit of the conditions in (8) is that no product term between the Lyapunov matrices $P_i$ and the filter parameters $\hat{A}_i$, $\hat{B}_i$, $\hat{C}_i$ and $\hat{D}_i$ exists in these conditions. It is known that such a feature can be made use of to relax the restrictions imposed on the Lyapunov matrices $P_i$ in the filter design problem with uncertain transition probabilities and/or the mode-independent filter design problem (see de Souza et al., 2006, Liu et al., 2008 for continuous-time MJLSS and de Souza, 2003, Gonçalves et al., 2011, Morais et al., 2014 and Zhang & Boukas, 2009a for discrete-time MJLSS).

It should be emphasized that the method for decoupling the mentioned product terms is different from those in the literature. To see the differences, we specifically compare Theorem 1 in this paper with Liu et al. (2008, Theorem 3.1) and with de Souza (2003, Theorem 3.1):

- Besides the Lyapunov matrices, Liu et al. (2008, Theorem 3.1) still includes $8n_x^2$ extra scalar variables (in terms of $G_i$ and $Z_i$ therein) for each $i \in \mathbb{M}$, while Theorem 1 includes only $2n_x^2$ extra scalar variables (in terms of $K_i$ here), much fewer than the former.
- Similarly, de Souza (2003, Theorem 3.1) includes $4n_x^2$ extra scalar variables (in terms of $G_i$ therein) for each $i \in \mathbb{M}$, which is still more than the number of the extra scalar variables in Theorem 1 in this paper.

The above differences show that, although the existing conditions, Liu et al. (2008, Theorem 3.1) and de Souza (2003, Theorem 3.1), and Theorem 1 in this paper all are necessary and sufficient for $H_\infty$ filtering analysis for MJLSSs with exactly known transition probabilities, Theorem 1 in this paper has fewer variables for $H_\infty$ filtering analysis, thanks to which, a straightforward consequence is that our filter design method in the sequel is less computationally demanding than some of the existing ones in the literature. Besides, it will be shown later that our filter design method has an advantage with respect to conservatism reduction when compared with some existing ones.

### 3.2. $H_\infty$ filter design

The following theorem presents a sufficient condition for the existence of filters that guarantee the stochastic stability and an $H_\infty$ filtering performance level for MJLSSs with uncertain transition probabilities. It can be obtained by further imposing a constraint on the extra variables $K_i$ in Theorem 1.

**Theorem 2.** Consider the system in (1) with uncertain transition probabilities $P_{ij}(\lambda)$ for $\lambda_i \in \Lambda_i$ and $i, j \in \mathbb{M}$. Given scalars $\gamma > 0$, $\varepsilon_1$ and $\varepsilon_2$, a filter in (3) exists such that the filtering error system in (4) is stochastically stable with an $H_\infty$ performance level $\gamma$ if there exist matrices $P_i(\lambda) > 0$, $\hat{X}_i$, $\hat{X}_i^-$, $\hat{X}_i^+$, $\hat{X}_i^-$ and $\hat{X}_i^+$ for $i \in \mathbb{M}$ such that

$$\Psi_i^T \Phi_i(\lambda) \Psi_i + \hat{X}_i^+ \hat{X}_i^- + \hat{X}_i^+ \hat{X}_i^- \Psi_i < 0, \quad \forall (i, \lambda) \in \mathbb{M} \times \Lambda \quad (18)$$

where $\Phi_i(\lambda)$ is defined in (7) with $P_i$, $\hat{C}_i$ and $\hat{D}_i$ replaced by $P_i(\lambda)$, $\hat{C}_i$ and $\hat{D}_i$, respectively. $\Psi_i$ is defined in (8) and $\hat{X}_i = \begin{bmatrix} I & \varepsilon_1 I & \varepsilon_2 I & B_i \end{bmatrix}$, $\hat{X}_i^- = \begin{bmatrix} -I & \hat{C}_i & \hat{C}_i & B_i \end{bmatrix}$, $\hat{X}_i^+ = \hat{X}_i^-$.

Moreover, if the conditions in (18) are feasible, the filter parameters can be given by

$$\hat{A}_i = \hat{X}_i^+ \hat{X}_i^- \hat{X}_i^-, \quad \hat{B}_i = \hat{X}_i^+ \hat{X}_i^- \hat{X}_i^-, \quad \hat{C}_i = \hat{C}_i, \quad \hat{D}_i = \hat{D}_i. \quad (19)$$

**Proof.** Suppose that the conditions in (18) are feasible for some matrices $P_i(\lambda) > 0$ and $\hat{X}_i$, $\hat{X}_i^-$, $\hat{X}_i^+$, $\hat{X}_i^-$, $\hat{X}_i^+$ for $i \in \mathbb{M}$. Construct a candidate filter in (3) with the parameters given by (19), then we have

$$\Psi_i^T \Phi_i(\lambda) \Psi_i + \hat{X}_i^+ \hat{X}_i^- \hat{X}_i^+ \hat{X}_i^- \Psi_i < 0, \quad \forall (i, \lambda) \in \mathbb{M} \times \Lambda \quad (20)$$

where $G_i$ and $H_i$ are defined as in (8) and $K_i$ is specified as

$$K_i = \begin{bmatrix} \hat{X}_i^+ \hat{X}_i^- \hat{X}_i^- \hat{X}_i^- \end{bmatrix}. \quad (21)$$

According to Theorem 1, the conditions in (20) guarantee that, for the system in (1) and the filter parameters given by (19), the filtering error system in (4) is stochastically stable with an $H_\infty$ performance level $\gamma$ for all $\lambda \in \mathbb{M}$. The proof is completed.

The conditions in (18) are linear with respect to $P_i(\lambda)$, $\hat{X}_i$, $\hat{X}_i^-$, $\hat{X}_i^+$, $\hat{X}_i^-$ and $\hat{X}_i^+$. Due to the dependence on the uncertain parameter $\lambda$,
there are an infinite number of LMs in (18) needing to be checked. To make these conditions numerically tractable, we in the next employ a relaxation method similar to Morais et al. (2014, 2013) to obtain an alternative set of a finite number of LMs such that those in (18) are guaranteed for all $\lambda \in \Lambda$. Consider the parameter-dependent Lyapunov matrix $P(\lambda)$ given by

$$P(\lambda) = \sum_{k \in \mathcal{F}(g)} \lambda^k P_{k,l}$$

(22)

where $g = (g_1, g_2, \ldots, g_M) \in \mathbb{N}^M$ is the degree vector, $k = (k_1, k_2, \ldots, k_M)$ with $k_l = (k_{l1}, k_{l2}, \ldots, k_{lM}) \in \mathbb{N}^M$ is the exponent vector, $\lambda^k = \lambda_{k_1}^{k_{l1}} \lambda_{k_2}^{k_{l2}} \cdots \lambda_{k_M}^{k_{lM}}$ with $\lambda_{k_l} = \lambda_{k_{l1}}^{k_{l1}} \lambda_{k_{l2}}^{k_{l2}} \cdots \lambda_{k_{lM}}^{k_{lM}}$. $P_{k,l}$ is a homogeneous monomial in $\lambda$ and $P_{k,l}$ is the corresponding coefficient matrix. The set $\mathcal{F}(g)$ is defined as $\mathcal{F}(g) = \{k_1 \times \mathcal{F}(g_2) \times \cdots \times \mathcal{F}(g_M) | \mathcal{F}(g_l) \subseteq \mathbb{N}^M\}$ with $\mathcal{F}(g_l) \subseteq \mathbb{N}^M$ being the set of all possible $S_l$-tuple vectors such that for any $k \in \mathcal{F}(g)$, $\sum_{i=1}^S k_i = g_i$, $i \in M$. In other words, $P(\lambda)$ given by (22) represents a homogeneous polynomial matrix with $g$, as the partial degree of variable $x_i$. For two exponent vectors $k, l \in \mathcal{F}(g)$, operation $k \geq l$ or $k \leq l$ indicates that no element of $k - l$ is smaller than zero. For later use, denote by $I$ a vector of ones, and for a vector $k$, with one non-zero element, denote by $v(k)$ the position of the non-zero element. In addition, define a function $\phi(h)$ as $\phi(h) = h_1! h_2! \cdots h_n!$ for $h = (h_1, h_2, \ldots, h_n) \in \mathbb{N}^n$, and further define a symbol $S_{i,l,k,h}$ as

$$S_{i,l,k,h}^{(1)} \triangleq \begin{bmatrix} 0 & P_{k,l-i-h} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$S_{i,l,k,h}^{(2)} \triangleq \begin{bmatrix} 0 & P_{k,l-i-h} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

With the above preparation, we can obtain the following theorem which provides a sufficient condition consisting of a finite number of LMs for the existence of a filter with a guaranteed $H_\infty$ filtering performance.

**Theorem 3.** Consider the system in (1) with uncertain transition probabilities $p_{i,j}(\lambda)$ for $\lambda \in \Lambda$, and $i, j \in M$. Given scalars $r > 0, \epsilon_1, \epsilon_2$, and vectors $g, d \in \mathbb{N}^M$, a filter in (3) exists such that the filtering error system in (4) is stochastically stable with an $H_\infty$ performance level $\gamma$ if there exist matrices $P_{k,l} \in \mathcal{F}(g), \mathcal{L}_i, \mathcal{S}_i, \mathcal{R}_i$ and $\mathcal{S}_i$ for $i \in M$ that are feasible, the filter parameters can be given by (19).

**Proof.** Using the homogeneous polynomial representation of $P(\lambda)$ in (22) and noting the fact that

$$\prod_{r=1}^M \left( \sum_{l=1}^{S_r} \lambda^l \right) P_{i,r} = \left( \sum_{k \in \mathcal{F}(g)} \phi(k) P_{i,k} \right)$$

for any degree $d \in \mathbb{N}^M$, we have

$$P(\lambda) = \prod_{r=1}^M \left( \sum_{l=1}^{S_r} \lambda^l \right) P_{i,r}$$

(23)

$$\prod_{r=1}^M \left( \sum_{l=1}^{S_r} \lambda^l \right) P_{i,r} = \left( \sum_{k \in \mathcal{F}(g+b+1)} \phi(b+1) P_{i,k} \right)$$

(24)

$$\prod_{r=1}^M \left( \sum_{l=1}^{S_r} \lambda^l \right) P_{i,r} = \left( \sum_{k \in \mathcal{F}(g+b+1)} \phi(b) P_{i,k} \right)$$

(25)

Decompose the matrix $\Phi_i(\lambda)$ in (18) as $\Phi_i(\lambda) = \Gamma_i + \gamma_i$, where $\gamma_i$ is defined in (25) and $\Gamma_i(\lambda)$ consists of the remaining terms in $\Phi_i(\lambda)$. By combining (26) and (27), it can be verified that

$$\Psi_i^T \Phi_i(\lambda) \Psi_i + \gamma_i^T \mathcal{L}_i + \mathcal{R}_i^T \gamma_i = \Psi_i^T \mathcal{S}_i(\lambda) \Psi_i + (\Psi_i^T \mathcal{T}_i \Psi_i + \gamma_i^T \mathcal{L}_i + \mathcal{R}_i^T \gamma_i)$$

(26)

$$\Psi_i^T \Gamma_i + \gamma_i^T \mathcal{L}_i + \mathcal{R}_i^T \gamma_i = \sum_{k \in \mathcal{F}(g+b+1)} \lambda^k \left( \sum_{l \in \mathcal{F}(g+b+1)} \phi(b) \Psi_i^T \mathcal{S}_i \Psi_i \right)$$

(27)
Thus the inequalities in (24) guarantee the satisfaction of those in (18) for all $i \in \mathbb{M}$ and $\lambda \in \Lambda$. Finally, by virtue of Theorem 2, the proof can be completed.

The conditions in (23) and (24) include three degree vectors $g$, $d$ and $h$, of which $g$ is the degree of the Lyapunov matrices as in (22) while $d$ and $h$ are the levels of Polya’s relaxation (see Morais et al., 2015, 2013). By increasing the degrees, Theorem 3 becomes less conservative but requires more computational demand.

In order to highlight the contribution of this paper, we here are also interested in the case that the Lyapunov matrices $P_i(\lambda)$ are fixed as an uncertainty-independent form, $P_0$ for each $i \in \mathbb{M}$. Under this setting, the conditions in (18) are simplified to a finite number of LMIs as the following corollary shows.

**Corollary 1.** Consider the system in (1) with uncertain transition probabilities $\pi_{ij}(\lambda)$ for $\lambda \in \Lambda$, and $i, j \in \mathbb{M}$. Given scalars $\gamma > 0$, $\epsilon_1$, and $\epsilon_2$, a filter in (3) exists such that the filtering error system in (4) is stochastically stable with an $H_\infty$ performance level $\gamma$ if there exist matrices $P_0 > 0$, $\bar{X}_i$, $\bar{Y}_i$, $\bar{R}_i$, $\bar{H}_i$, $\bar{G}_i$ and $\bar{D}_i$ for $i \in \mathbb{M}$ such that

$$
\psi_i^{\left( i \right)} \psi_1 + \psi_i^{\left( i \right)} + \psi_i^{\left( n \right)} \psi_i + \psi_i^{\left( n \right)} = 0, \quad \forall (i, s) \in \mathbb{M} \times s_i
$$

where $\bar{X}_i$, $\bar{Y}_i$, $\bar{R}_i$, $\bar{H}_i$, $\bar{G}_i$ and $\bar{D}_i$ are as in (18).

Moreover, if the conditions in (28) are feasible, then the filter parameters can be given by (19).

**Proof.** Constraining $P(\lambda)$ as $P(\lambda) = P_0$ for all $\lambda \in \Lambda$ and noting $\sum_{i=1}^{n_i} \lambda_i^{(i)} = 1$, we have

$$
\psi_i^{\left( i \right)} \psi_1 + \psi_i^{\left( i \right)} + \psi_i^{\left( n \right)} \psi_i + \psi_i^{\left( n \right)} = \sum_{i=1}^{s_i} \lambda_i^{(i)} \left( \psi_i^{\left( i \right)} \psi_1 + \psi_i^{\left( i \right)} + \psi_i^{\left( n \right)} \psi_i + \psi_i^{\left( n \right)} \right)
$$

that is, the inequalities in (18) for each $i \in \mathbb{M}$ are affine with respect to $\lambda_i$. Therefore, under the setting $P(\lambda) = P_0$, the conditions in (18) are equivalent to those in (28). Then applying Theorem 2 can complete the proof.

The proposed method can be modified for designing partly-mode-dependent (or mode-cluster-dependent) filters (Gonçalves et al., 2009) or mode-independent filters (de Souza et al., 2006). To show the two cases, suppose that the modes of the original MJLS are grouped into $N$ clusters and the modes indices in the $n$th cluster are denoted by a set $M_n$ that satisfies $\mathbb{M} = \bigcup_{n=1}^{N} M_n$ and $M_i \cap M_j = \emptyset$ for $i \neq j$. A mode-cluster-dependent filter is concerned with a filter that, for the $n$th cluster, satisfies $\hat{A}_n = \hat{A}_n$, $\hat{B}_n = \hat{B}_n$, $\hat{C}_n = \hat{C}_n$ and $\hat{D}_n = \hat{D}_n$ for $i \in M_n$, which means that the filter has the same parameters, $\hat{A}_n$, $\hat{B}_n$, $\hat{C}_n$ and $\hat{D}_n$ for the switching signal $\tilde{r}_k \in \mathbb{M}_n$. To apply the proposed method, for instance, Corollary 1, we only need to replace $\hat{X}_i$, $\hat{Y}_i$, $\hat{R}_i$, $\hat{H}_i$, $\hat{G}_i$ and $\hat{D}_i$ by the new matrix variables $\hat{X}_n$, $\hat{Y}_n$, $\hat{R}_n$, $\hat{H}_n$ and $\hat{D}_n$, respectively, for all $n \in \{1, 2, \ldots, N\}$ and $i \in M_n$. A mode-independent filter is a special case of a mode-cluster-dependent one with $N = 1$ and $M_1 = \mathbb{M}$. Let $\hat{X}_i = \hat{X}_i$, $\hat{Y}_i = \hat{Y}_i$, $\hat{R}_i = \hat{R}_i$, $\hat{H}_i = \hat{H}_i$ and $\hat{G}_i = \hat{G}_i$ and $\hat{D}_i = \hat{D}_i$ for all $i \in \mathbb{M}$, then the proposed method can be applied to design mode-independent filters.

**Remark 1.** It has been shown that, for filtering performance analysis with known transition probabilities, appropriately fixing the scalars $\epsilon_1$ and $\epsilon_2$ in Theorem 1 will not introduce conservatism. However, this is not the case for Theorem 2 and Corollary 1 because they only give sufficient conditions for the existence of a required filter. In other words, given different values of $\epsilon_1$ and $\epsilon_2$, Corollary 1 usually leads to different design results. Thus, one may tune the scalars for Corollary 1 so as to further improve the guaranteed filter performance, that is, reduce the $H_\infty$ filtering performance level $\gamma$.

According to the existing numerical results in the literature, the settings $\epsilon_1 = 1$ and $\epsilon_1 = 2$ are usually chosen for the continuous- and discrete-time cases, respectively, which will also be adopted for the numerical examples in this paper. A straightforward way to achieve better results is trial and error (for instance, check some grid points of a given $(\epsilon_1, \epsilon_2)$ domain); another approach is to utilize the function `fminsearch` in Matlab with $(\epsilon_1, \epsilon_2)$ as the argument and the minimization of $\gamma$ subject to the LMIs in (23) and (24) or (28) as the objective.

### 3.3. Comparisons with some existing results

In this section, we theoretically compare the proposed filter design method with some existing representatives in order to show that our method is either less conservative than or equivalent to but less computationally demanding than them.

#### 3.3.1. Comparison with Liu et al. (2008)

In Liu et al. (2008), the design problem of mode-independent filters for continuous-time MJLSs was considered. Instead of directly comparing the LMI approaches to filter design, that is, Corollary 1 in this paper and Theorem 3.2 in the reference, we resort to the $H_\infty$ filtering analysis conditions for the sake of statement convenience. Keep in mind that the filter discussed in this subsection is strictly proper, mode-independent and the MJLS has exactly known transition probabilities.

On one hand, note that Theorem 3.2 in Liu et al. (2008) was obtained by applying a congruence transformation to the $H_\infty$ filtering analysis condition, Theorem 3.1 therein, and further restricting $G_i = Z_i = G$ for all $i \in \mathbb{M}$. Hence, it is not difficult to show that the optimal $H_\infty$ filtering performance obtained by Theorem 3.2 in the reference is equivalent to

$$
\gamma^*_i = \min_{\gamma, \varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4} \gamma : \text{s.t. (29)}
$$

where

$$
\Theta_i = \begin{bmatrix}
\bar{A}_i^T G + C_i^T \bar{A}_i + \bar{P}_i & \bar{A}_i^T G + P_i - G^T \bar{D}_i & \bar{G}_i & \bar{C}_i^T \\
* & -G - G^T & 0 & 0 \\
* & * & -\gamma I & 0 \\
* & * & * & -I
\end{bmatrix}
$$

with

$$
\gamma^*_i = \min_{\gamma, \varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4} \gamma : \text{s.t. (29)}
$$

For convenience, the above condition has been re-written with the notation in this paper. Introduce a matrix $U_i$ and partition the matrix $G$ as

$$
U_i = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad G = \begin{bmatrix}
\bar{X}_1 & \bar{X}_2 \\
\bar{X}_1 & \bar{X}_2
\end{bmatrix}
$$

where $\bar{X}_1$ and $\bar{X}_2$ are $n_x \times n_x$ matrix blocks and “*” denotes matrices that do not matter with the following derivations. Then multiplying $\Theta_i$ on the left and right by $U_i^T$ and $U_i$, respectively, we can show

$$
U_i^T \Theta_i U_i = \psi_i^{\left( i \right)} \psi_i^{\left( i \right)} + \psi_i^{\left( i \right)} K_i H_i + H_i^T K_i^T G_i
$$

where $\hat{A}_i = \hat{A}_i$, $\hat{B}_i = \hat{B}_i$, $\hat{C}_i = \hat{C}_i$ in $\Phi_i$ and $H_i, K_i = [\bar{X}_1^T \bar{X}_2^T]^T$ and $\epsilon_1 = \epsilon_2 = 1$ in $G_i$ (see (8) for symbols $\Phi_i, \psi_i, G_i, H_i$ and $K_i$). The above relation implies that the solvability of the inequalities in (29) guarantee that of those in (8) with $\bar{K}_i = [\bar{X}_1^T \bar{X}_2^T]^T$ and $\epsilon_1 = \epsilon_2 = 1$. However, the converse is not always true, because...
Therefore, there must exist $P_i > 0$, $K_i = [X_i^T \ X_i^T]^T$, $\hat{A}$, $\hat{B}$ and $\hat{C}$ such that $\eqref{8}$ under $\varepsilon_1 = \varepsilon_2 = 1$ holds. Define a matrix

$$X = \text{diag}(X_i^T X_i, I, X_i^T X_i, I, I).$$

$\eqref{33}$

Then performing a congruence transformation to the inequalities in $\eqref{8}$ by $X$, we have

$$X^T (\Psi_i (\lambda) + C_i^T K_i H_i + H_i^T K_i^T G_i) X < 0.$$

Note that the above inequality can be re-formulated as $\eqref{8}$ with the following change of variables:

$$X_i^T X_i \hat{A} X_i X_i^T \to \hat{A}, \quad X_i^T X_i \hat{B} B \to \hat{B}, \quad \hat{C} X_i X_i^T \to \hat{C}$$

$$\begin{bmatrix} X_i X_i^T \end{bmatrix} \to \begin{bmatrix} X \end{bmatrix} = K_i,$$

\text{diag}(l, X_i X_i, l) \to \text{diag}(l, X_i^T X_i).$$

Therefore, there must exist $P_i > 0$, $K_i$, $\hat{A}$, $\hat{B}$ and $\hat{C}$ and the same $\gamma$ such that $\eqref{8}$ also holds, implying that $\gamma^* \leq \gamma^*_2$. The proof is completed.

According to the above discussions, one sees that $\gamma^* = \gamma^*_2 \leq \gamma^*$, showing that for mode-independent filter design, our method, Corollary 1, is less conservative than the one in Liu et al. (2008). Moreover, as aforementioned, Corollary 1 is also computationally advantageous over the one in Liu et al. (2008) because fewer variables are needed by our method for computing the filter parameters.

**Remark 2.** Specifically, the system considered in Liu et al. (2008) has no transition probability uncertainty. However, the method therein can also be extended to the case with transition probability uncertainty described by $\eqref{2}$. In a general setting, this can be achieved by relaxing matrices $W_{111}$, $W_{211}$, and $W_{221}$ in the reference to be $\lambda$-dependent, which is the same treatment used in deriving Theorem 2 in this paper. It is worth pointing out that, even compared with this adaptation, Theorem 2 in this paper still has advantages in both conservatism reduction and variable reduction. To see this, it is needed to relax the matrix $P_i$, in $\eqref{8}$ and $\eqref{29}$ as $P_i(\lambda)$ and to note that all the derivations in this subsection still hold because the transformation matrices $U_i$ in $\eqref{30}$ and $X$ in $\eqref{33}$ are independent of the uncertain parameter $\lambda$. In Section 5, the simplest adaptation, $P_i(\lambda) = P_0$, for the uncertain probability case is specifically used for numerical illustration, which follows the same treatment of Corollary 1. As can be seen from Table 1, the advantages of Corollary 1 are obvious.

| Table 1 | $\gamma^*$ and NoV for different methods in Example 1 ($H_{\infty}$ filtering). |
|---------|-------------------|-------------------|
| Method  | Strictly proper | Proper |
| $\gamma^*$ | NoV | $\gamma^*$ | NoV |
| (Known $\Pi$): | | | |
| de Souza et al. (2006) | 0.7404 | 85 | Inapplicable |
| Liu et al. (2008) | 0.3028 | 85 | Inapplicable |
| Morais et al. (2015) | 0.2019 | 151 | 0.1880 | 152 |
| Corollary 1 | 0.2019 | 67 | 0.1880 | 68 |
| (Uncertain $\Pi$): | | | |
| de Souza et al. (2006) | 0.9264 | 85 | Inapplicable |
| Liu et al. (2008) | 0.3111 | 85 | Inapplicable |
| Morais et al. (2015) | 0.2076 | 151 | 0.1925 | 152 |
| Theory 3 ($g = 0$) | 0.2076 | 67 | 0.1925 | 68 |
| Morais et al. (2015) | 0.2059 | 277 | 0.1914 | 278 |
| Morais et al. (2015) | 0.2059 | 529 | 0.1914 | 530 |
| Theory 3 ($g = 1$) | 0.2059 | 193 | 0.1914 | 194 |

$^\dagger$ The reference Liu et al. (2008) does not address this case directly, and the result obtained from an adaptation according to Remark 2.

3.3.2. Comparison with Morais et al. (2014)

In Morais et al. (2014), the discrete-time case of a similar problem to the one in the paper was considered based on the multi-simplex representation method. As we mentioned, Theorem 2 in this paper combined with the multi-simplex representation method can produce a series of LMI conditions for filter design, leading to Theorem 3. Regardless of the degrees of the polynomials, the general formulation of the condition in Morais et al. (2014) used for computing the filtering performance level and the filter parameters is given by (15) therein, with which the notation in this paper is re-written as (34) which is given in Box 1, where

$$Q(\lambda) = \begin{bmatrix} \bullet & X_i \end{bmatrix}, \quad X(\lambda) = \begin{bmatrix} \bullet & e_1 X_i \end{bmatrix}, \quad Y(\lambda) = \begin{bmatrix} \bullet & 0 \end{bmatrix}, \quad Z(\lambda) = \begin{bmatrix} \bullet & 0 \end{bmatrix}$$

with “•” denoting matrices that do not matter with the following discussion. Multiplying $\Sigma_i(\lambda)$ on the left and right by $U_i^T$ and $U_i$ (see $\eqref{30}$), respectively, we have

$$U_i^T \Sigma_i(\lambda) U_i = \Psi_i (\lambda) \Psi_i + C_i^T K_i H_i + H_i^T K_i^T G_i$$

$\eqref{36}$

where $K_i = [X_i^T \ X_i^T]^T$ (see $\eqref{8}$ for other symbols). The above relation implies that the solvability of the inequalities in $\eqref{34}$ for $P_i(\lambda)$, $Q(\lambda)$, $X(\lambda)$, $Y(\lambda)$, $Z(\lambda)$, $\hat{A}$, $\hat{B}$, $\hat{C}$ and $D_i$ guarantees that of those in $\eqref{8}$ for some $P_i(\lambda)$, $K_i$, $\hat{A}$, $\hat{B}$, $\hat{C}$ and $D_i$ with $K_i = [X_i^T \ X_i^T]^T$. These discussions show that for the discrete-time case, Theorem 2 and the method in Morais et al. (2014) can attain the same best $H_{\infty}$ filtering performance level, but obviously the former involves much fewer variables for computing the filter parameters.

For the polynomial relaxation conditions of a specific degree, that is, Theorem 1 in Morais et al. (2014) and Theorem 3 in this paper, a similar argument also follows. Actually, multiplying $\eqref{8}$ of Morais et al. (2014) by $U_i^T$ on the left and $U_i$ on the right, respectively, one sees that the resulting term $U_i^T \Psi_i U_i$ can be
eliminated and the remaining two additive terms $U_j^T(\Theta_k + \Phi_k)U_i$ are finally reduced to (24) in this paper with the relaxation level $b = w - g - 1$. Furthermore, the continuous-time case of a similar problem has been studied in Morais et al. (2015) most recently. However, when system matrices are known, it is found that the slack matrices $G_i$, $G_{i'}, K_i$, $K_{i'}$, $Q_i$ and $F_i$ in the reference correspond to those in (35) denoted by "•" and thus can be eliminated using the same method. Consequently, Theorem 3 is less computationally demanding than the conditions in Morais et al. (2014, 2015) for filter design of MJLSs with known system matrices at each mode.

Remark 3. It should be clarified that the presented comparisons in this subsection are valid only when the matrix $U_i$ in (30) are invariant for all $\lambda \in \Lambda$, which implies that system matrices $A_i$ and $B_i$ are exactly known, as is previously assumed. If system matrices also contain uncertainty, it is known that introducing extra slack matrices are useful to reduce conservatism (de Oliveira, Bernussou, & Gerome, 1999). In this paper, we focus on coping with uncertainties in the transition matrix only, and aim to exploit more efficient conditions for filter design for this specific case. If the system matrices at each mode in (1) are also uncertain, it is still suggested to apply the results, for instance, in Morais et al. (2015) to solve the corresponding filter design problem.

4. $H_2$ filtering

In this section, we consider the $H_2$ filtering counterpart for MJLSs with uncertain transition probabilities. New LMI conditions will be presented for $H_2$ filtering performance analysis and filter design, respectively.

4.1. $H_2$ filtering analysis

To deal with the $H_2$ performance of the filtering error system, we first introduce the following lemma, which for the continuous- and discrete-time cases can be found in, e.g., Costa et al. (1997) and de Farias et al. (2000), respectively.

Lemma 2. Given initial mode probabilities $\mu_i$, transition probabilities $\pi_{ij}$ for $i, j \in M$ and a scalar $\rho$, the system in (4) (suppose $\bar{D}(r_k) = 0$ for the continuous-time case) is stochastically stable with an $H_2$ performance level $\rho$ if and only if there exist symmetric matrices $P_i > 0, \forall i \in M$ such that for the continuous-time case,

$$\sum_{i=1}^{M} \mu_i \text{Tr} (\tilde{B}_i^T P_i \bar{B}_i) < \rho^2,$$  
(37)

and for the discrete-time case,

$$\sum_{i=1}^{M} \mu_i \text{Tr} (\tilde{B}_i^T P_i \bar{B}_i + \bar{D}_i^T \bar{D}_i) < \rho^2,$$  
(38)

where $P_i \equiv \sum_{j=1}^{M} \pi_{ij} P_j$.

According to Lemma 2, the task of computing the $H_2$ performance level of the filtering error system in (4) is cast into the solvability of the inequalities in (37) or (38) for some matrices $P_i > 0$. By introducing an auxiliary matrix $R_i$ such that $\tilde{B}_i^T P_i \bar{B}_i < R_i$ and applying Schur Complement Equivalence, the inequalities in (37) can be re-written as $\sum_{i=1}^{M} \mu_i \text{Tr} (R_i) - \rho^2 < 0$ and

$$\tilde{B}_i^T P_i \bar{B}_i < R_i, \quad \tilde{A}_i^T P_i \bar{A}_i + \bar{C}_i^T \bar{C}_i < 0, \quad \tilde{A}_i^T P_i \bar{A}_i < 0.$$

Similarly, by introducing the matrix $R_i$ such that $\tilde{B}_i^T P_i \bar{B}_i + \bar{D}_i^T \bar{D}_i < R_i$, the inequalities in (37) are converted into $\sum_{i=1}^{M} \mu_i \text{Tr} (R_i) - \rho^2 < 0$ and

$$\tilde{B}_i^T P_i \bar{B}_i < R_i, \quad \tilde{A}_i^T P_i \bar{A}_i - \tilde{C}_i^T \tilde{C}_i < 0, \quad \tilde{A}_i^T P_i \bar{A}_i < 0.$$

Define symbols $\tilde{\Phi}_i$ and $\tilde{\Phi}_i^d$ as

$$\tilde{\Phi}_i^c \equiv \begin{bmatrix} P_i & 0 \\ 0 & R_i \end{bmatrix}, \quad \tilde{\Phi}_i^{dt} \equiv \begin{bmatrix} P_i & 0 \\ 0 & -R_i \end{bmatrix}$$

$$\tilde{\Phi}_i^e \equiv \begin{bmatrix} 0 & P_i \\ P_i & \tilde{C}_i \end{bmatrix}, \quad \tilde{\Phi}_i^{et} \equiv \begin{bmatrix} 0 & P_i \\ 0 & -\tilde{C}_i \end{bmatrix}.$$

(41)

Note that the inequalities in (39) and (40) are not easy-to-handle if the transition probabilities are uncertain. To cope with this difficulty, following a similar technique that is used to derive Theorem 1, we can obtain the following new condition for analyzing the $H_2$ performance of the filtering error system in (4).

Theorem 4. Consider the system in (1) and the filter in (3) with known initial mode probabilities $\mu_i$, transition probabilities $\pi_{ij}$ for $i, j \in M$. Given a scalar $\rho > 0$, the filtering error system in (4) (suppose $\bar{D}(r_k) = 0$ for the continuous-time case) is stochastically stable with an $H_2$ performance level $\rho$ if and only if there exist symmetric matrices $P_i > 0, R_i$ and matrices $K_i$ for $i \in M$ such that

$$\sum_{i=1}^{M} \mu_i \text{Tr} (R_i) - \rho^2 < 0,$$

$$\tilde{A}_i^T \tilde{\Phi}_i^c \tilde{\Phi}_i^c + \tilde{C}_i^T K_i \tilde{H}_i^T \tilde{H}_i^T \tilde{C}_i < 0, \quad \forall i \in M,$$

$$\tilde{A}_i^T \tilde{\Phi}_i^{dt} \tilde{\Phi}_i^{dt} + \tilde{C}_i^T K_i \tilde{H}_i^T \tilde{H}_i^T \tilde{C}_i < 0, \quad \forall i \in M,$$

(42)

where $\tilde{\Phi}_i$ and $\tilde{\Phi}_i^d$ are defined in (41) and

$$\tilde{\Phi} = \begin{bmatrix} 0 & A_i & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 & I \end{bmatrix}, \quad \tilde{\Phi}^d = \begin{bmatrix} 0 & B_i & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$
where $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 > 0$ sufficiently large for the continuous-time case and $\epsilon_1 = \epsilon_3 = 0, \epsilon_3 = \epsilon_4 = 1$ for the discrete-time case, respectively.

**Proof.** We can re-write the conditions in (39) or (40) in the following unified form:

$$\tilde{W}^T_i \tilde{\Phi}_i \tilde{W}_i < 0, \quad \tilde{W}^T_i \tilde{\Phi}_i \tilde{W}_i < 0$$

(46)

where $\tilde{\Phi}_i$ and $\tilde{\Phi}_i$ are defined in (41) and

$$\tilde{W}^i_0 = \begin{bmatrix} \tilde{B}_i & 0 \\ 1 & 0 \end{bmatrix}, \quad \tilde{W}^i_0 = \begin{bmatrix} \tilde{B}_i & 0 \\ 1 & 0 \end{bmatrix}, \quad \tilde{W}_i = \begin{bmatrix} \tilde{A}_i & 0 \\ 0 & 0 \end{bmatrix}.$$

(43) and (44) → (46): Suppose that the conditions in (43) and (44) are satisfied. Define two symbols $\tilde{T}_i$ and $\tilde{T}_i$, for $i \in M$ as

$$\tilde{T}_i^\ast = \begin{bmatrix} \tilde{B}_iD_i & 0 \\ 1 & 0 \end{bmatrix}, \quad \tilde{T}_i^\ast = \begin{bmatrix} \tilde{B}_i & 0 \\ 1 & 0 \end{bmatrix}, \quad \tilde{T}_i = \begin{bmatrix} \tilde{A}_i & 0 \\ 0 & 0 \end{bmatrix}.$$

for which the row dimensions are such that $\tilde{H}_i \tilde{T}_i = 0$ and $\tilde{H}_i \tilde{T}_i = 0$ with $\tilde{H}_i$ and $\tilde{H}_i$ in (45). With $\tilde{A}_i$ and $\tilde{B}_i$ in $\tilde{W}_i$ substituted by the expressions in (4), it can be shown that $\tilde{W}_i \tilde{W}_i = \tilde{W}_i$ and $\tilde{W}_i \tilde{W}_i = \tilde{W}_i$. Noting that $\tilde{T}_i$ and $\tilde{T}_i$ have full column rank, we have

$$\tilde{W}^T_i \tilde{\Phi}_i \tilde{W}_i = \tilde{T}_i^T \left( \tilde{W}^T_i \tilde{\Phi}_i \tilde{W}_i + \tilde{C}_i^T \tilde{K}_i \tilde{H}_i + \tilde{H}_i^T \tilde{K}_i \tilde{C}_i \right) \tilde{T}_i < 0$$

$$\tilde{W}^T_i \tilde{\Phi}_i \tilde{W}_i = \tilde{T}_i^T \left( \tilde{W}^T_i \tilde{\Phi}_i \tilde{W}_i + \tilde{C}_i^T \tilde{K}_i \tilde{H}_i + \tilde{H}_i^T \tilde{K}_i \tilde{C}_i \right) \tilde{T}_i < 0$$

that is, the conditions in (43) and (44) imply those in (46).

(46) → (43) and (44): Define symbols $\tilde{T}_i$ and $\tilde{T}_i$, respectively, as

$$\tilde{T}_i^\ast = \begin{bmatrix} \tilde{B}_iD_i & 0 \\ 1 & 0 \end{bmatrix}, \quad \tilde{T}_i^\ast = \begin{bmatrix} \tilde{B}_i & 0 \\ 1 & 0 \end{bmatrix}, \quad \tilde{T}_i = \begin{bmatrix} \tilde{A}_i & 0 \\ 0 & 0 \end{bmatrix}.$$

which satisfy the following relations:

$$\tilde{W}_i \tilde{T}_i = \tilde{W}_i, \quad \tilde{W}_i \tilde{T}_i = \tilde{W}_i, \quad \tilde{H}_i \tilde{T}_i = \begin{bmatrix} 0 & -I \end{bmatrix}, \quad \tilde{H}_i \tilde{T}_i = \begin{bmatrix} 0 & -I \end{bmatrix}$$

(47)

where

$$\tilde{W}^T_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{W}^T_i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{j} = \begin{bmatrix} \epsilon_1 I & 0 \\ 0 & \epsilon_1 I \end{bmatrix}.$$

Combining these relations, we have

$$\tilde{F}_i^T \left( \tilde{W}^T_i \tilde{\Phi}_i \tilde{W}_i + \tilde{C}_i^T \tilde{K}_i \tilde{H}_i + \tilde{H}_i^T \tilde{K}_i \tilde{C}_i \right) \tilde{T}_i$$

$$= \tilde{W}^T_i \tilde{\Phi}_i \tilde{W}_i + \tilde{C}_i^T \tilde{K}_i \tilde{H}_i + \tilde{H}_i^T \tilde{K}_i \tilde{C}_i > 0$$

for the continuous-time case, where $\tilde{\Phi}_i$ and $\tilde{\Phi}_i$ are invertible, it follows from the above equations that the inequalities in (43) and (44) are satisfied if

$$\tilde{W}^T_i \tilde{\Phi}_i \tilde{W}_i + \tilde{C}_i^T \tilde{K}_i \tilde{H}_i + \tilde{H}_i^T \tilde{K}_i \tilde{C}_i > 0$$

(48)

(49)

For the continuous-time case, (48) and (49) can be explicitly expressed, respectively, as

$$\tilde{W}^T_i \tilde{\Phi}_i \tilde{W}_i + \tilde{C}_i^T \tilde{K}_i \tilde{H}_i + \tilde{H}_i^T \tilde{K}_i \tilde{C}_i < 0$$

(50)

$$\tilde{W}^T_i \tilde{\Phi}_i \tilde{W}_i + \tilde{C}_i^T \tilde{K}_i \tilde{H}_i + \tilde{H}_i^T \tilde{K}_i \tilde{C}_i < 0$$

(51)

where

$$\tilde{L}_i = \begin{bmatrix} P_{11} - \epsilon_1 K_{11} & P_{12} - \epsilon_4 K_{12} \\ P_{21} - \epsilon_3 K_{21} & P_{22} - \epsilon_4 K_{22} \end{bmatrix}$$

and $K_{11}$ and $K_{22}$ are the upper and lower right $n_r \times n_r$ blocks of $P_i$, respectively, and $P_{12}$ and $P_{22}$ are the upper and lower right $n_r \times n_r$ blocks of $P_i$, respectively. Using similar arguments as in the proof of Theorem 1, it can be verified that if the two inequalities in (46) hold for some $P_i > 0$, those in (50) and (51) are also satisfied for the following assignment:

$$K_{11} = \epsilon_1^{-1} P_{11}, \quad K_{22} = \epsilon_4^{-1} P_{22}, \quad \epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = \epsilon^{-1}$$

where $\epsilon > 0$ is a sufficiently small scalar. Consequently, the inequalities in (43) and (44) can be obtained.

For the discrete-time case where $\epsilon_1 = \epsilon_3 = 0$ and $\epsilon_3 = \epsilon_4 = 1$, (48) and (49) can be explicitly expressed, respectively, as

$$\tilde{W}^T_i \tilde{\Phi}_i \tilde{W}_i + \tilde{C}_i^T \tilde{K}_i \tilde{H}_i + \tilde{H}_i^T \tilde{K}_i \tilde{C}_i < 0$$

(52)

(53)

where

$$\tilde{L}_i = \begin{bmatrix} P_{11} - K_{11} & P_{12} - K_{12} \\ P_{21} - K_{21} & P_{22} - K_{22} \end{bmatrix}$$

and $P_{11}$ and $P_{22}$ are the upper and lower right $n_c \times n_c$ blocks of $P_i$, respectively. The above two inequalities can be directly obtained from those in (46) with $K_{11} = P_{11}, K_{22} = P_{22}$.

We have established the equivalence between the inequalities in (46) and those in (43) and (44), and further applying Lemma 2, can conclude the proof.
4.2. H₂ filter design

Based on Theorem 4, we can obtain the following result which provides a new parameter-dependent LMI condition for the existence of a filter with a guaranteed H₂ filtering performance for all transition uncertainties. The theorem can be proved by applying Theorem 4 and specifying $K_i$ as in (21). The proof follows similar lines as that of Theorem 3 and is thus omitted for saving space.

**Theorem 5.** Consider the system in (1) with known initial mode probabilities $\mu_i$ and uncertain transition probabilities $\pi_{ij}(\lambda)$ for $\lambda_i \in \Lambda_i$ and $i, j \in M$. Given scalars $\rho > 0$, $\epsilon_1$, $\epsilon_2$, $\epsilon_3$ and $\epsilon_4$, a filter in (3) exists such that the filtering error system in (4) (suppose $D(r_k) = 0$ for the continuous-time case) is stochastically stable with an $H_2$ performance level $\rho$ if there exist symmetric matrices $P_i(\lambda) > 0, R_i(\lambda)$ and matrices $X_i, A_i, B_i, C_i, D_i$ for $i \in M$ such that

$$
\sum_{i=1}^{M} \mu_i \text{Tr}(R_i(\lambda)) - \rho^2 < 0, \quad \forall \lambda \in \Lambda
$$

(52)

The meaning of the symbol $\mathcal{F}(g)$ is the same as that in the previous section. Define some symbols as follows:

$$
\tilde{\mathcal{E}}_{i,k,l,h}^{ct} \triangleq \text{diag} \left\{ P_{i,k-1-h,-R_{i,k-1-h}} \right\}
$$

$$
\tilde{\mathcal{E}}_{i,k,l,h}^{dt} \triangleq \text{diag} \left\{ \sum_{j=1}^{M} \pi_{ij}(\lambda) P_{j,k-1-h,-R_{i,k-1-h},0} \right\}
$$

$$
\tilde{\mathcal{E}}_{i,k,l,h}^{ct} \triangleq \left[ \begin{array}{cccc} 0 & 0 & \ldots & 0 \\ P_{i,k-1-h,0} & 0 & \ldots & 0 \\ 0 & P_{i,k-1-h,0} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \\ \end{array} \right]
$$

$$
\tilde{\mathcal{E}}_{i,k,l,h}^{ct} \triangleq \text{diag} \left\{ \sum_{j=1}^{M} \pi_{ij}(\lambda) P_{j,k-1-h,-R_{i,k-1-h},0} \right\}
$$

Using the same polynomial relaxation method in deriving the previous $H_2$ filtering condition, we can obtain the following result for $H_2$ filter design. The proof can be completed by following similar lines as that of Theorem 3.

**Theorem 6.** Consider the system in (1) with known initial mode probabilities $\mu_i$ and uncertain transition probabilities $\pi_{ij}(\lambda)$ for $\lambda_i \in \Lambda_i$ and $i, j \in M$. Given scalars $\rho > 0$, $\epsilon_1$, $\epsilon_2$, $\epsilon_3$ and $\epsilon_4$, a filter in (3) exists such that the filtering error system in (4) (suppose $D(r_k) = 0$ for the continuous-time case) is stochastically stable with an $H_2$ performance level $\rho$ if there exist symmetric matrices $P_i, R_i, k \in \mathcal{F}(g)$ and matrices $X_i, A_i, B_i, C_i$ and $D_i$ for $i \in M$ such that the LMIs in (23) and (55)–(57) hold.

$$
\tilde{\mathcal{E}}_{i,k,l,h}^{ct} < 0, \quad \forall k \in \mathcal{F}(g + d)
$$

(55)

$$
\tilde{\mathcal{E}}_{i,k,l,h}^{dt} < 0, \quad \forall (i, k) \in M \times \mathcal{F}(g + b)
$$

(56-dt)

$$
\tilde{\mathcal{E}}_{i,k,l,h}^{ct} < 0, \quad \forall (i, k) \in M \times \mathcal{F}(g + b + 1)
$$

(56-dt)

where

$$
\tilde{\mathcal{E}}_{i,k,l,h}^{ct} = \sum_{l \in \mathcal{F}(g), l \neq k} \phi(l) \sum_{i=1}^{M} \mu_i \text{Tr}(R_{i,k-l}) + \frac{\phi(g + d)}{\phi(k)} \rho^2
$$

$$
= \sum_{l \in \mathcal{F}(g), l \neq k} \phi(l) \tilde{\mathcal{E}}_{i,k,l,h}^{ct} + \frac{\phi(g + b + 1)}{\phi(k)} \rho^2
$$

$$
\tilde{\mathcal{E}}_{i,k,l,h}^{dt} = \sum_{l \in \mathcal{F}(g), l \neq k} \phi(l) \tilde{\mathcal{E}}_{i,k,l,h}^{ct}
$$

$$
\tilde{\mathcal{E}}_{i,k,l,h}^{dt} = \sum_{l \in \mathcal{F}(g), l \neq k} \phi(l) \tilde{\mathcal{E}}_{i,k,l,h}^{ct} + \frac{\phi(g + b + 1)}{\phi(k)} \rho^2
$$

$$
\tilde{\mathcal{E}}_{i,k,l,h}^{ct} = \sum_{l \in \mathcal{F}(g), l \neq k} \phi(l) \tilde{\mathcal{E}}_{i,k,l,h}^{ct} + \frac{\phi(g + b + 1)}{\phi(k)} \rho^2
$$

$$
\tilde{\mathcal{E}}_{i,k,l,h}^{dt} = \sum_{l \in \mathcal{F}(g), l \neq k} \phi(l) \tilde{\mathcal{E}}_{i,k,l,h}^{ct} + \frac{\phi(g + b + 1)}{\phi(k)} \rho^2
$$

$$
\tilde{\mathcal{E}}_{i,k,l,h}^{dt} = \sum_{l \in \mathcal{F}(g), l \neq k} \phi(l) \tilde{\mathcal{E}}_{i,k,l,h}^{ct} + \frac{\phi(g + b + 1)}{\phi(k)} \rho^2
$$

Using the same polynomial relaxation method in deriving the previous $H_2$ filtering condition, we can obtain the following result for $H_2$ filter design. The proof can be completed by following similar lines as that of Theorem 3.

**Remark 4.** For the continuous-time case, when $P_i$ and $R_i$ are parameter-independent, the LMIs in (53) are also parameter-independent. Hence, the index “$s$” of the LMIs in (59) can be removed for the continuous-time case.
Remark 5. For $H_2$ filtering in the continuous-time case, the auxiliary matrix $R_k$ in (39) stems from the relation $\underbrace{B_1^T P B_1}_\gamma < R_k$, which is introduced to linearize the synthesis condition. Instead, we can employ an alternative relation $\overbrace{\mu \underbrace{B_1^T P B_1}_\gamma}^{\mu \gamma} < R_k$ for the same purpose. In particular, Theorem 1 in Morais et al. (2015) is based on this choice. Furthermore, we have the following two comments. First, the filter design conditions resulting from the two relations are not equivalent to each other. This is because the obtained design conditions are sufficient only. Consequently, we cannot specifically relate Theorem 6 in this paper with Theorem 1 in Morais et al. (2015). Second, both conditions can be easily modified such that the modified ones are based on the same relation. For instance, under the relation $\underbrace{B_1^T P B_1}_\gamma < R_k$, the matrix $\Theta_k$ in (15) of Morais et al. (2015) will not be multiplied by the coefficient $\mu$, but the matrix $W_{\gamma_k}$ in (14) therein should be multiplied by $\mu$ instead. Note that these modifications will not change the sufficiency of the design conditions. Then, as in Section 3.3.2 for $H_\infty$ filtering, we can exactly compare the modified design conditions and draw similar conclusions in principle. For numerical results on $H_2$ filtering in the sequel, the method in Morais et al. (2015) specifically indicates the modified version of Theorem 1 therein.

Remark 6. For $H_2$ filtering in the discrete-time case, the methods in Fioravanti et al. (2008) and Liu et al. (2012) do not address the uncertain situation described by (2). However, following the technique to obtain Corollary 2, it is easy to adapt the methods in Fioravanti et al. (2008) and Liu et al. (2012) to the situation considered in this paper. Moreover, using similar arguments as in Section 3.3.1 for $H_\infty$ filtering, one can see that, compared with the method in Liu et al. (2012), Corollary 2 in the paper is not only less conservative but also includes fewer variables.

5. Numerical examples

In the section, we present three numerical examples to show the merits of the proposed filter design method.

Example 1. Consider a two-mode continuous-time example in (1) with state-space matrices from de Souza et al. (2006) and also given as follows:

$A_1 = \begin{bmatrix} -3 & 1 & 0 \\ 0.3 & -2.5 & 1 \\ -0.1 & 0.3 & -3.8 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2.5 & 0.5 & -0.1 \\ 0.1 & -3.5 & 0.3 \\ -0.1 & 1 & -2 \end{bmatrix}$

$B_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.6 \\ 0.5 \\ 0 \end{bmatrix}, \quad D_1 = 0.2, D_2 = 0.5$

$C_1 = \begin{bmatrix} 0.8 & 0.3 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -0.5 & 0.2 & 0.3 \end{bmatrix}$

$E_1 = \begin{bmatrix} 0.5 & -0.1 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 & 0.6 \end{bmatrix}, \quad F_1 = F_2 = 0.$

For the case that the transition rates are exactly known, the matrix $\mathcal{H}$ is given by

$\mathcal{H} = \begin{bmatrix} -0.5 & 0.5 \\ 0.3 & -0.3 \end{bmatrix}.$

For the case that the transition rates are uncertain, the vertices $\pi^{(a)}_k$ of the rows of the matrix $\mathcal{H}$ is given by

$\pi^{(1)}_1 = [-0.35 \ 0.35], \quad \pi^{(2)}_1 = [-0.65 \ 0.65], \quad \pi^{(1)}_2 = [0.2 \ -0.2], \quad \pi^{(2)}_2 = [0.4 \ -0.4].$

$H_2$ filtering. Since the methods in de Souza et al. (2006) and Liu et al. (2008) can design mode-independent filters only, we consider the design of mode-independent filters for this example. Using Corollary 1 and Theorem 3 in this paper and the design methods in de Souza et al. (2006), Liu et al. (2008) and Morais et al. (2015), the results on the minimum $H_\infty$ filtering performance level $\gamma^*$ achieved by these methods are listed in Table 1, which also includes the information on the number of variables (NoV) of the LMI conditions for each method. The parameters in Corollary 1 and Theorem 3 are set as $\epsilon_1 = \epsilon_2 = 1, d = b = 0$, while those in the method in Morais et al. (2015) are set as $\lambda_1 = \lambda_2 = 1, d = 0$. It is seen from the table that, compared with de Souza et al. (2006) and Liu et al. (2008), Corollary 1 or Theorem 3 with $g = 0$ not only gives rise to better results on the minimum $H_\infty$ disturbance attenuation level, but also involves fewer variables for computation. Compared with Morais et al. (2015), the conditions in this paper are still less computationally demanding. For the case of the uncertain $\mathcal{H}$, the parameters of the mode-independent proper filter designed by Corollary 1 are given by

$\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} = \begin{bmatrix} -4.2382 & -2.3638 & 3.7667 & -0.4365 \\ -0.0305 & -2.6680 & 0.3480 & -0.6369 \\ -1.2147 & -2.5653 & -0.4671 & -0.9001 \\ 0.0921 & -1.0228 & -0.3247 & 0.2122 \end{bmatrix}.$

For the design case of mode-independent and strictly proper filters, Theorem 3 ($d = b = 0, g = 1$) gives rise to $\gamma^* = 0.1918$ for parameters $(\epsilon_1, \epsilon_2) = (4.001, 7.1570)$ (obtained by directly applying the fminsearch function in Matlab). This $H_\infty$ filtering performance level is smaller than $\gamma^* = 0.2059$ for $\epsilon_1 = \epsilon_2 = 1$ (see Table 1), showing that it is possible to improve a filter by adjusting the two scalars $\epsilon_1, \epsilon_2$.

$H_2$ filtering. Let the initial probabilities $\mu = [0.9 \ 0.1]$ and via this example, we compare Corollary 2 and Theorem 6 in this paper with Theorem 1 in Morais et al. (2015). Since matrices $D_1$ and $D_2$ are non-zero, filters to be computed should be strictly proper. The parameters of these methods, besides $\epsilon_1 = \epsilon_4 = 1$ and others to be indicated, are set the same as those for $H_\infty$ filtering. Table 2 lists the calculated results. It is clear that, under the same degree $g$, the methods in this paper and Morais et al. (2015) give rise to the same $H_2$ filtering performance level $\rho^*$ but those in this paper consist of much fewer variables.

Example 2. Consider a four-mode discrete-time example in (1) from Zhang and Boukas (2009a) with matrices given by

$A_1 = \begin{bmatrix} 0 & -0.405 \\ 0.81 & 0.81 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -0.2673 \\ 0.81 & 1.134 \end{bmatrix}$

$A_3 = \begin{bmatrix} 0 & -0.81 \\ 0.81 & 0.972 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & -0.1863 \\ 0.81 & 0.891 \end{bmatrix}$

$B_1 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \quad C_1 = [1 \ 0], \quad D_1 = E_1 = [0 \ 1]$.

$F_i = [0 \ 0], \quad i \in \{1, 2, 3, 4\}.$

Table 2  $\rho^*$ and NoV for different methods in Example 1 ($H_2$ filtering).

<table>
<thead>
<tr>
<th>Method</th>
<th>$\rho^*$</th>
<th>NoV</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Known $\mathcal{H}$)</td>
<td>Morais et al. (2015)</td>
<td>0.1054</td>
</tr>
<tr>
<td></td>
<td>Corollary 1</td>
<td>0.1054</td>
</tr>
<tr>
<td>(Uncertain $\mathcal{H}$)</td>
<td>Morais et al. (2015)</td>
<td>0.1152</td>
</tr>
<tr>
<td></td>
<td>Theorem 3 ($g = 0$)</td>
<td>0.1152</td>
</tr>
<tr>
<td></td>
<td>Morais et al. (2015)</td>
<td>0.1151</td>
</tr>
<tr>
<td></td>
<td>Morais et al. (2015)</td>
<td>0.1151</td>
</tr>
<tr>
<td></td>
<td>Theorem 3 ($g = 1$)</td>
<td>0.1151</td>
</tr>
</tbody>
</table>

* Theorem 1 in Morais et al. (2015) is modified as pointed out in Remark 5.
The following two cases are considered for the uncertain transition probabilities:

\[ \Pi_1 = \begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.4 \\ ? & 0.3 & 0.2 & ? \\ 0.1 & 0.1 & 0.5 & 0.3 \\ 0.2 & ? & ? & ? \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 0.3 & 0.2 & 0.1 & 0.4 \\ ? & 0.3 & 0.2 & ? \\ 0.1 & 0.1 & 0.5 & 0.3 \\ 0.2 & ? & ? & ? \end{bmatrix} \]

where "?" denotes the unknown transition probabilities. The two cases correspond to Case I and Case II of the example in Zhang and Boukas (2009a).

**H∞ filtering.** For this example, we first design mode-dependent filters by Corollary 1 in this paper, and compare the achieved minimum H∞ filtering performance level with that obtained by the methods in Morais et al. (2014) and Zhang and Boukas (2009a). Set \( \epsilon_1 = \epsilon_2 = 0 \) for Corollary 1 and set \( \lambda_1 = \lambda_2 = 0, \ d = g = h = 0 \) for the method in Morais et al. (2014). The first part of Table 3 summarizes the results on the achieved minimum H∞ filtering performance level as well as the data on NOV of each method. On one hand, it is shown that with fewer variables to be optimized, Corollary 1 still obtains better H∞ filtering performance levels than those by Zhang and Boukas (2009a). On the other hand, although Corollary 1 and the method in Morais et al. (2014) give rise to the same H∞ filtering performance level, Corollary 1 merits an obvious computational advantage, because the NOV for Corollary 1 is almost only a half of that for the method in Morais et al. (2014). This fact well verifies the discussions in Section 3.3.2 regarding the theoretical comparison between the two methods.

We further consider the design case of mode-independent filters. Set \( \epsilon_1 = \epsilon_2 = 0, \ d = b = 0 \) for Theorem 3 and \( \lambda_1 = \lambda_2 = 0, \ d = 0 \) for the method in Morais et al. (2014). The calculated results by different methods are presented in the second part of Table 3, which show that, using the same degree of polynomially parameter-dependent Lyapunov matrices, Theorem 3 and the method in Morais et al. (2014) can produce filters with the same guaranteed H∞ filtering performance level, but obviously the former needs much fewer variables.

**H2 filtering.** Suppose that the initial probabilities \( \mu = [0.22 \ 0.18 \ 0.22 \ 0.38] \). We compute the minimum H2 filtering performance level by Corollary 2 and Theorem 6 in this paper and the methods in Fioravanti et al. (2008); Liu et al. (2012). For Theorem 6, parameters \( \epsilon_i, i = 1, \ldots, 4, \ d \) and \( b \) are set as \( \epsilon_1 = \epsilon_2 = 0, \ \epsilon_3 = \epsilon_4 = 1 \) and \( d = b = 0 \). Table 4 shows the computed results. It is easy to see that Corollary 2 gives smaller H2 performance levels than those by the methods in Fioravanti et al. (2008) and Liu et al. (2012). Moreover, increasing the degree of polynomial Lyapunov matrices can further reduce the H2 filtering error level.

### Table 3

\begin{tabular}{|c|cc|cc|}
\hline
Method              & Case I$_1$ & Case I$_2$ & Case II$_1$ & Case II$_2$ \\
\hline
\textbf{(Mode-dependent filters:)} & & & & \\
Zhang and Boukas (2009a) & 3.8215 & 125 & 4.2793 & 125 \tabularnewline Morais et al. (2014) & 1.6422 & 181 & 1.6436 & 181 \tabularnewline Corollary 1 & 1.6422 & 93 & 1.6436 & 93 \tabularnewline \hline
\textbf{(Mode-independent filters:)} & & & & \\
Morais et al. (2014) (g = 0, h = 0) & 1.7361 & 142 & 1.7625 & 142 \tabularnewline Theorem 3 (g = 0) & 1.7361 & 54 & 1.7625 & 54 \tabularnewline Morais et al. (2014) (g = 1, h = 0) & 1.7336 & 342 & 1.7425 & 582 \tabularnewline Morais et al. (2014) (g = 1, h = 1) & 1.7336 & 782 & 1.7425 & 1550 \tabularnewline Theorem 3 (g = 1) & 1.7336 & 254 & 1.7425 & 494 \tabularnewline \hline
\end{tabular}

### Table 4

\begin{tabular}{|c|cc|cc|}
\hline
Method              & Case I$_1$ & Case I$_2$ & Case II$_1$ & Case II$_2$ \\
\hline
\textbf{(Mode-dependent filters:)} & & & & \\
Fioravanti et al. (2008) & 0.7092 & 73 & 0.7112 & 73 \tabularnewline Corollary 2 or Theorem 6 (g = 0) & 0.6876 & 105 & 0.6893 & 105 \tabularnewline Theorem 6 (g = 1) & 0.6726 & 365 & 0.6734 & 677 \tabularnewline \hline
\textbf{(Mode-independent filters:)} & & & & \\
Fioravanti et al. (2008) & 0.7410 & 37 & 0.7410 & 37 \tabularnewline Liu et al. (2012) & 0.7271 & 74 & 0.7315 & 74 \tabularnewline Corollary 2 or Theorem 6 (g = 0) & 0.7134 & 66 & 0.7199 & 66 \tabularnewline Theorem 6 (g = 1) & 0.6982 & 326 & 0.7004 & 638 \tabularnewline \hline
\end{tabular}

---

**Example 3.** This two-mode discrete-time example is borrowed from de Souza and Fragoso (2003) with matrices given by

\[
A_1 = \begin{bmatrix} 1 & 5.2529 \times 10^{-2} \\ 1.5146 \times 10^{-3} & 1.1022 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}
\]

\[
A_2 = \begin{bmatrix} 0.9955 & 4.9660 \times 10^{-2} \\ -0.2669 & 0.8075 \end{bmatrix}, \quad C_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]

\[
D_1 = E_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad i \in \{1, 2\}
\]

The transition probabilities are uncertain with the vertices \( \pi^{(i)} \) of the rows of the matrix \( \Pi \) given by

\[
\pi^{(1)} = [0.6 \ 0.4], \quad \pi^{(2)} = [0.5 \ 0.5],
\]

\[
\pi^{(1)} = [0.2 \ 0.8], \quad \pi^{(2)} = [0.3 \ 0.7].
\]

Mode-independent filters for this example are designed by Corollary 1 in this paper and the method in Gonçalves et al. (2011). The scalars \( \epsilon_1 \) and \( \epsilon_2 \) for Corollary 1 are specified as \( \epsilon_1 = \epsilon_2 = 0 \). The design results are shown in Table 5, where the results for both strictly proper and proper filters are calculated. Apparently the proposed method, Corollary 1, generates filters with improved guaranteed H∞ filtering performance bounds than that obtained by the method in Gonçalves et al. (2011), although the former needs to solve an optimization problem with more variables. The proper filter obtained by Corollary 1 has the following state-space realization:

\[
\begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} = \begin{bmatrix} 0.1710 & 1.0022 \\ 0.2544 & 0.4892 \\ -0.3744 & -0.6256 \end{bmatrix}, \quad \begin{bmatrix} -0.4422 \ 0.3744 \end{bmatrix}
\]

To further show the effectiveness of the obtained filter, we do some time-domain simulations under the following disturbance

\[
w(k) = \begin{bmatrix} e^{-0.5k} \sin(0.1k) \\ e^{-0.7k} \sin(3.0k) \end{bmatrix}
\]

The initial states of the system are set as \( x_0 = [0.2 \ 0.4]^T \) and that of the filter are set to zero. The system output \( z(k) \) and its estimation \( \hat{z}(k) \) are plotted in Fig. 1, where the switching signals \( r(k) \) are randomly generated according to the transition probabilities at the four vertices of the uncertain transition probability domain. It is shown that the filter output can effectively track the target signal \( z(k) \) for all the four cases of the transition probabilities.

6. Conclusion

The H∞ and H2 filtering problems of MJLSSs with uncertain transition probabilities have been investigated in the paper and new approaches have been systematically proposed for designing...
H∞ or H2 filters for both continuous- and discrete-time MJLSs. To overcome the drawbacks of the existing results, a new technique has been applied to decouple the product terms between the Lyapunov matrices and the filter parameters, leading to improved conditions for H∞ and H2 filtering analysis. LMI conditions have been proposed for checking the existence of filters that satisfy a guaranteed H∞ or H2 filtering performance, so that the filter design problems are cast into convex optimization problems that can be effectively solved. Theoretical comparisons between the proposed method and some existing ones have been presented, showing that the proposed one has advantages in conservatism reduction. Finally, numerical examples have been provided to further illustrate the effectiveness and improvements of the proposed method. In the future, the method developed in the paper may be used to study control of networked control systems (Wang, Gao & Qiu, 2015) or control of switching electrical devices (Napoles et al., 2013).

References


