Automatica 83 (2017) 58–64

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Properties and stability analysis of discrete-time negative imaginary systems☆

Mei Liu, Junlin Xiong

Department of Automation, University of Science and Technology of China, Hefei 230026, China

ARTICLE INFO

Article history:
Received 1 February 2016
Received in revised form 16 February 2017
Accepted 19 April 2017

Keywords:
Discrete-time systems
Negative imaginary lemma
Positive feedback
Internal stability

ABSTRACT

This paper is concerned with discrete-time negative imaginary (DT-NI) functions. First, a new definition of DT-NI functions is introduced. Then, by means of the relations between discrete-time positive real and DT-NI functions, two different versions of DT-NI lemmas are established to characterize the DT-NI properties based on state-space realizations. Also, a necessary and sufficient condition is presented to guarantee the internal stability of positive feedback interconnected DT-NI systems. Meanwhile, some other properties of DT-NI functions are studied. Several numerical examples are presented to illustrate the main results of this paper. Compared to the previous results, our results remove the symmetric assumption in rational case.

1. Introduction

The concept of positive real (PR) functions originated in network theory (Anderson & Vongpanitlerd, 1973). PR systems have obtained great achievements both in theory and in practice (Brogliato, Lozano, Maschke & Egeland 2007). One major limitation of PR functions is that their relative degree must be zero or one (Brogliato et al., 2007; Xiong, Petersen & Lanzon 2010). The theory of negative imaginary (NI) systems, who allowed a maximum relative degree of two (Lanzon & Petersen, 2008; Mabrok, Kallapur, Petersen & Lanzon 2014; Xiong et al., 2010), has appeared as a useful complement to PR theory. Since the NI theory was first proposed in Lanzon and Petersen (2008), a bunch of extensive study has arisen from the theory of NI systems to the application of NI theory, e.g., see Cai and Hagen (2010), Liu and Xiong (2016b), Mabrok et al. (2014), Patra and Lanzon (2011) and Petersen and Lanzon (2010). In particular, the internal stability results of positive feedback interconnected systems with NI response play an important role in engineering applications, see Lanzon and Petersen (2008), Mabrok et al. (2014) and Xiong et al. (2010).

It is noteworthy that all the present theory and applications of NI systems focus on the study of continuous-time (CT) systems except Ferrante, Lanzon & Ntogramatzidis (2014). In this paper, we are interested in presenting a similar development for discrete-time (DT) real-rational proper systems without the symmetric restriction. One should realize that this work is important in practice for the following reasons: (1) Almost all modern control schemes are digital signals in nature (Jiang, 1993). To analyse the closed-loop systems stability or properties of these control schemes, one should discretize the systems via a suitable sampling with a zero-order hold device (Jiang, 1993). This sample procedure leads to DT systems. (2) Although the generalized concept of DT-NI functions via z-domain has been proposed in Ferrante et al. (2014) to allow the DT-NI functions to be non-rational, all the transfer function matrices in Ferrante et al. (2014) are limited to be symmetric.

As is well-known, the bilinear transformation $s = \frac{z-1}{z+1}$ maps the open left half plane for CT systems to the open unit disc for DT systems (Anderson, Hitz & Diem, 1974; Ober & Montgomery-Smith, 1990). Under this bilinear transformation, a continuous-time positive real (CT-PR) transfer function $F(s)$ with $F(\infty) < \infty$ is transformed into a discrete-time positive real (DT-PR) transfer function $F(z)$ with $F(-1) < \infty$ and vice versa (Anderson et al., 1974; Hitz & Anderson, 1969). Also, the DT-PR lemma in Hitz and Anderson (1969) was derived by using this transformation. Therefore, our main techniques to handle the properties of DT-NI transfer functions in this paper are based on the bilinear transformation. Much as the continuous-time negative imaginary (CT-NI) systems can be defined in terms of their properties on the purely imaginary axis, the DT-NI systems can be related to their behaviours on the unit circle.

The contributions of this paper are as follows: (1) A new definition for DT-NI transfer function matrices that may be non-symmetric is introduced; (2) Under different assumptions, the

☆ This work was supported by National Natural Science Foundation of China under Grant 61374026. The material in this paper was not presented at any conference. This paper was recommended for publication in the revised form by Associate Editor Delin Chu under the direction of Editor Ian R. Petersen.

E-mail addresses: lmaymay@mail.ustc.edu.cn (M. Liu), Junlin.xiong@gmail.com (J. Xiong).

http://dx.doi.org/10.1016/j.automatica.2017.05.006
0005-1098/© 2017 Published by Elsevier Ltd.

relations between DT-PR and DT-NI functions are studied; (3) Two different DT-NI lemmas are derived by removing the symmetric assumption; (4) By checking the loop gain at $z = 1$ of the positive feedback system, a necessary and sufficient condition is derived for the internal stability of interconnected DT-NI systems. Compared to the results in Ferrante et al. (2014), our main contribution in this paper is that the real-rational transfer function matrix is allowed to be non-symmetric, that also develops the results on the real-rational DT-NI transfer function matrices. Meanwhile, a different version of DT-NI lemma is provided.

The rest of the paper is organized as follows. Section 2 provides the basic concept and some useful properties for DT-NI systems. Section 3 states the new relations between DT-PR and DT-NI functions. Two DT-NI lemmas are presented in Section 4. Section 5 presents the internal stability of positive feedback interconnected systems. Section 6 concludes the paper.

Notation: $\mathbb{R}^{m\times n}$ and $\mathcal{R}^{m\times n}$ denote the sets of $m\times n$ real matrices and real-rational proper transfer function matrices, respectively. A^T, A^* and $\bar{A}$ denote the transpose, the complex conjugate transpose and the complex conjugate of a complex matrix A, respectively. $\bar{\lambda}$ denotes the maximum eigenvalue for a square complex matrix with only real eigenvalues. $A > (\geq) 0$ denotes a symmetric positive (semi-)definite matrix. I denotes any identity matrix with compatible dimensions.

2. Discrete-time negative imaginary transfer functions

In this section, a new definition of DT-NI transfer function matrices is proposed, and some useful properties of such functions are studied.

Lemma 1 (*Hitz and Anderson, 1969*). *A square matrix $F(z)$ whose elements are real-rational functions analytic in $|z| > 1$ is DT-PR if, and only if, it satisfies all the following conditions*

(1) *poles of elements of $F(z)$ on $|z| = 1$ are simple;*
(2) *$F^*(e^{j\theta}) + F(e^{j\theta}) \geq 0$ for all real θ at which $F(e^{j\theta})$ exists;*
(3) *if $z_0 = e^{j\theta_0}$, θ_0 is real, is a pole of an element of $F(z)$, and if K_0 is the residue matrix of $F(z)$ at z_0, then the matrix $e^{-j\theta_0}K_0$ is a nonnegative definite Hermitian.*

Remark 1. Conditions 1–3 of Lemma 1 can be replaced by (a) $F^*(e^{j\theta}) + F(e^{j\theta}) \geq 0$ for all $\theta \in [0, 2\pi]$, with $e^{j\theta}$ not a pole of any element of $F(z)$; (b) If $z_0 = e^{j\theta_0}$, $\theta_0 \in [0, 2\pi]$, is a pole of an element of $F(z)$, then it is a simple pole (that is the poles of $F(z)$ on the unit circle, $|z| = 1$, are simple), and the corresponding residue matrix $K_0 = \lim_{z\to z_0}(z - z_0)F(z)$ satisfies that $e^{-j\theta_0}K_0$ is a nonnegative definite Hermitian.

By analogy with the CT case, we now present a new definition of DT-NI transfer function matrices.

Definition 1. A square real-rational proper transfer function matrix $G(z)$ is called DT-NI if

(1) $G(z)$ has no poles in $|z| > 1$;
(2) $j[G(e^{j\theta}) - G^*(e^{j\theta})] \geq 0$ for all $\theta \in (0, \pi)$ except values of θ where $e^{j\theta}$ is a pole of $G(z)$;
(3) if $z_0 = e^{j\theta_0}$, $\theta \in (0, \pi)$, is a pole of $G(z)$, then it is a simple pole and the corresponding residue matrix $K = \lim_{z\to z_0}(z - z_0)jG(z)$ satisfies that $e^{-j\theta_0}K$ is a positive semidefinite Hermitian;
(4) if $z = 1$ is a pole of $G(z)$, then $\lim_{z\to 1}(z - 1)^2G(z)$ is a positive semidefinite Hermitian, and $\lim_{z\to 1}(z - 1)^mG(z) = 0$ for all $m \geq 3$;
(5) if $z = -1$ is a pole of $G(z)$, then $\lim_{z\to -1}(z + 1)^2G(z)$ is a negative semidefinite Hermitian, and $\lim_{z\to -1}(z+1)^mG(z) = 0$ for all $m \geq 3$.

In order to analyse the properties of DT-NI systems, we define the following matrices for a given DT-NI transfer function matrix $G(z)$:

$$A_2 = \lim_{z\to 1}(z - 1)^2G(z), \qquad A_1 = \lim_{z\to 1}(z - 1)\left(G(z) - \frac{A_2}{(z - 1)^2}\right),$$
$$C_2 = \lim_{z\to -1}(z + 1)^2G(z), \qquad C_1 = \lim_{z\to -1}(z + 1)\left(G(z) - \frac{C_2}{(z + 1)^2}\right).$$

According to Conditions 4 and 5 in Definition 1, $A_2 = A_2^* \geq 0$ and $C_2 = C_2^* \leq 0$.

Remark 2. When $G(z)$ is real-rational non-proper, it means that $G(z)$ has poles in $|z| > 1$, which does not satisfy Condition 1 of Definition 1. So, the present definition of DT-NI functions focuses on the proper function. For example, consider $G(z) = z$. $G(z)$ has a simple pole in $|z| > 1$ and $j[G(e^{j\theta}) - G^*(e^{j\theta})] = -2\sin\theta \leq 0$ for all $\theta \in (0, \pi)$, which imply $G(z)$ is not DT-NI.

Remark 3. The difference between Ferrante et al. (2014, Lemma 11) and Definition 1 in this paper is that the transfer function matrices in Ferrante et al. (2014) are restricted to be symmetric, so that it requires that $A_1 \geq A_2$ and $C_1 \geq -C_2$ in Ferrante et al. (2014, Lemma 11), while Definition 1 in this paper does not have those restrictions.

Two useful lemmas are given as follows.

Lemma 2 (*Xiong et al., 2010*). *If $A = A^* \geq 0$, then $\bar{A} = \bar{A}^* \geq 0$.*

Lemma 3 (*Liu and Xiong, 2016a*). *A CT-NI transfer function matrix $G(s)$ transforms into a DT-NI transfer function matrix $G(z)$ by the bilinear transformation $s = \frac{z-1}{z+1}$. Conversely, a DT-NI transfer function matrix $G(z)$ transforms into a CT-NI transfer function matrix $G(s)$ by the bilinear transformation $z = \frac{1+s}{1-s}$.*

Then, we have the following result, which states one important property of DT-NI systems.

Lemma 4. *Given a square real-rational proper DT-NI transfer function matrix $G(z)$. Then, $A_1 + A_1^T \geq 0$, and $C_1 + C_1^T \geq 0$ hold.*

Proof. Since $G(z)$ is DT-NI, it follows that $G(z)$ has at most a double pole at 1 and -1. When $G(z)$ has no poles at 1 and -1, one has that $A_1 = 0$ and $C_1 = 0$, and hence $A_1 + A_1^T = 0$, and $C_1 + C_1^T = 0$.

Now, consider the case when $G(z)$ has poles at 1. Similar to the minor decomposition theory of CT case, we can write $G(z)$ in the form

$$G(z) = G_1(z) + \frac{A_1}{z - 1} + \frac{A_2}{(z - 1)^2}, \tag{1}$$

where $G_1(z)$ has no poles at 1. By means of the bilinear transformation

$$s = \frac{z - 1}{z + 1},\ z = \frac{1 + s}{1 - s}, \tag{2}$$

Eq. (1) transforms into

$$\begin{aligned} G(s) &= G_1\left(\frac{1+s}{1-s}\right) + \frac{A_1}{\frac{1+s}{1-s} - 1} + \frac{A_2}{(\frac{1+s}{1-s} - 1)^2} \\ &= G_1\left(\frac{1+s}{1-s}\right) - \frac{A_1}{2} + \frac{A_2}{4} + \frac{A_1 - A_2}{2s} + \frac{A_2}{4s^2}, \end{aligned}$$

where $G_1(\frac{1+s}{1-s}) - \frac{A_1}{2} + \frac{A_2}{4}$ has no poles at $s = 0$. It follows from Lemma 3 that $G(s)$ is CT-NI. $z = 1$ is a pole of $G(z)$ iff $s = 0$ is a pole of $G(s)$. According to Lemma 3 in Mabrok et al. (2014) and Lemma 2 in Liu and Xiong (2016b), we have $\frac{A_1-A_2}{2} + (\frac{A_1-A_2}{2})^T \geq 0$. It follows that $A_1 + A_1^T \geq A_2 + A_2^T \geq 0$, that is $A_1 + A_1^T \geq 0$.

Next, consider the case when $G(z)$ has poles at -1. $z = -1$ is a pole of $G(z)$ iff $s = \infty$ is a pole of $G(s)$. Decompose $G(z)$ to the form

$$G(z) = G_1(z) + \frac{C_1}{z+1} + \frac{C_2}{(z+1)^2}, \tag{3}$$

where $G_1(z)$ has no poles at -1. Similarly, using the same transformation as in (2), Eq. (3) transforms into a CT-NI transfer function matrix

$$\begin{aligned} G(s) &= G_1\left(\frac{1+s}{1-s}\right) + \frac{C_1}{\frac{1+s}{1-s}+1} + \frac{C_2}{(\frac{1+s}{1-s}+1)^2} \\ &= G_1\left(\frac{1+s}{1-s}\right) + \frac{C_1}{2} + \frac{C_2}{4} + \frac{-C_1-C_2}{2}s + \frac{C_2}{4}s^2, \end{aligned}$$

where $G_1(\frac{1+s}{1-s})+\frac{C_1}{2}+\frac{C_2}{4}$ has no poles at $s=\infty$. According to Lemma 2 in Liu and Xiong (2016b), we have $(-\frac{C_1+C_2}{2}) + (-\frac{C_1+C_2}{2})^T \le 0$. It follows that $C_1 + C_1^T \ge -(C_2 + C_2^T) \ge 0$, and hence $C_1 + C_1^T \ge 0$. □

Remark 4. When $G(z)$ is symmetric, $\frac{A_1-A_2}{2} \ge 0$ and $\frac{-C_1-C_2}{2} \le 0$ in view of Conditions 4 and 5 in Ferrante et al. (2014, Lemma 3), that is, $A_1 \ge A_2$ and $C_1 \ge -C_2$, which coincide with Conditions 4 and 5 in Ferrante et al. (2014, Lemma 11).

We now present the definition of discrete-time strictly negative imaginary (DT-SNI) transfer functions.

Definition 2. A square real-rational proper transfer function matrix $G(z)$ is called DT-SNI if

(1) $G(z)$ has no poles in $|z| \ge 1$;
(2) $j[G(e^{j\theta}) - G^*(e^{j\theta})] > 0$ for all $\theta \in (0, \pi)$.

The following two lemmas state two useful properties of DT-(S)NI transfer function matrices, respectively.

Lemma 5. *A square real-rational proper transfer function matrix $G(z)$ with no poles at -1 is DT-(S)NI if and only if $G(-1) = G^T(-1)$ and $\hat{G}(z) = G(z) - G(-1)$ is DT-(S)NI.*

Proof. (Necessity) Suppose $G(z)$ is DT-(S)NI (SNI is also NI). It follows from Condition 2 in Definition 1 that $\lim_{\theta\to\pi} j[G(e^{j\theta}) - G(e^{j\theta})] \ge 0$, that is $j[G(-1) - G^T(-1)] \ge 0$. According to Lemma 2, one obtains that $j[G(-1) - G^T(-1)] \ge 0$, that is $j[G(-1) - G^T(-1)] \le 0$. Thus, we have $G(-1) = G^T(-1)$, and hence $j[\hat{G}(e^{j\theta}) - \hat{G}^*(e^{j\theta})] = j[G(e^{j\theta}) - G^*(e^{j\theta})] \ge (>) 0$ for all θ with $e^{j\theta}$ not a pole of $G(z)$, and $e^{-j\theta_0}\lim_{z\to e^{j\theta_0}}(z-e^{j\theta_0})j\hat{G}(z) = e^{-j\theta_0}\lim_{z\to e^{j\theta_0}}(z-e^{j\theta_0})jG(z)$ is a positive semidefinite Hermitian for all $\theta_0 \in (0,\pi)$ with $e^{j\theta_0}$ being a pole of $G(z)$. Furthermore, $\lim_{z\to 1}(z-1)^m\hat{G}(z) = \lim_{z\to 1}(z-1)^m G(z)$ for all $m \ge 2$. Also, $\hat{G}(z)$ has no poles in $|z| > (\ge) 1$ since $G(z)$ has no poles in $|z| > (\ge) 1$. So, according to Definition 1 (Definition 2), $\hat{G}(z)$ is DT-(S)NI.

(Sufficiency) $G(z) = \hat{G}(z) + G(-1)$, and the sufficient part follows as a similar fashion to the necessity part. □

Lemma 6. *A square real-rational proper transfer function matrix $G(z)$ with no poles at 1 is DT-(S)NI if and only if $G(1) = G^T(1)$ and $G(z) - G(1)$ is DT-(S)NI.*

Proof. Suppose $G(z)$ is DT-NI. It follows from Condition 2 of Definition 1 that $\lim_{\theta\to 0} j[G(e^{j\theta}) - G^*(e^{j\theta})] \ge 0$, that is $j[G(1) - G^T(1)] \ge 0$. Then, it follows from Lemma 2 that $j[G(1) - G^T(1)] \le 0$. Thus, we have $G(1) = G^T(1)$. The rest of the proof and the Sufficiency proof are similar to that of Lemma 5. □

The following lemma will be used to derive the DT-NI lemma.

Lemma 7. *Let (A, B, C, D) be a minimal state-space realization of a square real-rational proper DT transfer function matrix $G(z)$. Suppose $G(z)$ has no poles at -1 and 1. Then*

(1) $F(z) = \frac{z-1}{z+1}(G(z) - G(-1)) \sim \left(\begin{array}{c|c} A & B \\ \hline C(A-I)(A+I)^{-1} & C(A+I)^{-1}B \end{array}\right)$ is a minimal state-space realization;
(2) $F(z) = \frac{1+z}{1-z}(G(z) - G(1)) \sim \left(\begin{array}{c|c} A & B \\ \hline C(I-A)^{-1}(I+A) & C(I-A)^{-1}B \end{array}\right)$ is a minimal state-space realization.

Proof. The proof of Part 1 can be found in the proof of Ferrante et al. (2014, Lemma 17); details are omitted here.

The proof of Part 2: A realization of $G(z) - G(1)$ is given by $(A, B, C, -C(I-A)^{-1}B)$, and a realization of $\frac{1+z}{1-z}I$ is given by $(I, I, -2I, -I)$. Then, a realization of $F(z) = \frac{1+z}{1-z}(G(z) - G(1))$ is given by $\left(\begin{array}{cc|c} I & C & -C(I-A)^{-1}B \\ 0 & A & B \\ \hline -2I & -C & C(I-A)^{-1}B \end{array}\right)$. Change the state representation via a similarity transformation, and let $P = \begin{pmatrix} I & C(I-A)^{-1} \\ 0 & I \end{pmatrix}$, we obtain the equivalent system representation

$$F(z) \sim \left(\begin{array}{cc|c} I & 0 & 0 \\ 0 & A & B \\ \hline -2I & 2C(I-A)^{-1} - C & C(I-A)^{-1}B \end{array}\right), \tag{4}$$

where $2C(I-A)^{-1} - C = C(I-A)^{-1}(I+A)$. Then, the system in (4) has a transfer function matrix $F(z) = C(I-A)^{-1}(I+A)(zI-A)^{-1}B + C(I-A)^{-1}B$. One obtains the following realization

$$F(z) \sim \left(\begin{array}{c|c} A & B \\ \hline C(I-A)^{-1}(I+A) & C(I-A)^{-1}B \end{array}\right). \tag{5}$$

Because (A, B, C, D) is a minimal realization, one has that (A, B) is controllable and (A, C) is observable. Note that $A(I-A)^{-1} = (A-I)^{-1}A = (I-A)^{-1} - I$, $A^n(I-A)^{-1} = (A-I)^{-1}A^n$ and $(I-A)^{-1}(I+A) = [2(I-A)^{-1} - I]$. Then, the observability of (A, C) implies that

$$\begin{aligned} \begin{pmatrix} C(I-A)^{-1}(I+A) \\ C(I-A)^{-1}(I+A)A \\ \vdots \\ C(I-A)^{-1}(I+A)A^{n-1} \end{pmatrix} &= \begin{pmatrix} 2C(I-A)^{-1} - C \\ 2CA(I-A)^{-1} - CA \\ \vdots \\ 2CA^{n-1}(I-A)^{-1} - CA^{n-1} \end{pmatrix} \\ &= \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} [2(I-A)^{-1} - I] = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} [(I-A)^{-1}(I+A)] \end{aligned}$$

is of full column rank as $[(I-A)^{-1}(I+A)]$ is nonsingular. Therefore, the state-space realization in (5) is minimal. □

The following lemma characterizes the DT-NI functions in z-domain. This lemma could be considered as the DT counterpart of Lemma 7 in Liu and Xiong (2016b).

Lemma 8. *Let $G(z)$ be a square real-rational proper DT transfer function matrix. Then, $G(z)$ is DT-NI and symmetric if and only if*

(1) $G(z)$ has no poles in $|z| > 1$;
(2) $j[G(z) - G^*(z)] \ge 0$ for all $|z| > 1$ and $\text{Im}[z] > 0$.

Proof. (Necessity) It follows from Definition 9 in Ferrante et al. (2014) that Conditions 1 and 2 are satisfied.

(Sufficiency) According to Lemma 10 in Ferrante et al. (2014), if $G(z)$ satisfies Conditions 1 and 2, then Conditions 3 and 4 of Definition 9 in Ferrante et al. (2014) hold. Under the assumption that $G(z)$ is a real-rational transfer function matrix, Condition 3 in Ferrante et al. (2014, Definition 9) implies that $G(z)$ is symmetric. So $G(z)$ is

DT-NI and symmetric according to Definition 9 in Ferrante et al. (2014). □

3. Relations between DT-PR and DT-NI functions

This section presents two results that establish the relations between DT-PR and DT-NI functions under different assumptions. The first lemma can be seen as an extension of Lemma 16 in Ferrante et al. (2014) by removing the symmetric assumption.

Lemma 9. *Given a DT transfer matrix $G(z) \in \mathcal{R}^{m\times m}$. Suppose $G(z)$ has no poles at -1. Then, $G(z)$ is DT-NI if and only if*

(1) $G(-1) = G(-1)^T$;
(2) $F(z) = \frac{z-1}{z+1}(G(z) - G(-1))$ *is DT-PR.*

Proof. (Necessity) Suppose $G(z)$ is DT-NI. It follows from Lemma 5 that $G(-1) = G^T(-1)$. Note that $G(z)$ and $F(z)$ have the same set of poles except at $z = 1$. Condition 1 of Definition 1 implies that $F(z)$ is analytic in $|z| > 1$. When $e^{j\theta}$, $\theta \in (0,\pi)$, is not a pole of $G(z)$ and $F(z)$, Condition 2 of Definition 1 implies that $F(e^{j\theta}) + F^*(e^{j\theta}) = \left[\frac{j\sin\theta}{1+\cos\theta}(G(e^{j\theta}) - G(-1))\right] + \left[\frac{-j\sin\theta}{1+\cos\theta}(G^*(e^{j\theta}) - G^T(-1))\right] = j\frac{\sin\theta}{1+\cos\theta}[G(e^{j\theta}) - G^T(e^{j\theta})] \geq 0$.

If $G(z)$ has no poles at 1 ($\theta = 0$), then $F(z)$ has no poles at 1 and $F(1) = 0$. As a result, $F(1) + F^T(1) = 0$. If $G(z)$ has a simple pole at 1, then $F(z)$ has no poles at 1. Let $G(z) = \frac{A_1}{z-1} + G_1(z)$, where $A_1 + A_1^T \geq 0$ and $G_1(z)$ is analytic in $|z| > 1$ and at $z = \pm 1$. Then, $F(z) = \frac{A_1}{z+1} + \frac{z-1}{z+1}G_1(z) - \frac{z-1}{z+1}G(-1)$. It follows that $F(1) + F^T(1) = \frac{A_1 + A_1^T}{2} \geq 0$. Therefore, $F(e^{j\theta}) + F^*(e^{j\theta}) \geq 0$ for all $\theta \in [0,\pi)$ with $e^{j\theta}$ not a pole of $F(z)$. According to Lemma 2, we have $F(e^{j\theta}) + F^*(e^{j\theta}) \geq 0$ for all $\theta \in [0,\pi)$. That is $F(e^{j\theta}) + F^*(e^{j\theta}) \geq 0$ for all $\theta \in (-\pi, 0]$ with $e^{j\theta}$ not a pole of $F(z)$. Note that $G(z) - G(-1)$ has a blocking zero at -1. So $F(z)$ has no poles at -1 ($\theta = \pm\pi$), and $F(-1) + F^*(-1) \geq 0$ in view of the continuity of $F(z)$. Furthermore, $\theta \in [0,\pi] \cup [-\pi, 0]$ is equal to $\theta \in [0, 2\pi]$. Hence, it follows that $F(e^{j\theta}) + F^*(e^{j\theta}) \geq 0$ for all $\theta \in [0, 2\pi]$ with $e^{j\theta}$ not a pole of $F(z)$.

If $G(z)$ has a double pole at 1, then $F(z)$ has a simple pole at 1. Let $G(z) = \frac{A_2}{(z-1)^2} + \frac{A_1}{z-1} + G_1(z)$, where $A_2 = A_2^* \geq 0$, $A_1 + A_1^T \geq 0$, and $G_1(z)$ is analytic in $|z| > 1$ and at $z = \pm 1$. Then, $F(z) = \frac{A_2}{(z-1)(z+1)} + \frac{A_1}{z+1} + \frac{z-1}{z+1}G_1(z) - \frac{z-1}{z+1}G(-1)$. The residue matrix of $F(z)$ at 1 given by $K_0 = \lim_{z\to 1}(z-1)F(z) = \frac{A_2}{2}$ is a positive semidefinite Hermitian. Also, the matrix $e^{-j\theta}K_0|_{\theta=0} = \frac{A_2}{2}$ is a positive semidefinite Hermitian.

If $z_0 = e^{j\theta_0}$, $\theta_0 \in (0,\pi)$, is a pole of $G(z)$, then $e^{\pm j\theta_0}$, $\theta_0 \in (0,\pi)$, are simple poles of $G(z)$ and $F(z)$. In this case, $G(z)$ can be factored as $G(z) = \frac{1}{(z-e^{j\theta_0})(z-e^{-j\theta_0})}G_1(z)$. The residue matrix of $G(z)$ at $e^{j\theta_0}$ is given by $K = \lim_{z\to e^{j\theta_0}}(z - e^{j\theta_0})jG(z) = \frac{1}{2\sin\theta_0}G_1(e^{j\theta_0})$. Then, Condition 3 of Definition 1 implies that $e^{-j\theta_0}K = \frac{e^{-j\theta_0}}{2\sin\theta_0}G_1(e^{j\theta_0})$ is a positive semidefinite Hermitian. This implies that $\overline{e^{-j\theta_0}G_1(e^{j\theta_0})} = \overline{e^{j\theta_0}G_1^*(e^{j\theta_0})} \geq 0$. In view of Lemma 2, we have $\overline{e^{-j\theta_0}G_1(e^{j\theta_0})} = \overline{e^{j\theta_0}G_1^*(e^{j\theta_0})} \geq 0$, that is, $e^{j\theta_0}G_1(e^{-j\theta_0}) = e^{-j\theta_0}G_1^*(e^{-j\theta_0}) \geq 0$. Now, the residue matrix of $F(z)$ at $e^{j\theta_0}$ is given by

$$K_0 = \lim_{z\to e^{j\theta_0}}(z - e^{j\theta_0})\frac{z-1}{z+1}\left[\frac{G_1(z)}{(z-e^{j\theta_0})(z-e^{-j\theta_0})} - G(-1)\right] = \frac{\sin\theta_0}{1+\cos\theta_0}\lim_{z\to e^{j\theta_0}}\frac{j}{(z-e^{-j\theta_0})}G_1(z) = \frac{1}{2(1+\cos\theta_0)}G_1(e^{j\theta_0}).$$

Then, the matrix $e^{-j\theta_0}K_0 = \frac{e^{-j\theta_0}}{2(1+\cos\theta_0)}G_1(e^{j\theta_0})$ is a positive semidefinite Hermitian. Also, the residue of $F(z)$ at $e^{-j\theta_0}$ is given by

$$K_0 = \lim_{z\to e^{-j\theta_0}}(z - e^{-j\theta_0})\frac{z-1}{z+1}\left[\frac{G_1(z)}{(z-e^{j\theta_0})(z-e^{-j\theta_0})} - G(-1)\right] = \frac{-\sin\theta_0}{1+\cos\theta_0}\lim_{z\to e^{-j\theta_0}}\frac{j}{(z-e^{j\theta_0})}G_1(z) = \frac{1}{2(1+\cos\theta_0)}G_1(e^{-j\theta_0}).$$

Then, the matrix $e^{j\theta_0}K_0 = \frac{e^{j\theta_0}G_1(e^{-j\theta_0})}{2(1+\cos\theta_0)}$ is also a positive semidefinite Hermitian. Thus, according to Lemma 1, $F(z)$ is DT-PR.

(Sufficiency) Suppose $F(z)$ is DT-PR and $G(-1) = G^T(-1)$. We have $G(z) = \frac{z+1}{z-1}F(z) + G(-1)$, and Condition 1 of Definition 1 is immediate. Condition 2 of Lemma 1 implies that $j[G(e^{j\theta}) - G^*(e^{j\theta})] = j\left[\frac{e^{j\theta}+1}{e^{j\theta}-1}F(e^{j\theta}) + G(-1) - \frac{e^{-j\theta}+1}{e^{-j\theta}-1}F^*(e^{j\theta}) - G^T(-1)\right] = j\left[\frac{-\sin\theta j}{1-\cos\theta}F(e^{j\theta}) - \frac{\sin\theta j}{1-\cos\theta}F^*(e^{j\theta})\right] = \frac{\sin\theta}{1-\cos\theta}[F(e^{j\theta}) + F^*(e^{j\theta})] \geq 0$ for all $\theta \in (0,\pi)$ with $e^{j\theta}$ not a pole of $G(z)$.

If $z_0 = e^{j\theta_0}$, $\theta_0 \in (0,\pi)$, is a pole of $F(z)$, then $z_0 = e^{j\theta_0}$ is also a pole of $G(z)$. The residue matrix of $G(z)$ at z_0 is given by $K = \lim_{z\to z_0}(z - z_0)jG(z) = \lim_{z\to z_0}(z - z_0)j\left[\frac{z+1}{z-1}F(z) + G(-1)\right] = \lim_{z\to z_0}\frac{\sin\theta_0}{1-\cos\theta_0}(z - z_0)F(z)$. Then, the matrix $e^{-j\theta_0}K = \frac{\sin\theta_0}{1-\cos\theta_0}e^{-j\theta_0}\lim_{z\to z_0}(z-z_0)F(z)$ is a positive semidefinite Hermitian in view of Condition 3 in Lemma 1.

If $F(z)$ has no poles at 1 and $F(1) = 0$, then $G(z)$ has no poles at 1. If $F(z)$ has no poles at 1 but $F(1) \neq 0$, then $G(z)$ has a simple pole at 1. The residue matrix of $G(z)$ at 1 is given by $A_1 = \lim_{z\to 1}(z-1)G(z) = 2F(1)$, which satisfies $A_1 + A_1^T \geq 0$, because $F(z)$ is real-rational PR. If $F(z)$ has a simple pole at 1, then $G(z)$ has a double pole at 1, and the residue matrix of $G(z)$ at 1 is given by $\lim_{z\to 1}(z-1)^2G(z) = \lim_{z\to 1}(z-1)^2\left[\frac{z+1}{z-1}F(z) + G(-1)\right] = 2\lim_{z\to 1}(z-1)F(z)$, which is a positive semidefinite Hermitian in view of Condition 3 in Lemma 1. Thus, according to Definition 1, $G(z)$ is DT-NI. □

Remark 5. The differences between Lemma 9 in this paper and Lemma 16 in Ferrante et al. (2014) are twofold: (1) $G(z)$ in this paper is allowed to be non-symmetric while $G(z)$ in Ferrante et al. (2014) is required to be symmetric; (2) the condition of $G(\infty) = G^T(\infty)$ in Ferrante et al. (2014, Lemma 16) is replaced by $G(-1) = G(-1)^T$ in this paper. When $G(z)$ is symmetric, Condition 1 in Lemma 9 is redundant, because $G(-1) = G(-1)^T$ can be directly derived by the symmetric assumption.

Example 1. As an illustration of Lemma 9, consider the non-symmetric transfer function matrix $G(z) = \begin{pmatrix} \frac{(z+1)^2}{2(z^2+1)} & \frac{1-z^2}{2(z^2+1)} \\ \frac{z^2-1}{2(z^2+1)} & \frac{(z+1)^2}{2(z^2+1)} \end{pmatrix}$. $j[G(e^{j\theta}) - G^*(e^{j\theta})] = 0$ for all $\theta \in (0,\pi)$ with $e^{j\theta}$ not a pole of $G(z)$. The residue matrix of $G(z)$ at $z = e^{j\frac{\pi}{2}} = j$ is given by $K = \begin{pmatrix} \frac{1}{2}j & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}j \end{pmatrix}$, and the matrix $e^{-j\frac{\pi}{2}}K = -jK = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}j \\ \frac{1}{2}j & \frac{1}{2} \end{pmatrix}$ is a positive semidefinite Hermitian. So, according to Definition 1, $G(z)$ is DT-NI. Then, a calculation shows that $G(-1) = G^T(-1)$ and $F(z) = \frac{z-1}{z+1}[G(z) - G(-1)] = \begin{pmatrix} \frac{z^2-1}{2(z^2+1)} & \frac{-(z-1)^2}{2(z^2+1)} \\ \frac{(z-1)^2}{2(z^2+1)} & \frac{z^2-1}{2(z^2+1)} \end{pmatrix}$ satisfying all the conditions in Lemma 1. Hence, it can be seen that $G(z)$ is DT-NI if and only if $G(-1) = G^T(-1)$ and $F(z)$ is DT-PR. This verifies the results in Lemma 9.

Under the assumption that $G(z)$ has no poles at $z = 1$, the following lemma gives a different relation between DT-PR and DT-NI transfer functions.

Lemma 10. *Given a DT transfer matrix $G(z) \in \mathcal{R}^{m\times m}$. Suppose $G(z)$ has no poles at 1. Then, $G(z)$ is DT-NI if and only if*

(1) $G(1) = G(1)^T$;
(2) $F(z) = \frac{1+z}{1-z}(G(z) - G(1))$ is DT-PR.

Proof. (Necessity) Suppose $G(z)$ is DT-NI. It follows from Lemma 6 that $G(1) = G^T(1)$. Note that $G(z)$ and $F(z)$ have the same set of poles except at $z = -1$. So, $F(z)$ is analytic in $|z| > 1$ according to Condition 1 of Definition 1. When $e^{j\theta}$, $\theta \in (0, \pi)$, is not a pole of $G(z)$ and $F(z)$, Condition 2 of Definition 1 implies that $F(e^{j\theta}) + F^*(e^{j\theta}) = \left[\frac{j\sin\theta}{1-\cos\theta}(G(e^{j\theta}) - G(1))\right] + \left[\frac{-j\sin\theta}{1-\cos\theta}(G^*(e^{j\theta}) - G^T(1))\right] = j\frac{\sin\theta}{1-\cos\theta}[G(e^{j\theta}) - G^*(e^{j\theta})] \geq 0$.

If $G(z)$ has no poles at -1 ($\theta = \pi$), then $F(z)$ has no poles at -1 and $F(-1) + F^T(-1) = 0$. If $G(z)$ has a simple pole at -1, then $F(z)$ has also no poles at -1. Let $G(z) = \frac{C_1}{z+1} + G_1(z)$, where $C_1 + C_1^T \geq 0$ and $G_1(z)$ is analytic in $|z| > 1$ and at $z = \pm 1$. Then, $F(z) = \frac{C_1}{1-z} + \frac{1+z}{1-z}G_1(z) - \frac{1+z}{1-z}G(1)$. It follows that $F(-1) + F^T(-1) = \frac{C_1 + C_1^T}{2} \geq 0$. Therefore, $F(e^{j\theta}) + F^*(e^{j\theta}) \geq 0$ for all $\theta \in (0, \pi]$ with $e^{j\theta}$ not a pole of $F(z)$. It follows from Lemma 2 that $F(e^{j\theta}) + F^*(e^{j\theta}) \geq 0$ for all $\theta \in [-\pi, 0)$ with $e^{j\theta}$ not a pole of $F(z)$. Note that $G(z) - G(1)$ has a blocking zero at 1. So $F(z)$ has no poles at 1 ($\theta = 0$), and $F(1) + F^*(1) \geq 0$ in view of the continuity of $F(z)$. Furthermore, $\theta \in [0, \pi] \cup [-\pi, 0]$ is equal to $\theta \in [0, 2\pi]$. Hence, we have that $F(e^{j\theta}) + F^*(e^{j\theta}) \geq 0$ for all $\theta \in [0, 2\pi]$ with $e^{j\theta}$ not a pole of $F(z)$.

If $G(z)$ has a double pole at -1, then $F(z)$ has a simple pole at -1. Let $G(z) = \frac{C_2}{(z+1)^2} + \frac{C_1}{z+1} + G_1(z)$, where $C_2 = C_2^* \leq 0$, $C_1 + C_1^T \geq 0$, and $G_1(z)$ is analytic in $|z| > 1$ and at $z = \pm 1$. Then, $F(z) = \frac{C_2}{(1-z)(z+1)} + \frac{C_1}{1-z} + \frac{1+z}{1-z}G_1(z) - \frac{1+z}{1-z}G(1)$. The residue matrix of $F(z)$ at -1 is given by $K_0 = \lim_{z\to -1}(z+1)F(z) = \frac{C_2}{2}$, which is a negative semidefinite Hermitian. Then, the matrix $e^{-j\theta}K_0|_{\theta=\pi} = -\frac{C_2}{2}$ is a positive semidefinite Hermitian.

If $z_0 = e^{j\theta_0}$, $\theta_0 \in (0, \pi)$, is a pole of $G(z)$ and $F(z)$, then $e^{-j\theta_0}$, $\theta_0 \in (0, \pi)$, is also a pole of $F(z)$ and $G(z)$. Let K be the residue matrix of $G(z)$ at z_0. Then, the residue matrix of $F(z)$ at z_0 is given by

$$K_0 = \lim_{z\to z_0}(z - z_0)F(z) = \lim_{z\to z_0}(z - z_0)\frac{1+z}{1-z}(G(z) - G(1))$$
$$= \frac{\sin\theta_0}{1-\cos\theta_0}\lim_{z\to z_0}(z - z_0)jG(z) = \frac{\sin\theta_0}{1-\cos\theta_0}K.$$

This implies that the matrix $e^{-j\theta_0}K_0 = \frac{\sin\theta_0}{1-\cos\theta_0}e^{-j\theta_0}K$ is a positive semidefinite Hermitian in view of Condition 3 in Definition 1. Using the similar argument as in the proof of Lemma 9, it can be shown that the residue matrix of $F(z)$ at $e^{-j\theta_0}$, $\theta_0 \in (0, \pi)$, is the complex conjugate of the residue matrix of $F(z)$ at z_0. The matrix $e^{j\theta_0}\lim_{z\to e^{-j\theta_0}}(z - e^{-j\theta_0})F(z)$ is also a positive semidefinite Hermitian. Thus, according to Lemma 1, $F(z)$ is DT-PR.

(Sufficiency) Suppose $F(z)$ is DT-PR and $G(1) = G^T(1)$. We have $G(z) = \frac{1-z}{1+z}F(z) + G(1)$, and Condition 1 of Definition 1 is immediate. Condition 2 of Lemma 1 implies that $j[G(e^{j\theta}) - G^*(e^{j\theta})] = j\left[\frac{-\sin\theta j}{1+\cos\theta}F(e^{j\theta}) - \frac{\sin\theta j}{1+\cos\theta}F^*(e^{j\theta})\right] = \frac{\sin\theta}{1+\cos\theta}[F(e^{j\theta}) + F^*(e^{j\theta})] \geq 0$ for all $\theta \in (0, \pi)$ with $e^{j\theta}$ not a pole of $G(z)$.

If $z_0 = e^{j\theta_0}$, $\theta_0 \in (0, \pi)$, is a pole of $F(z)$ and $G(z)$, then the residue matrix of $G(z)$ at z_0 is given by $K = \lim_{z\to z_0}(z - z_0)jG(z) = \lim_{z\to z_0}(z - z_0)j[\frac{1-z}{1+z}F(z) + G(-1)] = \lim_{z\to z_0}\frac{\sin\theta_0}{1+\cos\theta_0}(z - z_0)F(z) = \frac{\sin\theta_0}{1+\cos\theta_0}K_0$, where K_0 is the residue matrix of $F(z)$ at z_0. Hence, the matrix $e^{-j\theta_0}K = \frac{\sin\theta_0}{1+\cos\theta_0}e^{-j\theta_0}K_0$ is a positive semidefinite Hermitian in view of Condition 3 of Lemma 1.

If $F(z)$ has no poles at -1 and $F(-1) = 0$, then $G(z)$ has no poles at -1. If $F(z)$ has no poles at -1 but $F(-1) \neq 0$, then $G(z)$ has a simple pole at -1. The residue matrix of $G(z)$ at -1 is given by $C_1 = \lim_{z\to -1}(z+1)G(z) = 2F(-1)$. Since $F(z)$ is a real-rational DT-PR, it follows that $C_1 + C_1^T = 2[F(-1) + F^T(-1)] \geq 0$. If $F(z)$ has a simple pole at -1, then $G(z)$ has a double pole at -1. The residue matrix of $G(z)$ at -1 is given by $\lim_{z\to -1}(z+1)^2G(z) = \lim_{z\to -1}(z+1)^2\left[\frac{1-z}{1+z}F(z) + G(1)\right] = 2\lim_{z\to -1}(z+1)F(z)$, which is a negative semidefinite Hermitian in view of Condition 3 in Lemma 1. Thus, according to Definition 1, $G(z)$ is DT-NI. □

Remark 6. The difference between Lemmas 10 and 9 is that $G(z)$ in Lemma 10 is allowed to have poles at -1 but no poles at 1, while $G(z)$ in Lemma 9 is allowed to have poles at 1 but no poles at -1. Similar to Remark 5, when $G(z)$ is symmetric, Condition 1 in Lemma 10 is redundant as $G(1) = G^T(1)$ can be directly derived via the symmetric assumption.

Example 2. As an application of Lemma 10, consider $G(z) = \frac{1+z}{1+3z}$. We can say that $G(z)$ is DT-NI if and only if $F(z) = \frac{1+z}{1-z}[G(z) - G(1)] = \frac{1+z}{2(1+3z)}$ is DT-PR. A calculation shows that $F(z)$ and $G(z)$ satisfy all the conditions in Lemma 1 and Definition 1, respectively.

4. Discrete-time negative imaginary lemma

In this section, two DT-NI lemmas are developed to give an algebraic characterization of linear DT-NI transfer functions.

Lemma 11. *Let (A, B, C, D) be a minimal state-space realization of a transfer matrix $G(z) \in \mathcal{R}^{m\times m}$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, and $m \leq n$. Suppose $\det(I + A) \neq 0$ and $\det(I - A) \neq 0$. Then, $G(z)$ is DT-NI if and only if*

(1) $C(I + A)^{-1}B - D = B^T(I + A^T)^{-1}C^T - D^T$;
(2) *There exists a matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, such that*

$$Y - A^TYA \geq 0 \text{ and } C = B^T(I - A^T)^{-1}Y(I + A). \quad (6)$$

Proof. The proof follows along the following equivalences.

$G(z) \sim (A, B, C, D)$ is DT-NI.

$\Leftrightarrow$ $G(-1) = G^T(-1)$, and $F(z) = \frac{z-1}{z+1}(G(z) - G(-1))$ is DT-PR (see Lemma 9).

$\Leftrightarrow$ $C(I + A)^{-1}B - D = B^T(I + A^T)^{-1}C^T - D^T$, and $F(z) \backsim \left(\begin{array}{c|c} A & B \\ \hline C(A-I)(A+I)^{-1}) & C(A+I)^{-1}B \end{array}\right)$ is DT-PR (via Lemma 7).

$\Leftrightarrow$ $C(I + A)^{-1}B - D = B^T(I + A^T)^{-1}C^T - D^T$, and there exist matrices $Y = Y^T > 0$, Q, W such that

$$Y - A^TYA = Q^TQ$$
$$(A^T + I)^{-1}(A^T - I)C^T - A^TYB = Q^TW$$
$$C(A + I)^{-1}B + B^T(I + A^T)^{-1}C^T - B^TYB = W^TW.$$

This equivalence is according to the DT-PR lemma in Hitz and Anderson (1969). The rest of the proof follows along similar lines of the proof of Ferrante et al. (2014, Theor.7). □

For DT-SNI transfer function matrices, we have the following property.

Corollary 1. *Let (A, B, C, D) be a minimal state-space realization of a DT-SNI transfer matrix $G(z) \in \mathcal{R}^{m\times m}$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, and $m \leq n$. Then, $\mathrm{rank}(B) = \mathrm{rank}(C) = m$.*

Proof. We know that $\mathrm{rank}(B) \leq m$ as $m \leq n$. Suppose $\mathrm{rank}(B) < m$. Then, there exists a nonzero vector $x \in \mathbb{R}^m$ such that $Bx = 0$. Therefore, $x^*j[G(e^{j\theta}) - G^*(j\theta)]x = jx^TC(e^{j\theta} - A)^{-1}Bx - jx^TB^T(e^{-j\theta} - A^T)^{-1}C^Tx = 0$, for any $\theta \in (0, \pi)$. This contradicts condition 2 of Definition 2. Thus, it can be concluded that the only case is $\mathrm{rank}(B) = m$. Similarly, we have $\mathrm{rank}(C) = m$. □

The following corollary relates the gain at $z = 1$ and the gain at $z = -1$ for DT-NI and DT-SNI transfer function matrices.

Corollary 2.

(1) *Given a DT-NI transfer function matrix* $G(z)$*. Then,* $G(1) - G(-1) \geq 0$.
(2) *Given a DT-SNI transfer function matrix* $G(z)$*. Then,* $G(1) - G(-1) > 0$.

Proof. The proof of Part one is the same as the proof of Lemma 18 in Ferrante et al. (2014).

The proof of Part two: According to Lemma 5, if $G(z)$ is DT-SNI, then $\hat{G}(z) = G(z) - G(-1)$ is DT-SNI. Let (A, B, C, D) be a minimal realization of $G(z)$. Applying Lemma 11 and Lemma 18 in Ferrante et al. (2014), we obtain that $\hat{G}(1) = G(1) - G(-1) = 2B^T(I-A)^{-T}Y(I-A)^{-1}B \geq 0$. Suppose there exists an $x \in \mathbb{R}^{n\times n}$ such that $2B^T(I-A)^{-T}Y(I-A)^{-1}Bx = 0$, which implies that $Bx = 0$ as $(I-A)^{-T}Y(I-A)^{-1} > 0$. It follows from Corollary 1 that B is of full column rank. Thus, the only case such that $Bx = 0$ is $x = 0$, which implies that $B^T(I-A)^{-T}Y(I-A)^{-1}B$ is nonsingular. Therefore, $G(1) - G(-1) > 0$. □

The following lemma gives a new version of DT-NI lemma according to Lemma 10.

Lemma 12. *Let* (A, B, C, D) *be a minimal state-space realization of a transfer matrix* $G(z) \in \mathcal{R}^{m\times m}$*, where* $A \in \mathbb{R}^{n\times n}$*,* $B \in \mathbb{R}^{n\times m}$*,* $C \in \mathbb{R}^{m\times n}$*,* $D \in \mathbb{R}^{m\times m}$*, and* $m \leq n$*. Suppose* $\det(I + A) \neq 0$ *and* $\det(I - A) \neq 0$*. Then,* $G(z)$ *is DT-NI if and only if*

(1) $C(I-A)^{-1}B + D = B^T(I-A^T)^{-1}C^T + D^T$;
(2) *There exists a matrix* $Y = Y^T > 0$*,* $Y \in \mathbb{R}^{n\times n}$*, such that*

$$Y - A^TYA \geq 0 \text{ and } C = B^T(I+A^T)^{-1}Y(I-A). \tag{7}$$

Proof. The equivalence follows along the following sequence of equivalent reformulations.

$G(z) \sim (A, B, C, D)$ is DT-NI.

$\Leftrightarrow$ $G(1) = G^T(1)$, and $F(z) = \frac{1+z}{1-z}(G(z) - G(1))$ is DT-PR (see Lemma 10).

$\Leftrightarrow$ $C(I-A)^{-1}B + D = B^T(I-A^T)^{-1}C^T + D^T$, and $F(z) \backsim \left(\begin{array}{c|c} A & B \\ \hline C(I-A)^{-1}(I+A)) & C(I-A)^{-1}B \end{array}\right)$ is DT-PR (see Lemma 7).

$\Leftrightarrow$ $C(I-A)^{-1}B + D = B^T(I-A^T)^{-1}C^T + D^T$, and there exist matrices $Y = Y^T > 0, Q, W$ such that

$$Y - A^TYA = Q^TQ$$
$$(A^T + I)(I-A^T)^{-1}C^T - A^TYB = Q^TW$$
$$C(I-A)^{-1}B + B^T(I-A^T)^{-1}C^T - B^TYB = W^TW.$$

This equivalence is via the DT-PR lemma in Hitz and Anderson (1969).

$\Leftrightarrow$ $C(I-A)^{-1}B + D = B^T(I-A^T)^{-1}C^T + D^T$, and there exist matrices $Y = Y^T > 0, Q, W$ such that

$$Y - A^TYA = Q^TQ \tag{8}$$
$$C = (W^TQ + B^TYA)(I+A)^{-1}(I-A) \tag{9}$$
$$\begin{aligned} &B^TYB - B^TY(A+I)^{-1}B - B^T(A^T+I)^{-1}YB \\ &\quad + B^T(I+A^T)^{-1}Q^TQ(I+A)^{-1}B \\ &= [W - Q(I+A)^{-1}B]^T[W - Q(I+A)^{-1}B]. \end{aligned} \tag{10}$$

$\Leftrightarrow$ $C(I-A)^{-1}B+D = B^T(I-A^T)^{-1}C^T+D^T$, and there exist matrices $Y = Y^T > 0, Q, W$ such that (8)–(9) hold and $W = Q(I+A)^{-1}B$ (the left of equality (10) is equal to zero).

$\Leftrightarrow$ $C(I-A)^{-1}B + D = B^T(I-A^T)^{-1}C^T + D^T$, and there exist matrices $Y = Y^T > 0, Q$ such that $Y - A^TYA = Q^TQ$ and $C = [B^T(I+A^T)^{-1}Q^TQ + B^TYA](I+A)^{-1}(I-A)$.

$\Leftrightarrow$ $C(I-A)^{-1}B + D = B^T(I-A^T)^{-1}C^T + D^T$, and there exists matrix $Y = Y^T > 0$ such that $Y - A^TYA \geq 0$ and $C = B^T(I+A^T)^{-1}Y(I-A)$. □

Remark 7. When $G(z)$ is symmetric, Condition 1 of Lemmas 11 and 12 are redundant as pointed out in Remarks 5 and 6. In addition, there is another method to prove Lemma 11 by using the bilinear transformation, see Liu and Xiong (2016a, Lemma 6).

5. Stability of interconnected DT-NI systems

This section presents the internal stability results of a positive feedback interconnection of two DT-NI systems, denoted by $[M(z), N(z)]$. The following theorem provides a necessary and sufficient condition for the stability of the interconnected systems $[M(z), N(z)]$ in terms of the loop gain at $z = 1$.

Theorem 1. *Given a square real-rational proper DT-NI transfer function matrix* $M(z)$*, and a square real-rational proper DT-SNI transfer function matrix* $N(z)$*. Suppose* $M(z)$ *and* $N(z)$ *have no poles at* -1 *and* 1*, and that also satisfy* $M(-1)N(-1) = 0$ *and* $N(-1) \geq 0$*. Then,* $[M(z), N(z)]$ *is internally stable if and only if* $\bar{\lambda}(M(1)N(1)) < 1$*.*

Proof. The proof is the same as the proof of Ferrante et al. (2014, Theor. 8). □

Remark 8. Theorem 1 could be considered as the DT version of Theorem 5 in Lanzon and Petersen (2008) and Theorem 1 in Xiong et al. (2010). Also, the result in Theorem 1 is simply "restatement" of Theorem 8 in Ferrante et al. (2014) with the new definitions in this paper. Moreover, all the results in the paper allow the transfer function matrix to be non-symmetric.

The following two corollaries are restatements of Theorem 1, written in the same form as the small gain theorem.

Corollary 3. *Given* $\gamma > 0$ *and a DT-SNI transfer function matrix* $N(z)$ *with* $N(-1) \geq 0$*. Then,* $[M(z), N(z)]$ *is internally stable for all DT-NI transfer function matrix* $M(z)$ *satisfying* $M(-1)N(-1) = 0$*, and* $\bar{\lambda}(M(1)) < \gamma$ *(respectively* $\leq \gamma$*) if and only if* $\bar{\lambda}(N(1)) \leq \frac{1}{\gamma}$ *(respectively* $< \frac{1}{\gamma}$*).*

Proof. Without loss of generality, assume $\gamma = 1$.

(Necessity) Suppose $\bar{\lambda}(N(1)) > 1$. We will show that there exists a $M(z)$ with $M(-1)N(-1) = 0$ and $\bar{\lambda}(M(1)) < 1$ such that $[M(z), N(z)]$ is unstable. $M(z)$ can be chosen as $M(z) = \frac{1}{\bar{\lambda}(N(1))(\frac{z-1}{z+1}+1)}I = \frac{z+1}{\bar{\lambda}(N(1))2z}I$ such that $\bar{\lambda}(N(1)) > 1$, $\bar{\lambda}(M(1)) < 1$ and $M(-1)N(-1) = 0$. Then, it follows that $M(1)N(1) = \frac{N(1)}{\bar{\lambda}(N(1))}I$, and $\bar{\lambda}(M(1)N(1)) = 1$, which contradicts $\bar{\lambda}(M(1)N(1)) < 1$, and hence $[M(z), N(z)]$ is unstable. So, $\bar{\lambda}(N(1)) \leq 1$.

(Sufficiency) It follows by noting that $\bar{\lambda}(M(1))\bar{\lambda}(N(1)) < 1$ implies that $\bar{\lambda}(M(1)N(1)) < 1$. □

Corollary 4. *Given* $\gamma > 0$ *and a DT-NI transfer function matrix* $N(z)$*. Then,* $[M(z), N(z)]$ *is internally stable for all DT-SNI transfer function matrix* $M(z)$ *satisfying* $M(-1) \geq 0$*,* $M(-1)N(-1) = 0$*, and* $\bar{\lambda}(M(1)) < \gamma$ *(respectively* $\leq \gamma$*) if and only if* $\bar{\lambda}(N(1)) \leq \frac{1}{\gamma}$ *(respectively* $< \gamma$*).*

Proof. The proof is the same as the proof of Corollary 3. □

Example 3. To illustrate the stability result in Theorem 1, consider an uncertain plant $M(z)$ as $M(z) = \frac{2k}{2z+1}$, where $k > 0$ is uncertain. A calculation shows that $j[M(e^{j\theta}) - M^*(e^{j\theta})] = \frac{8k\sin\theta}{(2\cos\theta+1)^2+4\sin^2\theta}$. According to Definition 1, $M(z)$ is DT-NI for all $k > 0$. Now, let a controller $N(z)$ be chosen as $N(z) = \frac{2z+2}{2z-1}$. It can be checked that $N(z)$ is DT-SNI transfer function satisfying $M(-1)N(-1) = 0$ and $N(-1) \geq 0$. Application of Theorem 1 shows that $[M(z), N(z)]$ is internally stable if and only if $\bar{\lambda}(M(1)N(1)) < 1$, that is, $0 < k < \frac{3}{8}$. This can be verified by directly using the stability criterion for DT systems. The transfer function of $[M(z), N(z)]$ is given by

$F(z) = \frac{M(z)}{1-M(z)N(z)} = \frac{2k(2z-1)}{4z^2-4kz-4k-1}$. Using the bilinear transformation $z = \frac{1+s}{1-s}$ and the Routh stability criterion, it can be found that $[M(z), N(z)]$ is internally stable if and only if $0 < k < \frac{3}{8}$.

6. Conclusions

This paper has studied the DT-NI properties of square real-rational proper transfer function matrices. Two different relations between DT-PR and DT-NI transfer function matrices were established, and two versions of DT-NI lemmas were developed to test DT-NI properties. Then, a necessary and sufficient condition, expressed as the loop gain at $z = 1$ being less than unity, was established for the internal stability of positive feedback interconnection of two DT-NI systems. Also, the developed results were illustrated by several numerical examples.

References

Anderson, B. D. O., Hitz, K. L., & Diem, N. D. (1974). Recursive algorithm for spectral factorization. *IEEE Transactions on Circuits and Systems*, *21*(6), 742–750.

Anderson, B. D. O., & Vongpanitlerd, S. (1973). *Network analysis and synthesis: a modern systems theory approach*. Upper Saddle River, NJ: Prentice-Hall.

Brogliato, B., Lozano, R., Maschke, B., & Egeland, O. (2007). *Dissipative systems analysis and control: theory and applications*. (2nd ed.). New York: Springer-Verlag.

Cai, C., & Hagen, G. (2010). Stability analysis for a string of coupled stable subsystems with negative imaginary frequency response. *IEEE Transactions on Automatic Control*, *55*(8), 1958–1963.

Ferrante, A., Lanzon, A., & Ntogramatzidis, L. (2014) Foundations of negative imaginary systems theory and relations with positive real systems. ArXiv preprint arXiv:1412.5709.

Hitz, L., & Anderson, B. D. O. (1969). Discrete positive-real functions and their application to system stability. *Proceedings of the Institution of Electrical Engineers*, *116*(1), 153–155.

Jiang, J. (1993). Preservations of positive realness under discretizations. *Journal of the Franklin Institute*, *330*(4), 721–734.

Lanzon, A., & Petersen, I. R. (2008). Stability robustness of a feedback interconnection of systems with negative imaginary frequency response. *IEEE Transactions on Automatic Control*, *53*(4), 1042–1046.

Liu, M. & Xiong, J. (2016a) Connection between discrete- and continuous-time results for positive real and negative imaginary systems. In *the proceedings of IEEE conference on decision and control* (pp. 4931–4936).

Liu, M., & Xiong, J. (2016b). On non-proper negative imaginary systems. *Systems & Control Letters*, *88*, 47–53.

Mabrok, M. A., Kallapur, A. G., Petersen, I. R., & Lanzon, A. (2014). Generalizing negative imaginary systems theory to include free body dynamics: control of highly resonant structures with free body motion. *IEEE Transactions on Automatic Control*, *59*(10), 2692–2707.

Ober, R., & Montgomery-Smith, S. (1990). Bilinear transformation of infinite-dimensional state-space systems and balanced realizations of nonrational transfer functions. *SIAM Journal on Control and Optimization*, *28*(2), 438–465.

Patra, S., & Lanzon, A. (2011). Stability analysis of interconnected systems with mixed negative-imaginary and small-gain properties. *IEEE Transactions on Automatic Control*, *56*(6), 1395–1400.

Petersen, I. R., & Lanzon, A. (2010). Feedback control of negative-imaginary systems. *IEEE Control Systems Magazine*, *30*(5), 54–72.

Xiong, J., Petersen, I. R., & Lanzon, A. (2010). A negative imaginary lemma and the stability of interconnections of linear negative imaginary systems. *IEEE Transactions on Automatic Control*, *55*(10), 2342–2347.

Mei Liu received her B.Sc. degree in Mathematics from China University of Mining and Technology in 2012. She is currently a Ph.D. Candidate in Department of Automation, University of Science and Technology of China. Her research interests include negative imaginary systems and positive real systems.

Junlin Xiong received his B.Eng. and M.Sc. degrees from Northeastern University, China, and his Ph.D. degree from the University of Hong Kong, Hong Kong, in 2000, 2003 and 2007, respectively. From November 2007 to February 2010, he was a research associate at the University of New South Wales at the Australian Defence Force Academy, Australia. In March 2010, he joined the University of Science and Technology of China where he is currently a professor in the Department of Automation. Currently, he is an associate editor for the IET Control Theory and Application. His current research interests are in the fields of negative imaginary systems, large-scale systems and networked control systems.