Connections between Discrete- and Continuous-Time Results for Positive Real and Negative Imaginary Systems

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Abstract—This paper studies the connection between discrete-time negative imaginary and continuous-time negative imaginary functions. Firstly, we analyse differences between two restatements that are claimed to provide necessary and sufficient conditions for systems to be discrete-time positive real. The conclusion is that one is equivalent to the definition of discrete-time positive real while the other one is not. Then, by means of the bilinear transformation, a connection between discrete-time negative imaginary and continuous-time negative imaginary functions is established. A new proof is provided for the discrete-time negative imaginary lemma by applying the connection. Several numerical examples illustrate the developed theory.

I. INTRODUCTION

Positive real systems, originated from network theory [1]–[3], have achieved great success both in continuous-time and discrete-time case. Positive real systems could model many realistic systems, and hence have been widely used in various control systems [2], [3]. For example, the single-input single-output positive real systems can be realized with electric circuits using only resistors, inductors and capacitors [2]; lightly damped flexible structures with force inputs and collocated velocity outputs often lead to positive real transfer functions [3]. Moreover, positive real systems could be used to establish the internal stability of a negative feedback interconnection with positive real frequency responses [3].

The concept of discrete-time positive real systems was firstly introduced in [4]. Meanwhile, the authors of [4] related discrete-time positive real properties on the unit circle as continuous-time case, and developed the well-known discrete-time positive real lemma. Except in [4], other versions of discrete-time positive real functions can be found in [5]–[7]. All of them were claimed to be equivalent to the definition of discrete-time positive real functions. However, it was shown in [8] that two versions of discrete-time positive real systems in [5]–[7] were disproved by three counter-examples. Until now, some researchers still adopt the incorrect one in [5], [6]; e.g., see [9, Def.3], [10, Theor.4]. In this paper, we shall discuss the main differences between those versions, and present a detail proof why the restatement in [5], [6] is not equivalent to the definition of discrete-time positive real functions.

In recent years, negative imaginary theory, emerged as a complement to positive real theory [11], has attracted much attention of many researchers; see [12]–[16]. This theory was firstly introduced in [17]. Subsequently, the concept of negative imaginary functions was further extended to allow poles on the imaginary axis [18]–[20], symmetric case [21], non-proper case [20] and discrete-time systems [10], [22], respectively. In [10], the concept of discrete-time symmetric negative imaginary functions was firstly proposed. In [22], the concept of discrete-time negative imaginary functions was extended by removing the symmetric restriction. Meanwhile, a new definition of discrete-time negative imaginary functions was given in [22].

In this paper, we are also interested in establishing the connection between continuous-time and discrete-time negative imaginary functions in terms of the bilinear transformation $s = \frac{z-1}{z+1}$. It is well-known that a continuous-time positive real function transforms into a discrete-time positive real function by such transformation [4], [23], and vice verse. This result motivates the question: for negative imaginary functions, does this result remain true? The answer is “Yes”, and a proof will be given in this paper. According to the connection, a new proof is presented for the discrete-time negative imaginary lemma.

The rest of the paper is organized as follows: Section II reviews the definition of discrete-time positive real functions, the discrete-time positive real lemma and two frequently-used equivalent conditions. Then, we analyze the differences between those lemmas. Section III studies the connection between continuous-time and discrete-time negative imaginary functions. Based on the connection, we present a new proof for the discrete-time negative imaginary lemma. Section IV concludes the paper.

Notation: $\mathbb{R}^{m \times n}$ and $\mathbb{R}^{m \times n}$ denote the sets of $m \times n$ real matrices and real-rational proper transfer function matrices, respectively. $\text{Re}[]$ denotes the real part of complex numbers. $A^T$ and $A^*$ denote the transpose and the complex conjugate transpose of a complex matrix $A$, respectively. $A > 0$ or $A \geq 0$ denotes symmetric positive definite or symmetric positive semidefinite matrix.

II. DISCRETE-TIME POSITIVE REAL TRANSFER FUNCTION MATRICES

In this section, our goal is to discuss the differences between the concept and equivalent conditions of discrete-time positive real functions. Firstly, we briefly recall the definition of discrete-time positive real functions in terms of $z$-domain [4], two different versions of discrete-time positive real functions in terms of properties on the unit cycle [4]–[6], and the discrete-time positive real lemma in terms of
state-space realization [4].

**Definition 1:** [4] A square real-rational proper transfer function matrix \(F(z)\) is called discrete-time positive real if

1) all elements of \(F(z)\) are analytic in \(|z| > 1\);
2) \(F^*(z) + F(z) \geq 0\) for all \(|z| > 1\).

**Remark 1:** When \(F(z)\) is non-proper, it means that \(F(z)\) has any poles in \(|z| > 1\), which contradicts Condition 1) in above definition. So, the present discrete-time positive real functions focus on proper functions.

The following lemma provides a necessary and sufficient condition for a real-rational transfer function matrix to be continuous-time positive real.

**Lemma 1:** [2, Theorem 2.7.2] Let \(F(s)\) be a square real-rational matrix of functions of \(s\). Then \(F(s)\) is continuous-time positive real if and only if

1) no element of \(F(s)\) has a pole in \(\text{Re}[s] > 0\);
2) \(F(j\omega) + F^*(j\omega) \geq 0\) for all real \(\omega\), with \(j\omega\) not a pole of any element of \(F(s)\);
3) if \(j\omega_0\) is a pole of any element of \(F(s)\), it is at most a simple pole, and the residue matrix, \(K_0 = \lim_{\omega \to j\omega_0} (s - j\omega_0)F(s)\) in case \(\omega_0\) is finite, and \(K_0 = \lim_{\omega \to j\omega_0} F(s)\) is positive definite Hermitian.

The following lemma is a restatement of Definition 1 in terms of properties on \(|z| = 1\).

**Lemma 2:** [4, Lemma 2] A square matrix \(F(z)\) whose elements are real-rational proper functions analytic in \(|z| > 1\) is discrete-time positive real if and only if

1) poles of elements of \(F(z)\) on \(|z| = 1\) are simple;
2) \(F^*(e^{j\theta}) + F(e^{j\theta}) \geq 0\) for all real \(\theta\) at which \(F(e^{j\theta})\) exists;
3) if \(\theta_0 = e^{j\theta_0}\) \(\theta_0\) is real, is a pole of an element of \(F(z)\), and if \(K_0\) is the residue matrix of \(F(z)\) at \(\theta_0\), then the matrix \(e^{-j\theta_0}K_0\) is positive semidefinite Hermitian.

**Remark 2:** One should note that the following two conditions are contained in Condition 3) of Lemma 2:

1) if \(F(z)\) has a simple pole at 1, then the corresponding residue matrix \(\lim_{z \to 1}(z - 1)F(z)\) is positive semidefinite Hermitian;
2) if \(F(z)\) has a simple pole at \(-1\), then the corresponding residue matrix \(\lim_{z \to -1}(z + 1)F(z)\) is negative semidefinite Hermitian.

This fact follows by a direct calculation that \(e^{-j\theta}|_{\theta = 0} = 1\) and \(e^{-j\theta}|_{\theta = \pi} = -1\).

The following lemma is another equivalent conditions of discrete-time positive real functions in terms of properties on \(|z| = 1\), which is similar to the form of continuous-time case in Lemma 1.

**Lemma 3:** [5, 6] A square real-rational proper transfer function matrix \(F(z)\) is discrete-time positive real if and only if

1) \(F(z)\) is analytic in \(|z| > 1\);
2) \(F^*(e^{j\theta}) + F(e^{j\theta}) \geq 0\) for all real \(\theta\) at which \(F(e^{j\theta})\) exists;
3) the poles of \(F(z)\) on \(|z| = 1\) are simple and the corresponding residue matrices of \(F(z)\) at those poles are positive semidefinite Hermitian.

**Remark 3:** According to Condition 3) in Lemma 3, we have the following results:

1) if \(F(z)\) has a simple pole at 1, then the corresponding residue matrix \(\lim_{z \to 1}(z - 1)F(z)\) is positive semidefinite Hermitian;
2) if \(F(z)\) has a simple pole at \(-1\), then the corresponding residue matrix \(\lim_{z \to -1}(z + 1)F(z)\) is positive semidefinite Hermitian.

Obviously, it follows from Remarks 2 and 3 that Condition 3) of Lemma 2 is not equivalent to Condition 3) of Lemma 3. When \(F(z)\) has poles on \(|z| = 1\), Condition 3) of Lemma 2 requires that \(e^{-j\theta_0}K_0\) be positive semidefinite Hermitian, while Condition 3) of Lemma 3 requires that the associated residue matrix \(K_0\) be positive semidefinite Hermitian. Hence, Lemmas 2 and 3 are not equivalent each other.

The following lemma is the classical discrete-time positive real lemma in terms of minimal state-space realisation, which has been widely used in virus areas.

**Lemma 4:** [4] Let \(F(z)\) be a square real-rational proper transfer function matrix of \(z\) with no poles in \(|z| > 1\) and simple poles only on \(|z| = 1\), and let \((A,B,C,D)\) be a minimal realisation of \(F(z)\). Then necessary and sufficient conditions for \(F(z)\) to be discrete-time positive real are that there exist a real matrix \(P = P^T > 0\) and real matrices \(L\) and \(W\) such that

\[
A^TPA - P = -L^TL \quad C^TA^TPB = L^TW \quad D^T + D - B^TPB = W^TW
\]

**Remark 4:** In [8], it has been pointed out that Definition 1, Lemmas 2 and Lemma 4 are agreeable with each other. In addition, Lemma 3 which was claimed to provide necessary and sufficient conditions for a system to be discrete-time positive real, is not equivalent to Definition 1, Lemmas 2 and 4. The authors of [8] utilized three examples to show this result.

In this section, we provide the detail reasons why Lemma 3 is not equivalent to Definition 1, Lemmas 2 and 4. As was mentioned in [8], Lemma 4 plays an important role in the research of discrete-time positive real systems. The authors of [4] proved Lemma 4 by applying the bilinear transformation

\[
s = \frac{z - 1}{z + 1}
\]

So, if we use Lemma 4, one should admit that, via the bilinear transformation in (1), the continuous-time positive real transfer function matrix \(F(s)\) is transformed into a discrete-time positive real transfer function matrix \(F(z)\), and vice verse. Next, we will show that: under the transformation in (1), Lemma 3 is not equivalent to Lemma 1.

According to Lemmas 2 and 3, it can be found that the main distinctions lie in Condition 3) of Lemmas 2 and 3 when the transfer function matrix \(F(z)\) has poles on \(|z| = 1\). Hence, without loss of generality, assume that the continuous-time positive real transfer function matrix \(F(s)\) has any simple poles on the purely imaginary axis. Then,
according to the minor decomposition theory in [2, pp. 216],

\[ F(s) = \sum_i \frac{K_i}{s - j\omega_i} + sA + \frac{C}{s} + F_0(s), \]  

(2)

where \( F_0(s) \) is analytic in \( \text{Re}[s] > 0 \), \( K_i = K_i^* \geq 0 \), \( A = A^* \geq 0 \) and \( C = C^* \geq 0 \) are the associated residue matrix at \( j\omega_i \) \((\omega_i > 0), 0 \) and \( \infty \), respectively. By means of the bilinear transformation in (1), equation (2) transforms into

\[ F(z) = \sum_i \frac{(z + 1)K_i}{(1 - j\omega_i)z - (1 + j\omega_i)} + \frac{z - 1}{z + 1}A + \frac{z + 1}{z - 1}C + F_1(z), \]

where \( F_1(z) = F_0 \left( \frac{z + 1}{z - 1} \right) \). Because \( F(s) \) is continuous-time positive real, it follows from [23, Theorem 1] and Lemma 4 that \( F(z) \) is discrete-time positive real. However, when \( e^{j\theta_i} \) is a simple pole of \( F(z) \), the associated residue matrix of \( F(z) \) at \( e^{j\theta_i} \) is given by

\[ K_0 = \lim_{z \to e^{j\theta_i}} \left( \frac{z - 1 + j\theta_i}{z - 1 - j\theta_i} \right) F(z) = \frac{2K_1}{(1 - j\theta_i)^2} - \frac{2K_1}{1 + \theta_i^2 - 2j\theta_i}, \]

which is not positive semidefinite Hermitian. This contradicts Condition 3) in Lemma 3. In addition, the matrix

\[ e^{-j\theta_i}K_0 = \left( \frac{1 - \theta_i^2}{1 + \theta_i^2} - \frac{2\theta_i}{1 + \theta_i^2} e^{j\theta_i} \right) \]

\[ = \frac{2K_1}{1 + \theta_i^2} - \frac{2K_1}{1 + \theta_i^2 - 2j\theta_i}, \]

is positive semidefinite Hermitian. This coincides with Condition 3) in Lemma 2. Similarly, when \( F(z) \) has poles at \( z = -1 \), we have the same results. Hence, Lemma 3 can not be used to test discrete-time positive real functions when the system has poles on \( |z| = 1 \).

\[ \text{Example 1:} \] To illustrate the main results in this section, consider a continuous-time positive real transfer function matrix \( F(s) = \left( \begin{array}{cc} \frac{x}{s+1} & \frac{1}{s+1} \\ \frac{x^2+1}{s+1} & \frac{x}{s+1} \end{array} \right) \). Using the bilinear transformation in (1) transforms the \( s \)-domain transfer function matrix \( F(s) \) to the \( z \)-domain transfer function matrix \( F(z) = \left( \begin{array}{cc} \frac{z^2-1}{2(z^2+1)} & \frac{(z+1)^2}{2(z^2+1)} \\ \frac{z^2-1}{2(z^2+1)} & \frac{z^2-1}{2(z^2+1)} \end{array} \right) \). A calculation shows that the residue matrix of \( F(z) \) at \( z = j \theta = \left( \theta = \frac{x}{2} \right) \) is given by \( K = \left( \begin{array}{cc} \frac{1}{2} & \frac{x}{2} \\ -\frac{x}{2} & \frac{1}{2} \end{array} \right) \), which is not positive semidefinite Hermitian and contradicts Condition 3) in Lemma 3. However, the matrix \( e^{-j\theta}K = -jK = \left( \begin{array}{cc} \frac{1}{2} & -\frac{x}{2} \\ -\frac{x}{2} & \frac{1}{2} \end{array} \right) \) is positive semidefinite Hermitian and satisfies Condition 3) in Lemma 2.

\[ \text{Remark 5:} \] Although some researchers adopt Lemma 3 in their research and the form of Lemma 3 is similar to the continuous-time case, it follows from above theoretical analysis that Lemma 3 is incorrect while Lemma 2 is correct. This discovery leads to that the definition of discrete-time negative imaginary functions in [22] is different to that one in [10]. Also, the research in the next section is partially motivated by this discovery.

### III. DISCRETE-TIME NEGATIVE IMAGINARY TRANSFER FUNCTION MATRICES

In this section, our goal is to establish the connection between discrete-time and continuous-time negative imaginary functions, and give a new proof of the discrete-time negative imaginary lemma. Firstly, the definition of continuous-time and discrete-time negative imaginary functions are introduced, respectively.

The definition of continuous-time negative imaginary functions is introduced in the following.

\[ \text{Definition 2: [20]} \] A square real-rational transfer function matrix \( G(s) \) is called continuous-time negative imaginary if

1. \( G(s) \) has no poles in \( \text{Re}[s] > 0 \);
2. \( j[G(j\omega) - G^*(j\omega)] \geq 0 \) for all \( \omega > 0 \) except values of \( \omega \) where \( j\omega \) is a pole of \( G(s) \);
3. if \( s = 0 \) is a pole of \( G(s) \), then \( \lim_{s \to 0} s^m G(s) \) is positive semidefinite Hermitian, and \( \lim_{s \to 0} s^m G(s) = 0 \) for all \( m \geq 3 \);
4. if \( s = j\omega_0 \) with \( \omega_0 > 0 \) is a pole of \( G(s) \), \( \omega_0 \) is finite, it is at most a simple pole and the residue matrix \( K = \lim_{s \to j\omega_0} (s - j\omega_0)G(s) \) is positive semidefinite Hermitian;
5. if \( s = j\infty \) is a pole of \( G(s) \), then \( \lim_{\omega \to \infty} \frac{G(j\omega)}{(j\omega)^m} \) is negative semidefinite Hermitian, and \( \lim_{\omega \to \infty} \frac{G(j\omega)}{(j\omega)^m} = 0 \) for all \( m \geq 3 \).

The definition of discrete-time negative imaginary functions is introduced in the following.

\[ \text{Definition 3: [22]} \] A square real-rational proper transfer function matrix \( G(z) \) is called discrete-time negative imaginary if

1. \( G(z) \) has no poles in \( |z| > 1 \);
2. \( j[G(e^{j\theta}) - G^*(e^{j\theta})] \geq 0 \) for all \( \theta \in (0, \pi) \) except values of \( \theta \) where \( e^{j\theta} \) is a pole of \( G(z) \);
3. if \( z_0 = e^{j\theta_0}, \theta_0 \in (0, \pi) \), is a pole of \( G(z) \), then it is a simple pole and the corresponding residue matrix \( K = \lim_{z \to z_0} (z - z_0)G(z) \) satisfies that \( e^{-j\theta_0}K \) is positive semidefinite Hermitian;
4. if \( z = 1 \) is a pole of \( G(z) \), then \( \lim_{z \to 1} (z - 1)^m G(z) = 0 \) for all \( m \geq 3 \);
5. if \( z = -1 \) is a pole of \( G(z) \), then \( \lim_{z \to -1} (z + 1)^2 G(z) = 0 \) is negative semidefinite Hermitian, and \( \lim_{z \to -1} (z + 1)^m G(z) = 0 \) for all \( m \geq 3 \).

The following lemma characterizes the connection between continuous-time and discrete-time negative imaginary transfer function matrices.

\[ \text{Lemma 5:} \] A continuous-time negative imaginary transfer function matrix \( G(s) \) transforms into a discrete-time negative imaginary transfer function matrix \( G(z) \) by the bilinear transformation \( s = \frac{z - 1}{z + 1} \). Conversely, a discrete-time negative
imaginary transfer function matrix $G(z)$ transforms into a continuous-time negative imaginary transfer function matrix $G(s)$ by the bilinear transformation $z = \frac{1 + js}{1 - js}$.

**Proof:** Assume that $G(s)$ is a continuous-time negative imaginary transfer function matrix. Then, under the bilinear transformation $s = \frac{z - 1}{z + 1}$, we will show that the five conditions in Definition 3 are satisfied.

1. Because $G(s)$ has no poles in Re[$s$] > 0, Condition 1) of Definition 3 is immediate.

2. If $s = j\omega$, $\omega > 0$, is not a pole of $G(s)$, then $z = \frac{1 + js}{1 - js} = \frac{1 + j\omega}{1 - j\omega} = \frac{1 - 0^2 + j2\omega}{1 + 0^2}$ is also not a pole of $G(z)$. Define $z = e^{j\theta}$, where $\cos \theta = \frac{1 - 0^2}{1 + 0^2}$, $\sin \theta = \frac{2\omega j}{1 + 0^2}$, and $\theta \in (0, \pi)$. Moreover, for all $\omega > 0$ with $j\omega$ not a pole of $G(s)$, $jG(j\omega) - G^*(j\omega) \geq 0$ implies that $jG(\frac{1 - 0^2 + j2\omega}{1 + 0^2}) - G^*(\frac{1 - 0^2 + j2\omega}{1 + 0^2}) \geq 0$ for all $\theta \in (0, \pi)$ with $e^{j\theta} = \frac{1 - 0^2 + j2\omega}{1 + 0^2}$, not a pole of $G(z)$.

3. If $s = 0$ is a pole of $G(s)$, then $z = 1$ is also a pole of $G(z)$. According to the minor decomposition theory in [20], $G(s)$ is of the form

$$G(s) = \frac{A_2}{s} + \frac{A_1}{s} + G_0(s), \quad (3)$$

where $A_2 = A_2^* \geq 0$, $A_1 + A_1^T \geq 0$, and $G_0(s)$ has no poles in Re[$s$] > 0 and at $s = 0$. By means of the bilinear transformation $s = \frac{z - 1}{z + 1}$, equation (3) transforms into

$$G(z) = \left(\frac{z + 1}{z - 1}\right)^2 A_2 \frac{z + 1}{z - 1} A_1 + G_0 \left(\frac{z - 1}{z + 1}\right),$$

where $G_0(\frac{z + 1}{z - 1})$ has no poles in $|z| > 1$ and at $z = 1$. Then, $\lim_{z \to 1} (z - 1)^m G(z) = 4A_2$ is positive semidefinite Hermitian, and $\lim_{z \to 1} (z - 1)^m G(z) = 0$ for all $m \geq 3$.

4. If $s = j\omega$, $\omega > 0$, is a pole of $G(s)$, then $z = \frac{1 + js}{1 - js} = \frac{1 + j\omega}{1 - j\omega} = \frac{1 - 0^2 + 2j\omega}{1 + 0^2}$ is also a pole of $G(z)$. Decompose $G(s)$ to the form

$$G(s) = \frac{-jK}{s - j\omega} + \frac{jK^*}{s + j\omega} + G_0(s), \quad (4)$$

where $K$ is the associated residue matrix at $j\omega$, $K = K^*$, $0$, and $G_0(s)$ has no poles in Re[$s$] > 0 and at $\pm j\omega$. By means of the transformation $s = \frac{z - 1}{z + 1}$, equation (4) transforms into

$$G(z) = \frac{-jK(z + 1)}{(1 - j\omega)z - (1 + j\omega)} + jK^*(z + 1) + G_0 \left(\frac{z - 1}{z + 1}\right),$$

where $G_0(\frac{z + 1}{z - 1})$ has no poles in $|z| > 1$ and at $\frac{1 + j\omega}{1 - j\omega}$, $\frac{1 - j\omega}{1 + j\omega}$. The corresponding residue matrix of $G(z)$ at $e^{j\theta} = \frac{1 - j\omega}{1 + j\omega}$ is given by

$$\begin{align*}
K &= \lim_{z \to \frac{1 - j\omega}{1 + j\omega}} \left(\frac{z - 1 + j\omega}{1 - j\omega} \right) G(z) \\
&= \lim_{z \to \frac{1 - j\omega}{1 + j\omega}} \left(\frac{(1 - j\omega)z - (1 + j\omega)}{1 - j\omega} \right) G(z) \\
&= \frac{K(z + 1)}{1 - j\omega} \left(\frac{z - 1 + j\omega}{1 - j\omega} \right) G(z) \\
&= \frac{K}{1 - j\omega} \left(\frac{1 + j\omega}{1 - j\omega} + \frac{2K}{1 - 0^2 - 2j\omega} \right),
\end{align*}$$

which is not positive semidefinite Hermitian unless $\omega_0 = 0$. Then, we have $e^{j\theta} = \frac{1 - 0^2 - 2j\omega}{1 + 0^2}$, and the matrix $e^{j\theta}K$ is positive semidefinite Hermitian. Similarly, the corresponding residue matrix of $G(z)$ at $\frac{1 - 0^2 + 2j\omega}{1 + 0^2}$ has the same property.

5. If $s = j\omega$ is a pole of $G(s)$, then $z = -1$ is also a pole of $G(z)$. Decompose $G(s)$ to the form

$$G(s) = s^2 C_2 + s C_1 + G_0(s), \quad (5)$$

where $C_2 = C_2^* \leq 0$, $C_1 + C_1^T \leq 0$, and $G_0(s)$ has no poles in Re[$s$] > 0 and at infinity. Under the transformation $s = \frac{z - 1}{z + 1}$, equation (5) transforms into

$$G(z) = \left(\frac{z - 1}{z + 1}\right)^2 C_2 + \frac{z - 1}{z + 1} C_1 + G_0 \left(\frac{-1}{1 + 1}\right),$$

where $G_0(\frac{z - 1}{z + 1})$ has no poles in $|z| > 1$ and at $z = -1$. Then, $\lim_{z \to -1} (z + 1)^m G(z) = 4C_2$ is negative semidefinite Hermitian, and $\lim_{z \to -1} (z + 1)^m G(z) = 0$ for all $m \geq 3$.

Conversely, assume that $G(z)$ is discrete-time negative imaginary transfer function matrix. We will show that the five conditions in Definition 2 are satisfied by means of the bilinear transformation $z = \frac{1 + js}{1 - js}$. $G(z)$ has no poles in $|z| > 1$, so that Condition 1) of Definition 2 is satisfied.

1. If $z = e^{j\theta}$, $\theta \in (0, \pi)$, is not a pole of $G(z)$, then $s = \frac{z - 1}{z + 1} = \frac{e^{j\theta}}{e^{j\theta} + 1} > 0$, is also not a pole of $G(s)$. Furthermore, $jG(e^{j\theta}) - G^*(e^{j\theta}) \geq 0$ implies that $jG(e^{j\theta}) - G^*(e^{j\theta}) \geq 0$ for all $\theta > 0$ with $j\omega = j\sin \theta$ not a pole of $G(s)$.

2. If $z = e^{j\theta}$, $\theta_0 \in (0, \pi)$, is a pole of $G(z)$, then $s = \frac{z - 1}{z + 1} = \frac{e^{j\theta_0}}{e^{j\theta_0} + 1} = j\sin \theta_0$ has no poles in $|z| > 1$ and at $e^{j\theta_0}$. Consider the transformation $z = \frac{1 + js}{1 - js}$. Equation (6) transforms into

$$G(s) = \frac{-jK_0(1 - s)}{(1 + e^{j\theta_0})s + (1 - e^{j\theta_0})s + (1 - e^{j\theta_0})} + jK_0^*(1 - s) \left(\frac{1 + s}{1 - s}\right),$$

where $K_0$ is the corresponding residue matrix at $e^{j\theta_0}$, $e^{-j\theta_0}K_0$ is positive semidefinite Hermitian, and $G_0(z)$ has no poles in $|z| > 1$ and at $e^{j\theta_0}$. The corresponding residue matrix of $G(z)$ at $e^{j\theta_0} = \frac{1 + js}{1 - js}$ is given by
where $G_0(\frac{1}{2+i\omega})$ has no poles in $\text{Re}[s] > 0$ and at $\pm \frac{i}{\sin \theta_0}$, then the residue matrix of $G(s)$ at $s = \frac{\sin \theta_0}{\cos \theta_0}$ is given by

$$K = \lim_{s \to \frac{\sin \theta_0}{\cos \theta_0}} \left( \frac{s - e^{j\theta_0} - 1}{e^{j\theta_0} + 1} \right) jG(s).$$

Remark 6: To illustrate the usefulness of Lemma 5, consider the following non-symmetric continuous-time transfer matrix $G(s) = \begin{pmatrix} \frac{1}{s + 1} & \frac{s}{s + 1} \\ \frac{s}{s + 1} & \frac{s^2}{s + 1} \end{pmatrix}$. $G(s)$ has no poles in $\text{Re}[s] > 0$, and $jG(j\omega) - G^*(j\omega)$ = 0 for all $\omega > 0$ where $j\omega$ is not a pole of $G(s)$. The residue matrix at $s = j$ is positive semidefinite Hermitian. It follows from Definition 2 that $G(s)$ is negative imaginary. By the bilinear transformation $s = \frac{\omega - \theta_0}{\omega + \theta_0}$, $G(s)$ transforms to $G(z) = \begin{pmatrix} \frac{(z+1)^2}{2(z^2+1)} & \frac{1 - z^2}{2(z^2+1)} \\ \frac{1 - z^2}{2(z^2+1)} & \frac{(z+1)^2}{2(z^2+1)} \end{pmatrix}$. Conditions 1) and 2) of Definition 3 are immediate after a direct calculation. The residue matrix of $G(z)$ at $j(\theta = \frac{\pi}{2})$ is given by $K = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, which is not positive semidefinite Hermitian. However, the matrix $e^{-j\theta}K = -jK = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ is positive semidefinite Hermitian and satisfies Condition 3) in Definition 3. Hence $G(z)$ is discrete-time negative imaginary.

In the following lemma, we present a new method to prove the discrete-time negative imaginary lemma.

Lemma 6: [22] Let $(A,B,C,D)$ be a minimal state-space realization of a real-rational proper discrete-time transfer function matrix $G(z) \in \mathbb{R}^{m \times m}$, where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times m}$, and $m \leq n$. Suppose $\det(I + A) \neq 0$, and $\det(I - A) \neq 0$. Then, $G(z)$ is negative imaginary if and only if

1) $C(I + A)^{-1}B - D = B^T(I + A^T)^{-1}C^T - D^T$,
2) there exists a matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n \times n}$, such that $Y - A^TYA \geq 0$ and $C = B^T(I - A^{-1})^{-1}Y(I + A)$.

Proof: By means of the bilinear transformations $s = \frac{z - 1}{z + 1}$ and $z = \frac{1 + \frac{i}{\pi}}{1 - \frac{i}{\pi}}$, the discrete-time transfer function matrix $G(z) = C(I - A)^{-1}B + D$ is transformed into a transfer function matrix $G(s) = H(sI - F)^{-1}G + J$, and vice versa, where

$$F = (A + I)^{-1}(A - I), \quad G = \sqrt{2}(I + A)^{-1}B, \quad H = \sqrt{2}(I + A)^{-1}, \quad J = D - C(A + I)^{-1}B.$$ (9)

According to Theorem 1 in [23] and Lemma 5, it follows that $(F,G,H,J)$ is a minimal realization of $G(s)$, and that $G(z)$ is discrete-time negative imaginary if and only if $G(s)$ is continuous-time negative imaginary. Then, the proof follows along the following sequence of equivalent reformulations:

$$\begin{aligned}
G(z) &\sim \begin{pmatrix} A & B \\ C & D \end{pmatrix} \\
&\sim \begin{pmatrix} F & G \\ H & J \end{pmatrix} \\
&\sim (A + I)^{-1}(A - I)Y + Y(A^T - I)(A^T + I)^{-1}C^T - D^T.
\end{aligned}$$

This equivalence is via the continuous-time negative imaginary lemma in [18, Lemma 7].

$\Leftrightarrow D - C(I + A)^{-1}B = D^T - B^T(I + A^T)^{-1}C^T$ and there exists a real matrix $Y = Y^T > 0$ such that

$$FY + YF^T \leq 0 \quad \text{and} \quad G + FYH^T = 0.$$
\[ D - C(I + A)^{-1}B = D^T - B^T(I + A^T)^{-1}C^T \] and there exists a real matrix \( Y = Y^T > 0 \) such that
\[ (A - I)Y(A^T + I) + (A + I)Y(A^T - I) \leq 0, \]
\[ B + (A - I)Y(A^T + I)^{-1}C^T = 0. \]
\[ \Rightarrow D - C(I + A)^{-1}B = D^T - B^T(I + A^T)^{-1}C^T \] and there exists a matrix \( Y = Y^T > 0, Y \in \mathbb{R}^{n \times n} \), such that
\[ Y - A^TYA \geq 0 \] and \( C = B^T(I - A^T)^{-1}Y(I + A). \)

Remark 7: It is worthwhile to note that the proof of Lemma 6 is based on the connection between continuous-time and discrete-time negative imaginary functions developed in Lemma 5. However, the proof of Lemma 6 in [22] is based on the relation between discrete-time positive real and negative imaginary functions. Moreover, Lemma 5 can be used to develop other properties of discrete-time negative imaginary systems.

Example 3: To illustrate the difference between [10, Lemma 11] and Definition 3, consider the following symmetric continuous-time transfer matrix \( G(s) = \begin{pmatrix} 2 & 1 \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix} \). \( G(s) \) has no poles in \( \text{Re}[s] > 0 \), and \( j[G(j\omega) - G^*(j\omega)] = 0 \) for all \( \omega > 0 \) where \( j \) is not a pole of \( G(s) \). The residue matrix of \( G(s) \) at \( j \) given by \( K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is positive semidefinite Hermitian. So, according to [21, Lemma 3.1], [10, Lemma 3] or Definition 2, \( G(s) \) is negative imaginary. Using the bilinear transformation \( z = \frac{1+j}{1-j} \), \( G(s) \) transforms to \( G(z) = \begin{pmatrix} \frac{(z+1)^2}{z+1} & \frac{(z+1)^2}{2(z+1)} \\ \frac{(z+1)^2}{2(z+1)} & \frac{(z+1)^2}{z+1} \end{pmatrix} \). According to the proof of Lemma 11 in [10] and Lemma 5, \( G(z) \) is discrete-time negative imaginary, which satisfies all five conditions in [10, Lemma 11]. \( G(z) \) has no poles in \( \{ z \mid |z| > 1 \} \) and a calculation shows that \( j[G(e^{j\theta}) - G^*(e^{j\theta})] = 0 \) for all \( \theta \in (0, \pi) \) with \( e^{j\theta} \) not a pole of \( G(z) \). However, the residue matrix of \( G(z) \) at \( j \) given by \( K = \lim_{\omega \to j}(z - j)jG(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) is not positive semidefinite Hermitian, which contradicts Condition 3) in [10, Lemma 11]. In addition, the matrix \( e^{-j\theta}K = -jk = \begin{pmatrix} 1 & \frac{1}{j} \\ \frac{1}{j} & 1 \end{pmatrix} \) is positive semidefinite Hermitian, which satisfies Condition 3) of Definition 3.

IV. CONCLUSIONS

This paper has studied two related problems. Firstly, it was shown by theoretical analysis that only the original necessary and sufficient conditions were equivalent to the definition of discrete-time positive real functions and discrete-time positive real lemma. This result is in line with conclusions in [8]. Secondly, motivated by the discrete-time positive real functions case, it was found that discrete-time and continuous-time negative imaginary functions were equivalent by bilinear transformations. A new method was proposed to prove the discrete-time negative imaginary lemma.

REFERENCES