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A study of interval analysis for cold-standby system reliability optimization under parameter uncertainty

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ABSTRACT

This paper presents a study of interval analysis for solving cold-standby system reliability optimization problems with considering parameter uncertainty. Most works reported in existing literature have been based on the assumption that the probabilistic properties and statistical parameters have a known functional form, which is usually not the case. Very often the parameters are presented in form of an interval-valued number or bounds/tolerance from the engineering design. In this paper, interval analysis is used to incorporate this in the system optimization problems. A definition of interval order relation reflecting decision makers' preference is proposed for comparing interval numbers. A computational algorithm is developed to evaluate the system reliability and expected mission cost, in which a discrete approximation approach and a technique of interval universal generating function are used. For illustration, an application to sequencing optimization for heterogeneous cold-standby system is given; a modified genetic algorithm is developed to solve the proposed optimization problem with interval-valued objective. The results indicate that the interval analysis exhibits a good performance for dealing with parameter uncertainty of cold-standby system optimization problems.

1. Introduction

The development of industrial technology involves an increasing amount of design of complex and interrelated systems. Reliability is an important performance measure of industrial systems, especially when it is of safety–critical concerns. Extensive research has been carried out on system reliability optimization; survey papers (Kuo & Prasad, 2000; Kuo & Wan, 2007; Tillman, Hwang, & Kuo, 1977) have summarized many earlier studies on reliability optimal problems.

In the existing literature of system reliability optimization, most results are based on assumptions that the probabilistic properties or parameters of time-to-failure are deterministic. However, due to observation difficulties, resource limits and system complexity, uncertainties are usually unavoidable while modeling real industrial systems. For many engineering problems, it is overly difficult or costly to collect sufficient data about the uncertainties, especially at the very beginning of design processes. Stakeholders and decision makers have to deal with a variety of uncertainty issues when making decisions without sufficient information. In fact, many parameters are specified as intervals of some kind in engineering design.

Bayesian approach (Howson & Urbach, 2006) could be used to study the uncertainties associated with the estimation of parameters of a probability distribution (Pasanisi, Keller, & Parent, 2012; Srivastava & Deb, 2013; Troffaes, Walter, & Kelly, 2014). The unknown parameters are assumed to be random variables. With the Bayesian approach, subjective judgments are required to estimate the Bayesian random variables. The estimation of the Bayesian random variables can be improved when more data become available. Before receiving more data, however, the Bayesian approach remains a subjective representation of uncertainty. Fuzzy theory is another commonly used method for analysing uncertainty issues (Dotoli, Epicoco, Falagario, & Sciancalepore, 2015; Hanss & Turrin, 2010; Wang & Watada, 2009). In the fuzzy approach, the imprecise parameters are represented as fuzzy numbers. However, the fuzzy sets and their membership functions are required to be known. It is a formidable task for decision makers to specify the appropriate membership functions in advance.

In order to overcome the drawbacks of probabilistic methods and fuzzy approaches, interval analysis first developed by Moore (1966) has recently received some attention. The interval analysis has been used to deal with problems of uncertainty in diverse

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Nomenclature

Notations

$[a]$	interval-valued number
$\underline{a}$	the lower bound of $[a]$
$\bar{a}$	the upper bound of $[a]$
ρ_0	level of decision maker's preference
ρ	ratio of two interval numbers in the inclusion type relation
$R_j(t), R(t)$	reliability of component j and system, respectively
$E_C(t)$	expected cost of system
N	number of components in the system
T_j	random variable representing the time-to-failure of component j
$F_j(\cdot)$	cumulative distribution function of T_j
$f_j(\cdot)$	probability density function of T_j
V_j	start-up cost of component j
w_j	running cost of component j per time unit
Y_j	cumulative working time of first j components
$s(j)$	index of component j in the predetermined order
τ	mission time of system
Δ	duration of each time interval
m	number of mission time intervals
$[p_j(i)]$	probability that component j fails in the time interval $[\Delta i, \Delta(i+1)]$
$t_{j,i}$	i-th realization of T_j
$u_j(z)$	u-function representing discrete distribution of T_j
$U_j(z)$	u-function representing discrete distribution of Y_j

Table 1
Component parameters for the optimization example.

Component	$[\eta_{1,\cdot}]$	$[\eta_{2,\cdot}]$	V	w
1	[57, 62]	[1.00, 1.05]	100	3.0
2	[78, 82]	[1.70, 1.90]	80	3.5
3	[65, 75]	[1.20, 1.40]	210	1.2
4	[34, 36]	[1.00, 1.10]	150	3.0
5	[130, 145]	[2.30, 2.50]	220	2.0
6	[77, 82]	[1.75, 1.85]	60	3.7
7	[78, 83]	[1.15, 1.25]	120	1.8
8	[46, 54]	[1.05, 1.20]	70	2.5
9	[72, 77]	[1.15, 1.25]	100	1.5
10	[69, 71]	[1.35, 1.65]	180	2.0

fields, such as circuit analysis (Kolev, 1993), damage identification (Wang, Yang, Wang, & Qiu, 2012), structure safety analysis (Impollonia & Muscolino, 2011; Wang, Gao, Song, & Zhang, 2014; Zhang, Dai, Beer, & Wang, 2013), electric power system (Pereira & Da Costa, 2014), and so on. In these studies, interval variables were used to quantitatively describe the uncertain parameters in the face of limited information. Up to now, research on interval uncertainty problems has concentrated mainly in the aforementioned fields, while the application of interval analysis to system reliability optimization for complex industrial systems is relatively new.

In the existing literature of system reliability optimization, Feizollahi and Modarres (2012) suggested a robust deviation framework to deal with uncertain component reliabilities in constrained redundancy allocation problems; they addressed uncertainty by assuming that the component reliabilities belong to interval uncertainty sets. However, the interval numbers are not incorporated directly. Gupta, Bhunia, and Roy (2009) and Bhunia, Sahoo, and Roy (2010) dealt with optimization problems for series systems; the reliability of each component was represented as an interval number. Sahoo, Bhunia, and Kapur (2012) studied the constrained multi-objective reliability optimization problem of systems with interval-valued component reliabilities. However, these studies dealt specifically with the series or series–parallel systems with active redundancy and given interval-valued component reliability, or placed greater attention on optimization algorithms.

In this paper, we present a study of interval analysis for cold-standby system optimization problems considering uncertain probabilistic parameters. Our study focuses on the evaluation and optimization of system reliabilities and expected mission costs. A discrete approximation approach based on Levitin et al. (2013) and the interval universal generating function (IUGF) technique (Li, Chen, Yi, & Tao, 2011) are used in the evaluation procedures for estimating the system reliability and the expected mission cost. IUGF is a technique which extends the universal generating function (UGF) (Levitin, 2005) for the situations with interval-valued parameters.

In solving the optimization problem with interval-valued objective, a set of interval values appear during the selection of the best alternative, which leads to a question related to the comparison of two arbitrary interval numbers. In this paper, we define a new order relation for two arbitrary interval numbers considering different levels of decision maker's preference. The level of decision maker's preference is measured by the ratio ρ_0, where $\rho_0 = 1$ stands for neutrality; $\rho_0 > 1$ stands for optimistic preference; otherwise pessimistic.

For purposes of illustration, we propose the application of interval analysis theory to the sequencing optimization problem for heterogeneous cold-standby systems (Levitin et al., 2013a). In this

Table 2
Best obtained components sequences for $\rho_0 = 1$[a] and different τ.

Mission time τ	Optimal initiation sequence	Expected mission cost $[E_C]$	System reliability $[R]$
400	**9, 7, 5, 8, 6, 2, 1, 10, 3, 4**	**[1551.5, 1641.6]**	[0.9677, 0.9839]
	1, 2, 3, 4, 5, 6, 7, 8, 9, 10[b]	[1859.3, 1924.2]	[0.9682, 0.9838]
500	**9, 7, 3, 5, 8, 6, 2, 1, 10, 4**	**[1963.8, 2075.5]**	[0.8451, 0.9035]
	1, 2, 3, 4, 5, 6, 7, 8, 9, 10[b]	[2220.1, 2275.1]	[0.8440, 0.9045]
600	**3, 9, 7, 5, 8, 10, 1, 6, 2, 4**	**[2342.0, 2424.1]**	[0.5955, 0.7269]
	1, 2, 3, 4, 5, 6, 7, 8, 9, 10[b]	[2517.7, 2540.4]	[0.5955, 0.7262]

[a] When $\rho_0 = 1$, the proposed order relation are the same with the definition in Bhunia and Samanta (2014).
[b] Comparative trials with initiation sequence of (1, 2, 3, 4, 5, 6, 7, 8, 9, 10).

Table 3
Best obtained components sequences for $\tau = 500$ and different ρ_0.

ρ_0	Optimal initiation sequence	$[E_C]$	$[R]$
$\rho_0 \leqslant 0.3$	9, 7, 5, 3, 8, 6, 2, 1, 10, 4	[1965.062, 2075.128]	[0.8451, 0.9035]
$0.31 \leqslant \rho_0 \leqslant 2.76$	9, 7, 3, 5, 8, 6, 2, 1, 10, 4	[1963.850, 2075.494]	
$\rho_0 \geqslant 2.77$	3, 9, 7, 5, 8, 6, 2, 1, 10, 4	[1963.831, 2075.545]	

paper, we model the parameters of component time-to-failure distributions as interval-valued numbers. A genetic algorithm (GA) is developed to solve the proposed sequencing optimization problem. In order to avoid premature convergence and to increase computational efficiency, dual mutation (Wang, Ma, & Wang, 2008) and the random keys technique (Bean, 1994) are introduced in the GA.

The rest of this paper is organized as follows. Section 2 provides some basics of interval analysis, gives the definitions of interval order relation and presents a general formulation of cold-standby system optimization problem with interval-valued objective functions. Section 3 proposes the computation procedure of the system reliability and the expected mission cost. Section 4 shows a study of the interval analysis for solving sequencing optimization problem for heterogeneous cold-standby systems, and a GA-based searching approach is developed. A numerical example is given in Section 5. Finally, conclusions are presented in Section 6.

2. Interval analysis for cold-standby system reliability optimization – a general formulation

2.1. Interval arithmetic

Interval arithmetic was introduced by Moore in its modern form as an extension of real arithmetic (Moore, 1979; Moore, Kearfott, & Cloud, 2009). In interval arithmetic, an uncertain variable a is represented as an interval number $[a] = [\underline{a}, \bar{a}]$ with $\underline{a} \leqslant a \leqslant \bar{a}$, where $\underline{a}$ and $\bar{a}$ are the lower and upper bounds of a, respectively. If $\underline{a} = \bar{a}$, then a is a real number. The basic arithmetical operations of interval variables $[a] = [\underline{a}, \bar{a}]$ and $[b] = [\underline{b}, \bar{b}]$ $(\underline{a} \geqslant 0, \underline{b} \geqslant 0)$ are defined as

$$[a] + [b] = [\underline{a} + \underline{b}, \bar{a} + \bar{b}], \tag{1}$$

$$[a] - [b] = [\underline{a} - \bar{b}, \bar{a} - \underline{b}], \tag{2}$$

$$[a] \cdot [b] = [\underline{a}\underline{b}, \bar{a}\bar{b}], \tag{3}$$

$$[a]/[b] = \left[\frac{\underline{a}}{\bar{b}}, \frac{\bar{a}}{\underline{b}}\right]. \tag{4}$$

It is assumed in the case of division that $\bar{b} \neq 0$ and $\underline{b} \neq 0$. The distribution law also holds for interval numbers ($\underline{a} \geqslant 0$, $\underline{b} \geqslant 0$ and $\underline{c} \geqslant 0$); that is,

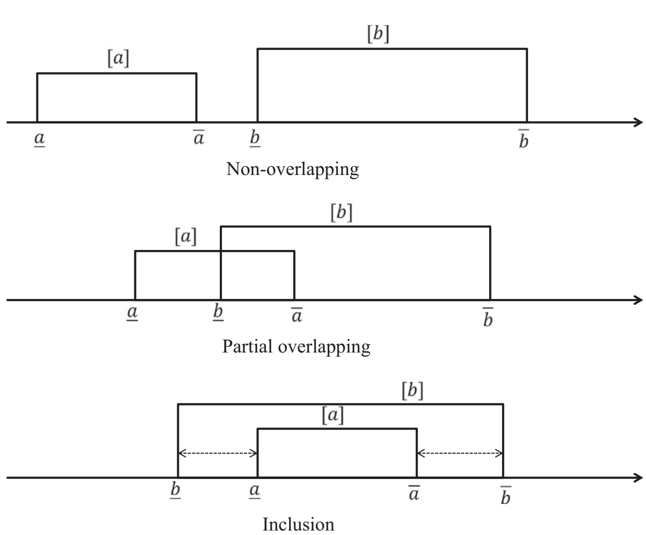

Fig. 1. Three types of relation between two interval numbers.

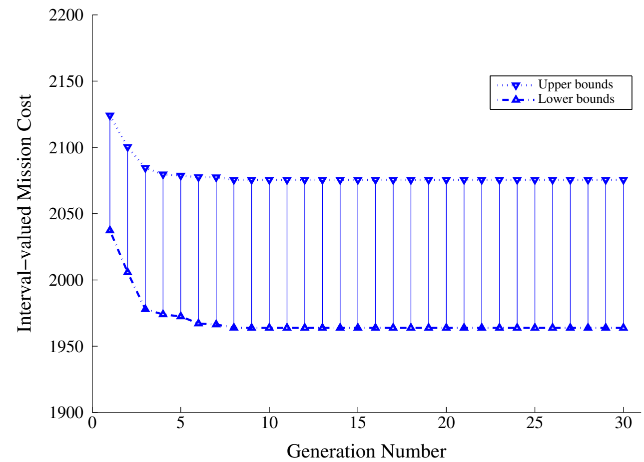

Fig. 2. Convergence process of the best fitness for $\tau = 500$ and $\rho_0 = 1$.

$$[a] \cdot ([b] + [c]) = [a] \cdot [b] + [a] \cdot [c]. \tag{5}$$

It may be noted that $-[a]$ is not an additive inverse for $[a]$ in the system of intervals,

$$[a] + (-[a]) = [\underline{a}, \bar{a}] + [-\bar{a}, -\underline{a}] = [\underline{a} - \bar{a}, \bar{a} - \underline{a}]. \tag{6}$$

This equals $[0, 0]$ only if $\underline{a} = \bar{a}$. For more details of interval arithmetic, one can refer Moore et al. (2009).

2.2. Interval order relations

To solve the optimization problems with parameters specified by interval numbers, a set of interval values may appear during the selection of the best alternative. Thus, an order relation needs to be defined for the comparison of two arbitrary interval numbers. There are three types of relations between any two intervals $[a]$ and $[b]$: non-overlapping $(\bar{a} < \underline{b})$, partial overlapping $(\underline{a} \leqslant \underline{b} \leqslant \bar{a} \leqslant \bar{b})$ and inclusion $(\underline{b} < \underline{a} \leqslant \bar{a} < \bar{b})$, see Fig. 1. For purposes of simplification, here we only represent the situation $\bar{a} \leqslant \bar{b}$ for interval numbers $[a]$ and $[b]$. For the case that $\bar{a} \geqslant \bar{b}$, one can also use the above three types by exchanging the positions of $[a]$ and $[b]$.

In the existing literature, most of the definitions of interval order relation are either incomplete or inconvenient. Moore (1979) and Ishibuchi and Tanaka (1990) have suggested several order relations of intervals. However, these order relations cannot cover all the three types of relations between two intervals. Levin (2004) and Sevastjanov and Róg (2006) have developed a remoteness function and a complicated comparison technique, respectively. However, the processes are very much complicated and turn out to be inconvenient for decision makers seeking to find out the best alternative.

Actually, for the non-overlapping type and the partial overlapping type in Fig. 1, one can easily make the decision that $[a]$ is less than $[b]$ when $(\underline{a} \leqslant \underline{b}) \cap (\bar{a} \leqslant \bar{b})$. However, for the inclusion type in Fig. 1, $\underline{b} < \underline{a}$ and $\bar{a} < \bar{b}$ may make one feel confused to select an optimal alternative from $[a]$ and $[b]$. Therefore, we need to consider the preference of decision makers.

In Karmakar, Mahato, and Bhunia (2009), Sahoo et al. (2012), and Bhunia and Samanta (2014), the decision maker's point of view for the maximization and minimization problems has been considered in the order relations. However, in these definitions, the preferences of different decision makers were not distinguished quantitatively.

Accordingly, we propose two definitions of order relations while considering the measurement of preference. Let ρ_0 denote the level that measures the decision maker's preference. The definitions of order relations for minimization and maximization problems are presented as follows.

Definition 2.1. The order relation $\leqslant_{min}$ between the intervals $[a] = [\underline{a}, \bar{a}]$ and $[b] = [\underline{b}, \bar{b}]$ for minimization problems

(i) For the non-overlapping and the partial overlapping types,

$$[a]\leqslant_{min}[b] \text{ if } \underline{a} \leqslant \underline{b} \text{ and } \bar{a} \leqslant \bar{b};$$

(ii) For the inclusion type ($\underline{b} < \underline{a} \leqslant \bar{a} < \bar{b}$),

$$[a]\leqslant_{min}[b] \text{ if } \rho \geqslant \rho_0, \text{ where } \rho = \frac{\bar{b} - \bar{a}}{\underline{a} - \underline{b}}.$$

Definition 2.2. The order relation $\geqslant_{max}$ between the intervals $[a] = [\underline{a}, \bar{a}]$ and $[b] = [\underline{b}, \bar{b}]$ for maximization problems

(i) For the non-overlapping and the partial overlapping types,

$$[a]\geqslant_{max}[b] \text{ if } \underline{a} \geqslant \underline{b} \text{ and } \bar{a} \geqslant \bar{b};$$

(ii) For the inclusion type ($\underline{b} < \underline{a} \leqslant \bar{a} < \bar{b}$),

$$[a]\geqslant_{max}[b] \text{ if } \rho \geqslant \rho_0, \text{ where } \rho = \frac{\underline{a} - \underline{b}}{\bar{b} - \bar{a}}.$$

For example that an optimistic decision maker ($\rho_0 = 2$) is solving a minimization problem, if $\rho = \frac{\bar{b}-\bar{a}}{\underline{a}-\underline{b}} = 1.5 < \rho_0, [b]$ is regarded as less than $[a]$, though the mean value of $[b], (\underline{b} + \bar{b})/2$, is slightly greater than the mean value of $[a], (\underline{a} + \bar{a})/2$. It means that this decision maker is willing to take risk for a possible minimum objective. Notice that if $\rho_0 = 1$, the proposed Definitions 2.1 and 2.2 are equivalent to the definitions of interval order relations in Bhunia and Samanta (2014).

2.3. General formulation of system optimization problem with interval parameters

For the system optimization problems, the objective is always to improve system reliability $R(t;\boldsymbol{x})$ or minimize the system cost $E_C(t;\boldsymbol{x})$ or both while considering certain constraints, see review papers (Kuo & Prasad, 2000; Kuo & Wan, 2007; Tillman et al., 1977). The following formulations are widely adopted,

$$\begin{array}{ll} \max & R(t;\boldsymbol{x}) \\ \text{subject to} & E_C(t;\boldsymbol{x}) \leqslant E_C^*, \\ & g_i(t;\boldsymbol{x}) \leqslant b_i, \ \forall i \in I \end{array}$$

or

$$\begin{array}{ll} \min & E_C(t;\boldsymbol{x}) \\ \text{subject to} & R(t;\boldsymbol{x}) \geqslant R^*, \\ & g_i(t;\boldsymbol{x}) \leqslant b_i, \ \forall i \in I \end{array}$$

where $\boldsymbol{x}$ is the solution to the optimization problem, including the component choices, numbers or their corresponding optimal redundancy levels; $g_i(\cdot)$ is a function of $\boldsymbol{x}$; b_i is the maximum value of the ith resource constraints; and I is the set of all possible resource constraints; E_C^* is the cost constraint; R^* is the desired level of system reliability.

In practical engineering systems, due to the observation difficulties, resources limitations, and system complexity, uncertainties are unavoidable in the system model. Using the aforementioned interval analysis theory to deal with the uncertainty issues, the uncertain parameters are represented as interval-valued numbers. Based on the interval athematic, the system reliability and mission cost are also obtained as interval values. Thus,

$$\begin{array}{ll} \max & [R(t;\boldsymbol{x})] \\ \text{subject to} & [E_C(t;\boldsymbol{x})] \leqslant E_C^*, \\ & g_i(t;\boldsymbol{x}) \leqslant b_i, \ \forall i \in I \end{array}$$

or

$$\begin{array}{ll} \min & [E_C(t;\boldsymbol{x})] \\ \text{subject to} & [R(t;x)] \geqslant R^*. \\ & g_i(t;\boldsymbol{x}) \leqslant b_i, \ \forall i \in I \end{array}$$

where the order relations in Definitions 2.1 and 2.2 are used during the selection of the best alternative. The procedures of incorporating interval analysis in system reliability and mission cost evaluation are discussed in the next section.

3. Computational procedures of incorporating interval analysis in cold-standby system reliability and cost evaluation

3.1. Cold-standby redundancy

Consider a cold-standby system that consists of N statistically independent components in parallel. The components can be similar or dissimilar with equivalent functionality. Let $F_j\left(t; \left[\eta_{1,j}\right], \left[\eta_{2,j}\right], \ldots\right)$ be the cumulative distribution function (*cdf*) of the time-to-failure T_j of component j, where the parameters $\left[\eta_{1,j}\right], \left[\eta_{2,j}\right], \ldots$ are uncertain but known as interval-valued numbers. The same situation suits the probability density function (*pdf*) $f_j\left(t; \left[\eta_{1,j}\right], \left[\eta_{2,j}\right], \ldots\right)$, if applicable.

In the cold-standby system, one of the N components is put into operation and other $N-1$ components are waiting in the cold-standby mode with a predetermined order. When one working component fails, one of the redundant components is activated. The components in cold-standby state are assumed to be not fail before used. Denote $Y_j = T_1 + T_2 + \cdots + T_j$ as the sum of the first j components' failure times. Reliability of the cold-standby system with perfect failure detection and switching is equal to the probability of $Y_N > t$; that is,

$$\begin{aligned} [R(t)] &= [\Pr\{Y_N > t\}] = [\Pr\{T_1 + T_2 + \cdots + T_N > t\}] \\ &= [\Pr\{T_1 > t\}] + \sum_{j=2}^{N} \left[\Pr\left\{Y_j > t \cap Y_{j-1} < t\right\}\right]. \end{aligned} \quad (7)$$

Note that it is difficult to determine a closed-form version of Eq. (7), especially when the time-to-failures $T_1, T_2, \ldots, T_N$ follow non-exponential distributions, and even worse non-identical distributions. Hence, a discrete approximation of time-to-failure distributions (Levitin et al., 2013) is used in this paper.

τ is defined as the mission time and be divided into m equal intervals with duration $\Delta = \tau/m$. The probability $[p_j(i)]$ that component j fails in the time interval $[\Delta i, \Delta(i+1)]$ can be obtained as

$$\begin{aligned} [p_j(i)] = F_j\left(\Delta(i+1); \left[\eta_{1,j}\right], \left[\eta_{2,j}\right], \ldots\right) - F_j\left(\Delta i; \left[\eta_{1,j}\right], \left[\eta_{2,j}\right], \ldots\right), \\ 0 \leqslant i \leqslant m. \end{aligned} \quad (8)$$

Following Levitin et al. (2013), we consider the random discrete time-to-failure T_j of component j. We assume that the probability mass function (*pmf*) of T_j presented in the form of pairs $(t_{j,i} = \Delta i, [p_j(i)] = [\Pr\{T_j = t_{j,i}\}])$ for $0 \leqslant i \leqslant m$ approximates the *pdf* of component j's time-to-failure, where $t_{j,i}$ is a realization of T_j

in the time interval $[\Delta i, \Delta(i+1)]$ and be assigned with a value of Δi in this paper. Since no component should work longer than the mission time $\tau = \Delta m$, for the first working component, the possible maximum realization of the discrete random variable T_1 is Δm, thus,

$$[p_1(m)] = 1 - \sum_{i=0}^{m-1}[p_1(i)]. \tag{9}$$

So T_1 follows a *pmf* of $([p_1(0)], [p_1(1)], \ldots, [p_1(m)])$. For the cold-standby component j which is activated at time Δl, the possible maximum realization of the discrete random variable T_j is $\Delta(m-l)$, thus

$$[p_j(m-l)] = 1 - \sum_{i=0}^{m-l-1}[p_j(i)]. \tag{10}$$

According to the *pmf* of the discrete random variable T_j, the IUGF of component j is defined as

$$u_j(z) = \sum_{i=0}^{M_j}[p_j(i)] \cdot z^{t_{j,i}}, \tag{11}$$

where M_j is corresponding to the possible maximum realization of T_j with $t_{j,M_j} = \Delta M_j$. The operator of the IUGF used in this paper is defined as

$$u_1(z) \otimes_\varphi u_2(z) = \sum_{i=0}^{M_1}[p_1(i)] \cdot z^{t_{1,i}} \otimes_\varphi \sum_{h=0}^{M_2}[p_2(h)] \cdot z^{t_{2,h}}$$

$$= \sum_{i=0}^{M_1}\sum_{h=0}^{M_2}[p_1(i)] \cdot [p_2(h)] \cdot z^{\varphi(t_{1,i}, t_{2,h})}, \tag{12}$$

where $\varphi(t_{1,i}, t_{2,h}) = min(\tau, t_{1,i} + t_{2,h})$.

Let Y_k denote the random cumulative work time of the first k components of the cold-standby system and note that $Y_1 = T_1$, the IUGF of Y_k can be obtained by using following recursive procedure:

$$\begin{aligned} U_k(z) = U_{k-1}(z) \otimes_\varphi u_k(z) &= \sum_{l=0}^{m}[P_{k-1}(l)] \cdot z^{y_{k-1,l}} \otimes_\varphi \sum_{h=0}^{M_k}[p_k(h)] \cdot z^{t_{k,h}} \\ &= \sum_{l=0}^{m}\sum_{h=0}^{m-l}[P_{k-1}(l)] \cdot [p_k(h)] \cdot z^{\varphi(y_{k-1,l}, t_{k,h})} \\ &= \sum_{l=0}^{m}[P_k(l)] \cdot z^{y_{k,l}}, \quad \text{for } k = 1, 2, \ldots, N, \end{aligned} \tag{13}$$

where $U_0(z)$ is assigned as z^0, $y_{k,l}$ is the k-th realization of Y_k and $y_{k,l} = \Delta l$, and $[P_k(l)] = [\Pr\{Y_k = y_{k,l}\}]$. Thus, the IUGF of the system can be obtained by

$$U_N(z) = \sum_{l=0}^{m}[P_N(l)] \cdot z^{y_{N,l}}. \tag{14}$$

Therefore, the term of $[P_N(m)]$ in Eq. (14) with $y_{N,m} = \Delta m = \tau$ is equal to the overall system reliability, that is, $[R] = [P_N(m)]$.

For the k-th component in the cold-standby system, its exploitation cost depends on its working time, $min\{t_{k,h}, \tau - y_{k-1,l}\}$. Thus, the expected cost of using component k is $\sum_{l=0}^{m}\sum_{h=0}^{m-l}[P_{k-1}(l)] \cdot [p_k(h)] \cdot (V_k + min\{t_{k,h}, \tau - y_{k-1,l}\} \cdot w_k)$, where V_k is the start-up cost and w_k is the exploitation cost per time unit. Therefore, the expected mission cost $[E_C]$ of the cold-standby system obtained by adding all the costs of N components together.

3.2. Improving the interval arithmetic evaluation

Due to the dependence problem that exists in the interval analysis theory (Degrauwe, Lombaert, & De Roeck, 2010), the ranges of the obtained interval valued reliability and cost are occasionally overestimated compared with the exact ranges. The dependence problem occurs because the interval variables are treated as stochastically independent rather than recognizing their possible correlations. Actually, in the proposed evaluation processes, the range of $[p_j(i)]$ in Eq. (8) is overestimated if we simply apply subtraction operator of interval arithmetic, because the term $F_j\left(\Delta(i+1); [\eta_{1,j}], [\eta_{2,j}], \ldots\right)$ and $F_j\left(\Delta i; [\eta_{1,j}], [\eta_{2,j}], \ldots\right)$ have the same variables $[\eta_{1,j}], [\eta_{2,j}], \ldots$. In order to reduce the overestimation, the evaluation of $[p_j(i)]$ can be improved as

$$\begin{aligned}[p_j(i)] = \{F_j(\Delta(i+1); \xi_1, \xi_2, \ldots) - F_j(\Delta i; \xi_1, \xi_2, \ldots) | \underline{\eta_{1,j}} \leqslant \xi_1 \\ \leqslant \overline{\eta_{1,j}}, \underline{\eta_{2,j}} \leqslant \xi_2 \leqslant \overline{\eta_{2,j}}, \ldots\}.\end{aligned} \tag{15}$$

Furthermore, in the *pmf* of component j, the probability of the maximum realization of T_j in (9) and (10) can be improved by

$$[p_j(M_j)] = \left\{1 - F_j(\Delta M_j; \xi_1, \xi_2, \ldots) \middle| \underline{\eta_{1,j}} \leqslant \xi_1 \leqslant \overline{\eta_{1,j}}, \underline{\eta_{2,j}} \leqslant \xi_2 \leqslant \overline{\eta_{2,j}}, \ldots\right\}. \tag{16}$$

4. A study of cold-standby sequencing optimization problem with interval analysis

4.1. Problem formulation

In order to illustrate the proposed interval analysis method, a cold-standby sequencing optimization problem is studied in this section. Since the components are activated one by one in a cold-standby system, the system reliability equals to the probability that the sum of all components' working time reaches mission time. When the set of components is fixed, the system reliability does not depend on initiation sequence of components. However, if the system consists of dissimilar components, the mission cost depends on the different initiation sequence of the components. So the objective of sequencing optimization problem is to determine the optimal initiation sequence of system components with an objective of minimizing the expected system mission cost.

Consider a cold-standby system that consists of N dissimilar components in parallel with equivalent functionality. The sequencing optimization problem for the cold-standby system is formulated as follows,

$$\begin{aligned} &\min\ [E_C(\tau; \boldsymbol{s})] \\ &\text{subject to } \boldsymbol{s} \in D \end{aligned} \tag{17}$$

where $\boldsymbol{s} = (s(1), s(2), \ldots, s(N))$, and D is the set of all permutations of the N components, the system reliability $[R(\tau; \boldsymbol{s})]$ does not depend on the sequence of components $\boldsymbol{s}$. This is a minimization problem, so the order relation in Definition 2.1 is used. A recursive algorithm for evaluating the objective value of the problem (17) and the system reliability is given in the following.

4.2. Recursive algorithm for computing system reliability and mission cost

Based on the considerations in Section 3, the following recursive Algorithm 1 determines the expected mission cost $[E_C(\tau; \boldsymbol{s})]$ and system reliability $[R(\tau; \boldsymbol{s})]$ for any given initiation sequence of components $(s(1), s(2), \ldots, s(N))$ and given *pmf* of T_j for $j = 1, 2, \ldots, N$.

Algorithm 1. Evaluating the system reliability and expected mission cost

Initialization: $U_0 = z^0, [R] = [0,0], [E_C] = [0,0]$;
Main loop: repeat the following steps for $k = 1, \ldots, N$.
(i) *Obtain* $U_k(z) = U_{k-1}(z) \otimes_{\varphi} u_k(z)$;
(ii) *Add the value*

$$\sum_{l=0}^{m} \sum_{h=0}^{m-l} [P_{k-1}(l)] \cdot [p_k(h)] \cdot \left(V_{s(k)} + min\{t_{k,h}, \tau - y_{k-1,l}\} \cdot w_{s(k)}\right)$$

to $[E_C]$;
(iii) *Add the value* $[P_k(m)]$ *to* $[R]$;
(iv) *Remove the term* $[P_k(m)] \cdot z^{\tau}$ *from* $U_k(z)$.

4.3. GA based approach

Searching the optimal initiation sequence of components for a cold-standby system is a combinatorial optimization problem. It is not realistic to enumerate all the possible solutions. Heuristic algorithms have been widely suggested in reliability engineering field for solving this type of optimization problems. A GA based approach is used for the minimization problem (17). A brief description of the algorithm is given in the following. More detailed information about GA can be found in Gen and Cheng (2000).

4.3.1. Solution encoding and GA operators

Here we define the solution encoding and the specific GA operators used in this paper. Each solution is a permutation of N integer numbers, in which each number stands for a component and should appear in the sequence only once. For each permutation, the expected mission cost can be evaluated using the Algorithm 1.

A crossover operator that was suggested in Levitin et al. (2013) is used in our algorithm. The offspring copies all the numbers in the same position of first parent. Then the fragment of the offspring, defined as a set of adjacent numbers between two random sites, is reallocated by following the order that the numbers of fragment appear in the second parent. Following is an example of crossover operator, in which the fragment is marked in bold.

$$\begin{pmatrix} \text{1st parent}: & 1 & 2 & \mathbf{3} & \mathbf{4} & \mathbf{5} & \mathbf{6} & \mathbf{7} & 8 & 9 & 10 \\ \text{2nd parent}: & 10 & 2 & 9 & 8 & \mathbf{6} & \mathbf{5} & 1 & \mathbf{3} & \mathbf{4} & \mathbf{7} \\ \text{Offspring}: & 1 & 2 & \mathbf{6} & \mathbf{5} & \mathbf{3} & \mathbf{4} & \mathbf{7} & 8 & 9 & 10 \end{pmatrix}$$

A technique called random keys that was first suggested in Bean (1994) is adapted into the mutation operator of our algorithm. We first generate a uniform $(0,1)$ random deviate (random key) for each number of the candidate sequence. Then the numbers whose corresponding random key is less than the mutation probability are sorted in ascending order of random keys. Following is an example of mutation operator, in which the mutation probability is 0.45.

$$\begin{pmatrix} \text{Candidate solution}: & \mathbf{1} & 2 & 3 & 4 & 5 & \mathbf{6} & \mathbf{7} \\ \text{Random keys}: & \mathbf{0.24} & 0.97 & 0.75 & 0.48 & 0.80 & \mathbf{0.42} & \mathbf{0.15} \\ \text{New solution}: & \mathbf{7} & 2 & 3 & 4 & 5 & \mathbf{1} & \mathbf{6} \end{pmatrix}$$

4.3.2. Structure of GA

The main structure of the proposed GA is as follows:

Algorithm 2. General Framework of GA used to solve the proposed problem

1. Generate an initial population V_{pop} with a population size of *pop_size*. Set the maximum generation as *max_gen*, crossover probability p_c, dual mutation probabilities p_{m1} and p_{m2};
2. Calculate the fitness values of all individuals in V_{pop}, preserve the best individual;
3. Perform Ranking selection on V_{pop} according to the order relation $\leqslant_{min}$, put the selected individuals into V_{sel};
4. For any two individuals (parents) in V_{sel}, calculate their Hamming distance, if the Hamming distance is greater than a fixed threshold (0.2), perform crossover operation and generate an offspring, put the offspring into a temporary population V_{temp}; Otherwise, perform local mutation operation on the parents with probability of p_{m2}, and also put the generated individuals into V_{temp}.
5. Perform global mutation operation on V_{temp} with global mutation probability of p_{m1};
6. Select *pop_size* best individuals from the set of $V_{sel} \cup V_{temp}$ according to the order relation $\leqslant_{min}$, put them into V_{new};
7. Use the preserved best individual in Step 2 to replace the worst individual in the V_{new};
8. Let V_{new} replace V_{pop} to be the new generation;
9. If the number of generations reaches *max_gen*, the algorithm is terminated, output the final results; Otherwise, goto Step 2.

5. Numerical example of cold-standby sequencing optimization

The example is adapted from Levitin et al. (2013), the probabilistic parameters of the system components are known as interval numbers. Ten components with Weibull time-to-failure distribution $(F(t) = 1 - exp(-(t/\eta_1)^{\eta_2}), t \geqslant 0)$ are given in this example, and the parameters of distribution and cost are listed in Table 1. The GA parameters are set as: $pop\_size = 20$, $max\_gen = 30, p_c = 0.95, p_{m1} = 0.3$ and $p_{m2} = 0.45$. To run the GA by MATLAB, a computer with Intel Core i5-4590 CPU 3.3 GHz processor and 8 GB RAM under Window 7 operating system is used. Due to the stochastic nature of GA, the proposed Algorithm 2 is run 10 times and the best one is selected as the final solution.

The best initiation sequences obtained by the proposed GA for $\rho_0 = 1$ and three different mission times $\tau = 400, \tau = 500$, and $\tau = 600$, are presented in Table 2. A simple initiation sequence $(1,2,3,4,5,6,7,8,9,10)$ is given for comparison. The mission time is divided into 1000 time intervals, that is, $m = 1000$. As discussed in Levitin et al. (2013), when $m = 1000$, the accuracy of obtained results is acceptable. Compared with the comparative trials, the expected mission cost is significantly reduced by achieving the optimal initiation sequence of components.

It can also be observed from Table 2 that the evaluated system reliabilities for $\tau = 400, 500$ and 600 are not exactly equal to that of the comparative trials. This is due to the lack of additive inverses in interval arithmetic. A component in different initiation sequence may has different sequencing index and activating time, then the component's working time T_j may has different maximum realization of $t_{j,M_j} = \Delta M_j$ with the probability of $[p_j(M_j)] = 1 - F_j\left(\Delta M_j; \left[\eta_{1,j}\right], \left[\eta_{2,j}\right]\right)$, but $\sum_{i=0}^{M_j}[p_j(i)]$ is not exactly equal to 1. The tiny difference between the obtained values of system reliability for different initiation sequences can be regarded as the calculation error generating from the approximation approach. And it does not affect the solution of the sequencing optimization.

To demonstrate the influence of decision makers' preferences on the optimal initiation sequence and expected mission cost, different values of ρ_0 are considered in the selection of the best alternative. Table 3 contains results obtained for mission time $\tau = 500$ with different ρ_0. It can be seen that with the increase of ρ_0 the decision makers become more optimistic and adventurous, the obtained mission cost reaches a smaller lower bound while suffering a higher upper bound. As ρ_0 ranging from 0.1 to 10, there are three different obtained optimal initiation sequences, which roughly correspond to pessimistic, neutral and optimistic decision makers.

Fig. 2 shows the convergence of the best fitness value in each generation for $\tau = 500$ and $\rho_0 = 1$. The minimum expected mission cost is achieved after about 9 generations. The results show that the proposed GA consistently converged to the optimal solution. This also means our proposed GA is stable with the parameters of $pop\_size = 20, max\_gen = 30, p_c = 0.95, p_{m1} = 0.3$ and $p_{m2} = 0.45$.

6. Conclusions

A study of interval analysis has been carried out in this paper to better deal with parameter uncertainties of cold-standby system optimization problems. An order relation considering the decision makers' preference was defined for comparing interval numbers. A general formulation was presented for cold-standby system optimization problems with considering uncertain-but-bounded probabilistic parameters by using interval arithmetic. Based on a discrete approximation approach of time-to-failure distributions of system components, a technique of IUGF was introduced to evaluate the system reliability and the expected mission cost. Then, a sequencing optimization problem for a heterogeneous cold-standby system was studied for the purposes of illustration. Then a GA based approach was developed to solve the optimization problem. The results have shown that the interval analysis is a useful tool to deal with cold-standby system optimization problems with parameter uncertainty.

Our future work includes incorporating interval analysis into many other redundancy optimization problems, such as *k*-out-of-*n* cold-standby redundancy optimization (Levitin et al., 2013b), warm-standby redundancy optimization (Levitin, Xing, & Dai, 2013c), and mixed redundancy strategy optimization (Ardakan & Hamadani, 2014; Levitin, Xing, & Dai, 2014). Since the uncertain issues can appear in many contexts in diverse system optimization problems, there are quite a few open problems and challenging topics emanating from the work we have presented in this paper for future researchers. For example, the interval approach can be applied to optimal backup problem (Levitin, Xing, & Dai, 2015a), optimal loading problem (Levitin, Xing, & Dai, 2015b), optimal testing resources problem (Levitin, 2002; Wang, Tang, & Yao, 2010), etc. With the rapidly upgrading of technologies and equipment in industrial engineering, incorporating interval analysis into system reliability problems, especially in the early procedures of system design and optimization, will have a significant promoting effect on shortening research and development cycle of products.

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References

Ardakan, M. A., & Hamadani, A. Z. (2014). Reliability optimization of series–parallel systems with mixed redundancy strategy in subsystems. *Reliability Engineering & System Safety, 130*, 132–139.
Bean, J. C. (1994). Genetic algorithms and random keys for sequencing and optimization. *ORSA Journal on Computing, 6*(2), 154–160.
Bhunia, A. K., & Samanta, S. S. (2014). A study of interval metric and its application in multi-objective optimization with interval objectives. *Computers & Industrial Engineering, 74*, 169–178.
Bhunia, A. K., Sahoo, L., & Roy, D. (2010). Reliability stochastic optimization for a series system with interval component reliability via genetic algorithm. *Applied Mathematics and Computation, 216*(3), 929–939.
Degrauwe, D., Lombaert, G., & De Roeck, G. (2010). Improving interval analysis in finite element calculations by means of affine arithmetic. *Computers & Structures, 88*(3), 247–254.
Dotoli, M., Epicoco, N., Falagario, M., & Sciancalepore, F. (2015). A cross-efficiency fuzzy data envelopment analysis technique for performance evaluation of decision making units under uncertainty. *Computers & Industrial Engineering, 79*, 103–114.
Feizollahi, M. J., & Modarres, M. (2012). The robust deviation redundancy allocation problem with interval component reliabilities. *IEEE Transactions on Reliability, 61*(4), 957–965.
Gen, M., & Cheng, R. (2000). *Genetic algorithms and engineering optimization* (Vol. 7). John Wiley & Sons.
Gupta, R. K., Bhunia, A. K., & Roy, D. (2009). A GA based penalty function technique for solving constrained redundancy allocation problem of series system with interval valued reliability of components. *Journal of Computational and Applied Mathematics, 232*(2), 275–284.
Hanss, M., & Turrin, S. (2010). A fuzzy-based approach to comprehensive modeling and analysis of systems with epistemic uncertainties. *Structural Safety, 32*(6), 433–441.
Howson, C., & Urbach, P. (2006). *Scientific reasoning: The Bayesian approach*. Open Court Publishing.
Impollonia, N., & Muscolino, G. (2011). Interval analysis of structures with uncertain-but-bounded axial stiffness. *Computer Methods in Applied Mechanics and Engineering, 200*(21), 1945–1962.
Ishibuchi, H., & Tanaka, H. (1990). Multiobjective programming in optimization of the interval objective function. *European Journal of Operational Research, 48*(2), 219–225.
Karmakar, S., Mahato, S. K., & Bhunia, A. K. (2009). Interval oriented multi-section techniques for global optimization. *Journal of Computational and Applied Mathematics, 224*(2), 476–491.
Kolev, L. V. (1993). *Interval methods for circuit analysis* (Vol. 1). World Scientific.
Kuo, W., & Prasad, V. R. (2000). An annotated overview of system-reliability optimization. *IEEE Transactions on Reliability, 49*(2), 176–187.
Kuo, W., & Wan, R. (2007). Recent advances in optimal reliability allocation. In G. Levitin (Ed.), *Computational intelligence in reliability engineering* (pp. 1–36). Berlin, Heidelberg: Springer-Verlag.
Levin, V. I. (2004). Ordering of intervals and optimization problems with interval parameters. *Cybernetics and Systems Analysis, 40*(3), 316–324.
Levitin, G. (2002). Allocation of test times in multi-state systems for reliability growth testing. *IIE Transactions, 34*(6), 551–558.
Levitin, G. (2005). *The universal generating function in reliability analysis and optimization*.London: Springer-Verlag.
Levitin, G., Xing, L., & Dai, Y. (2013a). Cold-standby sequencing optimization considering mission cost. *Reliability Engineering & System Safety, 118*, 28–34.
Levitin, G., Xing, L., & Dai, Y. (2013b). Sequencing optimization in k-out-of-n cold-standby systems considering mission cost. *International Journal of General Systems, 42*(8), 870–882.
Levitin, G., Xing, L., & Dai, Y. (2013c). Optimal sequencing of warm standby elements. *Computers & Industrial Engineering, 65*(4), 570–576.
Levitin, G., Xing, L., & Dai, Y. (2014). Cold vs. hot standby mission operation cost minimization for 1-out-of-N systems. *European Journal of Operational Research, 234*(1), 155–162.
Levitin, G., Xing, L., & Dai, Y. (2015a). Optimal backup distribution in 1-out-of-N cold standby systems. *IEEE Transactions on Systems, Man, and Cybernetics: Systems, 45*(4), 636–646.
Levitin, G., Xing, L., & Dai, Y. (2015b). Optimal completed work dependent loading of components in cold standby systems. *International Journal of General Systems, 44*(4), 471–484.
Li, C. Y., Chen, X., Yi, X. S., & Tao, J. Y. (2011). Interval-valued reliability analysis of multi-state systems. *IEEE Transactions on Reliability, 60*(1), 323–330.
Moore, R. E. (1966). *Interval analysis* (Vol. 4)Englewood Cliffs: Prentice-Hall.
Moore, R. E. (1979). *Methods and applications of interval analysis* (Vol. 2) Philadelphia: SIAM.
Moore, R. E., Kearfott, R. B., & Cloud, M. J. (2009). *Introduction to interval analysis*. Philadelphia: SIAM.
Pasanisi, A., Keller, M., & Parent, E. (2012). Estimation of a quantity of interest in uncertainty analysis: Some help from Bayesian decision theory. *Reliability Engineering & System Safety, 100*, 93–101.
Pereira, L. E. S., & Da Costa, V. M. (2014). Interval analysis applied to the maximum loading point of electric power systems considering load data uncertainties. *International Journal of Electrical Power & Energy Systems, 54*, 334–340.
Sahoo, L., Bhunia, A. K., & Kapur, P. K. (2012). Genetic algorithm based multi-objective reliability optimization in interval environment. *Computers & Industrial Engineering, 62*(1), 152–160.
Sevastjanov, P., & Róg, P. (2006). Two-objective method for crisp and fuzzy interval comparison in optimization. *Computers & Operations Research, 33*(1), 115–131.
Srivastava, R., & Deb, K. (2013). An evolutionary based Bayesian design optimization approach under incomplete information. *Engineering Optimization, 45*(2), 141–165.

Tillman, F. A., Hwang, C. L., & Kuo, W. (1977). Optimization techniques for system reliability with redundancy – A review. *IEEE Transactions on Reliability, 26*(3), 148–155.

Troffaes, M., Walter, G., & Kelly, D. (2014). A robust Bayesian approach to modeling epistemic uncertainty in common-cause failure models. *Reliability Engineering & System Safety, 125*, 13–21.

Wang, C., Gao, W., Song, C., & Zhang, N. (2014). Stochastic interval analysis of natural frequency and mode shape of structures with uncertainties. *Journal of Sound and Vibration, 333*(9), 2483–2503.

Wang, J., Ma, Y., & Wang, F. (2008). Study of improved genetic algorithm based on dual mutation and its simulation. *Computer Engineering & Applications, 44*(3), 57–59, 9.

Wang, S., & Watada, J. (2009). Modelling redundancy allocation for a fuzzy random parallel–series system. *Journal of Computational and Applied Mathematics, 232*(2), 539–557.

Wang, X., Yang, H., Wang, L., & Qiu, Z. (2012). Interval analysis method for structural damage identification based on multiple load cases. *Journal of Applied Mechanics, 79*(5), 051010.

Wang, Z., Tang, K., & Yao, X. (2010). Multi-objective approaches to optimal testing resource allocation in modular software systems. *IEEE Transactions on Reliability, 59*(3), 563–575.

Zhang, H., Dai, H., Beer, M., & Wang, W. (2013). Structural reliability analysis on the basis of small samples: An interval quasi-Monte Carlo method. *Mechanical Systems and Signal Processing, 37*(1), 137–151.