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Boundedness of discretised non-linear systems under fast terminal sliding mode control

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Abstract: This study analyses the steady-state boundedness property of discretised non-linear systems under fast terminal sliding mode (FTSM) control. First, the recursive FTSM variables and surfaces are introduced. Then, the FTSM control law is designed by enforcing the system trajectory to reach the last FTSM surface after one sample period. The boundedness of the FTSM variables and the system steady states are established and the corresponding bounds are provided. Finally, the theoretical results are illustrated by a numerical example and a comparison with terminal sliding mode control is presented.

1 Introduction

Sliding mode control has been widely studied and applied in linear and non-linear systems due to the advantages such as strong robustness, order reduction, invariance; see [1–6]. To implement a sliding mode control, both a sliding mode surface and a control law need to be designed so that the closed-loop system is asymptotically stable. Conventional sliding mode control with linear sliding mode surfaces can guarantee that the system state reaches the equilibrium asymptotically [4]. To reach the equilibrium within finite time, terminal sliding mode (TSM) control method was proposed in [7]. However, the TSM control provides bad convergence performance when the system state is far away from the equilibrium point [8]. To obtain faster transient response, fast TSM (FTSM) control method was proposed in [8]. The method of FTSM control exhibits the advantages of finite-time convergence of TSM control and exponential convergence rate of linear sliding mode control [8]. The FTSM control method has been studied extensively in continuous-time domain; see, e.g. [9–11].

In discrete-time domain, the authors of [12, 13] have found and demonstrated that the system trajectory can only be driven into a neighbourhood of the sliding mode surface. The findings in [12, 13] motivate the study of sliding mode control for discrete-time systems [14–19]. Recently, TSM has been studied in [20–23]. In [20], the continuous-time TSM was discretised and the system steady-states along the TSM surface were shown to have period-2 orbits. The stability of the periodic orbits was established [20]. In [21], the dynamical behaviour was studied for discretised second-order linear systems along the TSM surface. In particular, the periodic orbits were classified into four classes according to the size of the sampling period. In [22], the trajectory of the system was proved to have period-2 motion only in steady state. In [23], the boundedness of the TSM variables and the system steady states were established and the bounds are provided as well. On the other hand, FTSM control has been illustrated through simulations as an effective method to improve the transient response speed of the system in [24]. However, theoretical analysis was not conducted in [24]. This motivates the current study.

This paper considers a class of discretised single-input single-output (SISO) non-linear systems and studies the boundedness property of both the FTSM variables and the system steady states under the FTSM control. First, the FTSM variables are defined recursively. The FTSM control law is designed by letting the last FTSM variable zero. With the designed control law, the system trajectory can be driven exactly on to the last FTSM surface after one sampling period; that is, the last FTSM variable equals zero. Second, the other FTSM variables are shown to be bounded in the 'steady state' of the system. The explicit expressions of the bounds are found as well. Moreover, boundedness of the steady state of the system is established by a recursive analysis and the explicit expressions of the bounds are provided. Third, by choosing appropriate FTSM parameters, the bounds of both the FTSM variables and the system steady state can be made arbitrarily small. Finally, a single link manipulator example is used to illustrate the effectiveness of the developed theoretical results. When compared with the TSM control method, the advantages of the FTSM control method proposed in this paper are twofold: (i) faster transient response can be achieved when the sliding mode parameters are chosen such that the bounds of the sliding mode variables are approximately equal; and (ii) smaller bounds of the sliding mode variables can be achieved when the sliding mode parameters are chosen such that the transient responses are similar.

2 Problem formulation

Consider a class of n-order SISO continuous-time systems

$$\begin{aligned} \dot{z} &= a(z) + b(z)u, \\ y &= c(z), \end{aligned} \tag{1}$$

where $z \in \mathbb{R}^n$ is the system state, $y \in \mathbb{R}$ is the system output and $u \in \mathbb{R}$ is the control input. $a(z) \in \mathbb{R}^n$ and $b(z) \in \mathbb{R}^n$ are smooth vector functions and $c(z) \in \mathbb{R}$ is a smooth scalar function.

The non-linear system (1) is assumed to have relative degree n for $z \in \Omega \subset \mathbb{R}^n$ [25]. Under this assumption, it follows from Theorem 13.1 of [25] that there exists a mapping $\Phi(z) = x$, $z \in \Omega$, such that the system (1) can be transformed into the following canonical form:

$$\begin{aligned} \dot{x}_i &= x_{i+1}, \quad i \in \{1, 2, \ldots, n-1\}, \\ \dot{x}_n &= f(x) + g(x)u, \end{aligned} \tag{2}$$

where $x_i = \mathcal{L}_a^{i-1}c(z)$, $f(x) = \mathcal{L}_a^n c(z)$ and $g(x) = \mathcal{L}_b\mathcal{L}_a^{n-1}c(z)$, $g(x) \neq 0$ for $x = \Phi(z)$, $z \in \Omega$. Here, $\mathcal{L}_a^i c(z)$ is the ith order Lie derivative of c with respect to a, $i \in \{0, 1, \ldots, n-1\}$; see [25].

Let T be the sampling period and $\Delta x_i(k) \triangleq (x_i(k+1) - x_i(k))/T$, $i \in \{1, 2, \ldots, n\}$. Then the system (2) is discretised to

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$$\begin{aligned}\Delta x_i(k) &= x_{i+1}(k), \quad i \in \{1, 2, ..., n-1\},\\ \Delta x_n(k) &= f(x(k)) + g(x(k))u(k).\end{aligned} \tag{3}$$

Define the recursive discrete-time FTSM variables as

$$\begin{aligned}s_1(k) &= x_1(k),\\ s_i(k) &= \Delta s_{i-1}(k) + \alpha_{i-1}s_{i-1}(k)\\ &\quad + \beta_{i-1}s_{i-1}^{\lambda_{i-1}}(k), \quad i \in \{2, 3, ..., n\},\end{aligned} \tag{4}$$

where $\Delta s_i(k) = (s_i(k+1) - s_i(k))/T$, $i \in \{1, 2, ..., n\}$. The parameters α_i, β_i and λ_i are known and satisfy the following conditions: $0 < \alpha_i < 1/T$, $\beta_i > 0$, $\lambda_i = q_i/p_i$, p_i and q_i are odd positive integers such that $1/2 < \lambda_i < 1$, $i \in \{1, 2, ..., n-1\}$. For $i \in \{1, 2, ..., n-1\}$, $s_i^{\lambda_i}$ is the real solution to the equation $x^{1/\lambda_i} = s_i$. Since p_i and q_i are odd positive integers, the equation $(-s_i)^{\lambda_i} = -(s_i)^{\lambda_i}$ holds for all $i \in \{1, 2, ..., n-1\}$. The ith FTSM surface is defined to be $s_i = 0$ for $i \in \{1, 2, ..., n\}$.

Remark 1: The system (2) is general and includes a large class of non-linear physical models in practice, such as inverted pendulum systems [26] and single link manipulator systems [25].

Remark 2: Consider the multiple-input multiple-output (MIMO) version of the system (1). Under some assumptions on the control input parameter matrix and the relative degrees, Lemma 5.2.1 in [27] can be used to transform system (1) to a set of the systems (2). Each individual system in the set can be discretised and the recursive FTSM variables can be defined. As a result, the results to be developed in this paper should be extended to the MIMO case.

The objective of this paper is to study the boundedness property of the system (3) along the sliding mode surfaces.

3 Boundedness property of systems

In this section, an FTSM control law is designed by setting the last FTSM variable to zero. Then the boundedness of both the FTSM variables and the steady state of the system is established, and the expressions of the bounds are obtained.

3.1 FTSM control law

Our FTSM control law is designed by letting $s_n(k+1) = 0$. First, we need figure out the expression of $\Delta s_n(k)$.

Define $\Delta^i \triangleq \Delta^{i-1}(\Delta)$ for $i \in \{2, 3, ..., n\}$, it follows from (3) and (4) that

$$\begin{aligned}\Delta^n s_1(k) &= \Delta^n x_1(k) = \Delta^{n-1}(\Delta x_1(k)) = \Delta^{n-1}x_2(k)\\ &= \cdots = \Delta x_n(k) = f(x(k)) + g(x(k))u(k).\end{aligned} \tag{5}$$

For the FTSM variable $s_2(k)$, we obtain that

$$\begin{aligned}\Delta s_2(k) &= (s_2(k+1) - s_2(k))/T\\ &= \{[\Delta s_1(k+1) + \alpha_1 s_1(k+1) + \beta_1 s_1^{\lambda_1}(k+1)]\\ &\quad - [\Delta s_1(k) + \alpha_1 s_1(k) + \beta_1 s_1^{\lambda_1}(k)]\}/T\\ &= \Delta^2 s_1(k) + \alpha_1 \Delta s_1(k) + \beta_1 \Delta s_1^{\lambda_1}(k),\\ \Delta^2 s_2(k) &= (\Delta s_2(k+1) - \Delta s_2(k))/T\\ &= \Delta^3 s_1(k) + \alpha_1 \Delta^2 s_1(k) + \beta_1 \Delta^2 s_1^{\lambda_1}(k),\\ &\vdots\\ \Delta^{n-1} s_2(k) &= (\Delta^{n-2} s_2(k+1) - \Delta^{n-2} s_2(k))/T\\ &= \Delta^n s_1(k) + \alpha_1 \Delta^{n-1} s_1(k) + \beta_1 \Delta^{n-1} s_1^{\lambda_1}(k).\end{aligned} \tag{6}$$

For $s_i(k)$, $i \in \{3, 4, ..., n\}$, we have

$$\begin{aligned}\Delta^{n-i+1} s_i(k) &= \Delta^{n-i+2} s_{i-1}(k) + \alpha_{i-1}\Delta^{n-i+1} s_{i-1}(k)\\ &\quad + \beta_{i-1}\Delta^{n-i+1} s_{i-1}^{\lambda_{i-1}}(k).\end{aligned} \tag{7}$$

From (5)–(7), we obtain that

$$\begin{aligned}\Delta s_n(k) &= \Delta^2 s_{n-1}(k) + \alpha_{n-1}\Delta s_{n-1}(k) + \beta_{n-1}\Delta s_{n-1}^{\lambda_{n-1}}(k)\\ &= \Delta^3 s_{n-2}(k) + \alpha_{n-2}\Delta^2 s_{n-2}(k)\\ &\quad + \beta_{n-2}\Delta^2 s_{n-2}^{\lambda_{n-2}}(k) + \alpha_{n-1}\Delta s_{n-1}(k)\\ &\quad + \beta_{n-1}\Delta s_{n-1}^{\lambda_{n-1}}(k)\\ &= \cdots\\ &= \Delta^n s_1(k) + \sum_{i=1}^{n-1}[\alpha_i \Delta^{n-i} s_i(k) + \beta_i \Delta^{n-i} s_i^{\lambda_i}(k)]\\ &= f(x(k)) + g(x(k))u(k) + \sum_{i=1}^{n-1}[\alpha_i \Delta^{n-i} s_i(k)\\ &\quad + \beta_i \Delta^{n-i} s_i^{\lambda_i}(k)].\end{aligned}$$

Then, the expression of $s_n(k+1)$ is given by

$$\begin{aligned}s_n(k+1) &= s_n(k) + T\Delta s_n(k)\\ &= s_n(k) + T\{f(x(k)) + g(x(k))u(k)\\ &\quad + \sum_{i=1}^{n-1}[\alpha_i \Delta^{n-i} s_i(k) + \beta_i \Delta^{n-i} s_i^{\lambda_i}(k)]\}.\end{aligned}$$

Now, by letting $s_n(k+1) = 0$, the FTSM control law is designed to be

$$\begin{aligned}u(k) = &-(Tg(x(k)))^{-1}\{s_n(k) + Tf(x(k))\\ &+ T\sum_{i=1}^{n-1}[\alpha_i \Delta^{n-i} s_i(k) + \beta_i \Delta^{n-i} s_i^{\lambda_i}(k)]\}.\end{aligned} \tag{8}$$

Under the FTSM control law (8), the nth FTSM surface is reached in one sampling instant, i.e. $s_n(k+1) = 0$ for $k \geq 0$. In the following, the boundedness property of the FTSM variables $s_i(k)$, $i \in \{1, 2, ..., n-1\}$, and system states $x_i(k)$, $i \in \{1, 2, ..., n\}$, are studied.

Remark 3: If a smooth and bounded uncertainty term $d(k)$ appears in the system (3), then the control law $u(k)$ can also be designed by letting $s_n(k+1) = 0$ and estimating the uncertainty term $d(k)$; see [18] for more details. In this case, the surface $s_n(k) = 0$ is reached approximately.

3.2 Boundedness of the FTSM variables

The boundedness property of the FTSM variables $s_i(k)$, $i \in \{1, 2, ..., n-1\}$, are investigated and the bounds of the FTSM variables in steady state are provided.

Lemma 1: Consider the function

$$f(s) = (1-a)s - bs^{\lambda}, \tag{9}$$

where $a < 1$, $b > 0$, and $\lambda = q/p$, p and q are positive odd integers such that $1/2 < \lambda < 1$. Let $s_\circ = (b/(1-a))^{1/(1-\lambda)}$ and $\eta_\circ = (b\lambda/(1-a))^{1/(1-\lambda)}$, the function $f(s)$ has the following properties:

1. $f(s) < 0$ for $s \in (-\infty, -s_\circ) \cup (0, s_\circ)$,
2. $f(s) = 0$ for $s \in \{-s_\circ, 0, s_\circ\}$,
3. $f(s) > 0$ for $s \in (-s_\circ, 0) \cup (s_\circ, \infty)$,
4. $f(s)$ is monotone decreasing for $s \in [-\eta_\circ, \eta_\circ]$,
5. $f(s)$ is monotone increasing for $s \in (-\infty, -\eta_\circ] \cup [\eta_\circ, \infty)$.

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The proof of Lemma 1 is straightforward, hence omitted. In Lemma 1, $s_\circ = (b/(1-a))^{1/(1-\lambda)} = (b/(1-a))^{p/(p-q)}$. In view of the facts: (i) $b/(1-a) > 0$, (ii) p is odd, and (iii) $p-q$ is even, we conclude that the value of $s_\circ$ is real. Similarly, $\eta_\circ$ is real as well.

Based on the properties in Lemma 1, the following lemmas can be obtained.

Lemma 2: Consider the discrete-time dynamical system

$$s(k+1) = (1-a)s(k) - bs^\lambda(k), \tag{10}$$

where $0 \le a < 1$, $b > 0$, and $\lambda = q/p$, p and q are positive odd integers such that $1/2 < \lambda < 1$. Then there exists a positive integer K such that $|s(k)| \le (b\lambda/(1-a))^{1/(1-\lambda)}$ for $k \ge K$.

Proof: Let $\eta_\circ = (b\lambda/(1-a))^{1/(1-\lambda)}$. The proof consists of two steps. First, we prove that there exists a positive integer K such that $s(K) \in [-\eta_\circ, \eta_\circ]$. Second, we show that $s(k) \in [-\eta_\circ, \eta_\circ]$ for $k > K$.

Step 1: Three cases are considered: (1) $|s(0)| \le \eta_\circ$, (2) $s(0) > \eta_\circ$, and (3) $s(0) < -\eta_\circ$. For Case 1, we simply choose $K = 0$ so that $|s(K)| \le \eta_\circ$ holds. For Case 2, a Lyapunov function is introduced as $V(k) = s^2(k)$. It derives from (10) that

$$\begin{aligned} V(k+1) - V(k) &= s^2(k+1) - s^2(k) \\ &= [s(k+1) - s(k)][s(k+1) + s(k)] \\ &= -[as(k) + bs^\lambda(k)] \\ &\quad \times [(2-a)s(k) - bs^\lambda(k)]. \end{aligned} \tag{11}$$

We will prove that $V(k+1) - V(k) < -\sigma$ as long as $s(k) > \eta_\circ$ holds, where σ is a positive constant to be determined.

If $s(k) > \eta_\circ$ holds, then it follows from Property 5 in Lemma 1 that

$$\begin{aligned} (2-a)s(k) - bs^\lambda(k) &= s(k) + [(1-a)s(k) - bs^\lambda(k)] \\ &> \eta_\circ + [(1-a)\eta_\circ - b\eta_\circ^\lambda] \\ &= (2-a)\eta_\circ - b\eta_\circ^\lambda \\ &= \eta_\circ\left((2-a) - \frac{1-a}{\lambda}\right) \triangleq \rho. \end{aligned} \tag{12}$$

Also, $s(k) > \eta_\circ$ implies that

$$\begin{aligned} as(k) + bs^\lambda(k) &> a\eta_\circ + b\eta_\circ^\lambda \\ &= \eta_\circ(a + b\eta_\circ^{\lambda-1}) \\ &= \eta_\circ\left(a + \frac{1-a}{\lambda}\right) \\ &> \eta_\circ\left((2-a) - \frac{1-a}{\lambda}\right) = \rho. \end{aligned} \tag{13}$$

Let $\sigma \triangleq \rho^2$. Substituting (12) and (13) into (11) results in $V(k+1) - V(k) < -\sigma$. As a result, the values of $s(k)$ decreases as k increases, and there exists a positive integer K_1 such that $s(K_1) > \eta_\circ$ and $s(K_1+1) \le \eta_\circ$.

On the other hand, it derives from Property 5 in Lemma 1 that

$$\begin{aligned} s(K_1+1) &= (1-a)s(K_1) - bs^\lambda(K_1) \\ &> (1-a)\eta_\circ - b\eta_\circ^\lambda \\ &= \eta_\circ((1-a) - b\eta_\circ^{\lambda-1}) \\ &= -\eta_\circ\frac{(1-a)(1-\lambda)}{\lambda} > -\eta_\circ. \end{aligned} \tag{14}$$

Therefore, $s(K_1+1) \in [-\eta, \eta]$. We choose $K = K_1 + 1$ so that $|s(K)| \le \eta_\circ$ holds.

For Case 3, the existence of K such that $|s(K)| \le \eta_\circ$ can be proven similarly as for Case 2.

Step 2: Due to $s(K) \in [-\eta_\circ, \eta_\circ]$, it follows from Property 4 in Lemma 1 and similar derivations as in (14) that

$$\begin{aligned} s(K+1) &= (1-a)s(K) - bs^\lambda(K) \\ &\ge (1-a)\eta_\circ - b\eta_\circ^\lambda > -\eta_\circ. \end{aligned} \tag{15}$$

In view of Property 4 in Lemma 1 and (15), one has

$$\begin{aligned} s(K+1) &= (1-a)s(K) - bs^\lambda(K) \\ &\le (1-a)(-\eta_\circ) - b(-\eta_\circ)^\lambda \\ &= -[(1-a)\eta_\circ - b\eta_\circ^\lambda] < \eta_\circ. \end{aligned} \tag{16}$$

Combining (15) and (16) leads to $|s(k)| \le \eta_\circ$ for $k > K$. This completes the proof.□

Lemma 3: Consider the discrete-time dynamical system with disturbance

$$s(k+1) = (1-a)s(k) - bs^\lambda(k) + d(k), \tag{17}$$

where $0 \le a < 1$, $b > 0$, and $\lambda = q/p$, p and q are positive odd integers such that $1/2 < \lambda < 1$. Here, $d(k) \ne 0$ is the disturbance and satisfies $|d(k)| \le \gamma$, $\gamma > 0$. Let $\omega = 1 + (1-a)(\lambda^{\lambda/(1-\lambda)} - \lambda^{1/(1-\lambda)})$, $s_\circ = (b/(1-a))^{1/(1-\lambda)}$, and s_* be the solution to the equation $as_* + bs_*^\lambda = \gamma$. Then there exists a positive integer K such that $|s(k)| \le \omega \max\{s_\circ, s_*\}$ for $k \ge K$.

Proof: Let $\eta = \omega \max\{s_\circ, s_*\}$ and $\eta_\circ = (b\lambda/(1-a))^{1/(1-\lambda)}$. The proof consists of two steps and the idea is similar to that used in the proof of Lemma 2. We first prove the existence of a positive integer K such that $s(K) \in [-\eta, \eta]$, then show that $s(k) \in [-\eta, \eta]$ for $k > K$.

Step 1: Three cases are considered: (1) $|s(0)| \le \eta$, (2) $s(0) > \eta$, and (3) $s(0) < -\eta$. In Case 1, it is straightforward to find that $K = 0$. For Case 2, the Lyapunov function $V(k) = s^2(k)$ can be used to prove that $s(k)$ decreases as k increases. It follows from (17) that

$$\begin{aligned} V(k+1) - V(k) &= s^2(k+1) - s^2(k) \\ &= [s(k+1) - s(k)][s(k+1) + s(k)] \\ &= -[as(k) + bs^\lambda(k) - d(k)] \\ &\quad \times [(2-a)s(k) - bs^\lambda(k) + d(k)]. \end{aligned} \tag{18}$$

Next, we prove that $V(k+1) - V(k) < -\sigma$ as long as $s(k) > \eta$, where σ is a positive constant to be determined.

Due to $s(k) > \eta \ge \omega s_*$ and $\omega > 1$, we have

$$\begin{aligned} as(k) + bs^\lambda(k) - d(k) &\ge as(k) + bs^\lambda(k) - \gamma \\ &> a\omega s_* + b(\omega s_*)^\lambda - \gamma \\ &> \omega^\lambda(as_* + bs_*^\lambda) - \gamma \\ &= \omega^\lambda\gamma - \gamma \triangleq \rho. \end{aligned} \tag{19}$$

Since $s(k) > \eta > s_\circ$, it derives from Property 3 in Lemma 1 that $(1-a)s(k) - bs^\lambda(k) > 0$. Hence, one has (see (20)) Let $\sigma \triangleq \rho^2$. Substituting (19) and (20) into (18) results in $V(k+1) - V(k) < -\sigma$. Hence, there exists a positive integer K_1 such that $s(K_1) > \eta$ and $s(K_1+1) \le \eta$.

Now, we prove that $s(K_1+1) > -\eta$. If $s_* \ge s_\circ$, it follows from Property 3 in Lemma 1 that $(1-a)s_* - bs_*^\lambda \ge 0$. Hence, we have $s_* \ge as_* + bs_*^\lambda = \gamma$. If $s_\circ \ge s_*$, we have $s_\circ = as_\circ + bs_\circ^\lambda \ge as_* + bs_*^\lambda = \gamma$. In all, $\eta > \max\{s_\circ, s_*\} \ge \gamma$ holds. Then, in view of Property 5 in Lemma 1, one has that

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$$\begin{aligned} s(K_1+1) &= (1-a)s(K_1) - bs^{\lambda}(K_1) + d(K_1) \\ &\geq (1-a)s(K_1) - bs^{\lambda}(K_1) - \gamma \\ &> (1-a)s_{\circ} - bs_{\circ}^{\lambda} - \gamma \\ &= -\gamma > -\eta. \end{aligned}$$

Therefore, $s(K_1+1) \in [-\eta, \eta]$. Let $K = K_1 + 1$, we have $|s(K)| \leq \eta$.

For Case 3, the existence of K such that $|s(K)| \leq \eta$ can be proven similarly.

Step 2: When $s(K) \in [-\eta, \eta]$, there exist two cases: (1) $s(K) \in [0, \eta]$ and (2) $s(K) \in [-\eta, 0]$.

For Case 1, it follows from $\eta > s_{*}$ that $a\eta + b\eta^{\lambda} > \gamma$. In addition, because $\eta > s_{\circ}$, it derives from Properties 2, 4, and 5 in Lemma 1 that

$$\begin{aligned} s(K+1) &= (1-a)s(K) - bs^{\lambda}(K) + d(k) \\ &\leq (1-a)s(K) - bs^{\lambda}(K) + \gamma \\ &\leq (1-a)\eta - b\eta^{\lambda} + \gamma < \eta. \end{aligned} \tag{21}$$

It follows from Property 4 in Lemma 1 that:

$$\begin{aligned} s(K+1) &= (1-a)s(K) - bs^{\lambda}(K) + d(k) \\ &\geq (1-a)s(K) - bs^{\lambda}(K) - \gamma \\ &\geq (1-a)\eta_{\circ} - b\eta_{\circ}^{\lambda} - \gamma \\ &= (1-a)s_{\circ}[\lambda^{1/(1-\lambda)} - \lambda^{\lambda/(1-\lambda)}] - \gamma \\ &\geq (1-a)\max\{s_{\circ}, s_{*}\}[\lambda^{1/(1-\lambda)} - \lambda^{\lambda/(1-\lambda)}] - \gamma \\ &\geq (1-a)\max\{s_{\circ}, s_{*}\}[\lambda^{1/(1-\lambda)} - \lambda^{\lambda/(1-\lambda)}] \\ &\quad - \max\{s_{\circ}, s_{*}\} \\ &= -\max\{s_{\circ}, s_{*}\}\{1 + (1-a)[\lambda^{\lambda/(1-\lambda)} \\ &\quad - \lambda^{1/(1-\lambda)}]\} = -\eta. \end{aligned} \tag{22}$$

Here, the second last '$\geq$' inequality holds due to the fact that $\lambda^{1/(1-\lambda)} - \lambda^{\lambda/(1-\lambda)} < 0$, the last '$\geq$' inequality holds due to $\max\{s_{\circ}, s_{*}\} \geq \gamma$. Combining (21) and (22) leads to that $|s(K+1)| \leq \eta$. Then it is derived recursively that $|s(k)| \leq \eta$ for all $k > K$.

For Case 2, the proof is similar to that in Case 1. This finishes the proof.□

Remark 4: Lemma 2 cannot be considered as a corollary of Lemma 3 by letting $d(k) \equiv 0$. If one considers the system (17) with $d(k) \equiv 0$, then Lemma 3 also provides a bound on $s(k)$ in (10) and the bound is given by $\eta = \omega \max\{(b/(1-a))^{1/(1-\lambda)}, (-(b/a))^{1/(1-\lambda)}\}$. Due to $\omega > 1$ and $\lambda^{1/(1-\lambda)} < 1$, the bound η is strictly larger than the bound $\eta_{\circ} = (b\lambda/(1-a))^{1/(1-\lambda)}$ given by Lemma 2. Therefore, Lemma 2 provides a tighter bound than Lemma 3 when the disturbance vanishes.

Based on Lemmas 2 and 3, the boundedness of the FTSM variables is characterised in the following theorem.

Theorem 1: Consider the system (3) under the control law (8). Let

$$\eta_n = 0, \quad \eta_{n-1} = \left(\frac{\beta_{n-1}\lambda_{n-1}T}{1-\alpha_{n-1}T}\right)^{1/(1-\lambda_{n-1})}.$$

For $i \in \{n-2, n-3, \ldots, 1\}$, choose $\omega_i = 1 + (1-\alpha_i T)(\lambda_i^{\lambda_i/(1-\lambda_i)} - \lambda_i^{1/(1-\lambda_i)})$, $s_{\circ i} = (\beta_i T/(1-\alpha_i T))^{1/(1-\lambda_i)}$ and s_{*i} be the solution to the equation $\alpha_i s_{*i} + \beta_i s_{*i}^{\lambda_i} = \eta_{i+1}$, let $\eta_i = \omega_i \max\{s_{\circ i}, s_{*i}\}$. Then there exist positive integers K_i, $i \in \{n, n-1, \ldots, 1\}$, such that the FTSM variables in (4) satisfy $|s_i(k)| \leq \eta_i$ for all $k \geq K_i$.

Proof: First, Let $K_n = 1$. We have shown that $s_n(k) = 0$ for $k \geq K_n$ under the control law (8) in Section 3.1. Thus $|s_n(k)| \leq \eta_n$ for $k \geq K_n$.

Next, consider $s_{n-1}(k)$. It follows from $s_n(k) = 0$ and (4) that:

$$s_{n-1}(k+1) = (1-\alpha_{n-1}T)s_{n-1}(k) - \beta_{n-1}Ts_{n-1}^{\lambda_{n-1}}(k).$$

According to Lemma 2, there exists a positive integer K_{n-1} such that $|s_{n-1}(k)| \leq \eta_{n-1}$ for $k \geq K_{n-1}$.

For $s_{n-2}(k)$, it obtains from (4) that

$$\begin{aligned} s_{n-2}(k+1) &= (1-\alpha_{n-2}T)s_{n-2}(k) - \beta_{n-2}Ts_{n-2}^{\lambda_{n-2}}(k) \\ &\quad + Ts_{n-1}(k). \end{aligned}$$

Hence, it follows from Lemma 3 that there exists a positive integer $K_{n-2} \geq K_{n-1}$ such that $|s_{n-2}(k)| \leq \eta_{n-2}$ for $k \geq K_{n-2}$.

Recursively, there exist positive integers $K_i \geq K_{i+1}$ such that $|s_i(k)| \leq \eta_i$ for $k \geq K_i$, $i \in \{n-3, n-4, \ldots, 1\}$. □

Remark 5: By choosing the FTSM parameters α_i, β_i, and λ_i in (4), the bounds η_i of the FTSM variables s_i in Theorem 1 can be made arbitrarily small. For example, we first consider the value of η_{n-1} by noting that

$$\eta_{n-1} = \left(\frac{\beta_{n-1}\lambda_{n-1}T}{1-\alpha_{n-1}T}\right)^{1/(1-\lambda_{n-1})}.$$

For any given $\varepsilon > 0$, we choose $\beta_{n-1} = [\varepsilon^{1-\lambda_{n-1}}(1-\alpha_{n-1}T)]/(\lambda_{n-1}T) > 0$ so that $\eta_{n-1} = \varepsilon$. Therefore, η_{n-1} can be made arbitrarily small by choosing an appropriate value of β_{n-1}. Next, let us consider the value of η_{n-2}. On the one hand, the function $f(s) = \alpha_{n-2}s + \beta_{n-2}s^{\lambda_{n-2}}$ is monotonically increasing for $s \in (-\infty, \infty)$ and satisfies $f(0) = 0$. In view of $\alpha_{n-2}s_{*n-2} + \beta_{n-2}s_{*n-2}^{\lambda_{n-2}} = \eta_{n-1}$, we can conclude that s_{*n-2} approaches zero as η_{n-1} does so. Therefore, s_{*n-2} can be made arbitrarily small. On the other hand, for any given $\varepsilon > 0$, we may choose $\beta_{n-2} = [\varepsilon^{1-\lambda_{n-2}}(1-\alpha_{n-2}T)]/T > 0$ such that $s_{\circ n-2} = \varepsilon$. Therefore, $s_{\circ n-2}$ can be arbitrarily small as well. Due to $\eta_{n-2} = \omega_{n-2}\max\{s_{\circ n-2}, s_{*n-2}\}$, it is nature to conclude that η_{n-2} can be made arbitrarily small. Iteratively, for $i \in \{n-3, n-4, \ldots, 1\}$, the values of η_i can be made arbitrarily small by adjusting the values of β_i.

3.3 Boundedness of steady state

Based on Theorem 1, the boundedness property of the steady state of the system (3) is studied and the bounds are provided.

Theorem 2: Consider the system (3) under the FTSM control law (8). Let η_i and K_i be defined as in Theorem 1, $i \in \{1, 2, \ldots, n\}$.

$$\begin{aligned} (2-a)s(k) - bs^{\lambda}(k) + d(k) &\geq 2s(k) - as(k) - bs^{\lambda}(k) - \gamma \\ &> 2[as(k) + bs^{\lambda}(k)] - as(k) - bs^{\lambda}(k) - \gamma \\ &= as(k) + bs^{\lambda}(k) - \gamma > \rho. \end{aligned} \tag{20}$$

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Choose $\delta_{i1} = \eta_i$, $i \in \{1, 2, ..., n\}$, $\delta_{i2} = \delta_{(i+1)1} + \alpha_i \delta_{i1} + \beta_i \delta_{i1}^{\lambda_i}$, $i \in \{1, 2, ..., n-1\}$, and $\delta_{ij} = \delta_{(i+1)(j-1)} + \alpha_i \delta_{i(j-1)} + \beta_i (2/T)^{j-2-\lambda_i} \delta_{i2}^{\lambda_i}$, $j \in \{3, 4, ..., n\}$, $i \in \{1, 2, ..., n-j+1\}$. Let $K_{\max} \triangleq \max\{K_1, K_2, ..., K_n\}$. Then the system trajectory of (3) satisfies that $|x_j(k)| \le \delta_{1j}$ for all $k \ge K_{\max}$, $j \in \{1, 2, ..., n\}$.

The proof of Theorem 2 is a slight modification of the proof of Theorem 2 in [23], hence omitted here.

Remark 6: By choosing the FTSM parameters α_i, β_i, and λ_i in (4), the bounds δ_{1j} in Theorem 2 can be arbitrarily small for $j \in \{1, 2, ..., n\}$. Due to $\delta_{i1} = \eta_i$, $i \in \{1, 2, ..., n\}$, the values of η_i can be made arbitrarily small as shown in Remark 5. Next, by noting $\delta_{i2} = \delta_{(i+1)1} + \alpha_i \delta_{i1} + \beta_i \delta_{i1}^{\lambda_i}$, $i \in \{1, 2, ..., n-1\}$, the value of δ_{i2} can be arbitrarily small as long as the values of δ_{i1} and $\delta_{(i+1)1}$ are small enough. Similarly, the values of $\delta_{i3} = \delta_{(i+1)2} + \alpha_i \delta_{i2} + \beta_i (\frac{2}{T})^{j-2-\lambda_i} \delta_{i2}^{\lambda_i}$, $i \in \{1, 2, ..., n-2\}$, can be made arbitrarily small because the values of δ_{i2} and $\delta_{(i+1)2}$ could be small enough. By iteration, for $j \in \{4, 5, ..., n\}$, the values of δ_{ij} can be arbitrarily small for $i \in \{1, 2, ..., n-j+1\}$. In particular, the values of δ_{1j} can be chosen arbitrarily small for $i \in \{1, 2, ..., n\}$.

4 Illustrative example

The developed results are illustrated by a numerical example and some simulations in this section. First, the boundedness of the FTSM variables and the system steady states is demonstrated. Second, it is shown that the bounds of the FTSM variables and the steady state of the system could be reduced by adjusting the FTSM parameters. Third, the effect of FTSM control is compared with the effect of TSM control.

Consider a single link manipulator system [25] of the form

$$\dot{z} = a(z) + b(z)u, \tag{23}$$

where

$$a(z) = \begin{bmatrix} z_2 \\ -l\sin(z_1) - m(z_1 - z_3) \\ z_4 \\ n(z_1 - z_3) \end{bmatrix}, \quad b(z) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ r \end{bmatrix}$$

with the parameters $l = 5.6$, $m = 1.5$, $n = 0.8$, and $r = 3.2$. The system (23) can be transformed into the following canonical form:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= f(x) + g(x)u, \end{aligned} \tag{24}$$

where $f(x) = -(l\cos x_1 + m + n)x_3 + l(x_2^2 - n)\sin x_1$ and $g(x) = mr$. The system (24) can be discredited to

$$\begin{aligned} \Delta x_1(k) &= x_2(k), \\ \Delta x_2(k) &= x_3(k), \\ \Delta x_3(k) &= x_4(k), \\ \Delta x_4(k) &= f(x(k)) + g(x(k))u(k). \end{aligned} \tag{25}$$

Suppose the initial state $[x_1, x_2, x_3, x_4]^{\mathrm{T}} = [2, 2, 2, 2]^{\mathrm{T}}$ and the sampling period $T = 0.1$ s. The simulation results are given as follows.

4.1 Boundedness of the FTSM variables and system states

Boundedness of the FTSM variables and the system states are illustrated by the numerical example. Theoretical bounds of the FTSM variables in Theorem 1 and the system states in Theorem 2 are given.

Choose the FTSM parameters in (4) as $\alpha_1 = \alpha_2 = \alpha_3 = 0.2$, $\beta_1 = \beta_2 = \beta_3 = 1$, $\lambda_1 = 9/11$, $\lambda_2 = 7/9$, and $\lambda_3 = 5/9$. The recursive FTSM variables (4) are given by

$$\begin{aligned} s_1(k) &= x_1(k), \\ s_2(k) &= \Delta s_1(k) + 0.2 s_1(k) + s_1^{9/11}(k), \\ s_3(k) &= \Delta s_2(k) + 0.2 s_2(k) + s_2^{7/9}(k), \\ s_4(k) &= \Delta s_3(k) + 0.2 s_3(k) + s_3^{5/9}(k). \end{aligned}$$

The FTSM control law (8) is given by

$$\begin{aligned} u(k) = &-0.021\{s_4(k) + 0.1[(-5.6\cos(x_1(k)) + 2.3)x_3(k) \\ &+5.6(x_2^2(k) - 0.8)\sin(x_1(k))] \\ &+0.1[0.8\Delta^3 s_1^{9/11}(k) + \Delta^3 s_1(k) \\ &+0.8\Delta^2 s_2^{7/9}(k) + \Delta^2 s_2(k) \\ &+0.8\Delta s_3^{5/9}(k) + \Delta s_3]\}. \end{aligned}$$

Fig. 1 gives the response trajectories and the bounds of the FTSM variables s_i, $i \in \{1, 2, 3, 4\}$. First, the surface $s_4 = 0$ is reached after one sampling period. Second, s_3 enters in the band $[-1.6 \times 10^{-3}, 1.6 \times 10^{-3}]$ at the 45th sampling period. Third, s_2 enters in the band $[-1.8 \times 10^{-4}, 1.8 \times 10^{-4}]$ at the 71th sampling period. Finally, s_1 enters in the band $[-1.3 \times 10^{-5}, 1.3 \times 10^{-5}]$ at the 92th sampling period.

Fig. 2 shows the response trajectories and the bounds of the system states x_i, $i \in \{1, 2, 3, 4\}$. First, x_4 enters in the band $[-0.09, 0.09]$ at the 82th sampling period. Second, x_3 enters in the band $[-4.9 \times 10^{-3}, 4.9 \times 10^{-3}]$ at the 90th sampling period. Third, x_2 enters in the band $[-2.6 \times 10^{-4}, 2.6 \times 10^{-4}]$ at the 90th sampling period. Finally, x_1 enters in the band $[-1.3 \times 10^{-5}, 1.3 \times 10^{-5}]$ at the 92th sampling period.

This illustrates the correctness of the results in Theorems 1 and 2.

4.2 Reducing the bounds of the FTSM variables and the system states

Bounds of the FTSM variables and the system states can be reduced by adjusting the FTSM parameters in (4). For example, these parameters can be adjusted to $\alpha_1 = \alpha_2 = \alpha_3 = 0.1$, $\beta_1 = \beta_2 = \beta_3 = 0.8$, $\lambda_1 = \lambda_2 = 9/11$, and $\lambda_3 = 3/5$.

Fig. 3 shows the response trajectories of the FTSM variables s_i and their the bounds $\pm\eta_i$, $i \in \{1, 2, 3, 4\}$. First, the surface $s_4 = 0$ is reached in one sampling period, which implies that $\eta_4 = 0$. The FTSM variables s_3, s_2, and s_1 are bounded by $[-5.2 \times 10^{-4}, 5.2 \times 10^{-4}]$, $[-7.5 \times 10^{-5}, 7.5 \times 10^{-5}]$, and $[-3.7 \times 10^{-6}, 3.7 \times 10^{-6}]$, respectively. By comparing Fig. 3 with Fig. 1, it can be found that the bounds of the FTSM variables are reduced.

Fig. 4 shows the response trajectories of the system states x_i and their bounds $\pm\delta_{1i}$, $i \in \{1, 2, 3, 4\}$. The states x_4, x_3, x_2, and x_1 are bounded within $[-0.028, 0.028]$, $[-1.6 \times 10^{-3}, 1.6 \times 10^{-3}]$, $[-1 \times 10^{-4}, 1 \times 10^{-4}]$, and $[-3.7 \times 10^{-6}, 3.7 \times 10^{-6}]$, respectively. Comparing Fig. 4 with Fig. 2 shows that the bounds of the system states are reduced by adjusting the FTSM parameters.

4.3 Comparison between FTSM control and TSM control

For comparison purpose, FTSM control and TSM control are applied to the system (25), respectively. The TSM variables and the

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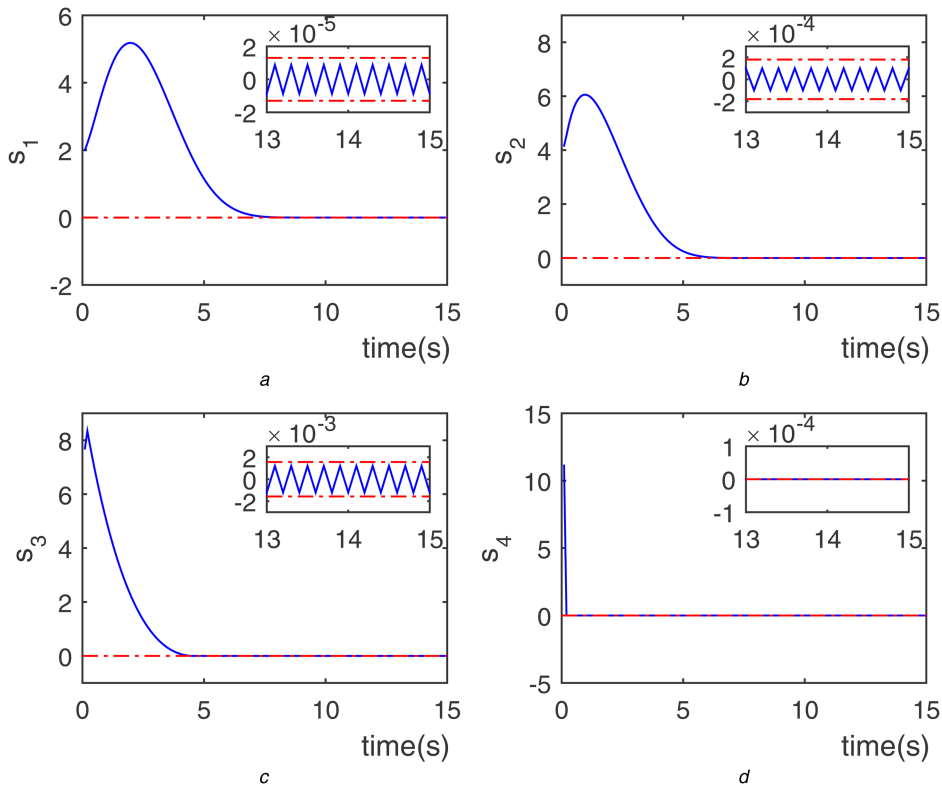

Fig. 1 *Response trajectories of FTSM variables*
(a) Response trajectory of s_1, ***(b)*** Response trajectory of s_2, ***(c)*** Response trajectory of s_3, ***(d)*** Response trajectory of s_4

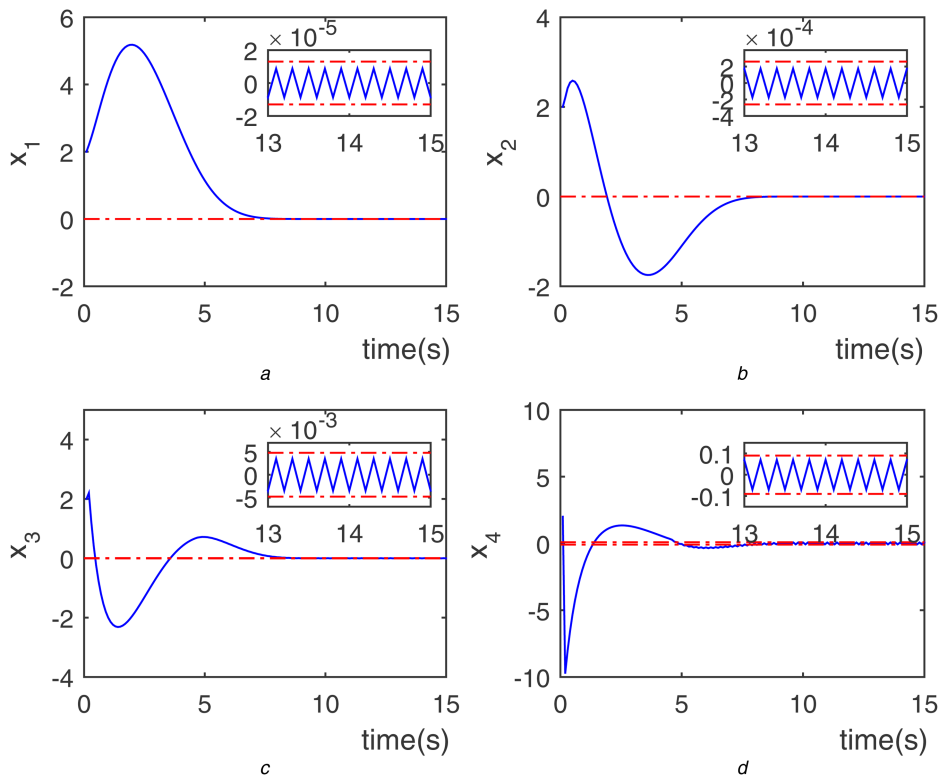

Fig. 2 *Response trajectories of system states under FTSM control*
(a) Response trajectory of x_1, ***(b)*** Response trajectory of x_2, ***(c)*** Response trajectory of x_3, ***(d)*** Response trajectory of x_4

TSM control law are designed by using the theory in [22]. The FTSM parameters are chosen as $\alpha_1 = \alpha_2 = \alpha_3 = 0.3$, $\beta_1 = \beta_2 = \beta_3 = 1$, $\lambda_1 = 9/11$, $\lambda_2 = 7/9$, and $\lambda_3 = 5/9$. The FTSM control and TSM control are compared from two perspectives.

Case 1: The TSM parameters are chosen as $\beta_1 = \beta_2 = \beta_3 = 1.03$, $\lambda_1 = 9/11$, $\lambda_2 = 7/9$, and $\lambda_3 = 5/9$. Fig. 5 gives the response trajectories of the FTSM variables and the TSM variables. Fig. 6 gives the response trajectories of the system states under FTSM control and under TSM control. It can be observed that FTSM control method possesses the advantage of faster transient response under similar bounds of the sliding mode variables and the system steady states.

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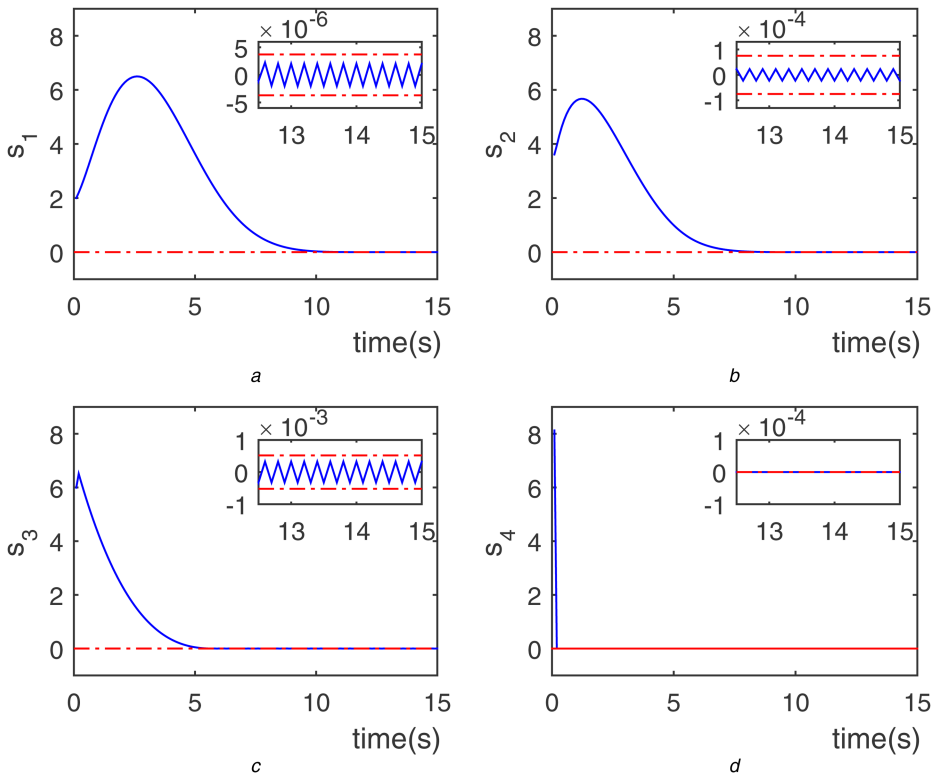

Fig. 3 *Response trajectories of FTSM variables*
(a) Response trajectory of s_1, ***(b)*** Response trajectory of s_2, ***(c)*** Response trajectory of s_3, ***(d)*** Response trajectory of s_4

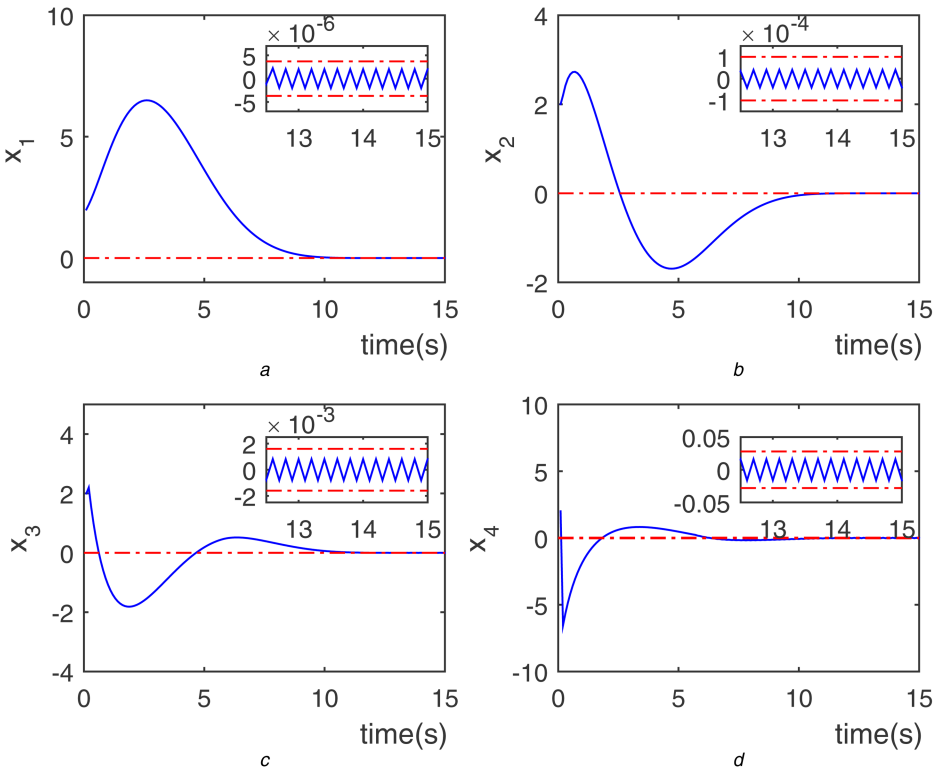

Fig. 4 *Response trajectories of system states under FTSM control*
(a) Response trajectory of x_1, ***(b)*** Response trajectory of x_2, ***(c)*** Response trajectory of x_3, ***(d)*** Response trajectory of x_4

Case 2: the TSM parameters are chosen as $\beta_1 = \beta_2 = \beta_3 = 1.5$, $\lambda_1 = 9/11$, $\lambda_2 = 7/9$, and $\lambda_3 = 5/9$. Fig. 7 gives the response trajectories of the FTSM variables and the TSM variables. Fig. 8 gives the response trajectories of the system states under FTSM control and the system states under TSM control. It can be observed that FTSM control method possesses smaller bounds of the sliding mode variables and the system steady states under similar transient responses.

5 Conclusions

In this paper, the boundedness property of the FTSM variables and the system state has been studied for discretised SISO non-linear systems. First, the FTSM variables and surfaces were defined and the control law was designed by enforcing the last FTSM surface to be reached after one sampling period. Second, the boundedness of the FTSM variables and the steady-state were established and the corresponding bounds were provided. Finally, a numerical example

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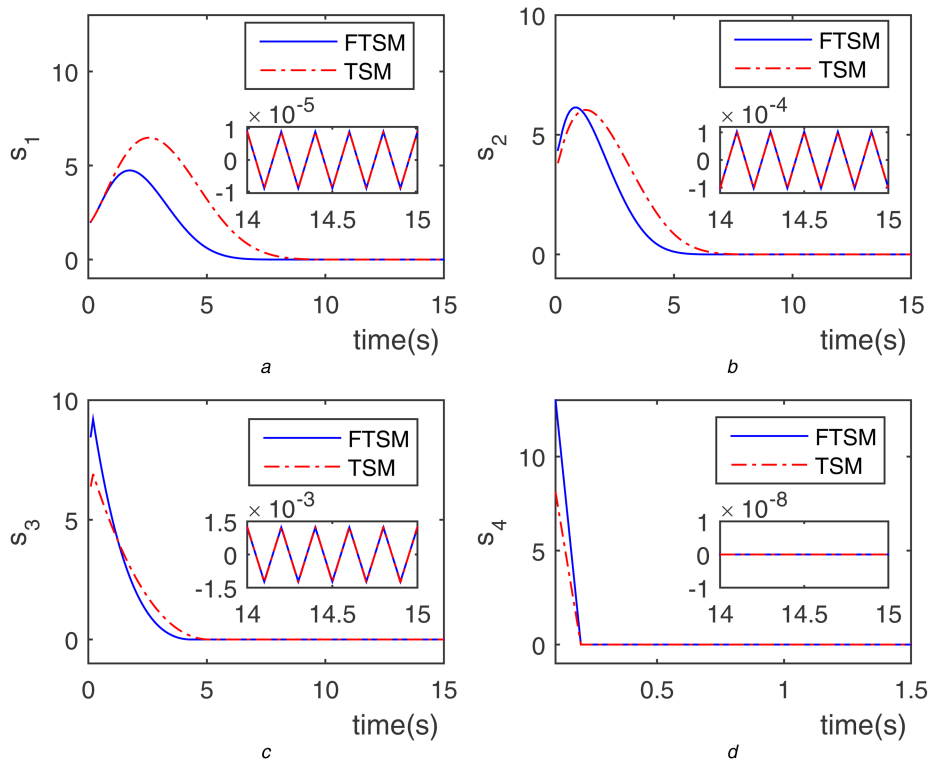

Fig. 5 *Response trajectories of FTSM variables and TSM variables*
(a) Response trajectories of s_1, ***(b)*** Response trajectories of s_2, ***(c)*** Response trajectories of s_3, ***(d)*** Response trajectories of s_4

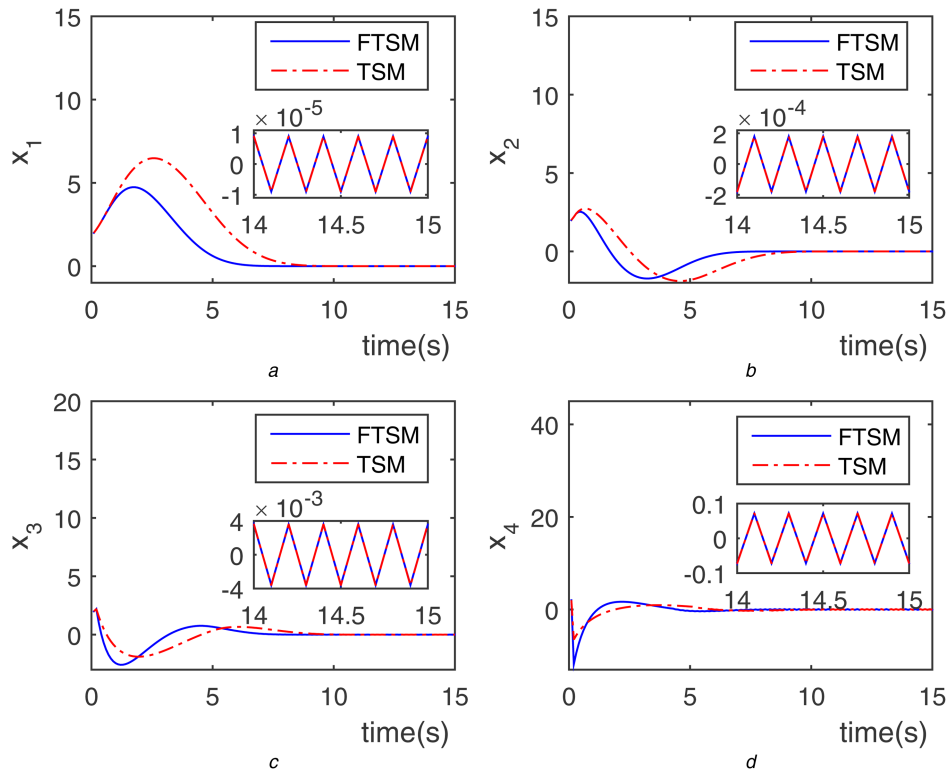

Fig. 6 *Response trajectories of system states under FTSM control and TSM control*
(a) Response trajectories of x_1, ***(b)*** Response trajectories of x_2, ***(c)*** Response trajectories of x_3, ***(d)*** Response trajectories of x_4

was used to illustrate the main results of this paper. Further work will be focused on the study of periodic behaviour of the system in steady state.

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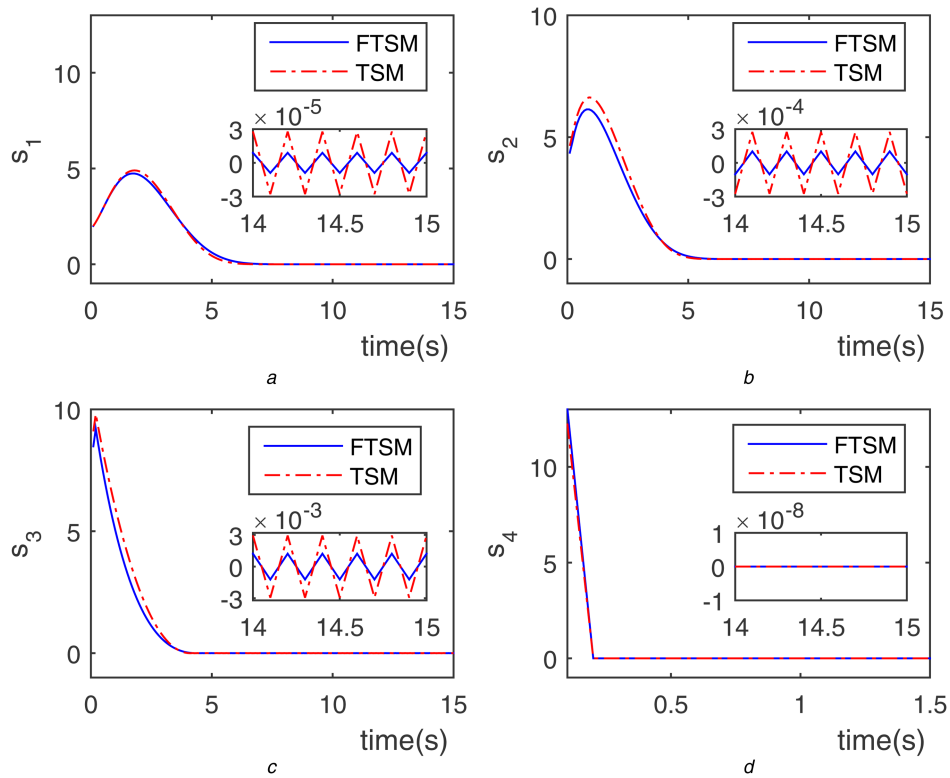

Fig. 7 *Response trajectories of FTSM variables and TSM variables*
(a) Response trajectories of s_1, ***(b)*** Response trajectories of s_2, ***(c)*** Response trajectories of s_3, ***(d)*** Response trajectories of s_4

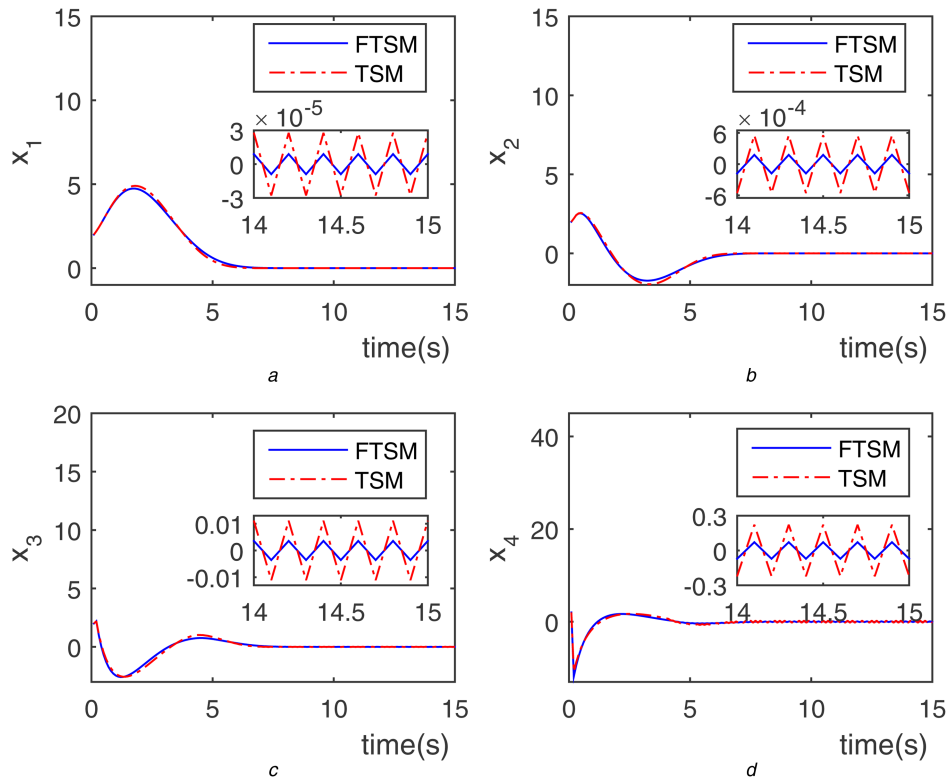

Fig. 8 *Response trajectories of system states under FTSM control and TSM control*
(a) Response trajectories of x_1, ***(b)*** Response trajectories of x_2, ***(c)*** Response trajectories of x_3, ***(d)*** Response trajectories of x_4

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