Boundedness of discretised non-linear systems under fast terminal sliding mode control

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Abstract: This study analyses the steady-state boundedness property of discretised non-linear systems under fast terminal sliding mode (FTSM) control. First, the recursive FTSM variables and surfaces are introduced. Then, the FTSM control law is designed by enforcing the system trajectory to reach the last FTSM surface after one sample period. The boundedness of the FTSM variables and the system steady states are established and the corresponding bounds are provided. Finally, the theoretical results are illustrated by a numerical example and a comparison with terminal sliding mode control is presented.

1 Introduction

Sliding mode control has been widely studied and applied in linear and non-linear systems due to the advantages such as strong robustness, order reduction, invariance; see [1–6]. To implement a sliding mode control, both a sliding mode surface and a control law need to be designed so that the closed-loop system is asymptotically stable. Conventional sliding mode control with linear sliding mode surfaces can guarantee that the system state reaches the equilibrium asymptotically [4]. To reach the equilibrium within finite time, terminal sliding mode (TSM) control method was proposed in [7]. However, the TSM control provides bad convergence performance when the system state is far away from the equilibrium point [8]. To obtain faster transient response, fast TSM (FTSM) control method was proposed in [8]. The method of FTSM control exhibits the advantages of finite-time convergence of TSM control and exponential convergence rate of linear sliding mode control [8]. The FTSM control method has been studied extensively in continuous-time domain; see, e.g. [9–11].

In discrete-time domain, the authors of [12, 13] have found and demonstrated that the system trajectory can only be driven into a neighbourhood of the sliding mode surface. The findings in [12, 13] motivate the study of sliding mode control for discrete-time systems [14–19]. Recently, TSM has been studied in [20–23]. In [20], the continuous-time TSM was discretised and the system steady-states along the TSM surface were shown to have period-2 orbits. The stability of the periodic orbits was established [20]. In [21], the dynamical behaviour was studied for discretised second-order linear systems along the TSM surface. In particular, the periodic orbits were classified into four classes according to the size of the sampling period. In [22], the trajectory of the system was proved to have period-2 motion only in steady state. In [23], the boundedness of the TSM variables and the system steady states were established and the bounds are provided as well. On the other hand, FTSM control has been illustrated through simulations as an effective method to improve the transient response speed of the system in [24]. However, theoretical analysis was not conducted in [24]. This motivates the current study.

This paper considers a class of discretised single-input single-output (SISO) non-linear systems and studies the boundedness property of both the FTSM variables and the system steady states under the FTSM control. First, the FTSM variables are defined recursively. The FTSM control law is designed by letting the last FTSM variable zero. With the designed control law, the system trajectory can be driven exactly on to the last FTSM surface after one sampling period; that is, the last FTSM variable equals zero.

Second, the other FTSM variables are shown to be bounded in the ‘steady state’ of the system. The explicit expressions of the bounds are found as well. Moreover, boundedness of the steady state of the system is established by a recursive analysis and the explicit expressions of the bounds are provided. Third, by choosing appropriate FTSM parameters, the bounds of both the FTSM variables and the system steady state can be made arbitrarily small. Finally, a single link manipulator example is used to illustrate the effectiveness of the developed theoretical results. When compared with the TSM control method, the advantages of the FTSM control method proposed in this paper are twofold: (i) faster transient response can be achieved when the sliding mode parameters are chosen such that the bounds of the sliding mode variables are approximately equal; and (ii) smaller bounds of the sliding mode variables can be achieved when the sliding mode parameters are chosen such that the transient responses are similar.

2 Problem formulation

Consider a class of n-order SISO continuous-time systems

\[ \begin{align*}
  \dot{x}_1 &= x_1 + u, \\
  \dot{x}_n &= f(x) + g(x)u,
\end{align*} \]

where \( x \in \mathbb{R}^n \) is the system state, \( y \in \mathbb{R} \) is the system output and \( u \in \mathbb{R} \) is the control input. \( a(x) \in \mathbb{R}^n \) and \( b(x) \in \mathbb{R}^n \) are smooth vector functions and \( c(x) \in \mathbb{R} \) is a smooth scalar function.

The non-linear system (1) is assumed to have relative degree \( n \) for \( x \in \Omega \subset \mathbb{R}^n \) [25]. Under this assumption, it follows from Theorem 13.1 of [25] that there exists a mapping \( \Phi(x) = x, z \in \Omega \), such that the system (1) can be transformed into the following canonical form:

\[ \begin{align*}
  \dot{x}_1 &= x_1 + v, & i \in \{1, 2, ..., n - 1\}, \\
  \dot{x}_n &= f(x) + g(x)u,
\end{align*} \]

where \( x_i = L_{\omega}^{-1}c(x), f(x) = L_{\omega}c(x) \) and \( g(x) = L_{\omega}L_{\omega}^{-1}c(x), g(x) \neq 0 \) for \( x = \Phi(x), z \in \Omega \). Here, \( L_{\omega}c(x) \) is the \( i \)th order Lie derivative of \( c \) with respect to \( a, i \in \{0, 1, ..., n - 1\} \); see [25].

Let \( T \) be the sampling period and \( \Delta x(k) = (x(k + 1) - x(k))/T, i \in \{1, 2, ..., n\}. \) Then the system (2) is discretised to
\[ \Delta x_i(k) = x_i(k), \quad i \in \{1, 2, \ldots, n-1\}, \]
\[ \Delta x_n(k) = f(x(k)) + g(x(k))u(k). \]  

(3)

Define the recursive discrete-time FTSM variables as

\[ s_i(k) = s_i(k), \]
\[ s_i(k) = \Delta s_i(k) + \alpha_i s_{i-1}(k) + \beta_i s_{i-1}^h(k), \quad i \in \{2, 3, \ldots, n\}, \]  

(4)

where \( \Delta s_i(k) = (s_i(k+1) - s_i(k))/T, \) \( i \in \{1, 2, \ldots, n\}. \) The parameters \( \alpha_i, \beta_i \) and \( \lambda_i \) are known and satisfy the following conditions: 0 < \( \alpha_i < 1/T, \) \( \beta_i > 0, \) \( \lambda_i = q_i/p_i, \) \( p_i \) and \( q_i \) are odd positive integers such that \( 1/2 < \lambda_i < 1, \) \( i \in \{1, 2, \ldots, n-1\}. \) For \( i \in \{1, 2, \ldots, n\}, \) \( s_i^h \) is the real solution to the equation \( s_i^h = s_i \).

Since \( p_i \) and \( q_i \) are odd positive integers, the equation \( (s_i^h)^2 = (s_i)^h \) holds for all \( i \in \{1, 2, \ldots, n\}. \) The 1th FTSM surface is defined to be \( s_1^h = 0 \) for \( i \in \{1, 2, \ldots, n\}. \)

Remark 1: The system (2) is general and includes a large class of non-linear physical models in practice, such as inverted pendulum systems [26] and single link manipulator systems [25].

Remark 2: Consider the multiple-input multiple-output (MIMO) version of the system (1). Under some assumptions on the control input parameter matrix and the relative degrees, Lemma 5.2.1 in [27] can be used to transform system (1) to system (2). Each individual system in the set can be discretised and the recursive FTSM variables can be defined. As a result, the results to be developed in this paper should be extended to the MIMO case.

The objective of this paper is to study the boundedness property of the system (3) along the sliding mode surfaces.

3 Boundedness property of systems

In this section, an FTSM control law is designed by setting the last FTSM variable to zero. Then the boundedness of both the FTSM variables and the steady state of the system is established, and the expressions of the bounds are obtained.

3.1 FTSM control law

Our FTSM control law is designed by letting \( s_n(k+1) = 0. \) First, we need figure out the expression of \( \Delta s_n(k). \)

Define \( \Delta^i \Delta^{i-1} \Delta (\Delta) \) for \( i \in \{2, 3, \ldots, n\}, \) it follows from (3) and (4) that

\[ \Delta^i s_i(k) = \Delta^i x_i(k) = \Delta^{i-1} (\Delta x_i(k)) = \Delta^{i-1} x_i^h(k) = \cdots = \Delta s_i(k) = f(x(k)) + g(x(k))u(k). \]  

(5)

For the FTSM variable \( s_i(k), \) we obtain that

\[ \Delta s_i(k) = (s_i(k+1) - s_i(k))/T \]
\[ = [\Delta s_i(k) + \alpha_i s_i(k+1) + \beta_i s_i^h(k), \]
\[ = \frac{\Delta s_i(k) + \alpha_i s_i(k) + \beta_i s_i^h(k)}{\alpha s_i(k) + \beta_i s_i^h(k)}, \]
\[ = \Delta^2 s_i(k) + \alpha \Delta s_i(k) + \beta \Delta s_i^h(k), \]
\[ = \Delta^3 s_i(k) + \alpha \Delta^2 s_i(k) + \beta \Delta^2 s_i^h(k), \]
\[ = \Delta^4 s_i(k) + \alpha \Delta^3 s_i(k) + \beta \Delta^3 s_i^h(k), \]
\[ = \Delta^{n-i} s_i(k) = (\Delta^{n-i} x_i(k+1) - \Delta^{n-i} x_i^h(k))/T \]
\[ = \Delta^{n-i} s_i(k) + \alpha \Delta^{n-i} s_i(k) + \beta \Delta^{n-i} s_i^h(k). \]  

(6)

For \( s_i(k), i \in \{3, 4, \ldots, n\}, \) we have

\[ \Delta^{n-i} s_i(k) = \Delta^{n-i} x_i(k) + \alpha_i \Delta^{n-i} x_{i-1}(k) + \beta_i \Delta^{n-i} x_{i-1}^h(k). \]  

(7)

From (5)–(7), we obtain that

\[ \Delta s_n(k) = \Delta^2 s_n(k) + \alpha \Delta s_n(k) + \beta \Delta s_n^h(k) \]
\[ = \Delta^{n-i} s_n(k) + \alpha \Delta^{n-i} s_{n-1}(k) + \beta \Delta^{n-i} s_{n-1}^h(k) \]
\[ + \beta \Delta^{n-i} s_{n-1}^h(k). \]  

Now, by letting \( s_n(k+1) = 0, \) the FTSM control law is designed to be

\[ u(k) = -(T g(x(k)))^{-1} [s_n(k) + T f(x(k)) \]
\[ + T \sum_{i=1}^{n} \beta (\Delta^{n-i} s_i(k) + \beta \Delta^{n-i} s_i^h(k))]. \]  

(8)

Under the FTSM control law (8), the nth FTSM surface is reached in one sampling instant, i.e. \( s_n(k+1) = 0 \) for \( k \geq 0. \) In the following, the boundedness property of the FTSM variables \( s_i(k), i \in \{1, 2, \ldots, n-1\}, \) and system states \( x_i(k), i \in \{1, 2, \ldots, n\}, \) are studied.

Remark 3: If a smooth and bounded uncertainty term \( d(k) \) appears in the system (3), then the control law \( u(k) \) can also be designed by letting \( s_n(k+1) = 0 \) and estimating the uncertainty term \( d(k); \) see [18] for more details. In this case, the surface \( s_n(k) = 0 \) is reached approximately.

3.2 Boundedness of the FTSM variables

The boundedness property of the FTSM variables \( s_i(k), i \in \{1, 2, \ldots, n-1\}, \) are investigated and the bounds of the FTSM variables in steady state are provided.

Lemma 1: Consider the function

\[ f(s) = (1 - a)s - bs^4, \]  

(9)

where \( a < 1, \) \( b > 0, \) \( \lambda = q/p, \) \( p \) and \( q \) are positive odd integers such that \( 1/2 < \lambda < 1. \) Let \( s_0 = (b/(1-a))^{1/(-1-[\lambda])} \) and \( s_\infty = (b/(1-a))^{1/[\lambda]}, \) the function \( f(s) \) has the following properties:

1. \( f(s) < 0 \) for \( s \in (-\infty, -s_\infty) \cup (0, s_0), \)
2. \( f(s) = 0 \) for \( s \in (-s_\infty, 0, s_0), \)
3. \( f(s) > 0 \) for \( s \in (-s_\infty, 0) \cup (s_0, \infty), \)
4. \( f(s) \) is monotone decreasing for \( s \in [-s_\infty, -\infty], \)
5. \( f(s) \) is monotone increasing for \( s \in (-\infty, -s_\infty] \cup [s_0, \infty), \)

The boundedness property of the FTSM variables \( s_i(k), i \in \{1, 2, \ldots, n-1\}, \) are investigated and the bounds of the FTSM variables in steady state are provided.
The proof of Lemma 1 is straightforward, hence omitted. In Lemma 1, \( s_i = (b/(1-a))^{1/(1-\lambda)} = (b/(1-a)^{\phi(p-q)} \). In view of the facts: (i) \( b/(1-a) > 0 \), (ii) \( p \) is odd, and (iii) \( p - q \) is even, we conclude that the value of \( s_i \) is real. Similarly, \( \gamma \) is real as well.

Based on the properties in Lemma 1, the following lemmas can be obtained.

**Lemma 2:** Consider the discrete-time dynamical system

\[
s(k+1) = (1-a) s(k) - b s^\lambda(k),
\]

where \( 0 \leq a < 1, \lambda > 0, \) and \( \lambda = q/p, p \) and \( q \) are positive odd integers such that \( 1/2 < \lambda < 1 \). Then there exists a positive integer \( K \) such that \( |s(k)| \leq (bl/(1-a))^{1/(1-\lambda)} \) for \( k \geq K \).

**Proof:** Let \( \eta = (bl/(1-a))^{1/(1-\lambda)} \). The proof consists of two steps. First, we prove that there exists a positive integer \( K \) such that \( s(k) \in [- \eta, \eta] \). Second, we show that \( s(k) \in [- \eta, \eta] \) for \( k > K \).

**Step 1:** Three cases are considered: (1) \( s(0) \leq \eta, \) (2) \( s(0) > \eta, \) and (3) \( s(0) < - \eta \). For Case 1, as \( s(k) \) is an increasing function, and \( s(k) > \eta \), then it follows from Property 5 in Lemma 1 that

\[
\lambda \geq \eta,
\]

so that \( |s(k)| \leq \eta \). Then, in view of Property 4 in Lemma 1, one has

\[
|s(k)| \leq \eta.
\]

Combining (15) and (16) leads to \( |s(k)| \leq \eta \), for \( k > K \). This completes the proof.
\[ s(K + 1) = (1 - a)s(k) - bs^2(k) + d(k) \geq (1 - a)s(k) - bs^2(k) - \gamma > (1 - a)s - bs^2 - \gamma = - \gamma > - \eta. \]

Therefore, \( s(K + 1) \in [-\eta, \eta] \). Let \( K = K_t + 1 \), we have \( |s(K)| \leq \eta \).

For Case 3, the existence of \( K \) such that \( |s(K)| \leq \eta \) can be proven similarly.

**Step 2:** When \( s(K) \in [-\eta, \eta] \), there exist two cases: (1) \( s(K) \in [0, \eta] \) and (2) \( s(K) \in [-\eta, 0] \).

For Case 1, it follows from \( \eta > s \), that \( a\eta + b\eta^3 > \gamma \). In addition, because \( \eta > s \), it derives from Properties 2, 4, and 5 in Lemma 1 that

\[ s(K + 1) = (1 - a)s(k) - bs^2(k) + d(k) \leq (1 - a)s(k) - bs^2(k) + \gamma \leq (1 - a)\eta - b\eta^3 + \gamma < \eta. \]

It follows from Property 4 in Lemma 1 that:

\[ s(K + 1) = (1 - a)s(k) - bs^2(k) + d(k) \geq (1 - a)s(k) - bs^2(k) - \gamma = (1 - a)s(k) - bs^2(k) - \gamma \geq (1 - a)\max{s(s,s)}(\lambda^{1/2 - 1} - \lambda^{1/2 - 1}) - \gamma \geq (1 - a)\lambda^{1/2 - 1} - \lambda^{1/2 - 1} - \gamma \geq (1 - a)\lambda^{1/2 - 1} - \lambda^{1/2 - 1} - \gamma \geq \max{s(s,s)}(\lambda^{1/2 - 1} - \lambda^{1/2 - 1}) - \gamma \geq \lambda^{1/2 - 1} - \lambda^{1/2 - 1} = - \lambda^{1/2 - 1} = - \eta. \]

Here, the second last \( \lambda \geq 0 \) inequality holds due to the fact that \( \lambda^{1/2 - 1} - \lambda^{1/2 - 1} \geq 0 \), the last \( \lambda \geq 0 \) inequality holds due to \( \lambda \geq 0 \) for all \( k > K \). Combining (21) and (22) leads to that \( |s(k + 1)| \leq \eta \). Then it is derived recursively that \( |s(k)| \leq \eta \) for all \( k > K \).

For Case 2, the proof is similar to that in Case 1. This finishes the proof.

**Remark 4:** Lemma 2 cannot be considered as a corollary of Lemma 3 by letting \( d(k) \equiv 0 \). If one considers the system (17) with \( d(k) \equiv 0 \), then Lemma 3 also provides a bound on \( s(k) \) in (10), and the bound is given by \( \eta = \omega \max{s(s,s)}(b/(1 - a))^{1/2 - 1} - (b/a)^{1/2 - 1} \). Due to \( \omega > 1 \) and \( \lambda^{1/2 - 1} < 1 \), the bound \( \eta \) is strictly larger than the bound \( \eta = (\omega \lambda^{1/2 - 1} - 1) \) given by Lemma 2. Therefore, Lemma 2 provides a tighter bound than Lemma 3 when the disturbance vanishes.

Based on Lemmas 2 and 3, the boundedness of the FTSM variables is characterised in the following theorem.

**Theorem 1:** Consider the system (3) under the control law (8). Let

\[ \eta_n = 0, \quad \eta_{n-1} = \left( \frac{\beta_{n-1}}{1 - \alpha_{n-1}T} \right)^{1/2\lambda_{n-1}}. \]

For any given \( \varepsilon > 0 \), we choose \( \beta_{n-1} = \left( \frac{\varepsilon^{1/\lambda_{n-1}}}{\lambda_{n-1}} \right)^{1/\lambda_{n-1}} \). If \( \alpha_{n-1} > 1 \), then \( \eta_{n-1} \) is monotonically increasing for \( s \in (0, \infty) \) and \( \varepsilon < 0 \). Therefore, \( \eta_{n-1} \) can be made arbitrarily small by choosing an appropriate value of \( \beta_{n-1} \). Next, let us consider the value of \( \eta_{n-1} \).

For any given \( \varepsilon > 0 \), we choose \( \beta_{n-1} = \varepsilon^{1/\lambda_{n-1}}(1 - \alpha_{n-1}T) \). If \( \alpha_{n-1} = 1 \), then \( \eta_{n-1} = \varepsilon \). Therefore, \( \eta_{n-1} \) can be made arbitrarily small by choosing an appropriate value of \( \beta_{n-1} \). Next, let us consider the value of \( \eta_{n-1} \).

On the other hand, the function \( f(s) = \alpha_{n-2}s^2 + \beta_{n-2}s^{1/2} + \beta_{n-2}s^{1/2} \) is monotonically increasing for \( s \in (0, \infty) \) and \( \varepsilon < 0 \). Therefore, \( \eta_{n-1} \) can be made arbitrarily small by choosing an appropriate value of \( \beta_{n-1} \). Next, let us consider the value of \( \eta_{n-1} \).

For any given \( \varepsilon > 0 \), we choose \( \beta_{n-1} = \varepsilon^{1/\lambda_{n-1}}(1 - \alpha_{n-1}T) \). If \( \alpha_{n-1} = 1 \), then \( \eta_{n-1} = \varepsilon \). Therefore, \( \eta_{n-1} \) can be made arbitrarily small by choosing an appropriate value of \( \beta_{n-1} \). Next, let us consider the value of \( \eta_{n-1} \).

For any given \( \varepsilon > 0 \), we choose \( \beta_{n-1} = \varepsilon^{1/\lambda_{n-1}}(1 - \alpha_{n-1}T) \). If \( \alpha_{n-1} = 1 \), then \( \eta_{n-1} = \varepsilon \). Therefore, \( \eta_{n-1} \) can be made arbitrarily small by choosing an appropriate value of \( \beta_{n-1} \). Next, let us consider the value of \( \eta_{n-1} \).

For any given \( \varepsilon > 0 \), we choose \( \beta_{n-1} = \varepsilon^{1/\lambda_{n-1}}(1 - \alpha_{n-1}T) \). If \( \alpha_{n-1} = 1 \), then \( \eta_{n-1} = \varepsilon \). Therefore, \( \eta_{n-1} \) can be made arbitrarily small by choosing an appropriate value of \( \beta_{n-1} \). Next, let us consider the value of \( \eta_{n-1} \).

**Remark 5:** By choosing the FTSM parameters \( \alpha, \beta, \lambda \) (4), the bounds \( \eta \) of the FTSM variables \( s \) is in Theorem 1 can be made arbitrarily small. For example, we first consider the value of \( \eta_{n-1} \) by noting that

\[ \eta_{n-1} = \left( \frac{\beta_{n-1}}{1 - \alpha_{n-1}T} \right)^{1/2\lambda_{n-1}}. \]

For any given \( \varepsilon > 0 \), we choose \( \beta_{n-1} = \varepsilon^{1/\lambda_{n-1}}(1 - \alpha_{n-1}T) \). If \( \alpha_{n-1} = 1 \), then \( \eta_{n-1} = \varepsilon \). Therefore, \( \eta_{n-1} \) can be made arbitrarily small by choosing an appropriate value of \( \beta_{n-1} \). Next, let us consider the value of \( \eta_{n-1} \).
Choose \( \delta_{ii} = \eta_i, \ i \in \{1, 2, \ldots, n\}, \ \delta_{ij} = \delta_{i(j+1)} + \alpha \delta_{ii} + \beta \delta_{ij}, \ i \in \{1, 2, \ldots, n-1\}, \) and
\[
\delta_{ij} = \delta_{i(j+1)} + \alpha \delta_{ij} + \beta \delta_{ij}, \ i \in \{1, 2, \ldots, n-1\}, \]
where \( \delta_{ij} \) is arbitrary small as shown in Remark 5. Next, by noting \( \delta_{ij} = \delta_{i(j+1)} + \alpha \delta_{ij} + \beta \delta_{ij}, \ i \in \{1, 2, \ldots, n-1\}, \) the system trajectory of (3) satisfies that \( |x_i(k)| \leq \delta_{ij} \) for all \( k \geq K_{\text{max}} \) (2), hence omitted here.

Remark 6: By choosing the FTSM parameters \( \alpha_i, \beta_i, \) and \( \lambda_i \) in (4), the bounds \( \delta_{ij} \) in Theorem 2 can be arbitrarily small for \( j \in \{1, 2, \ldots, n\}. \) Due to \( \delta_{ii} = \eta_i, \ i \in \{1, 2, \ldots, n\}, \) the values of \( \eta_i \) can be made arbitrarily small as shown in Remark 5. Next, by noting \( \delta_{ij} = \delta_{i(j+1)} + \alpha \delta_{ij} + \beta \delta_{ij}, \ i \in \{1, 2, \ldots, n-1\}, \) the value of \( \delta_{ij} \) can be made arbitrarily small as long as the values of \( \delta_{ij} \) and \( \delta_{ij} \) are small enough. Similarly, the values of \( \delta_{ij} = \delta_{i(j+2)} + \alpha \delta_{ij} + \beta \delta_{ij}, \ i \in \{1, 2, \ldots, n-2\}, \) can be made arbitrarily small because the values of \( \delta_{ij} \) and \( \delta_{ij} \) could be small enough. By iteration, for \( j \in \{4, 5, \ldots, n\}, \) the values of \( \delta_{ij} \) can be arbitrarily small for \( i \in \{1, 2, \ldots, n-j+1\}. \) In particular, the values of \( \delta_{ij} \) can be chosen arbitrarily small for \( i \in \{1, 2, \ldots, n\}. \)

4 Illustrative example
The developed results are illustrated by a numerical example and some simulations in this section. First, the boundedness of the FTSM variables and the system steady states is demonstrated. Second, it is shown that the bounds of the FTSM variables and the steady state of the system could be reduced by adjusting the FTSM parameters. Third, the effect of FTSM control is compared with the effect of TSM control.

Consider a single link planar manipulator [25] of the form
\[
\dot{z} = a(z) + b(z)u,
\]
where
\[
a(z) = \begin{bmatrix} z_2 \\ -\sin(z_2) - m(z_3 - z_3) \\ z_4 \\ n(z_3 - z_3) \end{bmatrix}, \quad b(z) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ r \end{bmatrix}
\]
with the parameters \( l = 5.6, \ m = 1.5, \ n = 0.8, \) and \( r = 3.2. \) The system (23) can be transformed into the following canonical form:
\[
x_1 = x_2, \quad x_2 = z_2, \quad x_3 = x_4, \quad x_4 = f(x) + g(x)u,
\]
where \( f(x) = -(l \cos(x_1 + m + n)x_1 + (l^2 - n) \sin(x_1)) \) and \( g(x) = m \). The system (24) can be discretized to
\[
\Delta x_1(k) = x_2(k), \quad \Delta x_2(k) = z_2(k), \quad \Delta x_3(k) = x_4(k), \quad \Delta x_4(k) = f(x(k)) + g(x(k))u(k).
\]
Suppose the initial state \( \{x_1, x_2, x_3, x_4\}^T = [2, 2, 2, 2]^T \) and the sampling period \( T = 0.1 \) s. The simulation results are given as follows.

4.1 Boundedness of the FTSM variables and system states
Boundedness of the FTSM variables and the system states are illustrated by the numerical example. Theoretical bounds of the FTSM variables in Theorem 1 and the system states in Theorem 2 are given.

Choose the FTSM parameters in (4) as \( \alpha_i = \alpha_1 = \alpha_2 = 0.2, \) \( \beta_i = \beta_1 = \beta_2 = 1, \) \( \lambda_1 = 9/11, \) \( \lambda_2 = 7/9, \) and \( \lambda_3 = 5/9. \) The recursive FTSM variables (4) are given by
\[
s_i(k) = x_i(k), \quad s_i(k) = \Delta s_i(k) + 0.2s_i(k) + s_i^{(i+1)}(k), \quad s_i(k) = \Delta s_i(k) + 0.2s_i(k) + s_i^{(i+2)}(k), \quad s_i(k) = \Delta s_i(k) + 0.2s_i(k) + s_i^{(i+3)}(k).
\]
The FTSM control law (8) is given by
\[
u(k) = -0.021[s_1(k) + 0.1([-5.6 \cos(x_1(k)) + 2.3] + 5.6(x_2(k) - 0.8 \sin(x_1(k))) \quad +0.1[0.8 \Delta x_1^{(i+1)}(k) + \Delta^2 x_2(k)] + 0.8 \Delta x_1^{(i+1)}(k) + \Delta x_2(k) + 0.8 \Delta x_1^{(i+1)}(k) + \Delta x_3(k))].
\]
Fig. 1 gives the response trajectories and the bounds of the FTSM variables \( s_i, i \in \{1, 2, 3, 4\}. \) First, the surface \( s_1 = 0 \) is reached after one sampling period. Second, \( s_2 \) enters in the band \([-1.6 \times 10^{-3}, 1.6 \times 10^{-3}]) \) at the 45th sampling period. Third, \( s_2 \) enters in the band \([-1.8 \times 10^{-3}, 1.8 \times 10^{-3}]) \) at the 71st sampling period. Finally, \( s_3 \) enters in the band \([-1.3 \times 10^{-3}, 1.3 \times 10^{-3}]) \) at the 92th sampling period.

Fig. 2 shows the response trajectories and the bounds of the system states \( x_i, i \in \{1, 2, 3, 4\}. \) First, \( x_1 \) enters in the band \([-0.09, 0.09]) \) at the 82th sampling period. Second, \( x_2 \) enters in the band \([-4.9 \times 10^{-3}, 4.9 \times 10^{-3}]) \) at the 90th sampling period. Third, \( x_3 \) enters in the band \([-2.6 \times 10^{-4}, 2.6 \times 10^{-4}]) \) at the 90th sampling period. Finally, \( x_4 \) enters in the band \([-1.3 \times 10^{-5}, 1.3 \times 10^{-5}]) \) at the 92th sampling period.

This illustrates the correctness of the results in Theorems 1 and 2.

4.2 Reducing the bounds of the FTSM variables and the system states
Bounds of the FTSM variables and the system states can be reduced by adjusting the FTSM parameters in (4). For example, these parameters can be adjusted to \( \alpha_i = \alpha_1 = \alpha_2 = 0.1, \) \( \beta_i = \beta_1 = \beta_2 = 0.8, \) \( \lambda_1 = \lambda_2 = 9/11, \) and \( \lambda_3 = 3/5. \)

Fig. 3 shows the response trajectories of the FTSM variables \( s_i \) and their bounds \( \pm \eta_i, i \in \{1, 2, 3, 4\}. \) First, the surface \( s_1 = 0 \) is reached in one sampling period, which implies that \( \eta_i = 0. \) The FTSM variables \( s_2, s_3, \) and \( s_4 \) are bounded by \([-5.2 \times 10^{-3}, 5.2 \times 10^{-3}] \), \([-7.5 \times 10^{-3}, 7.5 \times 10^{-3}] \), and \([-3.7 \times 10^{-3}, 3.7 \times 10^{-3}] \), respectively. By comparing Fig. 3 with Fig. 1, it can be found that the bounds of the FTSM variables are reduced.

Fig. 4 shows the response trajectories of the system states \( x_i \) and their bounds \( \pm \delta_i, i \in \{1, 2, 3, 4\}. \) The states \( x_1, x_3, x_2, \) and \( x_4 \) are bounded within \([-0.028, 0.028], [-1.6 \times 10^{-3}, 1.6 \times 10^{-3}] \), \([-1 \times 10^{-4}, 1 \times 10^{-4}] \), and \([-3.7 \times 10^{-3}, 3.7 \times 10^{-3}] \), respectively. Comparing Fig. 4 with Fig. 2 shows that the bounds of the system states are reduced by adjusting the FTSM parameters.

4.3 Comparison between FTSM control and TSM control
For comparison purpose, FTSM control and TSM control are applied to the system (25), respectively. The TSM variables and the
TSM control law are designed by using the theory in [22]. The FTSM parameters are chosen as $\alpha_1 = \alpha_2 = \alpha_3 = 0.3$, $\beta_1 = \beta_2 = \beta_3 = 1$, $\lambda_1 = 9/11$, $\lambda_2 = 7/9$, and $\lambda_3 = 5/9$. The FTSM control and TSM control are compared from two perspectives.

Case 1: The TSM parameters are chosen as $\beta_1 = \beta_2 = \beta_3 = 1.03$, $\lambda_1 = 9/11$, $\lambda_2 = 7/9$, and $\lambda_3 = 5/9$. Fig. 5 gives the response trajectories of the FTSM variables and the TSM variables. Fig. 6 gives the response trajectories of the system states under FTSM control and under TSM control. It can be observed that FTSM control method possesses the advantage of faster transient response under similar bounds of the sliding mode variables and the system steady states.

Fig. 1 Response trajectories of FTSM variables
(a) Response trajectory of $s_1$, (b) Response trajectory of $s_2$, (c) Response trajectory of $s_3$, (d) Response trajectory of $s_4$

Fig. 2 Response trajectories of system states under FTSM control
(a) Response trajectory of $x_1$, (b) Response trajectory of $x_2$, (c) Response trajectory of $x_3$, (d) Response trajectory of $x_4
Case 2: the TSM parameters are chosen as $\beta_1 = \beta_2 = \beta_3 = 1.5$, $\lambda_1 = 9/11$, $\lambda_2 = 7/9$, and $\lambda_3 = 5/9$. Fig. 7 gives the response trajectories of the FTSM variables and the TSM variables. Fig. 8 gives the response trajectories of the system states under FTSM control and the system states under TSM control. It can be observed that FTSM control method possesses smaller bounds of the sliding mode variables and the system steady states under similar transient responses.

5 Conclusions
In this paper, the boundedness property of the FTSM variables and the system state has been studied for discretised SISO non-linear systems. First, the FTSM variables and surfaces were defined and the control law was designed by enforcing the last FTSM surface to be reached after one sampling period. Second, the boundedness of the FTSM variables and the steady-state were established and the corresponding bounds were provided. Finally, a numerical example
was used to illustrate the main results of this paper. Further work will be focused on the study of periodic behaviour of the system in steady state.

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7 References


