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On α- and $\mathcal{D}$-negative imaginary systems

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This paper is concerned with α- and $\mathcal{D}$-negative imaginary systems. The definition of α-negative imaginary transfer functions is firstly introduced. Then, the relationship between negative imaginary and α-negative imaginary transfer functions is studied. By means of the generalised inverse, an α-negative imaginary lemma is proposed to test the α-negative imaginary property of transfer functions. Also, a necessary and sufficient condition is provided for the α-stability of interconnection of negative imaginary systems. A state-feedback controller design condition is established such that the resulting system is α-negative imaginary. Moreover, the concept of α-negative imaginary transfer functions is extended to that of $\mathcal{D}$-negative imaginary transfer functions. Finally, the developed results are illustrated by numerical examples.

Keywords: linear systems; negative imaginary properties; stabilisation

1. Introduction

Positive real systems are a class of systems which dissipate energy (Anderson & Vongpanitlerd, 1973; Brogliato, Lozano, Maschke, & Egeland, 2007) and satisfy the positive real condition: $F(s) + F^*(s) \geq 0$ for all $\Re[s] > 0$. A modified test for positive-realness of real-rational functions was presented in Chen and Smith (2009). Positive real systems could model many systems in practical applications. While for lightly damped or undamped flexible structures with collocated position sensors and force actuators, the obtained transfer functions often have the form $G(s) = \Sigma_{i=0}^{\infty} \frac{\psi_i^2}{s^2+2\zeta_i\omega_i s+\omega_i^2}$, where ω_i is the model frequency, $\zeta_i > 0$ is the damping coefficient and ψ_i corresponds to the boundary condition. The stability results in positive real theory (Anderson & Vongpanitlerd, 1973) are not applicable to such structures. The reason is that the relative degree of positive real systems must be zero or one (Brogliato et al., 2007). The transfer function $G(s)$ satisfies the negative imaginary condition: $j[G(j\omega) - G^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$. Such systems are called negative imaginary systems. For single-input singe-output negative imaginary systems, their Nyquist plots lie underneath the real axis; in other words, the phase of $G(s)$ belongs to interval $[-\pi, 0]$ for all $\omega \in (0, \infty)$.

In recent years, negative imaginary systems have attracted the attention of many researchers. The definition of negative imaginary transfer functions was extended by allowing simple poles on the imaginary axis except at the origin (Xiong, Petersen, & Lanzon, 2010). Also, lossless and finite frequency negative imaginary systems were studied in Xiong, Petersen, and Lanzon (2012a, 2012b), respectively. A further extension was proposed by Mabrok, Kallapur, Petersen, and Lanzon (2014) allowing poles at the origin. A generalised definition for negative imaginary transfer function matrices was proposed in Ferrante and Ntogramatzidis (2013), where the transfer function matrix needs to be symmetric. In addition, several further efforts have been devoted to study the properties and applications of negative imaginary systems (see Cai & Hagen, 2010; Lanzon & Petersen, 2008; Mabrok, Lanzon, Kallapur, & Petersen, 2013; Petersen, Lanzon, & Song, 2009; Song, Lanzon, Patra, & Petersen, 2010, 2012a, 2012b, and references cited therein).

For the active structural systems, highly resonant dynamics can degrade the system performance (Petersen & Lanzon, 2010). Moreover, the closer the poles to the imaginary axis are, the stronger the vibrations are. One effective way to suppress the vibration response is to place the poles in a suitable location, such as the region $\Re[s] < -\alpha$. In this paper, a new class of negative imaginary systems, which satisfy both the α-stability and the negative imaginary condition, will be introduced. This concept is inspired by the concept of α-strictly positive real (α-SPR) transfer function matrices in Lu, Ho, and Yeung (2003). Then, an α-negative imaginary lemma is derived by means of the generalised inverse. A necessary and sufficient condition is proposed for the α-stability of positive feedback interconnected systems. Moreover, a state feedback controller is designed such that the resulting closed-loop system satisfies the α-negative imaginary property. In addition, we extend the concept of α-negative imaginary transfer functions to $\mathcal{D}$-negative imaginary transfer functions, where all the poles lie in a linear matrix inequality (LMI) region.

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In general, pole placement in negative imaginary systems is useful. The transient response of a negative imaginary system is related to the location of its poles (Chilali & Gahinet, 1996; Chilali, Gahinet, & Apkarian, 1999; Faria, Assunção, Teixeira, Cardim, & Da Silva, 2009). Take a second-order negative imaginary system with poles $\lambda = -\zeta\omega_n \pm j\omega_m$ as an example. The step response is fully characterised by the undamped natural frequency ω_n, the damping coefficient ζ and the damped natural frequency ω_m (Chilali & Gahinet, 1996; Golnaraghi & Kuo, 2009). The study of α- and $\mathcal{D}$-negative imaginary systems is also inspired by the fact that pole-placement constrains are useful to avoid fast dynamics and high-frequency gain in practical systems (Scherer, Gahinet, & Chilali, 1997).

The organisation of the paper is as follows: preliminary results and the notion of α-negative imaginary transfer functions are introduced in Section 2. In Section 3, an α-negative imaginary lemma is established. The α-stability of positive feedback interconnected systems is studied. Section 4 presents a design of state-feedback controller such that the closed-loop system is α-negative imaginary. Section 5 studies $\mathcal{D}$-negative imaginary systems. Illustrative examples are presented in Section 6. Section 7 gives a conclusion of this paper.

Notation: $\mathcal{R}^{m\times n}$, $R^{m\times n}$ and $\mathcal{C}^{m\times n}$ denote the sets of $m \times n$ real-rational proper transfer function matrices, real and complex matrices, respectively. $\Re[\cdot]$ denotes the real part of complex numbers. $\lambda_{\max}(A)$ is the maximum eigenvalue of a square complex matrix A that only has real eigenvalues. A^T and A^* denote the transpose and complex conjugate transpose of a complex matrix A, respectively. A^- is the generalised inverse satisfying $AA^-A = A$. $A \otimes B$ is the Kronecker product of A and B. $A > 0$ (≥ 0) denotes symmetric positive (semi-)definite matrix.

2. Preliminaries and α-negative imaginary functions

In this section, we first recall the definitions of α-stability, α-SPR and negative imaginary transfer functions, then introduce the concept of α-negative imaginary transfer functions.

Definition 1 (Bourlès, 1987)**:** Given a real constant $\alpha \geq 0$. A linear time-invariant system with a real-rational proper transfer function $G(s)$ is said to be asymptotically α-stable, if every pole of $G(s)$ satisfies $\Re[\lambda_j] < -\alpha$.

Definition 2 (Lu et al., 2003)**:** Given a real constant $\alpha \geq 0$. A real-rational proper transfer function matrix $F(s) \in \mathcal{R}^{m\times m}$ is said to be strictly positive real with an α-stability (α-SPR) if

(1) $F(s)$ has no poles in $\Re[s] \geq -\alpha$.
(2) $F(j\omega) + F^*(j\omega) > 0$ for $\omega \in (-\infty, \infty)$.

Remark 1: When $\alpha = 0$, $F(s)$ is SPR. Condition 1 in Definition 2 implies that $F(s)$ is asymptotically α-stable.

Definition 3 (Xiong et al., 2010)**:** A real-rational proper transfer function matrix $G(s) \in \mathcal{R}^{m\times m}$ is said to be negative imaginary if

(1) $G(s)$ has no poles at the origin and in $\Re[s] > 0$.
(2) $j[G(j\omega) - G^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$ except values of ω where $j\omega$ is a pole of $G(s)$.
(3) If $s = j\omega_0$, $\omega_0 \in (0, \infty)$, is a pole of $G(s)$, it is at most a simple pole, and the residual matrix $K_0 = \lim_{s\to j\omega_0}(s - j\omega_0)jG(s)$ is positive semi-definite Hermitian.

Definition 3 has been generalised by allowing poles at the origin in Mabrok et al. (2014). We now present the definition of α-negative imaginary transfer functions.

Definition 4: Given a real constant $\alpha \geq 0$. A real-rational proper transfer function matrix $G(s) \in \mathcal{R}^{m\times m}$ is said to be α-negative imaginary if

(1) $G(s)$ is asymptotically α-stable.
(2) $j[G(j\omega) - G^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$.

Remark 2: When $\alpha = 0$, $G(s)$ in Definition 4 is stable negative imaginary (Petersen et al., 2009). Systems with poles on the imaginary axis are excluded in the definition of α-negative imaginary transfer functions.

A useful lemma is as follows.

Lemma 1 (Wang, Wei, Qiao, Lin, & Chen, 2004)**:** *Let $A \in \mathcal{C}^{m\times n}$, $B \in \mathcal{C}^{p\times q}$, $D \in \mathcal{C}^{m\times q}$. Then the matrix equation $AXB = D$ is consistent if and only if for some A^- and B^-,*

$$AA^-DB^-B = D.$$

If $AXB = D$ is consistent, the general solution is given by

$$X = A^-DB^- + Y - A^-AYBB^-$$

for arbitrary $Y \in \mathcal{C}^{n\times p}$.

The following lemma provides a new form of the negative imaginary lemma in terms of the generalised inverse.

Lemma 2: *Let (A, B, C, D) be a minimal state-space realisation of a real-rational proper transfer function matrix $G(s) \in \mathcal{R}^{m\times m}$, where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{m\times n}$, $D \in R^{m\times m}$. Then $G(s)$ is negative imaginary if and only if*

(1) $\det(A) \neq 0$, $D = D^T$.
(2) *There exists a $(C^T)^-$ such that $B(C^T)^-C^T = B$ holds.*
(3) *There exists a matrix $Y \in R^{n\times n}$ satisfying*

$$A^{-1}B(C^T)^- + YC^T(C^T)^- - Y$$
$$= [A^{-1}B(C^T)^- + YC^T(C^T)^- - Y]^T < 0 \quad (1)$$

$$Z = Z^T \leq 0, \quad (2)$$

where $(C^T)^-$ is the matrix in Condition 2, $Z = AY + YA^T - B(C^T)^- - A^{-1}B(C^T)^-A^T - AYC^T(C^T)^- - YC^T(C^T)^-A^T$.

Proof: The proof is completed via the following sequence of equivalent reformulations:

$G(s)$ is negative imaginary.

⇔ $\det(A) \neq 0$, $D = D^T$, and there exists a matrix $X > 0$ such that

$$B = -AXC^T \quad (3)$$

$$AX + XA^T \leq 0. \quad (4)$$

This equivalence is via Lemma 7 in Xiong et al. (2010).

⇔ $\det(A) \neq 0$, $D = D^T$, and it follows from Lemma 1 that there exists some $(C^T)^-$ satisfying

$$AA^{-1}(-B)(C^T)^-C^T = -B.$$

In this case, a symmetric positive definite solution of (3) is given by

$$X = Y - A^{-1}B(C^T)^- - YC^T(C^T)^- \quad (5)$$

for some $Y \in R^{n\times n}$. Substituting (5) into (4) results in

$$AY + YA^T - A^{-1}B(C^T)^-A^T - B(C^T)^- - AYC^T(C^T)^- - YC^T(C^T)^-A^T \leq 0.$$

⇔ $\det(A) \neq 0$, $D = D^T$, and there exists some $Y \in R^{n\times n}$ satisfying

$$\begin{aligned} &A^{-1}B(C^T)^- + YC^T(C^T)^- - Y \\ &\quad = [A^{-1}B(C^T)^- + YC^T(C^T)^- - Y]^T < 0 \\ &AY + YA^T - A^{-1}B(C^T)^-A^T - B(C^T)^- \\ &\quad - AYC^T(C^T)^- - YC^T(C^T)^-A^T \\ &= [AY + YA^T - A^{-1}B(C^T)^-A^T - B(C^T)^- \\ &\quad - AYC^T(C^T)^- - YC^T(C^T)^-A^T]^T \leq 0, \end{aligned}$$

where $(C^T)^-$ is the matrix which satisfies $B(C^T)^-C^T = B$. □

Remark 3: When C^T is full column rank, one has that $(C^T)^-C^T = I_m$, and that $B(C^T)^-C^T = B$. When C^T is not full column rank, the equality $B(C^T)^-C^T = B$ might not be true. Therefore, Condition 2 in Lemma 2 is necessary.

Remark 4: When (A, B, C, D) is a non-minimality realisation of $G(s)$, Conditions 1–3 in Lemma 2 become a sufficient condition to determine whether $G(s)$ is negative imaginary. The proof follows along similar lines to those in the proof of Corollary 1 in Song et al. (2012a).

Consider the following system

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1u(t) + B_2w(t) \\ z(t) = Cx(t) + D_1u(t) + D_2w(t), \end{cases} \quad (6)$$

where $x(t) \in R^n$, $w(t) \in R^k$, $z(t) \in R^m$ and $u(t) \in R^m$ are the state, disturbance input, output and control input, respectively. A, B_i, C_i, $i \in \{1, 2\}$, are known real constant matrices with appropriate dimensions. The transfer function from $u(t)$ to $z(t)$ of system (6) with $w(t) \equiv 0$ is given by

$$G(s) = C(sI - A)^{-1}B_1 + D_1.$$

Let $y(t) = e^{\alpha t}x(t)$, where $\alpha \geq 0$ is the decay rate defined as the maximum value of the real constant satisfying $\lim_{t\to\infty} e^{\alpha t}\|x(t)\| = 0$ (see Boyd, EI Ghaoui, Feron, & Balakrishnan, 1994). Then system (6) with $w(t) \equiv 0$ can be transformed to

$$\begin{cases} \dot{y}(t) = (A + \alpha I)y(t) + B_1e^{\alpha t}u(t) \\ z(t) = Ce^{-\alpha t}y(t) + D_1u(t). \end{cases} \quad (7)$$

The transfer function from $u(t)$ to $z(t)$ of system (7) is given by

$$R(s) = C[sI - (A + \alpha I)]^{-1}B_1 + D_1 = G(s - \alpha).$$

Then the asymptotic α-stability of $\dot{x}(t) = Ax(t)$ is equivalent to the asymptotic stability of $\dot{y}(t) = (A + \alpha I)y(t)$. The goal in the next section is to develop a sufficient condition to determine whether $G(s)$ is α-negative imaginary based on $G(s - \alpha)$.

3. α-negative imaginary lemma

The main results of this paper are presented in this section. The α-negative imaginary lemma extends the negative imaginary lemma in Lanzon and Petersen (2008) and Xiong et al. (2010) to the case where the transfer function matrices have all the poles lying in the set $\{s \in \mathcal{C} : \Re[s] < -\alpha\}$.

The following lemma provides a relationship between negative imaginary transfer function matrices and α-negative imaginary transfer function matrices.

Lemma 3: *For a given constant $\alpha \geq 0$, let $G(s)$ be a square real-rational proper transfer function matrix. If $G(s - \alpha)$ is stable negative imaginary, then $G(s)$ is α-negative imaginary.*

Proof: Let $(A + \alpha I, B, C, D)$ be a state-space realisation of the real-rational proper transfer function matrix $G(s - \alpha)$. Then $A + \alpha I$ is stable if and only if $G(s)$ is α-stable.

Let $s' = s - \alpha$. $G(s')$ is stable negative imaginary. So $G(s')$ has no poles on the imaginary axis and satisfies $G(\infty) = G^T(\infty)$. It follows from Lemma 3 in Xiong et al. (2010) that $s'[G(s') - G(\infty)]$ is positive real. That is, $F(s) = (s - \alpha)[G(s - \alpha) - G(\infty)]$ is positive real. Let $s = \alpha + j\omega$, $\omega > 0$. We have that $F(s) + F^*(s) = j\omega[G(j\omega) - G^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$. Then Condition 2 of Definition 4 is satisfied. Hence, according to Definition 4, $G(s)$ is α-negative imaginary. □

Remark 5: When $\alpha = 0$, $G(s)$ is stable negative imaginary.

The following corollary is a matrix inequality restatement of Lemma 3, which is analogous to the negative imaginary lemma in Lanzon and Petersen (2008) and Xiong et al. (2010).

Corollary 1 (α-negative imaginary lemma)**:** *For a given real constant $\alpha \geq 0$, let (A, B, C, D) be a state-space realisation of a real-rational proper transfer function matrix $G(s) \in \mathcal{R}^{m\times m}$, where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{m\times n}$, $D \in R^{m\times m}$. Then $G(s)$ is α-negative imaginary if the following conditions hold:*

(1) *$A + \alpha I$ is nonsingular, $D = D^T$.*
(2) *There exists a $(C^T)^-$ such that $B(C^T)^- C^T = B$ holds.*
(3) *There exists a matrix $Y \in R^{n\times n}$ such that*

$$(A+\alpha I)^{-1}B(C^T)^- + YC^T(C^T)^- - Y = [(A+\alpha I)^{-1}B(C^T)^- + YC^T(C^T)^- - Y]^T < 0 \quad (8)$$

$$\hat{Z} = \hat{Z}^T < 0, \quad (9)$$

where $(C^T)^-$ is the matrix in Condition 2, $\hat{Z} = (A+\alpha I)Y + Y(A+\alpha I)^T - B(C^T)^- - (A+\alpha I)^{-1}B(C^T)^-(A+\alpha I)^T - (A+\alpha I) YC^T(C^T)^- - YC^T(C^T)^-(A+\alpha I)^T$.

Proof: Let $(A + \alpha I, B, C, D)$ be a state-space realisation of $G(s - \alpha)$. According to Lemma 2, Conditions 1–3 imply $G(s - \alpha)$ is negative imaginary. Furthermore, Condition 3 implies that $X := Y - (A + \alpha I)^{-1}B(C^T)^- - YC^T(C^T)^- > 0$, and $(A + \alpha I)X + X(A + \alpha I)^T < 0$. Hence, $G(s - \alpha)$ is stable negative imaginary. It follows from Lemma 3 that $G(s)$ is α-negative imaginary. □

In addition, there is another method to test whether $G(s)$ is α-negative imaginary. If $D = D^T$ and there exists a matrix $Y > 0$ such that $AY + YA^T + 2\alpha Y < 0$ and $B = -AYC^T$, then $G(s)$ is α-negative imaginary. A similar form as in Lemma 2 can be obtained by substituting the general solution of $B = -AYC^T$ into $AY + YA^T + 2\alpha Y < 0$. This method is clearly different from the one in Lemma 3 and Corollary 1. This method depends on the properties of $G(s)$, while the one in Lemma 3 and Corollary 1 depends on $G(s - \alpha)$.

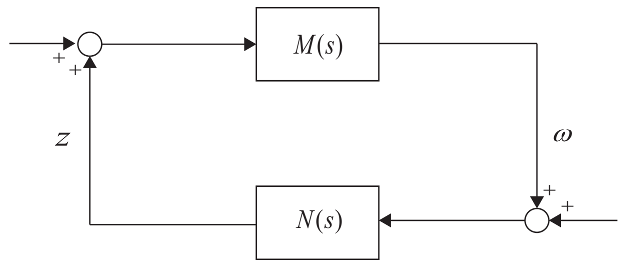

Figure 1. Positive feedback interconnection.

Remark 6: α-negative imaginary systems are actually negative imaginary systems with poles lying in the region $\Re[s] < -\alpha$. So all the properties about negative imaginary transfer functions in Lanzon and Petersen (2008) and Xiong et al. (2010) are also valid for α-negative imaginary transfer functions, including the sum computation property of Lemma 6 in Xiong et al. (2010) and the relationship between negative imaginary and positive real transfer functions of Lemma 3 in Xiong et al. (2010).

In the following, we consider the internal α-stability of positive feedback interconnection of two negative imaginary systems as shown in Figure 1, denoted by $[M(s), N(s)]$.

Theorem 1: *Given a real constant $\alpha \geq 0$. Suppose that $M(s - \alpha)$ is negative imaginary, $N(s - \alpha)$ is strictly negative imaginary, and that $M(\infty)N(\infty) = 0$, $N(\infty) \geq 0$. Then $[M(s), N(s)]$ is internally α-stable if and only if $\lambda_{\max}(M(-\alpha)N(-\alpha)) < 1$.*

Proof: Let $M(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, $N(s) \sim \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$, $M(s-\alpha) \sim \left[\begin{array}{c|c} A+\alpha I & B \\ \hline C & D \end{array}\right]$ and $N(s-\alpha) \sim \left[\begin{array}{c|c} \bar{A}+\alpha I & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$ be minimal realisations. Then, $D = D^T$, $\bar{D} = \bar{D}^T \geq 0$, $D\bar{D} = 0$, and the realisation is stabilisable and detectable.

$[M(s), N(s)]$ is internally α-stable.

$\Leftrightarrow$ $\begin{bmatrix} A+\alpha I & 0 \\ 0 & \bar{A}+\alpha I \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & \bar{B} \end{bmatrix}\begin{bmatrix} I & -\bar{D} \\ -D & I \end{bmatrix}^{-1}\begin{bmatrix} 0 & \bar{C} \\ C & 0 \end{bmatrix}$ is Hurwitz.

$\Leftrightarrow$ $[M(s - \alpha), N(s - \alpha)]$ is internally stable (according to Lemma 5.2 in Zhou, Doyle, & Glover, 1996).

$\Leftrightarrow$ $\lambda_{\max}(M(-\alpha)N(-\alpha)) < 1$ (according to Theorem 5 in Lanzon & Petersen, 2008). □

Remark 7: When $\alpha = 0$, it follows that $[M(s), N(s)]$ is internally stable if and only if $\lambda_{\max}(M(0)N(0)) < 1$, which recovers the result in Lanzon and Petersen (2008).

4. State-feedback controller synthesis

In this section, Lemma 3 is applied to design a state-feedback controller such that the resulting closed-loop system is α-negative imaginary. Consider system (6). Assume $D_1 = 0$ to simplify discussion.

Theorem 2: *Suppose $D_2 = D_2^T$ and there exist matrices $Y \in R^{n\times n}$, $Y > 0$, $M \in R^{m\times n}$ and a scalar $\varepsilon > 0$ satisfying*

$$(A+\alpha I)Y + Y(A+\alpha I)^T + B_1 M + M^T B_1^T + \varepsilon I \le 0 \tag{10}$$

$$B_2 + (A+\alpha I)YC^T + B_1 M C^T = 0. \tag{11}$$

Then the resulting closed-loop system of (6) *with the state-feedback control law $u(t) = MY^{-1}x(t)$ is α-negative imaginary.*

Proof: Applying the state-feedback control law $u(t) = MY^{-1}x(t)$ to system (6) yields the following closed-loop system

$$\begin{cases} \dot{x}(t) = (A + B_1 M Y^{-1})x(t) + B_2 w(t) \\ z(t) = Cx(t) + D_2 w(t). \end{cases} \tag{12}$$

The closed-loop transfer function from $w(t)$ to $z(t)$ is given by

$$T(s) = C(sI - A - B_1 M Y^{-1})^{-1} B_2 + D_2. \tag{13}$$

Because the LMI (10) and equality (11) are satisfied, we have

$$(A+\alpha I + B_1 M Y^{-1})Y + Y(A + \alpha I + B_1 M Y^{-1})^T + \varepsilon I \le 0 \tag{14}$$

$$B_2 + (A + \alpha I + B_1 M Y^{-1})YC^T = 0. \tag{15}$$

According to Lemma 7 in Xiong et al. (2010), the closed-loop transfer function $T(s - \alpha)$ is stable negative imaginary. Furthermore, according to Lemma 3, we have that $T(s)$ is α-negative imaginary. □

The conditions in Theorem 2 consist of both an LMI (10) and one equality (11). One way to handle the equality constraint is to use the generalised inverse as has been illustrated in Lemma 2. Another way is to solve an optimisation problem as follows.

Consider the following condition for $\beta \ge 0$

$$(B_2 + (A+\alpha I)YC^T + B_1 MC^T)^T (B_2 + (A+\alpha I)YC^T + B_1 MC^T) \le \beta I. \tag{16}$$

Applying Schur complement equivalence, the inequality (16) is equivalent to

$$\begin{bmatrix} -\beta I & B_2^T + CY(A+\alpha I)^T + CM^T B_1^T \\ B_2 + (A+\alpha I)YC^T + B_1 MC^T & -I \end{bmatrix} \le 0. \tag{17}$$

Thus, the problem of solving (10), (11) can be changed into the problem of finding a global solution of the minimisation problem:

$$\begin{aligned} &\min \beta \\ &\text{subject to (10), (17), } Y > 0 \text{ and } \beta \ge 0. \end{aligned} \tag{18}$$

A similar computational algorithm has been used in Chen, Niu, and Zou (2013) and more detailed computational procedure can be found in Niu, Ho, and Lam (2005). If the global infimum β is zero, the corresponding solution of the minimisation problem (18) will satisfy the LMI (10) and the equality (11). If the global infimum β is sufficiently small but not equal to zero, then $B_2 + (A + \alpha I)YC^T + B_1 MC^T \approx 0$. A sufficiently small β is enough to ensure the equality condition in practical applications.

5. $\mathcal{D}$-negative imaginary systems

In this section, we generalise the concept of α-negative imaginary transfer functions to negative imaginary transfer functions with pole placement in an LMI region.

Definition 5 (Chilali & Gahinet, 1996)**:** A subset $\mathcal{D}$ of the complex plane is said to be an LMI region if there exist a symmetric matrix $P \in R^{r\times r}$ and a matrix $Q \in R^{r\times r}$ such that

$$\mathcal{D} = \{s \in \mathcal{C} : P + sQ + \bar{s}Q^T < 0\}.$$

Definition 6: Given an LMI region $\mathcal{D}$. A real-rational proper transfer function matrix $G(s) \in \mathcal{R}^{m\times m}$ is said to be $\mathcal{D}$-negative imaginary if

(1) $G(s)$ is negative imaginary.
(2) All the poles of $G(s)$ lie in the LMI region $\mathcal{D}$.

Some typical LMI regions could be found in Chilali and Gahinet (1996), Chilali et al. (1999), Henrion, Bachelier, and Šebek (2001) and Faria et al. (2009). When $r \in \{1, 2\}$, we have the following results.

Lemma 4: *For a given region $\mathcal{D} = \{s \in \mathcal{C} : p + qs + q\bar{s} < 0,\ p \in R, q \in R\}$, let (A, B, C, D) be a state-space realisation of a real-rational proper transfer function matrix $G(s) \in \mathcal{R}^{m\times m}$, where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{m\times n}$, $D \in R^{m\times m}$. Then $G(s)$ is $\mathcal{D}$-negative imaginary if the following conditions hold:*

(1) *$pq > 0$, $D = D^T$.*
(2) *There exists a matrix $Y \in R^{n\times n}$, $Y > 0$, such that*
$$\begin{bmatrix} I \\ A \end{bmatrix}^T \begin{bmatrix} pY & qY \\ qY & 0 \end{bmatrix} \begin{bmatrix} I \\ A \end{bmatrix} < 0 \text{ and } B = -AYC^T.$$

Proof: Condition 2 implies that there exists a matrix $Y > 0$ such that $pY + qA^T Y + qYA < 0$ and $B = -AYC^T$.

Because $pq > 0$, we obtain that $A^TY + YA < 0$ and $B = -AYC^T$. According to Lemma 7 in Xiong et al. (2010), $G(s)$ is negative imaginary. In view of Theorem 1 in Henrion and Meinsma (2001), Condition 2 also implies that all the poles of $G(s)$ lie in the region $\mathcal{D}$. Hence, according to Definition 6, $G(s)$ is $\mathcal{D}$-negative imaginary. □

Lemma 5: *For a given region $\mathcal{D}$, assume $P \in R^{2\times 2}$, $Q \in R^{2\times 2}$, $P > 0$ and $Q > 0$. Let (A, B, C, D) be a state-space realisation of a real-rational proper transfer function matrix $G(s) \in \mathcal{R}^{m\times m}$, where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{m\times n}$, $D \in R^{m\times m}$. If $D = D^T$ and there exists a matrix $Y \in R^{n\times n}$, $Y > 0$, such that*

$$M_{\mathcal{D}}(A, Y) < 0, \text{and}\, B = -AYC^T,$$

where $M_{\mathcal{D}}(A, Y) = P \otimes Y + Q \otimes AY + Q^T \otimes (AY)^T$, then $G(s)$ is $\mathcal{D}$-negative imaginary.

Proof: Because $P > 0$, $Q > 0$, one has that $P + 2\Re[s]Q < 0$ for $\Re[s] < 0$. Therefore, the LMI region $\mathcal{D}$ lies in the open left-half plane. In view of Theorem 2.2 in Chilali and Gahinet (1996), if there exists a matrix $Y > 0$ such that $M_{\mathcal{D}}(A, Y) < 0$, then all the poles of A lie in the region $\mathcal{D}$. Furthermore, we have $AY + YA^T < 0$. It follows from Lemma 7 in Xiong et al. (2010) that $G(s)$ is negative imaginary. Therefore, according to Definition 6, $G(s)$ is $\mathcal{D}$-negative imaginary. □

Example 1: Consider $G(s) = \frac{1}{s+2}$. A state-space realisation is given by $(-2, 1, 1, 0)$. Given an LMI region $\mathcal{D} = \{s \in \mathcal{C} : 1 + s + \bar{s} < 0\}$. Then a solution $P = \frac{1}{2}$ satisfies the conditions in Lemma 4. It follows that $G(s)$ is negative imaginary and all the poles lie in the region $\Re[s] < -\frac{1}{2}$.

Lemma 6: *Let (A, B, C, D) be a state-space realisation of a real-rational proper transfer function matrix $G(s) \in \mathcal{R}^{m\times m}$, where $A \in R^{n\times n}$, $B \in R^{n\times m}$, $C \in R^{m\times n}$, $D \in R^{m\times m}$. Suppose $\theta \in (0, \frac{\pi}{2})$. If $D = D^T$ and there exists a matrix $Y \in R^{n\times n}$, $Y > 0$, satisfying*

$$\begin{bmatrix} \sin\theta(AY + YA^T) & \cos\theta(AY - YA^T) \\ \cos\theta(YA^T - AY) & \sin\theta(AY + YA^T) \end{bmatrix} < 0 \qquad (19)$$

$$B = -AYC^T, \qquad (20)$$

then $G(s)$ is negative imaginary and all the poles of $G(s)$ lie in the conic sector $\mathcal{S}(0, 0, \theta)$ as shown in Figure 2.

Proof: The (1, 1) block of inequality (19) implies that $\sin\theta(AY + YA^T) < 0$. Because $\theta \in (0, \frac{\pi}{2})$, we obtain $AY + YA^T < 0$. According to Lemma 7 in Xiong et al. (2010), $G(s)$ is negative imaginary. In view of Theorem 2.2 in Chilali and Gahinet (1996), the inequality (19) implies that all the poles of $G(s)$ lie in the conic sector $\mathcal{S}(0, 0, \theta)$. □

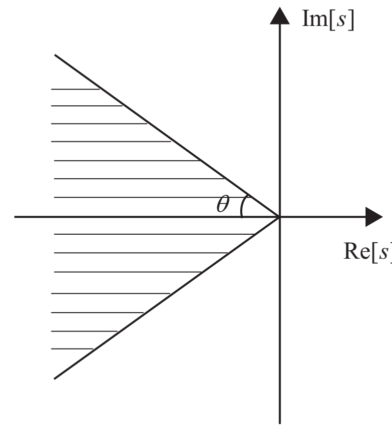

Figure 2. Conic sector $\mathcal{S}(0, 0, \theta)$.

When $r = 1$, the LMI regions correspond to vertical half-planes. When $r = 2$, the LMI regions may correspond to discs, ellipses, sectors and strips (Henrion & Meinsma, 2001; Henrion et al., 2001). Those two classes of LMI regions could cover most needs for control purpose in practice. For more complex LMI regions, we may find a common $Y > 0$ such that $B = -AYC^T$, $AY + YA^T < 0$ and $M_{\mathcal{D}_i}(A, Y) < 0$, $i \in \{1, \ldots, l\}$, where $\mathcal{D}_i$ are different LMI regions. An illustrative example is shown in Figure 3. This polygon region is the intersection of the left-half plane, two conic sectors and two horizontal strips. More details about the intersection of LMI regions can be found in Arzelier, Henrion, and Peaucelle (2002) and Chilali and Gahinet (1996). It is worth pointing out that all the results in Lemmas 4–6 hold without minimality assumption.

Take the conic sector $\mathcal{S}(0, 0, \theta)$ case as an example. Lemma 6 can be used to design a state-feedback controller such that the resulting closed-loop system is $\mathcal{D}$-negative imaginary.

Corollary 2: *Suppose $D_1 = 0$. If $D_2 = D_2^T$ and there exist matrices $Y \in R^{n\times n}$, $Y > 0$, $M \in R^{m\times n}$ and a scalar $\varepsilon > 0$ satisfying*

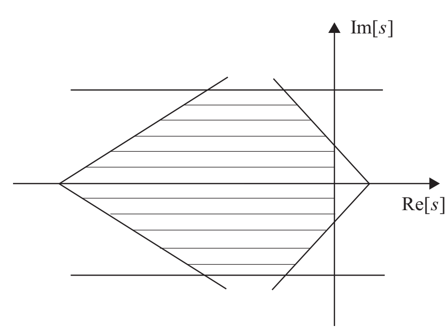

Figure 3. Convex polygon as an intersection of LMI regions.

$$\begin{bmatrix} \sin\theta(AY+YA^T+B_1M+M^TB_1^T) & \cos\theta(AY-YA^T+B_1M-M^TB_1^T) \\ \cos\theta(YA^T-AY+M^TB_1^T-B_1M) & \sin\theta(AY+YA^T+B_1M+M^TB_1^T) \end{bmatrix} + \varepsilon I \le 0 \quad (21)$$

$$B_2 + AYC^T + B_1MC^T = 0, \quad (22)$$

then the resulting closed-loop system of (6) *with the state-feedback control law* $u(t) = MY^{-1}x(t)$ *is* $\mathcal{D}$*-negative imaginary, where the region* $\mathcal{D}$ *is the conic sector* $\mathcal{S}(0, 0, \theta)$.

Proof: The result follows from the proofs of Theorem 2 and Lemma 6. □

A numerical method to solve the LMI (21) and the equality (22) is the same as the one suggested in Section 4. The problem of solving (21), (22) can be transformed into an optimisation problem as follows:

$$\begin{aligned} &\min \beta \\ &\text{subject to (21)},\ Y > 0,\ \beta \ge 0 \text{ and} \\ &\begin{bmatrix} -\beta I & B_2^T+CYA^T+CM^TB_1^T \\ B_2+AYC^T+B_1MC^T & -I \end{bmatrix} \le 0. \end{aligned} \quad (23)$$

Remark 8: For the other types of LMI regions, Lemmas 4 and 5 can also be applied to design a state-feedback controller such that the resulting closed-loop system is $\mathcal{D}$-negative imaginary. The design condition can be found similarly as in the conic sector case.

6. Illustrative examples

In this section, two examples are used to illustrate the main results of this paper.

Example 2: Consider the system (6) with coefficient matrices given by

$$A = \begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix}, B_1 = \begin{bmatrix} -3 \\ -5 \\ 3 \end{bmatrix}, B_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix},$$
$$C = \begin{bmatrix} -1 & 0 & 3 \end{bmatrix}, D_1 = D_2 = 0. \quad (24)$$

To illustrate Corollary 1, suppose $\alpha = 1$ and $w(t) = 0$. YALMIP (Lofbërg, 2004) and SeDuMi were used to find a solution of LMIs (8), (9) as

$$Y = \begin{bmatrix} 1.9724 & 2.2028 & 0.6425 \\ 1.9886 & 2.8755 & 0.6628 \\ 0.1568 & 0.0200 & 0.0523 \end{bmatrix}.$$

Therefore, the transfer function from $u(t)$ to $y(t)$ of the system in (6) and (24) is α-negative imaginary with $\alpha = 1$.

A state-feedback controller can be designed in the case when $w(t) \neq 0$. Given $\alpha = 1$. The optimisation problem (18) was solved to obtain a set of solutions as

$$M = \begin{bmatrix} -0.4478 & -0.6631 & 0.2814 \end{bmatrix},$$
$$Y = \begin{bmatrix} 1.1786 & 1.1370 & -0.2501 \\ 1.1370 & 1.7519 & -0.4468 \\ -0.2501 & -0.4468 & 0.2542 \end{bmatrix},$$
$$\beta = 2.8533 \times 10^{-12}, \varepsilon = 10^{-6}.$$

Hence, the LMI (10) and the equality (11) are satisfied. According to Theorem 2, the transfer function from $w(t)$ to $y(t)$ of the system in (6) and (24) is α-negative imaginary under the state-feedback controller

$$u(t) = MY^{-1}x(t) = \begin{bmatrix} -0.0733 & -0.2359 & 0.3725 \end{bmatrix} x(t).$$

Given $\theta = \frac{\pi}{12}$. The optimisation problem (23) was solved to obtain a set of solutions as

$$M = \begin{bmatrix} -28.4743 & -39.9705 & 22.5113 \end{bmatrix},$$
$$Y = \begin{bmatrix} 48.4809 & 50.4019 & -18.1522 \\ 50.4019 & 111.8582 & -24.4410 \\ -18.1522 & -24.4410 & 23.6423 \end{bmatrix},$$
$$\beta = 1.4672 \times 10^{-8}, \varepsilon = 10^{-6}.$$

According to Corollary 2, the transfer function from $w(t)$ to $y(t)$ of the system in (6) and (24) is negative imaginary, and all the poles lie in the conic sector $\mathcal{S}(0, 0, \frac{\pi}{12})$. The obtained state-feedback controller is given by

$$u(t) = MY^{-1}x(t) = \begin{bmatrix} -0.2214 & -0.1119 & 0.6664 \end{bmatrix} x(t).$$

Example 3: The example in Lanzon and Petersen (2008) was modified so that we can analyse the α-stability of an uncertain multi-input multi-output system. Consider the lightly damped uncertain mechanical plant as depicted in Figure 4. This two-degree-of-freedom spring mass system consists of two unit masses and three springs. The masses are attached to fixed walls via two springs of known unit stiffness and two dampers of known unit viscous resistance. Moreover, those two masses are coupled together via a third spring of unknown stiffness k and a third damper of known unit viscous resistance. Let $u(t) := \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}^T$ denote the control inputs, which are the forces applied to the masses; let $y(t) := \begin{bmatrix} y_1(t) & y_2(t) \end{bmatrix}^T$ denote the displacement

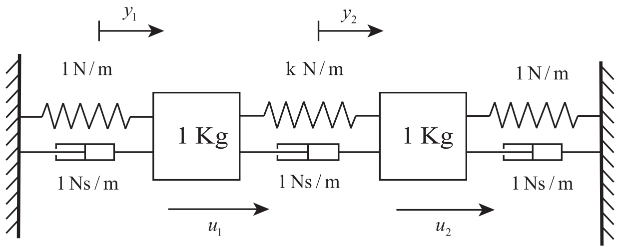

Figure 4. Lightly damped uncertain mechanical plant.

measurements of the masses. Let

$$p(s) := \frac{1}{s^2 + s + 1}, \quad \delta(s) := \frac{1}{s^2 + 3s + (2k + 1)}.$$

The transfer function of the uncertain plant from $u(t)$ to $y(t)$ is given by

$$P_\Delta(s) = P(s) + \Delta(s), \tag{25}$$

where $k > 0$ is unknown and represents the uncertainty in the system, $P(s) = p(s)\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$ is the nominal plant model and $\Delta(s) = \delta(s)\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$ is the uncertain remainder. We split the uncertain plant $P_\Delta(s)$ as in (25) for the purpose of control system design, see Figure 5.

Now, we choose

$$C(s) = \frac{1}{s+2}\begin{bmatrix} \dfrac{-3(s^2 + s + 1)(s + 2) - 1}{(s^2 + s + 1)(s + 2) + 1} & 1 \\ 1 & -1 \end{bmatrix}. \tag{26}$$

Define $R(s) := -C(s)(I + P(s)C(s))^{-1}$ as the transfer function matrix mapping from $w(t)$ to $z(t)$ so that the closed-loop system in Figure 5 can be rearranged into the form in Figure 6. Because of our particular choice of $C(s)$, we have $R(s - \alpha) = \frac{1}{s-\alpha+2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$. When $0 \le \alpha < 2$, $R(s - \alpha)$ is strictly negative imaginary. In addition, we have $\Delta(s - \alpha) = \frac{1}{s^2+(3-2\alpha)s+\alpha^2-3\alpha+2k+1}\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$. When $0 \le$

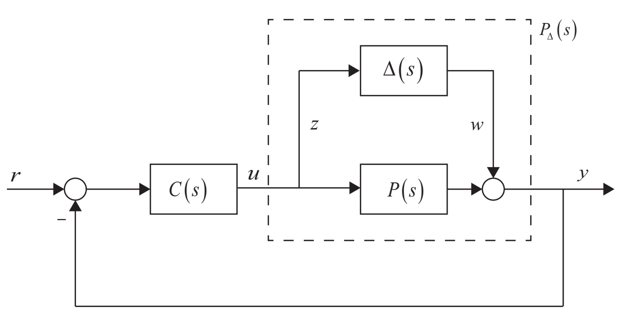

Figure 5. The closed-loop system.

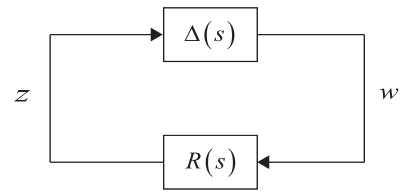

Figure 6. Rearranged closed-loop system.

$\alpha < \frac{3-\sqrt{5}}{2}$, $\Delta(s - \alpha)$ is negative imaginary for all $k > 0$. Note that $R(\infty) \ge 0$, $\Delta(\infty) \ge 0$ and $R(\infty)\Delta(\infty) = 0$.

For a given $\alpha = 0.3$, we have $\Delta(-\alpha)R(-\alpha) = \frac{1}{3.4k+0.323}\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$, $\lambda_{\max}(\Delta(-\alpha)R(-\alpha)) = \frac{3}{3.4k+0.323}$. Theorem 1 indicates that the uncertain plant is α-stable ($\alpha = 0.3$) by the controller (26) if and only if $k > 0.79$ (obtained through the condition $\frac{3}{3.4k+0.323} < 1$).

7. Conclusions

This paper has studied the α- and $\mathcal{D}$-negative imaginary properties of square real-rational proper transfer function matrices. The concept of α-negative imaginary transfer functions has been introduced. An α-negative imaginary lemma without minimality assumption has been established for transfer functions that are α-negative imaginary. A necessary and sufficient condition has been derived for the internal α-stability of positive feedback interconnected systems. Also, a state-feedback controller has been designed to ensure that the resulting closed-loop system is α-negative imaginary. Moreover, the $\mathcal{D}$-negative imaginary property has been introduced by considering pole placement in LMI regions. Finally, the main results of this paper have been illustrated by two examples.

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