Received: 3 January 2019 | Revised: 20 March 2019 | Accepted: 21 April 2019
DOI: 10.1002/rnc.4590

RESEARCH ARTICLE

WILEY

Razumikhin stability theorems for a general class of stochastic impulsive switched time-delay systems

Wei Ren[1] | Junlin Xiong[2]

[1]Decision and Control Systems, EECS, KTH Royal Institute of Technology, Stockholm, Sweden
[2]Department of Automation, University of Science and Technology of China, Hefei, China

Correspondence
Wei Ren, Decision and Control Systems, EECS, KTH Royal Institute of Technology, 100 44 Stockholm, Sweden.
Email: weire@kth.se

Funding information
National Natural Science Foundation of China, Grant/Award Number: 61773357

Summary
This paper studies stability of a general class of impulsive switched systems under time delays and random disturbances using multiple Lyapunov functions and fixed dwell-time. In the studied system model, the impulses and switches are allowed to occur asynchronously. As a result, the switching may occur in the impulsive intervals and the impulses can occur in the switching intervals, which have great effects on system stability. Since the switches do not bring about the change of the system state, we study two cases in terms of the impulses, ie, the stable continuous dynamics case and the stable impulsive dynamics case. According to multiple Lyapunov-Razumikhin functions and the fixed dwell-time, Razumikhin-type stability conditions are established. Finally, the obtained results are illustrated via a numerical example from the synchronization problem of chaotic systems.

KEYWORDS
fixed dwell-time, impulsive switched systems, Lyapunov-Razumikhin function, stochastic stability, stochastic systems, time-delay systems

1 | INTRODUCTION

Because of numerous applications in diverse fields of sciences and engineering, impulsive systems and switched systems are two important types of hybrid systems, and have been studied extensively in the past few decades. Impulsive systems are dynamical systems that combine continuous-time dynamics with instantaneous state jumps (ie, impulses).[1,2] Switched systems consist of a family of subsystems and a switching signal that orchestrates the switching among them.[3] Many physical systems can be modeled as impulsive or switched systems, such as networked control systems,[4] electronic circuit systems,[5] Lorenz systems,[6] aircraft,[7] and medical systems.[8] The readers can refer to other works[1,9-12] for a general introduction.

In the real world, impulses and switches coexist in many physical and man-made systems like chaotic systems[5] and networked control systems.[4] As a result, the impulses and switches can be studied together, thereby leading to impulsive switches systems. Many results can be found on stability and performances of impulsive switched systems.[13-15] However, in all the previous works, a common assumption is required a priori, ie, *the impulses and switches occur simultaneously*. Such an assumption is impractical and constrained. For instance, if the switching times between two subsystems cannot be ignored in circuit systems, then the switching bring about asynchronous changes of the currents and voltages. In addition, impulsive control of switched systems[16-18] and switched control of impulsive systems[19] lead to asynchronization between the switches and impulses. Therefore, the impulses and switches are coexistent in many physical systems, but do not necessarily occur synchronously. This is the main motivation for us to study this topic further. Furthermore, to the best of our knowledge, until now, there is no work in the literature on stability of such class of impulsive switched systems with asynchronous switches and impulses.

On the other hand, both time delays and external disturbances are frequently encountered in engineering systems,[20,21] and affect system performances in some extent. For instance, finite switching speeds of the amplifiers or information processing lead to the time delays in hardware implementations.[20] External random fluctuations in the transmission process or the probabilistic factors bring about the unavoidable stochastic perturbations in control systems. Recently, great efforts have been devoted to stochastic impulsive (switched) delayed systems, and the related results can be found; see other works[18,21-25] and references therein.

In this paper, we study stability of a general class of stochastic impulsive switched time-delay systems, where the impulses and switches occur asynchronously. According to a Lyapunov-based approach and the fixed dwell-time (FDT), sufficient conditions are established for the stability of such class of stochastic impulsive switched time-delay systems. The main contributions of this paper are two-fold. First, such class of stochastic impulsive switched time-delay systems is studied for the first time, which are more general and practical than those in previous works.[5,16,17,19,22] If the impulses and switches occur simultaneously, then the studied system is similar to those in the literature.[22] Second, Razumikhin-type stability conditions are established to guarantee stochastic stability. Asynchronous impulses and switches result in two discrete-time sequences, which induces additional difficulties in stability analysis. To obtain Razumikhin-type conditions, multiple Lyapunov-Razumikhin functions (LRFs) and FDT are applied. In contrast to exponential Lyapunov functions as in previous works,[25-27] general LRFs are implemented here. As a result, Razumikhin-type conditions can be applied to study stability of control systems that cannot be analyzed via exponential Lyapunov functions.

The reminder of this paper is organized as follows. In Section 2, the problem is formulated and some preliminaries are presented. Razumikhin-type stability conditions are derived for stochastic impulsive switched time-delay systems in the stable continuous dynamics case in Section 3 and in the stable impulsive dynamics case in Section 4. A numerical example from the synchronization problem of stochastic chaotic systems is provided in Section 5 to illustrate the obtained results. Conclusion and future works are presented in Section 6.

Notation. $\mathbb{R} := (-\infty, +\infty)$; $\mathbb{R}_t^+ := [t, +\infty)$ for a given $t \in \mathbb{R}$. $\mathbb{N} := \{0, 1, \dots\}$; $\mathbb{N}^+ := \{1, 2, \dots\}$. Given a vector or matrix P, $P^\top$ denotes its transpose. For a matrix $P \in \mathbb{R}^{n\times n}$, $\mathrm{tr}[P]$ denotes the trace of P; $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ are the minimal and maximal eigenvalues of P, respectively. I denotes the identity matrix with the appropriate dimension. $|\cdot|$ represents the Euclidean vector norm; $\mathbb{P}\{\cdot\}$ denotes the probability measure; $\mathbb{E}[\cdot]$ denotes the mathematical expectation; Id denotes the identity function. Given a function $f : \mathbb{R}_{t_0}^+ \to \mathbb{R}^n$ with $t_0 \geq \tau > 0$, $f(t^-) := \limsup_{s\to 0^-} f(t+s)$; $\|f\|_\tau := \sup_{s\in[t_0-\tau, t_0]}|f(s)|$; $|f(t)|_\tau := \sup_{s\in[t-\tau,t]}|f(s)|$. $C^{1,2}$ denotes the class of nonnegative functions on $\mathbb{R}_0^+ \times \mathbb{R}^n$, which are continuously differentiable on the first augment and continuously twice differentiable on the second augment. A function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is of class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing; $\alpha(t)$ is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded; $\alpha(t)$ is of class $\mathcal{VK}$ ($\mathcal{VK}_\infty$) if it is of class $\mathcal{K}$ ($\mathcal{K}_\infty$) and convex; $\alpha(t)$ is of class $\mathcal{CK}$ ($\mathcal{CK}_\infty$) if it is of class $\mathcal{K}$ ($\mathcal{K}_\infty$) and concave. A function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is of class $\mathcal{KL}$ if $\beta(s,t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s,t) \to 0$ as $t \to \infty$ for each fixed $s \geq 0$.

2 | PROBLEM FORMULATION

Consider the following stochastic impulsive switched time-delay system:

$$\begin{cases} dx(t) = f_{\sigma(t)}(t, x_t, u)dt + g_{\sigma(t)}(t, x_t, u)dB(t), & t \in \mathbb{R}_{t_0}^+ \backslash \mathcal{I}, \\ x(t) = h_{\sigma(t)}(x(t^-), u(t^-)), & t \in \mathcal{I}, \\ x(t) = x(t^-), & t \in \mathcal{S}\backslash\mathcal{I}, \\ x(t) = \xi(t), & t \in [t_0 - \tau, t_0], \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the system state, $u(t) \in \mathbb{R}^{n_u}$ is the external input, and $B(t) \in \mathbb{R}^{n_w}$ is an $\mathfrak{F}_t$-adapted Brownian motion defined on a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P}, \{\mathfrak{F}_t\}_{t\geq t_0})$. The time-delay state is denoted by $x_t := x(t - \tau(t))$, where the time delay $\tau(t)$ is bounded with a constant $\tau > 0$. $\mathcal{I} := \{i_1, i_2, \dots\}$ and $\mathcal{S} := \{s_1, s_2, \dots\}$ are, respectively, the impulsive time sequence and switching time sequence, which are strictly increasing. The function $\sigma : \mathbb{R}_{t_0}^+ \to \mathfrak{L} =: \{1, \dots, L\}$ is the switching signal, which is piecewise right-continuous. The initial function $\xi : [t_0 - \tau, t_0] \to \mathbb{R}^{n_x}$ is an $\mathfrak{F}_{t_0}$-adapted random variable with finite $\mathbb{E}[\|\xi\|_\tau^2]$. For all $l \in \mathfrak{L}$, the functions $f_l : \mathbb{R}_{t_0}^+ \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$, $g_l : \mathbb{R}_{t_0}^+ \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x\times n_w}$, and

$h_l : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ are assumed to be Lipschitz and Borel-measurable. Suppose that $f_l(t,0,0) \equiv 0$, $g_l(t,0,0) \equiv 0$, and $h_l(0,0) \equiv 0$ for all $t \in \mathbb{R}^+_{t_0}$. That is, $x(t) \equiv 0$ is a trivial solution of the system (1). It follows from other works[25,28,29] that the system (1) has a unique solution process for all the time.

Compared with impulsive switched systems studied in previous works[13,14,22] where the switches and impulses are synchronous, the switches and impulses are allowed to be asynchronous in the system (1), ie, $\mathcal{I} \neq \mathcal{S}$. As a result, there are two classes of discrete-time sequences, ie, the impulsive time sequence $\mathcal{I}$ and the switching time sequence $\mathcal{S}$, which need to be studied simultaneously. Due to the coexistence and asynchronization of such two discrete-time sequences, many phenomena are included in this paper, such as the overlapping (ie, $\mathcal{I} \bigcap \mathcal{S} \neq \emptyset$) and disjointness (ie, $\mathcal{I} \bigcap \mathcal{S} = \emptyset$). In particular, if either the impulses or switches do not exist, or both the impulsive and switching time sequences are the same, then the system (1) is similar to those studied in other works.[13,14,22,27]

Definition 1. Given a switching time sequence $\mathcal{S}$ and an impulsive time sequence $\mathcal{I}$, the system (1) is *stochastically input-to-state stable* (SISS), if for an arbitrary $\varepsilon \in (0,1)$, there exist $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$ such that for all $x(t) \in \mathbb{R}^{n_x}$ and $u \in \mathbb{R}^{n_u}$,

$$\mathbb{P}\left\{|x(t)| \leq \beta(\mathbb{E}[\|\xi\|_\tau], t-t_0) + \gamma(\|u\|)\right\} \geq 1-\varepsilon, \quad \forall t \in \mathbb{R}^+_{t_0}. \tag{2}$$

Remark 1. Definition 1 is an extension of the one in the work of Ren and Xiong[24] for stochastic impulsive systems, the one in the work of Wu et al[25] for impulsive systems, and those in the works of Ren and Xiong[23] and Zhao et al[30] for switched systems. In addition, if there are no external disturbances, then Definition 1 is reduced to be the one for stochastic asymptotic stability as in the work of Wu and Sun.[14]

The goal of this paper is to study the SISS property of the system (1) in two cases, ie, the stable continuous dynamics case and the stable impulsive dynamics case. The stability analysis is based on multiple Lyapunov functions and the FDT. In the following, an infinitesimal operator of multiple Lyapunov functions is defined.

Definition 2 (See the work of Mao[28]).
Given any $C^{1,2}$ function $V_l : \mathbb{R}^+_{t_0} \times \mathbb{R}^{n_x} \to \mathbb{R}^+_0$ with $l \in \mathfrak{L}$, the differential operator $\mathscr{L}$ associated with the continuous dynamics in the system (1), is defined as

$$\mathscr{L}V_l(t,x_t) := \frac{\partial V_l(t,x)}{\partial t} + \frac{\partial V_l(t,x)}{\partial x} f_l(t,x_t,u) + \frac{1}{2}\text{tr}\left[g_l^\top(t,x_t,u)\frac{\partial^2 V_l(t,x)}{\partial x^2} g_l(t,x_t,u)\right].$$

From Itô's differential formula in chapter 1 in the work of Mao,[28] we have that, for all $l \in \mathfrak{L}$ and $t \in \mathbb{R}^+_{t_0} \backslash \mathcal{T}$, the derivative of $V_l(t,x)$ is given by

$$dV_l(t,x) = \mathscr{L}V_l(t,x_t)dt + \frac{\partial V_l(t,x)}{\partial x} g_l(t,x_t,u)dB(t).$$

By taking expectation, we have from the proofs of Lemma 1 and Theorem 1 in the work of Zhao et al[30] that

$$d\mathbb{E}[V_l(t,x)] = \mathbb{E}[\mathscr{L}V_l(t,x)]dt, \quad \forall t \in \mathbb{R}^+_{t_0} \backslash \mathcal{T}.$$

3 | STABLE CONTINUOUS DYNAMICS CASE

In this section, the stable continuous-time dynamics case is studied, which means that the continuous dynamics in (1) is stable, whereas the impulsive dynamics is not. For this case, based on multiple LRFs and the FDT, Razumikhin-type conditions are established to guarantee SISS of the system (1).

To begin with, some auxiliary notations are introduced. Denote by $\mathcal{T} := \{t_1, t_2, \dots\}$ the discrete time sequence combining both $\mathcal{I}$ and $\mathcal{S}$. Define the set $\mathfrak{I}_1(\theta_1) := \{\mathcal{I} = \{i_1, i_2, \dots\} | \mathcal{I} \subseteq \mathbb{R}^+_{t_0}, i_{k+1} - i_k \geq \theta_1, \forall k \in \mathbb{N}\}$ for some constant $\theta_1 > 0$, and thus, the impulsive intervals are not less than θ_1. With these auxiliary notations, the following theorem provides sufficient conditions for the SISS property of the system (1) in the stable continuous dynamics case.

Theorem 1. *Consider the system (1). Assume that there exist $C^{1,2}$ Lyapunov functions $V_l : \mathbb{R}^+_{t_0-\tau} \times \mathbb{R}^{n_x} \to \mathbb{R}^+_0$, $l \in \mathfrak{L}$, $\alpha_1, \varphi \in \mathcal{VK}_\infty$, $\alpha_2, \phi_1, \phi_2 \in \mathcal{CK}_\infty$, $\rho_2 \in \mathcal{K}_\infty$, and constants $\bar{\rho}_1 \in (0,1), \theta_1 > \delta > 0, 0 < N < \infty$ such that $\phi_1 > \text{Id}, \phi_2 \geq \text{Id}$, and*

(A.1) for all $t \in \mathbb{R}^+_{t_0-\tau}$ and all $l \in \mathfrak{L}$, $\alpha_1(|x(t)|) \leq V_l(t,x(t)) \leq \alpha_2(|x(t)|)$;

(A.2) *for all* $t \in \mathbb{R}^+_{t_0} \backslash \mathcal{T}$, $V_{\sigma(t)}(t, x(t)) \geq \max\{\bar{\rho}_1 |V_{\sigma(t)}(t, x(t))|_\tau, \rho_2(|u(t)|)\}$ *implies that*

$$\mathscr{L} V_{\sigma(t)}(t, x_t) \leq -\varphi(V_{\sigma(t)}(t, x(t)));$$

(A.3) *for all* $t \in \mathcal{I}$, $V_{\sigma(t)}(t, x(t)) \leq \phi_1(V_{\sigma(t^-)}(t^-, x(t^-)))$;

(A.4) *for all* $t \in \mathcal{S} \backslash \mathcal{I}$, $V_{\sigma(t)}(t, x(t)) \leq \phi_2(V_{\sigma(t^-)}(t^-, x(t^-)))$;

(A.5) *the FDT condition is satisfied, ie,*

$$\int_a^{\phi_1(a)} \frac{ds}{\varphi(s)} + N \int_a^{\phi_2(a)} \frac{ds}{\varphi(s)} \leq \theta_1 - \delta, \quad \forall a \in \mathbb{R}^+, \tag{3}$$

where $N \in \mathbb{N}^+$ *is the maximal number of the switches in an impulsive interval.*

Then, the system (1) is SISS for all the discrete-time sequence pair $(\mathcal{I}, \mathcal{S}) \in \mathfrak{I}_1(\theta_1) \times \mathfrak{S}(N)$, *where* $\mathfrak{S}(N)$ *is the set of all the admissible switching time sequences satisfying (A.5).*

Proof. We prove the SISS property of the system (1) by constructing the functions β and γ such that the inequality (2) holds. According to (A.2) to (A.4), the proof is divided into two cases, ie, the case that $V_{\sigma(t)}(t, x(t)) \geq \max\{\bar{\rho}_1 |V_{\sigma(t)}(t, x(t))|_\tau, \rho_2(|u(t)|)\}$ and the case that $V_{\sigma(t)}(t, x(t)) < \max\{\bar{\rho}_1 |V_{\sigma(t)}(t, x(t))|_\tau, \rho_2(|u(t)|)\}$.

Case 1: $V_{\sigma(t)}(t, x(t)) \geq \max\{\bar{\rho}_1 |V_{\sigma(t)}(t, x(t))|_\tau, \rho_2(|u(t)|)\}$ for all $t \in \mathbb{R}^+_{t_0}$. In this case, the conditions (A.2)-(A.4) are written directly as

$$\begin{aligned} &\mathscr{L} V_{\sigma(t)}(t, x) \leq -\varphi(V_{\sigma(t)}(t, x)), \quad t \in \mathbb{R}^+_{t_0} \backslash \mathcal{T}, \\ &V_{\sigma(t)}(t, x(t)) \leq \phi_1(V_{\sigma(t^-)}(t^-, x(t^-))), \quad t \in \mathcal{I}, \\ &V_{\sigma(t)}(t, x(t)) \leq \phi_2(V_{\sigma(t^-)}(t^-, x(t^-))), \quad t \in \mathcal{S} \backslash \mathcal{I}. \end{aligned} \tag{4}$$

Since $\varphi \in \mathcal{VK}$ and $\phi_1, \phi_2 \in \mathcal{CK}$, it follows from the expectation operator and Jensen's inequality in chapter 2, 18.3 in the work of Rogers and Williams[31] that

$$\mathbb{E}[\mathscr{L} V_{\sigma(t)}(t, x)] \leq -\varphi(\mathbb{E}[V_{\sigma(t)}(t, x)]), \quad t \in \mathbb{R}^+_{t_0} \backslash \mathcal{T}, \tag{5}$$

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \leq \phi_1(\mathbb{E}[V_{\sigma(t^-)}(t^-, x(t^-))]), \quad t \in \mathcal{I}, \tag{6}$$

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \leq \phi_2(\mathbb{E}[V_{\sigma(t^-)}(t^-, x(t^-))]), \quad t \in \mathcal{S} \backslash \mathcal{I}. \tag{7}$$

If there exists a $\bar{t} \in [t_k, t_{k+1})$ such that $\mathbb{E}[V_{\sigma(t)}(\bar{t}, x(\bar{t}))] = 0$, we obtain from (5) to (7) and the equilibrium point $x(t) = 0$ that $\mathbb{E}[V_{\sigma(t)}(t, x(t))] \equiv 0$ holds for all $t > \bar{t}$, which is a trivial case. In the following, we just need to study the case that $\mathbb{E}[V_{\sigma(t)}(t, x(t))] > 0$.

Integrating both sides of (5) yields that

$$\int_{t_k}^{t} \frac{\mathbb{E}[\mathscr{L} V_{\sigma(t)}(s, x(s))] ds}{\varphi(\mathbb{E}[V_{\sigma(t)}(s, x(s))])} \leq -(t - t_k), \quad \forall t \in [t_k, t_{k+1}). \tag{8}$$

Based on (8), define the following function: for any fixed $\nu > 0$,

$$F(\varrho) := \int_\nu^\varrho \frac{ds}{\varphi(s)}, \quad \forall \varrho > 0. \tag{9}$$

Since $\varphi \in \mathcal{VK}_\infty$, the function $F : \mathbb{R}^+_0 \to \mathbb{R}$ is continuous and strictly increasing, and its inverse $F^{-1} : \mathbb{R} \to \mathbb{R}^+_0$ is also continuous and strictly increasing.

Using Itô's formula in chapter IV, 3 in the work of Rogers and Williams[31] and Fubini's Theorem in chapter II, 12.2 in the work of Rogers and Williams,[31] the inequality (8) is equivalent to

$$\int_{\mathbb{E}[V_{\sigma(t)}(t_k, x(t_k))]}^{\mathbb{E}[V_{\sigma(t)}(t, x(t))]} \frac{ds}{\varphi(s)} \leq -(t - t_k), \quad \forall t \in [t_k, t_{k+1}), \tag{10}$$

which implies that

$$F(\mathbb{E}[V_{\sigma(t)}(t, x(t))]) - F(\mathbb{E}[V_{\sigma(t)}(t_k, x(t_k))]) \leq -(t - t_k), \quad \forall t \in [t_k, t_{k+1}).$$

For any impulsive interval $[i_k, i_{k+1}]$, $k \in \mathbb{N}$, since the switches may occur in the impulsive interval, we obtain from (6) to (7), (10), and the FDT condition (A.5) that

$$\begin{aligned}
&F(\mathbb{E}[V_{\sigma(i_{k+1})}(i_{k+1}, x(i_{k+1}))]) - F(\mathbb{E}[V_{\sigma(i_k)}(i_k, x(i_k))]) \\
&\leq F\left(\phi_1\left(\mathbb{E}\left[V_{\sigma\left(i_{k+1}^-\right)}\left(i_{k+1}^-, x\left(i_{k+1}^-\right)\right)\right]\right)\right) - F\left(\mathbb{E}\left[V_{\sigma\left(i_{k+1}^-\right)}\left(i_{k+1}^-, x\left(i_{k+1}^-\right)\right)\right]\right) \\
&\quad + F\left(\mathbb{E}\left[V_{\sigma\left(i_{k+1}^-\right)}\left(i_{k+1}^-, x\left(i_{k+1}^-\right)\right)\right]\right) - F\left(\mathbb{E}\left[V_{\sigma(s_{j_1})}\left(s_{j_1}, x(s_{j_1})\right)\right]\right) \\
&\quad + F\left(\mathbb{E}\left[V_{\sigma(s_{j_1})}\left(s_{j_1}, x(s_{j_1})\right)\right]\right) - F\left(\mathbb{E}\left[V_{\sigma\left(s_{j_1}^-\right)}\left(s_{j_1}^-, x\left(s_{j_1}^-\right)\right)\right]\right) \\
&\quad + F\left(\mathbb{E}\left[V_{\sigma\left(s_{j_1}^-\right)}\left(s_{j_1}^-, x\left(s_{j_1}^-\right)\right)\right]\right) - F\left(\mathbb{E}\left[V_{\sigma(s_{j_2})}\left(s_{j_2}, x(s_{j_2})\right)\right]\right) \\
&\quad \ldots \\
&\quad + F\left(\mathbb{E}\left[V_{\sigma\left(s_{j_N}^-\right)}\left(s_{j_N}^-, x\left(s_{j_N}^-\right)\right)\right]\right) - F\left(\mathbb{E}\left[V_{\sigma(i_k)}\left(i_k, x(i_k)\right)\right]\right) \\
&\leq F\left(\phi_1\left(\mathbb{E}\left[V_{\sigma\left(i_{k+1}^-\right)}\left(i_{k+1}^-, x\left(i_{k+1}^-\right)\right)\right]\right)\right) - F\left(\mathbb{E}\left[V_{\sigma\left(i_{k+1}^-\right)}\left(i_{k+1}^-, x\left(i_{k+1}^-\right)\right)\right]\right) \\
&\quad + \sum_{p=1}^{N}\left[F\left(\phi_2\left(\mathbb{E}\left[V_{\sigma\left(s_{j_p}^-\right)}\left(s_{j_p}^-, x\left(s_{j_p}^-\right)\right)\right]\right)\right) - F\left(\mathbb{E}\left[V_{\sigma\left(s_{j_p}^-\right)}\left(s_{j_p}^-, x\left(s_{j_p}^-\right)\right)\right]\right)\right] - \theta_1 \\
&\leq -\delta,
\end{aligned}$$

where $s_{j_1}, \ldots, s_{j_N}$ are the switching time instants in the impulsive interval $[i_k, i_{k+1}]$, $k \in \mathbb{N}$. As a result, for any impulsive interval $[i_k, i_{k+1}]$, $k \in \mathbb{N}$, we have that

$$\mathbb{E}[V_{\sigma(i_{k+1})}(i_{k+1}, x(i_{k+1}))] \leq F^{-1}(F(\mathbb{E}[V_{\sigma(i_k)}(i_k, x(i_k))]) - \delta). \tag{11}$$

Iterating (11) from i_1 to i_{k+1}, $k \in \mathbb{N}$, we obtain that

$$\mathbb{E}[V_{\sigma(i_{k+1})}(i_{k+1}, x(i_{k+1}))] \leq F^{-1}(F(\mathbb{E}[V_{\sigma(i_1)}(i_1, x(i_1))]) - k\delta),$$

which holds for all $k \in \mathfrak{K} := \{k \in \mathbb{N} | F(\mathbb{E}[V_{\sigma(i_1)}(i_1, x(i_1))]) - k\delta \geq \lim_{\rho \downarrow 0} F(\rho)\}$. Denote $k_1 := \max_{k \in \mathfrak{K}} k$ (if not exists, $k_1 := \infty$) and $r := \mathbb{E}[V(t_0, x(t_0))]$.

In the following, a class $\mathcal{KL}$ function β_1 is constructed as a bound of $\mathbb{E}[V_{\sigma(t)}(t, x(t))]$. For all $k \in \{1, \ldots, k_1\}$, define

$$\begin{aligned}
\beta_1(r, i_1 - t_0) &:= \max\left\{\mathbb{E}[V_{\sigma(i_1)}(i_1, x(i_1))], \phi_1(\mathbb{E}[V_{\sigma(i_1)}(i_1, x(i_1))])\right\}, \\
\beta_1(r, i_{k+1} - t_0) &:= F^{-1}(F(\beta_1(r, i_k - t_0)) - k\delta).
\end{aligned}$$

In the interval $(i_k - t_0, i_{k+1} - t_0)$, $\beta_1(r, s)$ is required to be continuously decreasing and to lie above every solution of (5). If $i_1 > t_0$, then $\beta_1(r, s)$ can be constructed to be continuously decreasing in $[0, i_1 - t_0)$. If $k_1 < \infty$, then $\beta_1(r, s)$ in the interval $[i_{k_1} - t_0, \infty)$ can be defined to decrease continuously to zero as $s \to \infty$. Therefore, from the construction of $\beta_1(r, s)$, we have that $\beta_1(r, s)$ is continuous and decreasing with respect to s, and

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \leq \beta_1(\mathbb{E}[V_{\sigma(t_0)}(t_0, x(t_0))], t - t_0), \quad \forall t \in \mathbb{R}_{t_0}^+.$$

If $k_1 = \infty$, then we need to prove that $\beta_1(r, s) \to 0$ as $s \to \infty$.

Claim that, if $\beta_1(r, i_k - t_0) \to 0$ as $k \to \infty$, then $\beta_1(r, s) \to 0$ as $s \to \infty$. Assume that such a claim is invalid, and thus, there exists $\epsilon > 0$ such that $\lim_{k \to \infty} \beta_1(r, i_k - t_0) = \epsilon > 0$, where ϵ is related to the choice of r. Denote $\vartheta := \min_{\epsilon \leq \nu \leq \beta_1(r,0)} \varphi(\nu)$. From the middle-value theorem, we obtain that

$$\begin{aligned}
\delta &\leq F(\beta_1(r, i_k - t_0)) - F(\beta_1(r, i_{k+1} - t_0)) \\
&\leq \frac{\beta_1(r, i_k - t_0) - \beta_1(r, i_{k+1} - t_0)}{\vartheta},
\end{aligned}$$

which further implies that

$$\beta_1(r, i_k - t_0) - \beta_1(r, i_{k+1} - t_0) \geq \delta\vartheta > 0. \tag{12}$$

From (12) and the construction of the function β_1, $\beta_1(r, i_k - t_0)$ decreases to zero as $k \to \infty$, which contradicts with the assumption that $\lim_{k\to\infty}\beta_1(r, i_k - t_0) = \epsilon > 0$. Therefore, we conclude that the aforementioned claim is valid. That is, given $r > 0$, $\beta_1(r, s) \to \infty$ as $s \to \infty$.

Define the functions $\beta_2(r, t) := \sup_{0\leq v\leq r}\beta_1(v, t)$ and $\beta_3(r, t) := \frac{1}{r}\int_r^{2r} \beta_2(s, t)ds + re^{-t}$. From the construction of the function β_1, we have that $\beta_3(r, t) \geq \beta_2(r, t) \geq \beta_1(r, t)$ for all $r, t > 0$ and that $\beta_3(r, t) \in \mathcal{KL}$. As a result,

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \leq \beta_3(\mathbb{E}[V_{\sigma(t_0)}(t_0, x(t_0))], t - t_0), \quad \forall t \in \mathbb{R}^+_{t_0}. \tag{13}$$

Case 2: $V_{\sigma(t)}(t, x(t)) \leq \max\{\bar{\rho}_1|V_{\sigma(t)}(t, x(t))|_\tau, \rho_2(|u(t)|)\}$ for all $t \in \mathbb{R}^+_{t_0}$. In this case, by taking expectation, we have that

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \leq \bar{\rho}_1\mathbb{E}[|V_{\sigma(t)}(t, x(t))|_\tau] + \rho_2(|u(t)|). \tag{14}$$

Using lemma 1 in the work of Mazenc and Malisoff,[32] we obtain from (14) that

$$\begin{aligned}\mathbb{E}[V_{\sigma(t)}(t, x(t))] &\leq e^{\frac{\ln\bar{\rho}_1}{\tau}(t-t_0)}\mathbb{E}[|V_{\sigma(t_0)}(t_0, x(t_0))|_\tau] + (1 - \bar{\rho}_1)^{-2}\rho_2(\|u\|)\\ &=: \beta_4(\mathbb{E}[|V_{\sigma(t_0)}(t_0, x(t_0))|_\tau], t - t_0) + \gamma_1(\|u\|),\end{aligned} \tag{15}$$

where $\ln\bar{\rho}_1 < 0$ holds from the fact that $\bar{\rho}_1 \in (0, 1)$, $\beta_4(v, t) := e^{\frac{\ln\bar{\rho}_1}{\tau}t}v$, and $\gamma_1(v) := (1 - \bar{\rho}_1)^{-2}\rho_2(v)$.

In aforementioned two cases, multiple Lyapunov functions are decreasing and convergent along the time line. In the following, we consider the combination of such two cases. Because of the decrease of multiple Lyapunov functions, there exists a time instant $t^* \in \mathbb{R}^+_{t_0}$ such that $V_{\sigma(t)}(t, x(t)) \leq \max\{\bar{\rho}_1|V_{\sigma(t)}(t, x(t))|_\tau, \rho_2(|u(t)|)\}$ holds for all $t \geq t^*$, which thus implies that (15) holds for all $t \geq t^*$. In the interval $[t_0, t^*]$, both aforementioned cases may hold in certain bounded subintervals. We divide the interval $[t_0, t^*]$ into finite subintervals $[T_k, T_{k+1})$ with $k \in \{0, \dots, 2K\}$ and finite $K \in \mathbb{N}$. That is, $[t_0, t^*] = \bigcup_{k\in\{0,\dots,2K\}}[T_k, T_{k+1})$ with $T_0 = t_0$ and $T_{2K+1} = t^*$. Without loss of generality, assume that the first case holds in $[T_{2j}, T_{2j+1})$ with $j \in \{0, \dots, K\}$ and the second case holds in $[T_{2j+1}, T_{2j+2})$ with $j \in \{0, \dots, K\}$. Hence, for all $j \in \{0, \dots, K\}$, we have that

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \leq \beta_3(\mathbb{E}[V_{\sigma(T_{2j})}(T_{2j}, x(T_{2j}))], t - T_{2j}), \quad \forall t \in [T_{2j}, T_{2j+1}), \tag{16}$$

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \leq \beta_4(\mathbb{E}[|V_{\sigma(T_{2j+1})}(T_{2j+1}, x(T_{2j+1}))|_\tau], t - T_{2j+1}) + \gamma_1(\|u\|), \quad \forall t \in [T_{2j+1}, T_{2j+2}), \tag{17}$$

where $T_{2K+2} := \infty$. Since $\mathbb{E}[V_{\sigma(t)}(t, x(t))]$ decreases in the intervals $[T_k, T_{k+1}]$ with $k \in \{0, \dots, 2K\}$, and the upper bound functions depend on the starting instants, we can follow (16) and (17) and construct the functions $\bar{\beta} : \mathbb{R}^+_0 \times \mathbb{R}^+_0 \to \mathbb{R}^+_0$ and $\bar{\gamma} : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ such that

- at the time instant $t = T_k$ and $k \in \{0, \dots, 2K + 1\}$,

$$\bar{\beta}(\mathbb{E}[V_{\sigma(t)}(t_0, x(t_0))], T_k - t_0) \geq \bar{\beta}(\mathbb{E}[V_{\sigma(t)}(t_0, x(t_0))], T_{k+1} - t_0); \tag{18}$$

- at the time instant $t = T_{2j}$ and $j \in \{0, \dots, K\}$,

$$\bar{\beta}(\mathbb{E}[V_{\sigma(t)}(t_0, x(t_0))], t - t_0) + \bar{\gamma}(\|u\|) \geq \beta_3(\mathbb{E}[V_{\sigma(T_{2j})}(T_{2j}, x(T_{2j}))], t - T_{2j}); \tag{19}$$

- at the time instant $t = T_{2j+1}$ and $j \in \{0, \dots, K\}$,

$$\bar{\beta}(\mathbb{E}[V_{\sigma(t)}(t_0, x(t_0))], t - t_0) + \bar{\gamma}(\|u\|) \geq \beta_4(\mathbb{E}[|V_{\sigma(T_{2j+1})}(T_{2j+1}, x(T_{2j+1}))|_\tau], t - T_{2j+1}) + \gamma_1(\|u\|). \tag{20}$$

Such functions $\bar{\beta}$ and $\bar{\gamma}$ exist due to (16) and (17) and finite $K \in \mathbb{N}$. In the interval $[T_k, T_{k+1}]$ with $k \in \{0, \dots, 2K\}$, we require $\bar{\beta}(v, s)$ to be continuously decreasing and larger than $\beta_3(v, s)$ and $\beta_4(v, s)$ in the corresponding intervals. Since (20) holds for the time instant $t = T_{2K+1}$, we have that (20) holds for all $t > T_{2K+1} = t^*$.

From (A.1) and the construction of the functions $\bar{\beta}$ and $\bar{\gamma}$, we obtain that

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \leq \bar{\beta}(\alpha_2(\mathbb{E}[\|\xi\|_\tau]), t - t_0) + \bar{\gamma}(\|u\|), \quad \forall t \in \mathbb{R}^+_{t_0}. \tag{21}$$

Furthermore, we can majorize the functions $\bar{\beta}$ and $\bar{\gamma}$ to be of classes $\mathcal{KL}$ and $\mathcal{K}_\infty$, respectively. It follows from (21) and Markov's inequality in chapter II, 18.1 in the work of Rogers and Williams[31] that, for any $\epsilon \in (0,1)$, there exist $\beta(v,t) := \alpha_1^{-1}(2\bar{\beta}(\alpha_2(v),t))/\epsilon$ and $\gamma(v) := \alpha_1^{-1}(2\bar{\gamma}(v))/\epsilon$ such that

$$\mathbb{P}\left\{|x(t)| \leq \beta(\mathbb{E}[\|\xi\|_\tau], t-t_0) + \gamma(\|u\|)\right\} \geq 1-\epsilon, \quad \forall t \in \mathbb{R}^+_{t_0}.$$

Thus, the proof is completed. □

Remark 2. In this section, we define the set $\mathfrak{I}_1(\theta_1)$ with some constant $\theta_1 > 0$, which thus implies that the impulsive intervals are not less than θ_1. In addition, we assume that $0 < N < \infty$, which implies that the maximal number of the switches is finite in any impulsive interval, and is reasonable due to the fact that the sequence $\mathcal{S}$ is strictly increasing. From $0 < N < \infty$ and the FDT condition (A.5), the switching intervals are positive. As a result, the Zeno and chattering phenomena are ruled out in this paper.

Remark 3. In Theorem 1, the conditions (A.2) to (A.4) are the nonlinear version, ie, the functions ψ, ϕ_1, ϕ_2 are nonlinear. Therefore, the obtained conditions are more general than the existing results in other works[13,16,22] based on exponential Lyapunov functions. The derived stability conditions in Theorem 1 can be applied to study dynamic systems that cannot be analyzed via exponential Lyapunov functions; see the numerical example in Section 5. As a result, Theorem 1 extends the existing results in the works of Liu et al[13] and Teel[33] to the case of stochastic impulsive switched time-delay systems, and can be applied to control systems that cannot be studied via exponential Lyapunov functions.

Remark 4. In comparison with the result obtained in the work of Ren and Xiong,[24] the FDT condition (A.5) implies that the sequences $\mathcal{I}$ and $\mathcal{S}$ are coupled. In the left-hand side of the inequality (3), the first item is for the impulses and the second item is for the switches. If the functions ψ, ϕ_1, ϕ_2 are linear, then the FDT condition (A.5) is reduced to the average dwell-time condition, which is also a coupled version for the sequences $\mathcal{I}$ and $\mathcal{S}$. The scenarios that $\mathcal{I}$ and $\mathcal{S}$ overlap or are disjoint are also included in Theorem 1, and the constant N is to constraint the number of the switches in the impulsive intervals. If the sequences $\mathcal{I}$ and $\mathcal{S}$ are the same, then $N \equiv 1$, and the FDT condition (A.5) is reduced to $\int_a^{\phi(a)} \psi^{-1}(s)ds \leq \theta_1 - \delta$ for all $a > 0$, where $\phi(a) = \max\{\phi_1(a), \phi_2(a)\}$. On the other hand, if both the continuous dynamics and the impulsive dynamics in (1) are stable, then $\psi \leq \mathrm{Id}$, which implies that $\int_a^{\phi_1(a)} \psi^{-1}(s)ds \leq 0$. Therefore, the FDT condition (A.5) is relaxed as

$$N \int_a^{\phi_2(a)} \frac{ds}{\varphi(s)} \leq \theta_1 - \delta, \quad \forall a > 0,$$

which only constrains the switching time sequence, and implies that the switching intervals are smaller than θ_1/N.

In Theorem 1, the Razumikhin condition is linear with respect to the time-delay item, which can be further relaxed. In the following, based on the relation between Razumikhin-type theorem and small gain theorem,[33] the next theorem provides an alternative for SISS of the system (1), and relaxes the Razumikhin condition in Theorem 1.

Theorem 2. *Consider the system (1). Assume that there exist $C^{1,2}$ Lyapunov functions $V_l : \mathbb{R}^+_{t_0-\tau} \times \mathbb{R}^{n_x} \to \mathbb{R}^+_0$, $l \in \mathfrak{L}$, $\alpha_1, \varphi \in \mathcal{VK}_\infty, \alpha_2, \phi_1, \phi_2 \in \mathcal{CK}_\infty, \rho_1, \rho_2 \in \mathcal{K}_\infty$ with $\phi_1 > \mathrm{Id}, \phi_2 \geq \mathrm{Id}, \rho_1 < \mathrm{Id}$, and constants $\theta_1 > \delta > 0, 0 < N < \infty$ such that (A.1) and (A.3) to (A.5) hold, and*

(A.2′) for all $t \in \mathbb{R}^+_{t_0} \backslash \mathcal{T}$, $V_{\sigma(t)}(t,x(t)) \geq \max\{\rho_1(|V_{\sigma(t)}(t,x(t))|_\tau), \rho_2(|u(t)|)\}$ implies that

$$\mathscr{L} V_{\sigma(t)}(t,x_t) \leq -\varphi(V_{\sigma(t)}(t,x(t))).$$

Then, the system (1) is SISS for all the discrete-time sequence pair $(\mathcal{I},\mathcal{S}) \in \mathfrak{I}_1(\theta_1) \times \mathfrak{S}(N)$, where $\mathfrak{S}(N)$ is defined in Theorem 1.

Proof. Similar to the proof of Theorem 1, the stability analysis is divided into two cases, ie, the case that $V_{\sigma(t)}(t,x(t)) \geq \max\{\rho_1(|V_{\sigma(t)}(t,x(t))|_\tau), \rho_2(|u(t)|)\}$ and the case that $V_{\sigma(t)}(t,x(t)) < \max\{\rho_1(|V_{\sigma(t)}(t,x(t))|_\tau), \rho_2(|u(t)|)\}$. The proof for Case 1 is the same as that of Theorem 1 and (12) holds. For Case 2, by taking expectation, we have that

$$\mathbb{E}[V_{\sigma(t)}(t,x(t))] \leq \max\{\rho_1(\mathbb{E}[\|V_{\sigma(t)}(t,x(t))\|_\tau]), \rho_2(|u(t)|)\},$$

combining which with (13) yields that, for all $t \in \mathbb{R}^+_{t_0}$,

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \le \max\{\rho_1(\mathbb{E}[\|V_{\sigma(t)}(t, x(t))\|_\tau]), \rho_2(\|u\|_\tau), \beta_3(\mathbb{E}[V_{\sigma(t_0)}(t_0, x(t_0))], t - t_0)\}, \tag{22}$$

$$\mathbb{E}[|V_{\sigma(t)}(t, x(t))|_\tau] \le \max\{\Upsilon(t - t_0)\mathbb{E}[|V_{\sigma(t)}(t_0, x(t_0))|_\tau], \mathbb{E}[\|V_{\sigma(t)}(t, x(t))\|_\tau]\}, \tag{23}$$

where $\|V_{\sigma(t)}(t, x(t))\|_\tau := \sup_{t \ge t_0} |V_{\sigma(t)}(t, x(t))|_\tau$ and $\Upsilon(t) := 0.5(1 - \mathrm{sgn}(t - \tau))$.

According to (A.2′), (22) and (23), and along the similar fashion as the proof of Theorem 1 in the work of Teel,[33] we obtain that there exist $\bar{\beta}(v, t) := \alpha_1^{-1}(\beta_3(\alpha_2(v), t))$ and $\bar{\gamma}(v) := \alpha_1^{-1}(\rho_2(v))$ such that

$$\mathbb{E}[\|x(t)\|_\tau] \le \bar{\beta}(\mathbb{E}[\|\xi\|_\tau], t - t_0) + \bar{\gamma}(\|u\|), \quad \forall t \in \mathbb{R}^+_{t_0}. \tag{24}$$

Note that $\mathbb{E}[\|x(t)\|_\tau] \ge \mathbb{E}[|x(t)|]$ for all $t \in \mathbb{R}^+_{t_0}$. From (24) and Markov's inequality in chapter II, 18.1 in the work of Rogers and Williams,[31] we have that, for any $\varepsilon \in (0, 1)$, there exist $\beta(v, t) := \bar{\beta}(v, t)/\varepsilon$ and $\gamma(v) := \bar{\gamma}(v)/\varepsilon$ such that

$$\mathbb{P}\{|x(t)| \le \beta(\mathbb{E}[\|\xi\|_\tau], t - t_0) + \gamma(\|u\|)\} \ge 1 - \varepsilon, \quad \forall t \in \mathbb{R}^+_{t_0}.$$

Thus, the proof is completed. □

Remark 5. The difference between Theorems 1 and 2 lies in the Razumikhin condition. In particular, the function ρ_1 in Theorem 2 is allowed to be arbitrary function satisfying the small gain condition, which is a relaxation of the Razumikhin condition in Theorem 1. Note that ρ_1 in Theorem 2 can be linear, and the SISS property is still established. However, if the Razumikhin condition in Theorem 1 is relaxed, then lemma 1 in the work of Mazenc and Malisoff[32] cannot be applied; see remark 1 in the work of the aforementioned authors[32] for more details.

4 | STABLE IMPULSIVE DYNAMICS CASE

In this section, we study the stable impulsive dynamics case, which means that the impulsive dynamics in (1) is stable, whereas the continuous dynamics is not. For such case, Razumikhin-type stability conditions are established. Before presenting the main results, define the set $\mathfrak{S}_1(\theta_2) := \{\mathcal{S} = \{s_1, s_2, \dots\} | \mathcal{S} \subseteq \mathbb{R}^+_{t_0}, 0 < s_{k+1} - s_k \le \theta_2, \forall k \in \mathbb{N}\}$ for some constant $\theta_2 > 0$. That is, all the switching intervals are less than θ_2. With the set $\mathfrak{S}_1(\theta_2)$, the following theorem establishes sufficient conditions for the system (1) in the stable impulsive dynamics case.

Theorem 3. *Consider the system (1). Assume that there exist $C^{1,2}$ Lyapunov functions $V_l : \mathbb{R}^+_{t_0-\tau} \times \mathbb{R}^{n_x} \to \mathbb{R}^+_0$, $l \in \mathfrak{L}$, $\alpha_1 \in \mathcal{VK}_\infty, \alpha_2, \varphi, \phi_1, \phi_2 \in \mathcal{CK}_\infty, \rho_2 \in \mathcal{K}_\infty$ with $\phi_1 < \mathrm{Id}, \phi_2 \ge \mathrm{Id}$, and constants $\bar{\rho}_1 \in (0, 1), \theta_2 > \delta > 0, 0 < M < \infty$ such that (A.1) holds, and*

(B.1) for all $t \in \mathbb{R}^+_{t_0} \backslash \mathcal{T}$, $V_{\sigma(t)}(t, x(t)) \ge \max\{\bar{\rho}_1 |V_{\sigma(t)}(t, x(t))|_\tau, \rho_2(|u(t)|)\}$ implies that

$$\mathscr{L} V_{\sigma(t)}(t, x_t) \le \varphi(V_{\sigma(t)}(t, x(t)));$$

(B.1) for all $t \in \mathcal{I}$, $V_{\sigma(t)}(t, x(t)) \le \phi_1(V_{\sigma(t^-)}(t^-, x(t^-)))$;

(B.3) for all $t \in \mathcal{S} \backslash \mathcal{I}$, $V_{\sigma(t)}(t, x(t)) \le \phi_2(V_{\sigma(t^-)}(t^-, x(t^-)))$;

(B.4) the FDT condition is satisfied, ie,

$$M \int_{\phi_1(a)}^{a} \frac{ds}{\varphi(s)} + \int_{a}^{\phi_2(a)} \frac{ds}{\varphi(s)} \ge \theta_2 - \delta, \quad \forall a > 0,$$

where $M \in \mathbb{N}^+$ is the maximal number of the impulses in an switching interval; then, the system is SISS for all the time sequence pair $(\mathcal{I}, \mathcal{S}) \in \mathfrak{I}(M) \times \mathfrak{S}_1(\theta_2)$, where $\mathfrak{I}(M)$ denotes the set of all the admissible impulsive time sequences satisfying (B.4).

Proof. For the case that $V_{\sigma(t)}(t, x(t)) \ge \max\{\bar{\rho}_1 |V_{\sigma(t)}(s, x(s))|_\tau, \rho_2(|u(t)|)\}$ for all $t \in \mathbb{R}^+_{t_0}$, since $\varphi, \phi_1, \phi_2 \in \mathcal{CK}$, it follows from (B.1) to (B.3) and Jensen's inequality that

$$\mathbb{E}[\mathscr{L} V_{\sigma(t)}(t, x(t))] \le \varphi(\mathbb{E}[V_{\sigma(t)}(t, x(t))]), \quad t \in \mathbb{R}^+_{t_0} \backslash \mathcal{T}, \tag{25}$$

$$\mathbb{E}[V_{\sigma(t)}(t, h(x, u))] \le \phi_1(\mathbb{E}[V_{\sigma(t)}(t, x(t))]), \quad t \in \mathcal{I}, \tag{26}$$

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \le \phi_2(\mathbb{E}[V_{\sigma(t^-)}(t^-, x(t^-))]), \quad t \in \mathcal{S} \backslash \mathcal{I}. \tag{27}$$

Integrating (25) implies that, for any $t \in [t_k, t_{k+1})$,

$$\int_{t_k}^{t} \frac{\mathbb{E}[\mathscr{L} V_{\sigma(t)}(s, x(s))]ds}{\varphi(\mathbb{E}[V_{\sigma(t)}(s, x(s))])} \le t - t_k.$$

Similar to the proof of Theorem 1, we define the following function: for any fixed $\nu > 0$,

$$F(\varrho) := \int_{\nu}^{\varrho} \frac{ds}{\varphi(s)}, \quad \forall \varrho > 0.$$

Thus, $F : \mathbb{R}_0^+ \to \mathbb{R}$ and its inverse $F^{-1} : \mathbb{R} \to \mathbb{R}_0^+$ are continuous and strictly increasing.

Along the similar fashion as in the proof of Theorem 1, it follows from the FDT condition (B.4) and (26) and (27) that, for the switching time instants $s_k, s_{k+1} \in \mathcal{S}$, $k \in \mathbb{N}$,

$$\begin{aligned} &F(\mathbb{E}[V_{\sigma(s_{k+1})}(s_{k+1}, x(s_{k+1}))]) - F(\mathbb{E}[V_{\sigma(s_k)}(s_k, x(s_k))]) \\ &\le F\left(\phi_2\left(\mathbb{E}\left[V_{\sigma(s_k)}\left(s_{k+1}^-, x\left(s_{k+1}^-\right)\right)\right]\right)\right) - F\left(\mathbb{E}\left[V_{\sigma(s_k)}\left(s_{k+1}^-, x\left(s_{k+1}^-\right)\right)\right]\right) \\ &\quad + F\left(\mathbb{E}\left[V_{\sigma(s_k)}\left(s_{k+1}^-, x\left(s_{k+1}^-\right)\right)\right]\right) - F(\mathbb{E}[V_{\sigma(s_k)}(i_{j_1}, x(i_{j_1}))]) \\ &\quad + F(\mathbb{E}[V_{\sigma(s_k)}(i_{j_1}, x(i_{j_1}))]) - F\left(\mathbb{E}\left[V_{\sigma(s_k)}\left(i_{j_1}^-, x\left(i_{j_1}^-\right)\right)\right]\right) \\ &\quad + F\left(\mathbb{E}\left[V_{\sigma(s_k)}\left(i_{j_1}^-, x\left(i_{j_1}^-\right)\right)\right]\right) - F(\mathbb{E}[V_{\sigma(s_k)}(i_{j_2}, x(i_{j_2}))]) \\ &\quad \dots \\ &\quad + F\left(\mathbb{E}\left[V_{\sigma(s_k)}\left(i_{j_M}^-, x\left(i_{j_M}^-\right)\right)\right]\right) - F(\mathbb{E}[V_{\sigma(s_k)}(s_k, x(s_k))]) \\ &\le \theta_2 - \delta - \theta_2 = -\delta, \end{aligned} \tag{28}$$

where $i_{j_1}, \dots, i_{j_M}$ are the impulsive time instants in the switching interval $[s_k, s_{k+1})$, $k \in \mathbb{N}$. As a result, we obtain from (28) that

$$\mathbb{E}[V_{\sigma(s_{k+1})}(s_{k+1}, x(s_{k+1}))] \le F^{-1}(F(\mathbb{E}[V_{\sigma(s_k)}(s_k, x(s_k))]) - \delta).$$

According to the similar construction of the function $\beta_1(r, s)$ as in the proof of Theorem 1, one has that

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \le \beta_1(V_{\sigma(t_0)}(t_0, x(t_0)), t - t_0), \quad \forall t \in \mathbb{R}_{t_0}^+. \tag{29}$$

The remaining is the same as the proof of Theorem 1, and hence, the inequality (13) holds for all $t \in \mathbb{R}_{t_0}^+$.

For the second case that $V_{\sigma(t)}(t, x(t)) \le \max\{\bar{\rho}_1 |V_{\sigma(t)}(t, x(t))|_\tau, \rho_2(|u(t)|)\}$ for all $t \in \mathbb{R}_{t_0}^+$, along the same fashion as Case 2 in Theorem 1, it obtains that

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \le e^{\frac{\ln \bar{\rho}_1}{\tau}(t-t_0)} \alpha_2(\mathbb{E}[\|\xi\|_\tau]) + (1 - \bar{\rho}_1)^{-2} \rho_2(\|u\|). \tag{30}$$

The remaining is the same as the proof of Theorem 1, and thus, we can conclude that the system (1) is SISS for all the time sequence pair $(\mathcal{I}, \mathcal{S}) \in \mathfrak{I}(M) \times \mathfrak{S}_1(\theta_2)$. □

Remark 6. Since the stable impulsive dynamics is studied in this section, we define the set $\mathfrak{S}_1(\theta_2)$ with some constant $\theta_2 > 0$, which implies that the switching intervals are positive and not larger than θ_1. In Theorem 3, the FDT condition (B.4) indicates that the number of the impulses is finite in any switching interval. In addition, the impulsive intervals are positive since the sequence $\mathcal{I}$ is strictly increasing. Similar to Remark 2, the Zeno and chattering phenomena are excluded in this paper.

Remark 7. From the proofs of Theorems 1 and 3, we observe that the sequence $\mathcal{I}$ is used as the beacons to construct the function β_3 in the proof of Theorem 1, whereas the sequence $\mathcal{S}$ is treated as the beacons to construct the function β_3 in the proof of Theorem 3. The essential reason lies in that the impulsive dynamics is assumed to be stable in Section 4, whereas the continuous dynamics is assumed to be stable in Section 3. In addition, the switches do not bring about the changes of the system state. As a result, in the proof of Theorem 3, we first construct a function to decrease along the switching times, and then make such a function decrease along the whole time line. From the FDT condition (B.4), the larger the number of the impulses in the switching intervals is, the faster the convergence of the system state is, which implies the effects of impulses on system stability and can be treated as a criterion in impulsive control of stochastic switched systems; see the works of Sun et al[6] and Rivadeneira and Moog.[8]

The counterpart of Theorem 2 is provided as follows.

Theorem 4. *Consider the system (1). Assume that there exist $C^{1,2}$ Lyapunov functions $V_l : \mathbb{R}^+_{t_0-\tau} \times \mathbb{R}^{n_x} \to \mathbb{R}^+_0$, $l \in \mathfrak{L}$, $\alpha_1 \in \mathcal{VK}_\infty, \alpha_2, \varphi, \phi_1, \phi_2 \in \mathcal{CK}_\infty$, $\rho_1, \rho_2 \in \mathcal{K}_\infty$ with $\phi_1 > \mathrm{Id}, \phi_2 \geq \mathrm{Id}, \rho_1 \leq \mathrm{Id}$ and constants $\theta > \delta > 0, 0 < M < \infty$ such that (A.1), (B.2)-(B.4) hold, and*

(B.1′) for all $t \in \mathbb{R}^+_{t_0} \backslash \mathcal{T}$, $V_{\sigma(t)}(t, x(t)) \geq \max\{\rho_1(|V_{\sigma(t)}(t, x(t))|_\tau), \rho_2(|u(t)|)\}$ implies that

$$\mathscr{L} V_{\sigma(t)}(t, x_t) \leq \varphi(V_{\sigma(t)}(t, x(t))).$$

Then, the system (1) is SISS for all the discrete-time sequence pair $(\mathcal{I}, \mathcal{S}) \in \mathfrak{I}(M) \times \mathfrak{S}_1(\theta_2)$, where $\mathfrak{I}(M)$ is defined in Theorem 3.

The proof is a combination of the strategies of the proofs of Theorems 2 and 3, and hence omitted here.

5 | NUMERICAL EXAMPLE

In this section, a numerical example is presented to illustrate the developed results in the previous sections. Consider the following stochastic switched delayed neural system (see also the work of Yang et al[18]):

$$dx(t) = \left[-C_{\sigma(t)}x(t) + A_{\sigma(t)}f_{\sigma(t)}(x(t)) + B_{\sigma(t)}f_{\sigma(t)}(x(t-\tau)) + D_{\sigma(t)} \int_{-\infty}^{t} p_{\sigma(t)}(t-s)f_{\sigma(t)}(x(s))ds + I_{\sigma(t)}(t) + u(t) \right] dt$$
$$+ g_{\sigma(t)}(x(t), x(t-\tau))dB(t), \tag{31}$$

where $x(t) \in \mathbb{R}^n$ is the neural system state, $f_{\sigma(t)}(x(t)) \in \mathbb{R}^n$ is the neuron activation function, $p_{\sigma(t)}(t)$ is the scalar function to describe the delay kernel, and $u(t) \in \mathbb{R}^{n_u}$ is the external input. $B(t) \in \mathbb{R}^{n_w}$ is Brownian motion and $g_{\sigma(t)}(x(t), x(t-\tau))$ is the noise perturbation, which is Borel-measurable. The switching function $\sigma(t) : \mathbb{R}^+_0 \to \mathfrak{L} = \{1, \ldots, L\}$ is piecewise continuous and the switching time sequence is denoted by $\mathcal{S}$. In addition, the weight matrices in (31) are of appropriate dimensions.

Let the system (31) be the driving system and assume that the switching signal is known to the receiver in priori. In the sequel, the controlled response systems with the same switching rule as in the system (31) can be designed as follows:

$$\begin{aligned} dy(t) = (&-C_{\sigma(t)}y(t) + A_{\sigma(t)}f_{\sigma(t)}(y(t)) + B_{\sigma(t)}f_{\sigma(t)}(y(t-\tau)) \\ &+ D_{\sigma(t)} \int_{-\infty}^{t} p_{\sigma(t)}(t-s)f_{\sigma(t)}(y(s))ds + I_{\sigma(t)}(t) + U_\sigma(t))dt \\ &+ G_{\sigma(t)}(y(t), y(t-\tau))dB(t), \end{aligned} \tag{32}$$

where $y(t) \in \mathbb{R}^n$ is the response neural system state. For each $l \in \mathfrak{L}$, $U_l(t) = \sum_{k=1}^{\infty} E_{lk}(y(t) - x(t))\delta(t - i_k)$ are the impulsive controllers, where $E_{lk} \in \mathbb{R}^{n\times n}$ are constant matrices, and $\delta(t - i_k)$ is the Dirac impulse function with discontinuous time sequence $\mathcal{I}$. That is, the control input is implemented in the impulsive time sequence $\mathcal{I}$. In addition, let $E_{lk} = E_l$.

Define the synchronization error $z(t) := y(t) - x(t)$. Based on (31) and (32), the stochastic impulsive switched error dynamical system $\mathcal{E}$ is obtained as

$$\begin{cases} dz(t) = \big[-C_{\sigma(t)}z(t) + A_{\sigma(t)}H_{\sigma(t)}(z(t)) + B_{\sigma(t)}H_{\sigma(t)}(z(t-\tau)) \\ \qquad\quad + D_{\sigma(t)} \int_{-\infty}^{t} p_{\sigma(t)}(t-s)H_{\sigma(t)}(z(s))ds - u(t)\big] dt \\ \qquad\quad + G_{\sigma(t)}(z(t), z(t-\tau))dB(t), \quad t \in \mathbb{R}^+_{t_0} \backslash \mathcal{T}, \\ z(t) = (I + E_{\sigma(t^-)})z(t^-), \quad t \in \mathcal{I}, \\ z(t) = z(t^-), \quad t \in \mathcal{S} \backslash \mathcal{I}, \end{cases}$$

where $H_{\sigma(t)}(z(t)) := f_{\sigma(t)}(y(t)) - f_{\sigma(t)}(x(t))$ and $G_{\sigma(t)}(z(t), z(t-\tau)) := g_{\sigma(t)}(y(t), y(t-\tau)) - g_{\sigma(t)}(z(t), z(t-\tau))$. Assume that there exists constants $J_l > 0$ such that $|H_l(z(t))| \leq J_l|z(t)|$ for all $l \in \mathfrak{L}$.

Choose the LRFs as $V_l(t, z(t)) = z^\top(t)z(t)$ for all $l \in \mathfrak{L}$. Thus, the condition (A.1) holds with $\alpha_1(v) = \alpha_2(v) := |v|^2$. For all $t \in \mathcal{I}$,

$$V_{\sigma(t)}(t, z(t)) \leq \lambda_{\max}^2(I + E_{\sigma(t^-)})V_{\sigma(t^-)}(t^-, z(t^-)),$$

and for all $t \in \mathcal{S}\backslash\mathcal{I}$, there exists $\mu_2 \geq 1$ such that

$$V_{\sigma(t)}(t, z(t)) \leq \mu_{2)}V_{\sigma(t^-)}(t^-, z(t^-)).$$

In the continuous interval, it obtains from detailed computation that, for all $t \in \mathbb{R}_{t_0}^+\backslash\mathcal{T}$, the differentials of the LRFs satisfy

$$\begin{aligned}
\mathscr{L}V_l(t, z(t)) = {} & 2z^\top(t)\left[-C_iz(t) + A_\sigma f_i(z(t)) + B_lf_i(z(t-\tau)) + D_l\int_{-\infty}^{t} p_l(t-s)f_l(z(s))ds - u(t)\right] \\
& + \mathrm{tr}\left[G_l^\top(z(t), z(t-\tau))G_l(z(t), z(t-\tau))\right] \\
\leq {} & \left[-2\lambda_{\min}(C_l) + \varrho_l^{-1} + \xi_l\lambda_{\max}\left(A_l^\top A_l\right) + \xi_l^{-1} + \varpi_l^{-1} + \epsilon_l^{-1}\right]V_l(t, z(t)) \\
& + \varrho_l\lambda_{\max}\left(B_l^\top B_l\right)V_l(t-\tau, z(t-\tau)) + \varpi_lJ_l^2K_l\lambda_{\max}\left(D_l^\top D_l\right)\int_{-\infty}^{t} p_i(t-s)V_l(s, z(s))ds \\
& + \mathrm{tr}\left[G_l^\top(z(t), z(t-\tau))G_l(z(t), z(t-\tau))\right] + \epsilon_lu^\top(t)u(t),
\end{aligned}$$

where $\varrho_l, \xi_l, \epsilon_l, \varpi_l$ are positive constants and $K_l = \int_0^\infty p_l(s)ds$.

Assume there are two subsystems, ie, $l \in \{1, 2\}$. Set $E_l = -0.3I$, $\mu_2 = 1.02$, $p_1(t) = \exp(-0.5t)$, $p_2(t) = \exp(-0.2t)$, and

$$\begin{aligned}
& C_1 = \begin{bmatrix} 7 & 0 \\ 0 & 6 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 3 & -0.3 \\ 6 & 5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1.4 & 1 \\ 0.4 & -8 \end{bmatrix}, \\
& C_2 = \begin{bmatrix} 0.7 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 2 & -0.3 \\ 5 & 4.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1.4 & 1 \\ 0.3 & -6 \end{bmatrix}, \\
& D_1 = \begin{bmatrix} -1.2 & -1 \\ -2.8 & -1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} -1.2 & -1 \\ -2.8 & -1.2 \end{bmatrix}, \quad u(t) = \begin{bmatrix} 4\sin(0.4t) \\ 5\cos(0.5t) \end{bmatrix}, \\
& g_1(x(t), x(t-\tau)) = a\begin{bmatrix} 2x_1(t) + x_1(t-\tau) \\ x_1(t-\tau) + x_2(t) + x_2(t-\tau) \end{bmatrix}, \\
& g_2(x(t), x(t-\tau)) = c\begin{bmatrix} x_1(t) + x_2(t-\tau) \\ x_1(t-\tau) + x_2(t) \end{bmatrix}.
\end{aligned}$$

Let $a = c = 0.5$, and it follows from Figure 1 that all the subsystems are unstable. If the condition $V_{\sigma(t)}(t, z(t)) \geq \max\{0.5|V_{\sigma(t)}(t, z(t))|_\tau, 0.5|u(t)|^2\}$ holds, then we have that

$$\mathscr{L}V_l(t, z(t)) \leq 32.0409V_l(t, z(t)) + 9.4014\int_{-\infty}^{t} V_l(s, z(s))ds, \quad \forall t \in \mathbb{R}_{t_0}^+\backslash\mathcal{T},$$

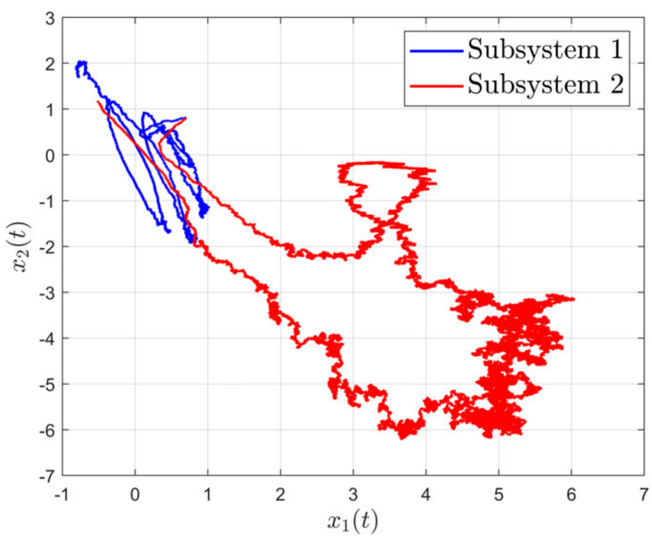

FIGURE 1 State trajectories of two subsystems with initial condition $x(t) = (0.7, 0.8)^\top$ for $t \in [-1, 0]$ [Colour figure can be viewed at wileyonlinelibrary.com]

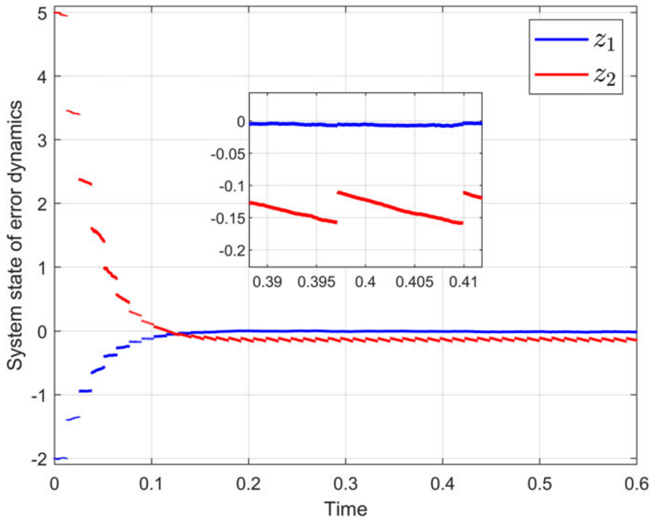

FIGURE 2 State response of the system $\mathcal{E}$. Both the impulsive time sequence and switching time sequence are periodic and overlapped, ie, $M = 3$ and $\theta_2 = 0.03843$ [Colour figure can be viewed at wileyonlinelibrary.com]

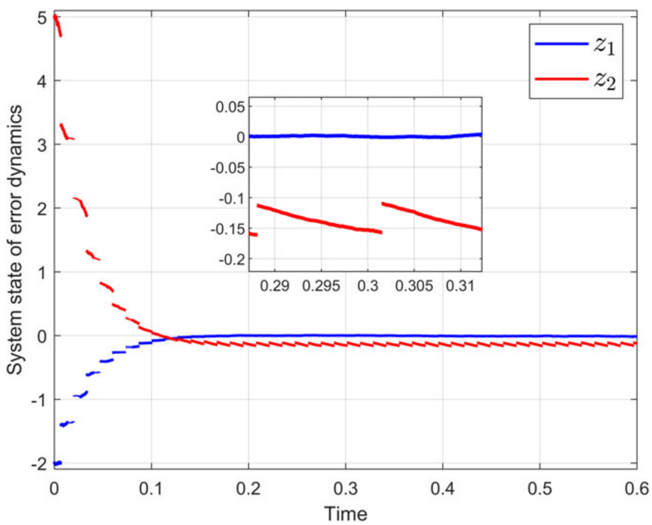

FIGURE 3 State response of the system $\mathcal{E}$. Both the impulsive time sequence and switching time sequence are periodic with the same period but not overlapped, ie, $M = 1$ and $\theta_2 = 0.0134$ [Colour figure can be viewed at wileyonlinelibrary.com]

which implies that Lyapunov functions are not exponential. As a result, it obtains from Theorem 3 that $\theta_2 - \delta \leq 0.0202M + 0.0126$, under which SISS of the error dynamical system $\mathcal{E}$ is guaranteed. Under the initial state $z(t) = [-2, 5]^\top$ for $t \in [-1, 0]$, $\tau = 0.05$, the Gaussian white noise with zero-mean and variance of 40, and the periodic impulsive time sequence and switching time sequence with $M = 3$ and $\theta_2 = 0.03843$, the state response of the estimate error system $\mathcal{E}$ is given in Figure 2. If the impulsive intervals are the same as the switching intervals but the impulsive time sequence and the switching time sequence are not overlapped ($M = 1$), then we have from Theorem 3 that $M = 1$ and $\theta_2 = 0.0134$. In the sequel, under the same conditions, the state response of the estimate error system $\mathcal{E}$ is given in Figure 3.

6 | CONCLUSION

In this paper, stochastic input-to-state stability was studied for a general class of stochastic impulsive switched time-delay systems, where the impulses and switches are allowed to occur asynchronously. Both the stable continuous dynamics case and the stable impulsive dynamics case were investigated. For such two cases, Razumikhin-type conditions were derived to guarantee system stability. Furthermore, the obtained results were illustrated via a numerical example. Future research can be directed to controller/observer design for stochastic impulsive systems and stability analysis for impulsive switched systems with state-dependent switching and impulses.

ACKNOWLEDGEMENT

This work was supported by National Natural Science Foundation of China, under Grant 61773357.

ORCID

Wei Ren https://orcid.org/0000-0002-3300-0993

REFERENCES

1. Haddad WM, Chellaboina VS, Nersesov SG. *Impulsive and Hybrid Dynamical Sytems: Stability, Dissipativity, and Control*. Princeton, NJ: Princeton University Press; 2014.
2. Dashkovskiy S, Mironchenko A. Input-to-state stability of nonlinear impulsive systems. *SIAM J Control Optim*. 2013;51(3):1962-1987.
3. Hespanha JP, Morse AS. Stability of switched systems with average dwell-time. In: Proceedings of the 38th IEEE Conference on Decision and Control; 1999; Phoenix, AZ.
4. Donkers MCF, Heemels WPMH, van de Wouw N, Hetel L. Stability analysis of networked control systems using a switched linear systems approach. *IEEE Trans Autom Control*. 2011;56(9):2101-2115.
5. Guan Z-H, Hill DJ, Yao J. A hybrid impulsive and switching control strategy for synchronization of nonlinear systems and application to Chua's chaotic circuit. *Int J Bifurc Chaos*. 2006;16(01):229-238.
6. Sun J, Zhang Y, Wu Q. Impulsive control for the stabilization and synchronization of Lorenz systems. *Phys Lett A*. 2002;298(2-3):153-160.
7. Lian J, Li C, Xia B. Sampled-data control of switched linear systems with application to an F-18 aircraft. *IEEE Trans Ind Electron*. 2017;64(2):1332-1340.
8. Rivadeneira PS, Moog CH. Impulsive control of single-input nonlinear systems with application to HIV dynamics. *Appl Math Comput*. 2012;218(17):8462-8474.
9. Wang D, Wang W, Shi P. Delay-dependent exponential stability for switched delay systems. *Optim Control Appl Methods*. 2009;30(4):383-397.
10. Feketa P, Bajcinca N. On robustness of impulsive stabilization. *Automatica*. 2019;104:48-56.
11. Jiao T, Zheng WX, Xu S. Stability analysis for a class of random nonlinear impulsive systems. *Int J Robust Nonlinear Control*. 2017;27(7):1171-1193.
12. Teel AR, Subbaraman A, Sferlazza A. Stability analysis for stochastic hybrid systems: a survey. *Automatica*. 2014;50(10):2435-2456.
13. Liu J, Liu X, Xie W-C. Class-$\mathcal{KL}$ estimates and input-to-state stability analysis of impulsive switched systems. *Syst Control Lett*. 2012;61(6):738-746.
14. Wu H, Sun J. p-Moment stability of stochastic differential equations with impulsive jump and Markovian switching. *Automatica*. 2006;42(10):1753-1759.
15. Ren W, Xiong J. Lyapunov conditions for stability of stochastic impulsive switched systems. *IEEE Trans Circuits Syst I Regul Pap*. 2018;65(6):1994-2004.
16. Zhang W, Tang Y, Miao Q, Du W. Exponential synchronization of coupled switched neural networks with mode-dependent impulsive effects. *IEEE Trans Neural Netw Learn Syst*. 2013;24(8):1316-1326.
17. Wang Y-W, Yang M, Wang HO, Guan Z-H. Robust stabilization of complex switched networks with parametric uncertainties and delays via impulsive control. *IEEE Trans Circuits Syst I Regul Pap*. 2009;56(9):2100-2108.
18. Yang X, Huang C, Zhu Q. Synchronization of switched neural networks with mixed delays via impulsive control. *Chaos Solit Fractals*. 2011;44(10):817-826.
19. Yao J, Guan Z-H, Chen G, Ho DW. Stability, robust stabilization and H_∞ control of singular-impulsive systems via switching control. *Syst Control Lett*. 2006;55(11):879-886.
20. Li Z, Soh Y, Wen C. *Switched and Impulsive Systems: Analysis, Design and Applications*. Vol 313. Berlin, Germany: Springer Science & Business Media; 2005.
21. Khadra A, Liu XZ, Shen X. Impulsively synchronizing chaotic systems with delay and applications to secure communication. *Automatica*. 2005;41(9):1491-1502.
22. Liu J, Liu X, Xie W-C. Input-to-state stability of impulsive and switching hybrid systems with time-delay. *Automatica*. 2011;47(5):899-908.
23. Ren W, Xiong J. Stability and stabilization of switched stochastic systems under asynchronous switching. *Syst Control Lett*. 2016;97:184-192.
24. Ren W, Xiong J. Stability analysis of impulsive stochastic nonlinear systems. *IEEE Trans Autom Control*. 2017;62(9):4791-4797.
25. Wu X, Tang Y, Zhang W. Input-to-state stability of impulsive stochastic delayed systems under linear assumptions. *Automatica*. 2016;66:195-204.
26. Chen W-H, Zheng WX. Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays. *Automatica*. 2009;45(6):1481-1488.

27. Gao L, Wang D. Input-to-state stability and integral input-to-state stability for impulsive switched systems with time-delay under asynchronous switching. *Nonlinear Anal Hybrid Syst*. 2016;20:55-71.
28. Mao X. *Stochastic Differential Equations and Applications*. 2nd ed. Cambridge, UK: Woodhead Publishing; 2007.
29. Alwan MS, Liu X, Xie W-C. Existence, continuation, and uniqueness problems of stochastic impulsive systems with time delay. *J Frankl Inst*. 2010;347(7):1317-1333.
30. Zhao P, Feng W, Kang Y. Stochastic input-to-state stability of switched stochastic nonlinear systems. *Automatica*. 2012;48(10):2569-2576.
31. Rogers LCG, Williams D. *Diffusions, Markov Processes and Martingales: Volume 1, Foundations*. Cambridge, UK: Cambridge University Press; 2000.
32. Mazenc F, Malisoff M. Trajectory based approach for the stability analysis of nonlinear systems with time delays. *IEEE Trans Autom Control*. 2015;60(6):1716-1721.
33. Teel AR. Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem. *IEEE Trans Autom Control*. 1998;43(7):960-964.

How to cite this article: Ren W, Xiong J. Razumikhin stability theorems for a general class of stochastic impulsive switched time-delay systems. *Int J Robust Nonlinear Control*. 2019;29:3988–4001. https://doi.org/10.1002/rnc.4590