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Robust stabilisation of time-varying delay systems with probabilistic uncertainties

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For robust stabilisation of time-varying delay systems, only sufficient conditions are available to date. A natural question is as follows: if the existing sufficient conditions are not satisfied, and hence no controllers can be found, what can one do to improve the stability performance of time-varying delay systems? This question is addressed in this paper when there is a probabilistic structure on the parameter uncertainty set. A randomised algorithm is proposed to design a state-feedback controller, which stabilises the system over the uncertainty domain in a probabilistic sense. The capability of the designed controller is quantified by the probability of stability of the resulting closed-loop system. The accuracy of the solution obtained from the randomised algorithm is also analysed. Finally, numerical examples are used to illustrate the effectiveness and advantages of the developed controller design approach.

Keywords: probabilistic uncertain parameters; randomised algorithms; robust stabilisation; time-varying delay systems

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1. Introduction

Time-varying delays are frequently encountered in various engineering systems such as biological systems, chemical systems, hydraulic systems and networked systems (Xia & Jia, 2002; Zhang, Wang, Xiao, & Chen, 2015). They are often a source of oscillations and instabilities in systems (Xu & Lam, 2008; Zhang, Lin, & Chen, 2015). Moreover, uncertainties are unavoidable in practical systems. Thus, robust stability analysis and synthesis problems for time-varying delay systems have attracted the attention of many researchers. For example, robust stability analysis problems were discussed in Xu and Lam (2005, 2008), He, Wang, Xie, and Lin (2007), Du, Lam, Shu, and Wang (2009) and Shao and Han (2012); and robust stabilisation problems were considered in Moon, Park, Kwon, and Lee (2001), Fridman and Shaked (2003) and Du, Lam, and Shu (2010). Recently, robust H_∞ control and filtering problems were studied in Xu, Lam, Yang, and Verriest (2003) and Xu, Lam, Chen, and Zou (2005). It should be noted that almost all of the works on the robust stability analysis and synthesis of time-varying delay systems require the uncertainties being of specific forms and provide sufficient conditions only.

Because of the sufficiency nature of the conditions, a time-varying delay system is considered to be robustly stable or can be robustly stabilised when one sufficient condition is feasible. However, when all the existing sufficient conditions fail to be satisfied, one cannot tell whether or not such a system is robustly stable or can be robustly stabilised. In other words, for the time-varying delay systems whose robust stability properties cannot be determined by solving the sufficient conditions (e.g., Gouaisbaut & Peaucelle, 2006; He et al., 2007; Jiang & Han, 2008; Xu & Lam, 2008), no information can be obtained about the level of stability over the uncertainty set. In addition, when the existing sufficient synthesis conditions (e.g., Fridman & Shaked, 2002; Moon et al., 2001) cannot provide a controller to robustly stabilise the systems over all the admissible uncertainties, nothing can be done to improve the closed-loop performance. Hence, a research question is how to analyse and synthesise these uncertain time-varying delay systems when all the sufficient conditions are not feasible. To answer this question, one approach is to develop less conservative design conditions. This paper will propose another approach.

In this paper, an approach based on randomised algorithm is proposed to robustly analyse and stabilise uncertain time-varying delay systems. The uncertainty set in the system is modelled as a parametric form and has a probability description. In the probabilistic setting, the robust stability analysis problem becomes to assess the probability of stability of the system, and the robust stabilisation problem becomes to design a controller that stabilises the system over the probabilistic parameter uncertainty set. Similar ideas have appeared in Fujisaki, Dabbene, and Tempo (2003) and Tempo, Calafiore, and Dabbene (2005), where the probabilistic analysis and synthesis problems have been studied for standard linear systems. One advantage of the

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randomised approach is that poor performance is allowed for the uncertainties that are not likely to occur (Tempo et al., 2005). Another advantage of the approach is that less conservative results might be achieved by exploiting the probabilistic characteristics of the uncertainty set. The reason leading to this advantage is twofold: the probabilistic information about the uncertainties is exploited to help design controllers (Tempo et al., 2005). A tight uncertainty set is allowed for the system, which could have any specific structure besides the norm-bounded structure (Han, 2004) and the polytopic structure (Xia & Jia, 2002). From the engineering point of view, the approach in this paper provides us an effective analysis and synthesis tool when the sufficient conditions mentioned above fail for uncertain time-varying delay systems.

This paper addresses a new research question about robust stability analysis and synthesis of uncertain time-varying delay systems. By modelling the uncertain parameters as random variables, the concept of probability of stability is first introduced to analyse the system stability property over the probabilistic uncertainties. Then, a robust state-feedback controller design problem is formulated as a stochastic optimisation problem. The stochastic optimisation problem can be solved by a randomised algorithm to be proposed in the paper. The proposed algorithm is based on the stochastic subgradient method (Calafiore & Polyak, 2001). After obtaining the controller by the proposed randomised algorithm, the stability property of the resulting closed-loop system is evaluated according to the probability of stability of the closed-loop system. Numerical examples are given to show the effectiveness of the approach. The examples also illustrate the advantages of exploiting the probabilistic structure of the parameter uncertainties in time-varying delay systems.

Notation: For $x \in \mathbb{R}$, the notation $\lceil x \rceil$ stands for the minimum integer greater than or equal to x. $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ denote, respectively, the n-dimensional Euclidean space and the set of all $m \times n$ real matrices. $\mathbb{R}^+$ and $\mathbb{S}^{n\times n}$ denote the set of all non-negative real numbers and the set of all $n \times n$ real symmetric positive definite matrices, respectively. For real symmetric matrices X and Y of the same order, the notation $X \geq Y$ (respectively, $X > Y$) means that $X - Y$ is positive semi-definite (respectively, positive definite). I is the identity matrix with compatible dimensions. The superscript 'T' denotes the transpose for vectors or matrices, and 'Tr' stands for the trace of a square matrix. The inner product of two matrices is defined as $\langle A, B\rangle \triangleq \mathrm{Tr}(A^{\mathrm{T}}B)$ and the Frobenius norm is defined as $\|A\|_{\mathrm{F}} \triangleq \sqrt{\mathrm{Tr}(A^{\mathrm{T}}A)}$. For any symmetric matrix $A \in \mathbb{R}^{n\times n}$, the matrix projection on the cone set of positive semi-definite matrices is defined as $[A]^+ \triangleq \arg\min_{X\in\Omega} \|A - X\|_{\mathrm{F}}$, where $\Omega \triangleq \{X \in \mathbb{R}^{n\times n} : X = X^T \geq 0\}$. Moreover, $\mathrm{E}(\cdot)$ stands for the mathematical expectation operator.

2. Problem formulation

Consider the time-varying delay system

$$\begin{aligned} \dot{x}(t) &= A(\theta)x(t) + A_d(\theta)x(t-d(t)) + B(\theta)u(t),\ t \geq 0, \\ x(t) &= \phi(t), \quad \forall t \in [-h, 0], \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the control input, $\phi(t)$ is the initial condition, $d(t)$ is the time-varying delay and satisfies $0 < d(t) \leq h$ and h is a given positive constant. The parameter θ is a random vector, and denotes the uncertain parameters in system matrices. The parameter θ takes values in a bounded uncertainty set Θ and its probability distribution over Θ is assumed to be known. The system matrices $A(\theta)$, $A_d(\theta)$ and $B(\theta)$ are bounded functions of $\theta \in \Theta$.

Remark 1: When a-priori information about the uncertain parameter θ is available, the probability distribution of θ can be estimated using either parametric methods or non-parametric methods. Otherwise, one reasonable assumption is that θ has a uniform distribution. Moreover, the uncertainty set Θ may have any specific structure besides the norm-bounded (Han, 2004) and the polytopic ones (Xia & Jia, 2002).

The concept of probability of stability introduced in Tempo et al. (2005) is adopted here to characterise the stability property of the system (1) over the probabilistic uncertainty set Θ. First, let us define an indicator function J: $\Theta \to \{0, 1\}$ given by

$$J(\hat{\theta}) \triangleq \begin{cases} 1, & \text{the system (1) is asymptotically stable} \\ & \text{under } \theta = \hat{\theta}, \text{ where } \hat{\theta} \in \Theta; \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

Then, the probability of stability p is defined in terms of $J(\hat{\theta})$ as follows:

Definition 1 (Probability of stability p (Tempo et al., 2005)):

$$p \triangleq \Pr\{\hat{\theta} \in \Theta \mid J(\hat{\theta}) = 1\}.$$

Remark 2: The probability of stability p gives us a quantitative measure of the system stability performance when the uncertainties are of a probabilistic nature. Such a measure could be useful when the deterministic methods such as those in Fridman and Shaked (2003), He et al. (2007) and Jiang and Han (2008) fail to determine the stability property of time-varying delay systems.

Consider the state-feedback controller

$$u(t) = Kx(t), \tag{3}$$

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where $K \in \mathbb{R}^{m\times n}$ is the control gain to be determined. The resulting closed-loop system is

$$\begin{aligned}\dot{x}(t) &= [A(\theta) + B(\theta)K]x(t) + A_d(\theta)x(t-d(t)),\\ x(t) &= \phi(t), \qquad \forall t \in [-h, 0].\end{aligned} \tag{4}$$

The objective of this paper is to improve the probability of stability of the closed-loop system (4) as high as possible by designing a state-feedback controller (3).

The following lemma provides us a sufficient stabilising condition for the system (1) and plays an essential role in the development of the main result.

Lemma 2.1: *Consider the uncertain time-varying delay system* (1) *under* $\theta = \hat{\theta}$, *where* $\hat{\theta} \in \Theta$. *For a given* $h > 0$ *and a given small enough scalar* $\varepsilon > 0$, *there exists a state-feedback controller* (3) *such that the closed-loop system* (4) *is asymptotically stable, if there exist matrices* $X > 0$, $Y > 0$, *and* L *such that*

$$V(X, Y, L, \hat{\theta}) \triangleq \begin{bmatrix} \Xi_1 & 2A_d(\hat{\theta})Y & \Xi_2 \\ * & -4Y & 2hYA_d^T(\hat{\theta}) \\ * & * & -Y \end{bmatrix} \le 0 \tag{5}$$

holds, where

$$\begin{aligned}\Xi_1 &= \big(A(\hat{\theta}) + A_d(\hat{\theta})\big)X + X\big(A(\hat{\theta}) + A_d(\hat{\theta})\big)^T \\ &\quad + B(\hat{\theta})L + L^TB^T(\hat{\theta}) + \varepsilon I,\\ \Xi_2 &= hX\big(A(\hat{\theta}) + A_d(\hat{\theta})\big)^T + hL^TB^T(\hat{\theta}).\end{aligned}$$

In this case, the control gain is given by $K = LX^{-1}$.

Proof: It follows from Theorem 1 of Gouaisbaut and Peaucelle (2006) that if there exist $P > 0$, $Z > 0$ such that

$$\begin{bmatrix} PA(\hat{\theta}) + A^{\mathrm{T}}(\hat{\theta})P - Z + \varepsilon I & PA_d(\hat{\theta}) + Z & hA^{\mathrm{T}}(\hat{\theta}) \\ A_d^{\mathrm{T}}(\hat{\theta})P + Z & -Z & hA_d^{\mathrm{T}}(\hat{\theta}) \\ hA(\hat{\theta}) & hA_d(\hat{\theta}) & -Z^{-1} \end{bmatrix} \le 0 \tag{6}$$

holds for $\hat{\theta} \in \Theta$, then the system (1) under $\theta = \hat{\theta}$ is asymptotically stable when $u(t) \equiv 0$.

Replacing $A(\hat{\theta})$ in Equation (6) with $A(\hat{\theta}) + B(\hat{\theta})K$ yields

$$\begin{bmatrix} H^{\mathrm{T}}\overline{A}(\hat{\theta}) + \overline{A}^{\mathrm{T}}(\hat{\theta})H + \varepsilon I & hA_K^{\mathrm{T}}(\hat{\theta}) \\ hA_K(\hat{\theta}) & -Z^{-1} \end{bmatrix} \le 0, \tag{7}$$

where

$$\begin{aligned}\overline{A}(\hat{\theta}) &= \begin{bmatrix} A(\hat{\theta}) + B(\hat{\theta})K & A_d(\hat{\theta}) \\ I & -I \end{bmatrix},\\ H &= \begin{bmatrix} P & 0 \\ -\frac{1}{2}Z & \frac{1}{2}Z \end{bmatrix},\\ A_K(\hat{\theta}) &= [\,A(\hat{\theta}) + B(\hat{\theta})K \;\; A_d(\hat{\theta})\,].\end{aligned}$$

Pre- and post-multiplying the inequality in Equation (7) by $\mathrm{diag}\{H^{-\mathrm{T}}, I\}$ and $\mathrm{diag}\{H^{-1}, I\}$, respectively, and applying the change of variables $X = P^{-1}$, $L = KP^{-1}$, $Y = Z^{-1}$, we have inequality (5). □

Remark 3: The condition in Theorem 1 of Gouaisbaut and Peaucelle (2006) is a strict inequality. Here, a small enough term εI is introduced in Ξ_1 in linear matrix inequality (LMI) (5) to get a non-strict inequality. This non-strict inequality will be used in the following results.

Remark 4: Although there exist less conservative sufficient conditions (e.g., Du et al., 2009; Jiang & Han, 2008; Xu & Lam, 2008) than the condition in Lemma 2.1, the reason of using Lemma 2.1 in this paper is that the inequality (5) has fewer matrix variables, and hence leads to a simpler presentation of our results. Lemma 2.1 can be replaced with any less conservative conditions such as those in Jiang and Han (2008), Xu and Lam (2008) and Du et al. (2009), and similar results could be derived with a less conservative but more complicated form by following the development in the paper.

3. Probabilistic robust stabilisation

In this section, we first formulate the probabilistic stabilisation problem as a stochastic optimisation problem. Then, a randomised algorithm is proposed to solve the stochastic optimisation problem, and a state-feedback controller can be obtained. After obtaining the controller, the stability property of the closed-loop system is analysed in terms of the probability of stability by another randomised algorithm.

Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{L}$ be bounded and closed convex subsets of $\mathbb{S}^{n\times n}$, $\mathbb{S}^{n\times n}$, $\mathbb{R}^{m\times n}$, respectively. To handle the design problem in a compact way, we introduce a scalar function $v : (\mathcal{X}, \mathcal{Y}, \mathcal{L}) \times \Theta \to \mathbb{R}^+$ given by

$$v(X, Y, L, \hat{\theta}) \triangleq \|[V(X, Y, L, \hat{\theta})]^+\|_{\mathrm{F}}, \tag{8}$$

where $V(X, Y, L, \hat{\theta})$ is defined in Equation (5).

For a particular $\hat{\theta} \in \Theta$, the triple (X, Y, L) is a feasible solution to the LMI in Equation (5), if and only if $v(X, Y, L, \hat{\theta}) = 0$. Thus, $v(X, Y, L, \hat{\theta})$ measures a 'distance' from the current solution (X, Y, L) to a feasible solution in some sense. With these premises, (X^*, Y^*, L^*) is robustly feasible for the LMI in Equation (5) for all $\hat{\theta} \in \Theta$, if and only if $v(X^*, Y^*, L^*, \hat{\theta}) = 0$ for all $\hat{\theta} \in \Theta$. Since θ is a random variable, we can define $f(X, Y, L) \triangleq \mathrm{E}\{v(X, Y, L, \theta)\}$, where the expectation is taken with respect to θ. The value of $f(X, Y, L)$ measures the 'expected distance' from the current solution (X, Y, L) to a robustly feasible solution over the uncertainty set Θ.

Our aim is to find an optimal solution $(X_{\min}, Y_{\min}, L_{\min})$ that minimises the 'expected distance' $f(X, Y, L)$ over the probabilistic uncertainty set Θ. In other words, we want

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to find a controller that tries to stabilise the system (1) with as high probability of stability as possible. This aim is equivalent to solving the stochastic optimisation problem:

$$\min_{X>0, Y>0, L} \quad f(X, Y, L). \tag{9}$$

Remark 5: Recall that the objective of the paper stated in the previous section is to design a state-feedback controller (3) such that the probability of stability p of the closed-loop system (4) is as high as possible. Let $f_{\min}$ be the minimum value of the objective function at the optimal solution $(X_{\min}, Y_{\min}, L_{\min})$ to Equation(9). If $f_{\min} = 0$, the LMI in Equation (5) has a robustly feasible solution almost surely for all the elements in the uncertainty set Θ. In this case, one may need to seek for the deterministic synthesis approaches. However, if $f_{\min} > 0$, the LMI in Equation (5) is not feasible for a subset of Θ that is not ignorable. In this case, the deterministic methods cannot find a controller. By solving the stochastic optimisation problem (9), a state-feedback controller can always be found and is optimal in the sense that the probability of stability of the closed-loop system is the highest.

To solve the optimisation problem (9), a randomised algorithm is to be proposed, which is based on the method in Calafiore and Polyak (2001). As in Calafiore and Polyak (2001), the following assumptions are needed to guarantee the accuracy of the solution returned by the algorithm.

Assumption 1: *The Frobenius norms of the subgradients of $v(X, Y, L, \hat{\theta})$ are uniformly bounded by constants R_1, R_2, R_3, respectively, for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$, $L \in \mathcal{L}$ and $\hat{\theta} \in \Theta$. That is, $\|\partial_X\{v(X, Y, L, \hat{\theta})\}\|_F \le R_1$, $\|\partial_Y\{v(X, Y, L, \hat{\theta})\}\|_F \le R_2$, $\|\partial_L\{v(X, Y, L, \hat{\theta})\}\|_F \le R_3$ for all $X \in \mathcal{X}$, $Y \in \mathcal{Y}$, $L \in \mathcal{L}$ and $\hat{\theta} \in \Theta$.*

Assumption 2: *The objective function $f(X, Y, L)$ has a minimum value f_{min} at point $(X_{\min}, Y_{\min}, L_{\min}) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{L}$.*

Assumption 3: $V_f \triangleq \max\{f(X, Y, L) - f_{\min} : X \in \mathcal{X}, Y \in \mathcal{Y}, L \in \mathcal{L}\} < \infty$.

In Lemma 3.1, the formulae to compute $\partial_X\{v(X, Y, L, \hat{\theta})\}$, $\partial_Y\{v(X, Y, L, \hat{\theta})\}$ and $\partial_L\{v(X, Y, L, \hat{\theta})\}$ are provided. The proof is similar to that of Lemma 3 in Fujisaki et al. (2003) and hence omitted here.

Lemma 3.1: *The subgradients of the function $v(X, Y, L, \hat{\theta})$ are given as follows:*

If $v(X, Y, L, \hat{\theta}) > 0$, then

$$\begin{aligned}
&\partial_X\{v(X, Y, L, \hat{\theta})\} \\
&= \begin{bmatrix} I & 0 & 0 \end{bmatrix} \frac{[V(X, Y, L, \hat{\theta})]^+}{v(X, Y, L, \hat{\theta})} \begin{bmatrix} (A(\hat{\theta}) + A_d(\hat{\theta})) \\ 0 \\ h(A(\hat{\theta}) + A_d(\hat{\theta})) \end{bmatrix} \\
&\quad + \begin{bmatrix} (A(\hat{\theta}) + A_d(\hat{\theta})) \\ 0 \\ h(A(\hat{\theta}) + A_d(\hat{\theta})) \end{bmatrix}^T \frac{[V(X, Y, L, \hat{\theta})]^+}{v(X, Y, L, \hat{\theta})} \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix}, \\
&\partial_Y\{v(X, Y, L, \hat{\theta})\} \\
&= \begin{bmatrix} 0 & 2I & 0 \end{bmatrix} \frac{[V(X, Y, L, \hat{\theta})]^+}{v(X, Y, L, \hat{\theta})} \begin{bmatrix} A_d(\hat{\theta}) \\ -I \\ hA_d(\hat{\theta}) \end{bmatrix} \\
&\quad + \begin{bmatrix} A_d^T(\hat{\theta}) & -I & hA_d^T(\hat{\theta}) \end{bmatrix} \frac{[V(X, Y, L, \hat{\theta})]^+}{v(X, Y, L, \hat{\theta})} \begin{bmatrix} 0 \\ 2I \\ 0 \end{bmatrix} \\
&\quad - \begin{bmatrix} 0 & 0 & I \end{bmatrix} \frac{[V(X, Y, L, \hat{\theta})]^+}{v(X, Y, L, \hat{\theta})} \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix}, \\
&\partial_L\{v(X, Y, L, \hat{\theta})\} \\
&= \begin{bmatrix} B^T(\hat{\theta}) & 0 & hB^T(\hat{\theta}) \end{bmatrix} \frac{[V(X, Y, L, \hat{\theta})]^+}{v(X, Y, L, \hat{\theta})} \begin{bmatrix} 2I \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

Otherwise, $v(X, Y, L, \hat{\theta}) = 0$, and

$$\partial_X\{v(X, Y, L, \hat{\theta})\} = 0, \quad \partial_Y\{v(X, Y, L, \hat{\theta})\} = 0, \quad \partial_L\{v(X, Y, L, \hat{\theta})\} = 0.$$

3.1. Probabilistic robust stabilisation via a randomised algorithm

In this subsection, algorithm SCDPU (Stabilising Controller Design under Probabilistic Uncertainties) is proposed to find a solution to the stochastic optimisation problem (9). The accuracy of the solution obtained by the randomizsed algorithm is analysed in Theorem 3.2. The convergence of the randomised algorithm is proved in Corollary 3.3.

The following result gives us some insight about the accuracy of the obtained solution.

Theorem 3.2: *Let the parameters N, M and $\eta^{(i)}$ in algorithm SCDPU be as follows:*

$$N = \left\lceil \frac{4u_0^2(R_1^2 + R_2^2 + R_3^2)}{\epsilon_1^2} \right\rceil, \quad M = \left\lceil \frac{2V_f^2}{\epsilon_1^2} \ln \frac{1}{1-\beta} \right\rceil,$$
$$\eta^{(i)} = \frac{u_0}{\sqrt{(R_1^2 + R_2^2 + R_3^2)N}}, \quad i = 0, \ldots, N-1,$$

where $\epsilon_1 > 0$, $\beta \in (0, 1)$, $u_0^2 = E\{\|X^{(0)} - X_{\min}\|_F^2 + \|Y^{(0)} - Y_{\min}\|_F^2 + \|L^{(0)} - L_{\min}\|_F^2\}$. Then, the solution $(\widehat{X}, \widehat{Y}, \widehat{L})$ generated by algorithm SCDPU satisfies the inequality

$$\Pr\left(f(\widehat{X}, \widehat{Y}, \widehat{L}) - f_{\min} \ge \epsilon_1\right) \le 1 - \beta. \tag{10}$$

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Algorithm SCDPU (Stabilising Controller Design under Probabilistic Uncertainties)

Set initial matrices $X^{(0)} \in \mathbb{S}^{n\times n}, Y^{(0)} \in \mathbb{S}^{n\times n}$, $L^{(0)} \in \mathbb{R}^{m\times n}$.
Set N, M and $\eta^{(i)}$.
Set $j = 0$.
while $j < M$ **do**
  Set $i = 0$.
  while $i < N$ **do**
    Generate $\hat{\theta}^{(i)} \in \Theta$ according to the given probability distribution.
    Compute

$$X^{(i+1)} = \left[X^{(i)} - \eta^{(i)}\partial_X\{v(X^{(i)}, Y^{(i)}, L^{(i)}, \hat{\theta}^{(i)})\}\right]^+,$$
$$Y^{(i+1)} = \left[Y^{(i)} - \eta^{(i)}\partial_Y\{v(X^{(i)}, Y^{(i)}, L^{(i)}, \hat{\theta}^{(i)})\}\right]^+,$$
$$L^{(i+1)} = L^{(i)} - \eta^{(i)}\partial_L\{v(X^{(i)}, Y^{(i)}, L^{(i)}, \hat{\theta}^{(i)})\}.$$

    Set $i = i + 1$.
  end while
  Compute

$$\overline{X}^{(j)} = \frac{\sum_{i=0}^{N-1}\eta^{(i)}X^{(i)}}{\sum_{i=0}^{N-1}\eta^{(i)}}, \quad \overline{Y}^{(j)} = \frac{\sum_{i=0}^{N-1}\eta^{(i)}Y^{(i)}}{\sum_{i=0}^{N-1}\eta^{(i)}},$$
$$\overline{L}^{(j)} = \frac{\sum_{i=0}^{N-1}\eta^{(i)}L^{(i)}}{\sum_{i=0}^{N-1}\eta^{(i)}}.$$

  Set $j = j + 1$.
end while
Compute

$$\widehat{X} = \frac{1}{M}\sum_{j=0}^{M-1}\overline{X}^{(j)}, \quad \widehat{Y} = \frac{1}{M}\sum_{j=0}^{M-1}\overline{Y}^{(j)},$$
$$\widehat{L} = \frac{1}{M}\sum_{j=0}^{M-1}\overline{L}^{(j)}.$$

return $K = \widehat{L}\widehat{X}^{-1}$.

Proof: We first prove that the following inequality

$$\mathrm{E}\{f(\overline{X}^{(j)}, \overline{Y}^{(j)}, \overline{L}^{(j)})\} - f_{\min} \le c(N) \tag{11}$$

holds, where $c(N) = \frac{u_0\sqrt{R_1^2+R_2^2+R_3^2}}{\sqrt{N}}$.

Define the distance from the current point $(X^{(i+1)}, Y^{(i+1)}, L^{(i+1)})$ to the optimal point $(X_{\min}, Y_{\min}, L_{\min})$ as

$$r_{i+1}^2 \triangleq \left\|X^{(i+1)} - X_{\min}\right\|_{\mathrm{F}}^2 + \left\|Y^{(i+1)} - Y_{\min}\right\|_{\mathrm{F}}^2 + \left\|L^{(i+1)} - L_{\min}\right\|_{\mathrm{F}}^2.$$

By the definition of the projection on the cone of positive semi-definite matrices (see e.g., Tempo et al., 2005, Chapter 11), we have that $\|[X]^+ - X_{\min}\|_{\mathrm{F}} \le \|X - X_{\min}\|_{\mathrm{F}}$ for any symmetric X. Therefore,

$$\begin{aligned}
r_{i+1}^2 \le{}& \left\|X^{(i)} - \eta^{(i)}\partial_X\{v(X^{(i)}, Y^{(i)}, L^{(i)}, \hat{\theta}^{(i)})\} - X_{\min}\right\|_{\mathrm{F}}^2 \\
&+ \left\|Y^{(i)} - \eta^{(i)}\partial_Y\{v(X^{(i)}, Y^{(i)}, L^{(i)}, \hat{\theta}^{(i)})\} - Y_{\min}\right\|_{\mathrm{F}}^2 \\
&+ \left\|L^{(i)} - \eta^{(i)}\partial_L\{v(X^{(i)}, Y^{(i)}, L^{(i)}, \hat{\theta}^{(i)})\} - L_{\min}\right\|_{\mathrm{F}}^2 \\
={}& r_i^2 + (\eta^{(i)})^2\Big(\left\|\partial_X\{v(X^{(i)}, Y^{(i)}, L^{(i)}, \hat{\theta}^{(i)})\}\right\|_{\mathrm{F}}^2 \\
&+ \left\|\partial_Y\{v(X^{(i)}, Y^{(i)}, L^{(i)}, \hat{\theta}^{(i)})\}\right\|_{\mathrm{F}}^2 \\
&+ \left\|\partial_L\{v(X^{(i)}, Y^{(i)}, L^{(i)}, \hat{\theta}^{(i)})\}\right\|_{\mathrm{F}}^2\Big) \\
&- 2\eta^{(i)}\Big[\mathrm{Tr}\Big((X^{(i)} - X_{\min})^{\mathrm{T}}\partial_X\{v(X^{(i)}, Y^{(i)}, L^{(i)}, \hat{\theta}^{(i)})\}\Big) \\
&+ \mathrm{Tr}\Big((Y^{(i)} - Y_{\min})^{\mathrm{T}}\partial_Y\{v(X^{(i)}, Y^{(i)}, L^{(i)}, \hat{\theta}^{(i)})\}\Big) \\
&+ \mathrm{Tr}\Big((L^{(i)} - L_{\min})^{\mathrm{T}}\partial_L\{v(X^{(i)}, Y^{(i)}, L^{(i)}, \hat{\theta}^{(i)})\}\Big)\Big].
\end{aligned}$$

In view of Lemmas 2–4 in Calafiore and Polyak (2001), $v(X, Y, L, \hat{\theta}^{(i)})$ is convex with respect to (X, Y, L) for each fixed $\hat{\theta}^{(i)}$, $i = 0, \ldots, N-1$. Therefore,

$$\begin{aligned}
&\mathrm{Tr}\Big((X^{(i)} - X_{\min})^{\mathrm{T}}\partial_X\{v(X^{(i)}, Y^{(i)}, L^{(i)}, \hat{\theta}^{(i)})\}\Big) \\
&\quad+\mathrm{Tr}\Big((Y^{(i)} - Y_{\min})^{\mathrm{T}}\partial_Y\{v(X^{(i)}, Y^{(i)}, L^{(i)}, \hat{\theta}^{(i)})\}\Big) \\
&\quad+\mathrm{Tr}\Big((L^{(i)} - L_{\min})^{\mathrm{T}}\partial_L\{v(X^{(i)}, Y^{(i)}, L^{(i)}, \hat{\theta}^{(i)})\}\Big) \\
&\ge v(X^{(i)}, Y^{(i)}, L^{(i)}, \hat{\theta}^{(i)}) - v(X_{\min}, Y_{\min}, L_{\min}, \hat{\theta}^{(i)}).
\end{aligned}$$

By Assumption 1, we have

$$\begin{aligned}
r_{i+1}^2 \le{}& r_i^2 + (\eta^{(i)})^2(R_1^2 + R_2^2 + R_3^2) \\
&- 2\eta^{(i)}\Big(v(X^{(i)}, Y^{(i)}, L^{(i)}, \hat{\theta}^{(i)}) \\
&- v(X_{\min}, Y_{\min}, L_{\min}, \hat{\theta}^{(i)})\Big).
\end{aligned} \tag{12}$$

Taking expectations with respect to $\hat{\theta}^{(i)}$ on both sides of Equation (12) yields

$$\begin{aligned}
\mathrm{E}\{r_{i+1}^2\} \le{}& \mathrm{E}\{r_i^2\} + (\eta^{(i)})^2(R_1^2 + R_2^2 + R_3^2) \\
&- 2\eta^{(i)}\Big(f(X^{(i)}, Y^{(i)}, L^{(i)}) - f_{\min}\Big).
\end{aligned} \tag{13}$$

Since $X^{(i)}$, $Y^{(i)}$ and $L^{(i)}$ are also random variables (see Remark 7 for details), we again take expectations with respect to $(X^{(i)}, Y^{(i)}, L^{(i)})$ on both sides of Equation (13) and have

$$\begin{aligned}
\mathrm{E}\{r_{i+1}^2\} \le{}& \mathrm{E}\{r_i^2\} + (\eta^{(i)})^2(R_1^2 + R_2^2 + R_3^2) \\
&- 2\eta^{(i)}\Big(\mathrm{E}\{f(X^{(i)}, Y^{(i)}, L^{(i)})\} - f_{\min}\Big).
\end{aligned}$$

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Summing the above inequalities up over $i = 0, \ldots, N-1$ and omitting $\mathrm{E}\{r_N^2\}$, we have

$$2\sum_{i=0}^{N-1}\eta^{(i)}\Big(\mathrm{E}\{f(X^{(i)}, Y^{(i)}, L^{(i)})\} - f_{\min}\Big) \le u_0^2 + \sum_{i=0}^{N-1}(\eta^{(i)})^2\big(R_1^2 + R_2^2 + R_3^2\big). \tag{14}$$

On the other hand, taking expectations with respect to $\hat{\theta}^{(i)}$ on both sides of the Jensen's inequality (Boyd & Vandenberghe, 2004) for the convex function $v(\overline{X}^{(j)}, \overline{Y}^{(j)}, \overline{L}^{(j)}, \hat{\theta}^{(i)})$, we obtain

$$f\Big(\overline{X}^{(j)}, \overline{Y}^{(j)}, \overline{L}^{(j)}\Big) \le \frac{\sum_{i=0}^{N-1}\eta^{(i)} f(X^{(i)}, Y^{(i)}, L^{(i)})}{\sum_{i=0}^{N-1}\eta^{(i)}}. \tag{15}$$

Then, taking expectations with respect to $(\overline{X}^{(j)}, \overline{Y}^{(j)}, \overline{L}^{(j)})$ and $(X^{(i)}, Y^{(i)}, L^{(i)})$ on both sides of (15), respectively, and subtracting $f_{\min}$ from both sides, we have

$$\mathrm{E}\Big\{f\Big(\overline{X}^{(j)}, \overline{Y}^{(j)}, \overline{L}^{(j)}\Big)\Big\} - f_{\min} \le \frac{\sum_{i=0}^{N-1}\eta^{(i)}\Big(\mathrm{E}\{f(X^{(i)}, Y^{(i)}, L^{(i)})\} - f_{\min}\Big)}{\sum_{i=0}^{N-1}\eta^{(i)}}.$$

Due to Equation (14), we get

$$\mathrm{E}\Big\{f\Big(\overline{X}^{(j)}, \overline{Y}^{(j)}, \overline{L}^{(j)}\Big)\Big\} - f_{\min} \le \frac{u_0^2 + \sum_{i=0}^{N-1}(\eta^{(i)})^2\big(R_1^2 + R_2^2 + R_3^2\big)}{2\sum_{i=0}^{N-1}\eta^{(i)}}. \tag{16}$$

Applying the definition of $\eta^{(i)}$ in Theorem 3.2 to Equation (16), we have the inequality (11).

Next, we prove that the inequality (10) holds. Consider the following random variables:

$$z^{(j)} \triangleq f\Big(\overline{X}^{(j)}, \overline{Y}^{(j)}, \overline{L}^{(j)}\Big) - f_{\min}, \quad \overline{z}^{(M)} \triangleq \frac{1}{M}\sum_{j=0}^{M-1} z^{(j)}.$$

Denote

$$\mu \triangleq \mathrm{E}\Big\{f\Big(\overline{X}^{(j)}, \overline{Y}^{(j)}, \overline{L}^{(j)}\Big)\Big\} - f_{\min}.$$

Clearly, $\mathrm{E}\{z^{(j)}\} = \mu$ and $\mathrm{E}\{\overline{z}^{(M)}\} = \mu$. By Assumption 3, we have $0 \le z^{(j)} \le V_f$ for all j. Thus, according to the Hoeffding inequality (Hoeffding, 1963), we have

$$e^{-2M(\frac{\delta_1}{V_f})^2} \ge \Pr\big(\overline{z}^{(M)} - \mu \ge \delta_1\big).$$

Then, from the inequality (11), the above inequality becomes

$$\begin{aligned} e^{-2M(\frac{\delta_1}{V_f})^2} &\ge \Pr\Bigg(\overline{z}^{(M)} - \mu \ge \delta_1 + c(N) - \Big(\mathrm{E}\big\{f(\overline{X}^{(j)}, \overline{Y}^{(j)}, \overline{L}^{(j)})\big\} - f_{\min}\Big)\Bigg) \\ &= \Pr\left(\frac{1}{M}\sum_{j=0}^{M-1} z^{(j)} - \mu \ge \delta_1 + c(N) - \mu\right) \\ &= \Pr\left(\frac{1}{M}\sum_{j=0}^{M-1} z^{(j)} \ge \delta_1 + c(N)\right) \\ &= \Pr\left(\frac{1}{M}\sum_{j=0}^{M-1} f(\overline{X}^{(j)}, \overline{Y}^{(j)}, \overline{L}^{(j)}) - f_{\min} \ge \delta_1 + c(N)\right). \end{aligned}$$

Moreover, we have

$$f(\widehat{X}, \widehat{Y}, \widehat{L}) \le \frac{1}{M}\sum_{j=0}^{M-1} f(\overline{X}^{(j)}, \overline{Y}^{(j)}, \overline{L}^{(j)})$$

holds for the same reason as the inequality (15). Thus,

$$\Pr(f(\widehat{X}, \widehat{Y}, \widehat{L}) - f_{\min} \ge \delta_1 + c(N)) \le \Pr\left(\frac{1}{M}\sum_{j=0}^{M-1} f(\overline{X}^{(j)}, \overline{Y}^{(j)}, \overline{L}^{(j)}) - f_{\min} \ge \delta_1 + c(N)\right).$$

Finally, we have

$$e^{-2M(\frac{\delta_1}{V_f})^2} \ge \Pr(f(\widehat{X}, \widehat{Y}, \widehat{L}) - f_{\min} \ge \delta_1 + c(N)). \tag{17}$$

Let $\delta_1 = \frac{1}{2}\epsilon_1 > 0$, and $\beta = 1 - e^{-2M(\frac{\delta_1}{V_f})^2} \in (0, 1)$. Because $c(N) \ge \frac{1}{2}\epsilon_1$, the inequality (17) implies the inequality (10). This completes the proof. □

Corollary 3.3: *When $N \to \infty$ in algorithm SCDPU, $E\big\{f(\overline{X}^{(j)}, \overline{Y}^{(j)}, \overline{L}^{(j)})\big\} \to f_{\min}$.*

Proof: In the proof of Theorem 3.2, we have shown that

$$\mathrm{E}\big\{f(\overline{X}^{(j)}, \overline{Y}^{(j)}, \overline{L}^{(j)})\big\} - f_{\min} \le c(N)$$

holds, where $c(N) = \frac{u_0\sqrt{R_1^2+R_2^2+R_3^2}}{\sqrt{N}}$. Because $c(N) \to 0$ as $N \to \infty$, one has that $\mathrm{E}\big\{f(\overline{X}^{(j)}, \overline{Y}^{(j)}, \overline{L}^{(j)})\big\} - f_{\min} \to 0$ for any j. □

Remark 6: In algorithm SCDPU, the initial matrices $X^{(0)}$, $Y^{(0)}$ and $L^{(0)}$ can be taken as a feasible solution to LMI (5)

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for a particular point $\hat{\theta} \in \Theta$. The parameter $\hat{\theta}$ might be chosen randomly within Θ. The values of N, M and the steps $\eta^{(i)}$ can be chosen according to Theorem 3.2.

Remark 7: The matrices $X^{(i)}$, $Y^{(i)}$, $L^{(i)}$, $\overline{X}^{(j)}$, $\overline{Y}^{(j)}$ and $\overline{L}^{(j)}$ obtained by algorithm SCDPU are all random variables. The reason is that they are generated according to the randomly chosen $\hat{\theta}^{(i)}$.

Remark 8: Algorithm SCDPU always returns a state-feedback control gain K, but it may not be a stabilising gain. It is particularly useful when the optimal value of the objective function $f_{\min} > 0$.

Remark 9: The parameters u_0, R_1, R_2 and R_3 are usually not known in practice, so is the value of $\eta^{(i)}$ in algorithm SCDPU. An alternative choice of the step sizes $\eta^{(i)}$ is that $\eta^{(i)} > 0$, $\eta^{(i)} \to 0$, and $\sum_{i=0}^{N-1} \eta^{(i)} = \infty$ as $N \to \infty$ (see Kushner & Yin, 1997, for details). This choice of $\eta^{(i)}$ also ensures that $c(N)$ in Equation (11) goes to zero as $N \to \infty$, and hence the convergence of algorithm SCDPU is guaranteed.

3.2. *Closed-loop probabilistic robust stability analysis*

Algorithm SCDPU gives us a control gain K. The capability of the obtained controller can be evaluated by algorithm PSA (Probabilistic Stability Analysis) presented in this subsection. The probability of stability p of the resulting closed-loop system is estimated, and its value is denoted by $\hat{p}$. Due to the difficulty of computing multidimensional probability, the algorithm is based on the Monte Carlo method.

Algorithm PSA (Probabilistic Stability Analysis)

1: Choose $\epsilon_2 > 0$, $\delta_2 \in (0, 1)$, and a large integer N.
2: Generate N samples $\hat{\theta}^{(1)}, \ldots, \hat{\theta}^{(N)}$according to the given probability distribution.
3: Compute $J(\hat{\theta}^{(1)}), \ldots, J(\hat{\theta}^{(N)})$.
4: **return** $\hat{p} = \frac{1}{N} \sum_{i=1}^{N} J(\hat{\theta}^{(i)})$.

Remark 10: The value of N in algorithm PSA could be chosen based on the Chernoff inequality (Chernoff, 1952). That is, one may choose $N \geq \frac{1}{2\epsilon_2^2} \log \frac{2}{\delta_2}$. As a result, we have that $\Pr\{|\hat{p} - p| \geq \epsilon_2\} \leq \delta_2$. The indicator function $J(\hat{\theta}^{(i)})$ is estimated by solving the LMI in Equation (5), that is, if the LMI in Equation (5) is feasible for a particular $\hat{\theta}^{(i)}$, then $J(\hat{\theta}^{(i)}) = 1$; otherwise, $J(\hat{\theta}^{(i)}) = 0$.

4. Illustrative examples

In this section, three examples are presented to illustrate the developed method. Example 1 is used to show the effectiveness of the probabilistic analysis method when the deterministic methods fail to tell the robust stability property of the system (1) under $u(t) \equiv 0$. Examples 2 and 3 focus on the probabilistic robust stabilisation problem. Example 2 illustrates the usefulness of our method to stabilise the system (1), where the parametric uncertainties are of non-linear nature. Example 3 has three cases. Case 1 shows the usefulness of our algorithm when the deterministic methods cannot find a robust stabilising controller. Case 2 illustrates the flexibility of using different probability distributions to model the uncertainties. Case 3 shows the advantage that the probabilistic uncertainty set is not required to have any specific structure.

Example 1: The system data are as follows:

$$A = \begin{bmatrix} 0 & -0.12 + 12\theta_1 \\ 1 & -0.465 - \theta_2 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.1 & -0.35 \\ 0 & 0.3 \end{bmatrix}, \tag{18}$$

and the uncertainty set is

$$\Theta = \{(\theta_1, \theta_2) : |\theta_1| \leq 0.041, |\theta_2| \leq 0.041\}.$$

It was verified that the robust stability conditions in Fridman and Shaked (2003), He et al. (2007) and Jiang and Han (2008) are not feasible for any $h > 0$. As a result, we cannot tell whether the system (18) is robustly stable or not over the uncertainty set Θ.

We now assume that (θ_1, θ_2) be uniformly distributed on Θ and compute the probability of stability of the system (18) by applying algorithm PSA. The number of samples N is 14,974 by choosing $\epsilon_2 = 0.01$ and $\delta_2 = 0.10$ (according to Remark 10). Table 1 lists the estimated probability of stability $\hat{p}$ for different delay bounds $h \in \{0.10, 0.20, \ldots, 0.60\}$. Here, the values of h are chosen by hand for illustration purpose. If $\hat{p} = 0.9943$ is acceptable in practice, we can say that the system (18) is 'almost' robustly stable with maximum allowable upper bound (MAUB) of delay $h_{\max} = 0.43$.

Example 2: The system data are as follows:

$$A = 0, \quad A_d = \begin{bmatrix} \theta_1\theta_2^3 & 1 \\ -1 + \theta_1 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} -1 + \theta_2 \\ 1 \end{bmatrix}, \tag{19}$$

and the uncertainty set is given by

$$\Theta = \{(\theta_1, \theta_2) : |\theta_1| \leq 0.93, |\theta_2| \leq 0.18\}.$$

Since the uncertain parameters (θ_1, θ_2) appear in the system matrix A_d in a non-linear form, the existing stabilising

Table 1. Estimated $\hat{p}$ for different h in Example 1.

h	0.10	0.20	0.30	0.40	0.43	0.45
$\hat{p}$	0.9944	0.9944	0.9943	0.9943	0.9943	0.9875
h	0.48	0.50	0.53	0.55	0.58	0.60
$\hat{p}$	0.9449	0.9020	0.8171	0.7379	0.6135	0.5267

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methods (e.g., Fridman & Shaked, 2002, 2003; Li, Guo, & Zhang, 2008) are not applicable to the system (19) directly.

We now apply algorithm SCDPU to obtain a state-feedback controller for the system (19) when $h = 0.10$. First, θ_1 and θ_2 are assumed to be uniformly distributed on Θ. According to Remark 6, a feasible solution to the LMI in Equation (5) is chosen as the initial matrices. For example, if we choose $\theta_1 = 0.93$, $\theta_2 = -0.18$, $h = 3.7080$ and $\varepsilon = 10^{-6}$, the initial matrices are given by

$$X^{(0)} = \begin{bmatrix} 1.3122 & 0.1170 \\ 0.1170 & 0.0634 \end{bmatrix}, \quad Y^{(0)} = \begin{bmatrix} 1.3127 & -0.3247 \\ -0.3247 & 0.0919 \end{bmatrix},$$
$$L^{(0)} = [\,0.1740 \;\; 0.0332\,].$$

Then, let $R_1 = 3$, $R_2 = 3$, $R_3 = 3$, $V_f = 10$, $u_0 = 10$, $\epsilon_1 = 0.20$, $\beta = 0.80$. Here, the values of R_1, R_2 and R_3 were obtained by estimating the upper bounds of $\|\partial_X\{v(X, Y, L, \hat{\theta})\}\|_{\mathrm{F}}$, $\|\partial_Y\{v(X, Y, L, \hat{\theta})\}\|_{\mathrm{F}}$ and $\|\partial_L\{v(X, Y, L, \hat{\theta})\}\|_{\mathrm{F}}$, respectively. In view of Theorem 3.2, the numbers of samples are $N = 2.70 \times 10^5$, $M = 8.05 \times 10^3$, respectively, and the step size $\eta^{(i)}$ is 0.0029. By executing algorithm SCDPU, we obtain a state-feedback controller

$$u(t) = [\,-26.0 \;\; -1487.4\,]x(t).$$

Finally, applying this controller, the stability property of the resulting closed-loop system is analysed by algorithm PSA with $\epsilon_2 = 0.01$ and $\delta_2 = 0.10$, the estimated probability of stability $\hat{p}$ is 1.0000.

We also designed controllers for the system (19) for different $h \in \{0.10, 0.20, \ldots, 0.60\}$, and computed the corresponding probability of stability of the resulting closed-loop systems. The results are given in Table 2. It can be seen that if the acceptable probability of stability is 0.9605, the controller designed by our method stabilises the system (19) with an MAUB of delay $h_{\max} = 0.50$. Also, it can be seen that when $h = 0.80$, the estimated probability of stability is 0.1472, which means that the designed controller can only stabilise the system with a low probability. This might be unacceptable in practice.

Example 3: The system data are as follows:

$$A = 0, \quad A_d = \begin{bmatrix} 0 & 1 \\ -1+\theta_1 & -0.5 \end{bmatrix}, \quad B = \begin{bmatrix} -1+\theta_2 \\ 1 \end{bmatrix}. \tag{20}$$

Table 2. Estimated $\hat{p}$ for different h in Example 2.

h	0.10	0.20	0.30	0.40	0.50	0.60	0.80
$\hat{p}$	1.0000	1.0000	1.0000	1.0000	0.9605	0.8186	0.1472

Case 1: The uncertainty set is

$$\Theta = \{(\theta_1, \theta_2) : |\theta_1| \le 1.00, |\theta_2| \le 0.68\}.$$

It was verified that the existing robust stabilisation conditions in Fridman and Shaked (2002, 2003) and Li et al. (2008) are not feasible for any $h > 0$. Thus, we cannot find a robust stabilising controller for the system (20) using the techniques in these references.

We followed the controller design steps described in Example 2, and obtained a state-feedback controller for $h = 0.10$. In this case, θ_1 and θ_2 are assumed to be uniformly distributed on Θ. We first obtained the initial matrices as

$$X^{(0)} = \begin{bmatrix} 1.7993 & -0.2712 \\ -0.2712 & 0.0957 \end{bmatrix}, \; Y^{(0)} = \begin{bmatrix} 1.0809 & -0.2684 \\ -0.2684 & 0.1705 \end{bmatrix},$$
$$L^{(0)} = [\,-0.0566 \;\; 0.0024\,]$$

by choosing $\theta_1 = 1.00$, $\theta_2 = 0.68$, $h = 1.9000$, $\varepsilon = 10^{-6}$ in Equation (5). The numbers of samples N, M and the step size $\eta^{(i)}$ are chosen the same as the ones in Example 2. Then, a state-feedback controller

$$u(t) = [\,-53.5874 \;\; -194.2362\,]x(t)$$

was obtained by algorithm SCDPU. Finally, the estimated probability of stability $\hat{p}$ is 0.9983 by algorithm PSA with the same numbers of samples as in Example 2.

We also designed controllers for the system (20) for different $h \in \{0.10, 0.15, 0.20, \ldots, 0.50\}$, and the values of $\hat{p}$ are depicted in Figure 1 (solid line). It can be seen that if the acceptable probability of stability is 0.9862, then the controller

$$u(t) = [\,-20.3739 \;\; -93.0156\,]x(t)$$

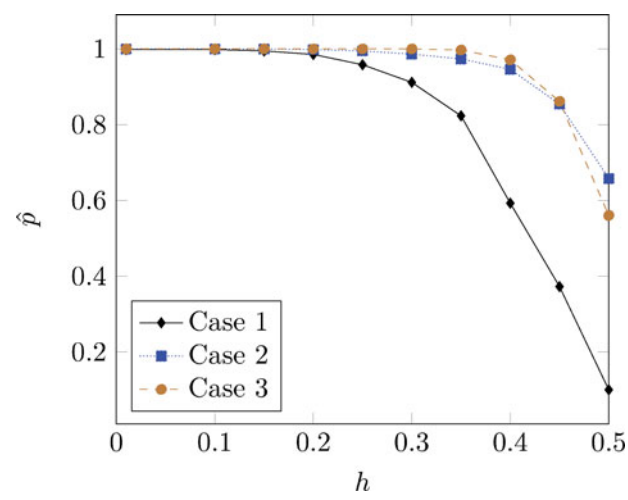

Figure 1. Estimated $\hat{p}$ for different h in Case 1, Case 2 and Case 3 of Example 3.

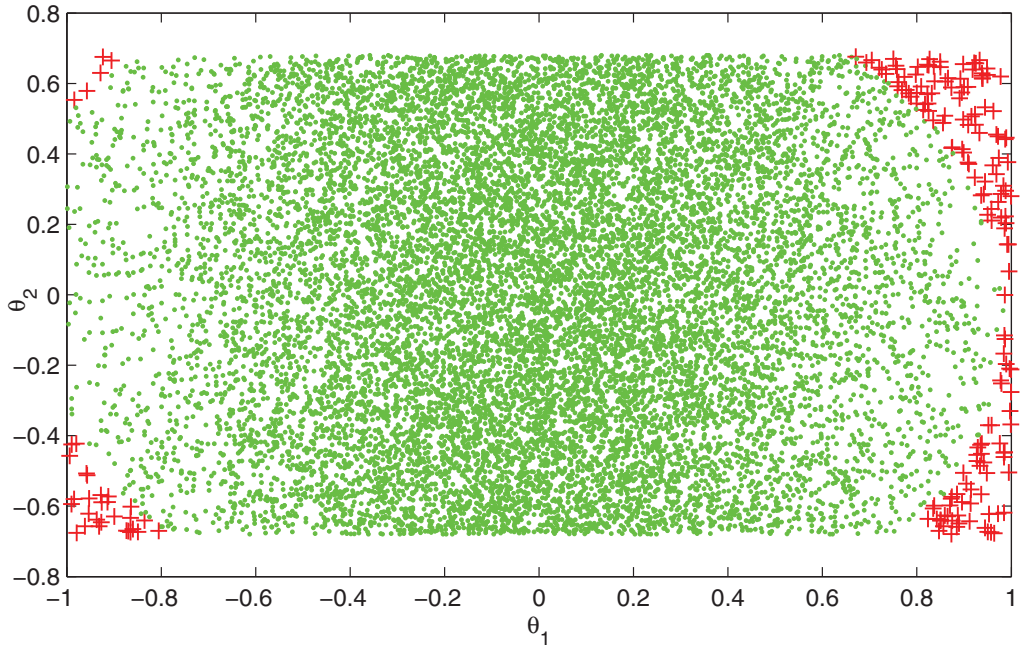

Figure 2. Sample results in Case 2.

designed by our method stabilises the system (20) with an MAUB of delay $h_{\max} = 0.1748$.

Case 2: The uncertainty set Θ is

$$\Theta = \{(\theta_1, \theta_2) : |\theta_1| \le 1.00, |\theta_2| \le 0.68\},$$

which is the same as the one in Case 1. However, the probability distribution of the uncertainties is assumed to be a truncated Gaussian distribution (Tempo et al., 2005). This assumption is reasonable in practice because it is quite possible that the probability of large variations in system parameters is relatively small.

Suppose $X \sim \mathrm{N}(\mu, \sigma^2)$. For $a \le x \le b$, the truncated probability density function (pdf) of X is

$$f_X(x; \mu, \sigma, a, b) = \begin{cases} \dfrac{\frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})}{\Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma})}, & a \le x \le b; \\ 0, & \text{otherwise}, \end{cases}$$

where ϕ and Φ are the pdf and the distribution function of the standard normal distribution, respectively. Thus, the pdfs of θ_1 and θ_2 are

$$f_{\theta_1}(\hat{\theta}_1; 0, 0.4, -1, 1) = 6.3\phi(2.5\hat{\theta}_1),$$
$$f_{\theta_2}(\hat{\theta}_2; 0, 0.4, -0.68, 0.68) = 4.8\phi(2.5\hat{\theta}_2),$$

respectively, when $(\hat{\theta}_1, \hat{\theta}_2) \in \Theta$; otherwise, the pdfs of θ_1 and θ_2 are 0.

The controller design procedure is the same as the one in Case 1, except that uncertainty sampling is based on truncated Gaussian distributions. Then, state-feedback controllers were found for different $h \in \{0.10, 0.15, 0.20, \ldots, 0.50\}$, and the values of $\hat{p}$ of the resulting closed-loop systems are depicted in Figure 1 (dotted line). It can be seen that, if $\hat{p} = 0.9862$ is acceptable, an MAUB of delay of the resulting closed-loop system is $h_{\max} = 0.30$ by applying the obtained controller

$$u(t) = \begin{bmatrix} -1.6590 & -55.5146 \end{bmatrix} x(t).$$

Figure 2 depicts the sample plot of the stability test of the closed-loop system over the uncertainty set when $h = 0.30$, where crosses represent the unstabilisable uncertainties of the closed-loop system. We can see that the unstabilisable uncertainties are less likely to occur and they mainly occur near the boundary of the uncertainty domain.

Case 3: The uncertainty set is assumed to be an ellipse given by

$$\Theta = \{(\theta_1, \theta_2) : \theta_1^2/1.00^2 + \theta_2^2/0.68^2 \le 1\}. \tag{21}$$

To use the existing robust stabilisation results in the above-mentioned references, the uncertainty set (21) has to be overbounded by a polytopic uncertainty set or a norm-bounded uncertainty set. The polytopic uncertainty sets to be considered in this example are polytopes with three, four and five vertices, respectively. These polytopes contain the ellipse in Equation (21). Figure 3 illustrates the overbounding of the polytopic uncertainty model. Three sets of testing vertices $\{A1, A2, A3\}$, $\{B1, B2, B3, B4\}$ and $\{C1, C2, C3,$

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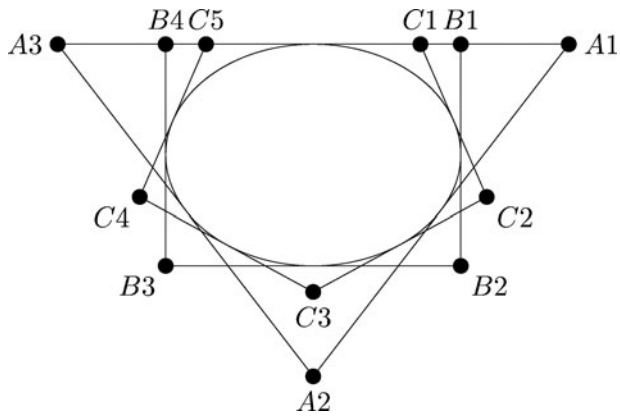

Figure 3. Ellipse uncertainty set and its overbounding sets.

$C4$, $C5\}$ will be used in the applications of the results in Fridman and Shaked (2002, 2003) and Li et al. (2008). On the other hand, the norm-bounded uncertainty model $\left[\,\Delta A_d \;\; \Delta B\,\right] = DF(t)\left[\,E \;\; E_1\,\right]$ with $F^{\mathrm{T}}(t)F(t) \leq I$ is also considered when $|\theta_1| \leq 1.00$, $|\theta_2| \leq 0.68$. Here, D, E, E_1 and $F(t)$ are appropriately chosen as $\left[\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 \\ 0.34 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \theta_1(t) & 0 \\ 0 & \tilde{\theta}_2(t) \end{smallmatrix}\right]$, respectively, where $|\tilde{\theta}_2(t)| \leq 1$. It was verified that no robust state-feedback controller can be found for any $h > 0$ using the results in the compared approaches because the conditions there are all infeasible.

We now perform the same design procedure as the one in Case 1, except that the generation of the random uncertainties is restricted to the ellipse (21). Then, state-feedback controllers were found for different h and the values of $\hat{p}$ of the closed-loop systems are depicted in Figure 1 (dashed line). Applying the obtained controller

$$u(t) = \left[\,-15.7847 \;\; -406.3448\,\right]x(t),$$

the MAUB of the delay of the closed-loop system is $h_{\max} = 0.3645$, when $\hat{p} = 0.9862$ is acceptable according to our computation.

5. Conclusions

This paper studied the robust stabilisation problem for time-varying delay systems with probabilistic uncertain parameters. When the deterministic methods could not find a robustly stabilising controller, the probabilistic robust stabilisation method developed in this paper provides a potential useful controller design technique because at least a controller could be obtained. We formulated the problem of designing a robust state-feedback controller as a stochastic optimisation problem, and proposed a randomised algorithm to solve the optimisation problem. The accuracy of the solution obtained from the algorithm was studied, and the convergence of the algorithm was guaranteed. Once a controller was obtained, the stability property of the closed-loop system could be analysed in terms of the probability of stability. Numerical examples were employed to show the effectiveness and advantages of the developed approach.

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