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H_∞ model reduction for negative imaginary systems

Lanlin Yu and Junlin Xiong

Department of Automation, University of Science and Technology of China, Hefei, China

ABSTRACT

This paper is concerned with the H_∞ model reduction for negative imaginary (NI) systems. For a given linear time-invariant system that is stable and NI, our goal is to find a stable reduced-order NI system satisfying a pre-specified H_∞ approximation error bound. Sufficient conditions in terms of matrix inequalities are derived for the existence and construction of an H_∞ reduced-order NI system. Iterative algorithms are provided to solve the matrix inequalities and to minimise the H_∞ approximation error. Finally, a numerical example is used to demonstrate the effectiveness of the proposed model reduction method.

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1. Introduction

A negative imaginary (NI) system is a stable system with equal number of inputs and outputs, having a real-rational, proper transfer function matrix $G(s)$ that satisfies the frequency domain condition $j[G(j\omega) - G^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$ (Lanzon & Petersen, 2008). Many practical systems can be modelled as NI systems by appropriately choosing the system inputs and outputs. For example, such systems arise when considering the transfer functions from force inputs to collocated position outputs in lightly damped structures (Lanzon & Petersen, 2008; Xiong, Petersen, & Lanzon, 2010), from voltage inputs to capacitance charge outputs in RLC circuit networks (Petersen, 2015; Xiong, Lanzon, & Petersen, 2016), from voltage inputs to shaft rotational velocity outputs in DC servo motor (Song, Lanzon, Patra, & Petersen, 2012). Recently, NI systems theory has been extended and emerged as an useful theory with practical applications. In Xiong et al. (2010), the definition of NI systems has been extended by allowing poles on the imaginary axis except at the origin. A further extension allowing poles at the origin has been proposed in Mabrok, Kallapur, Petersen, and Lanzon (2014). A more general definition about NI transfer functions which were not necessarily proper has been given in Liu and Xiong (2016). Furthermore, NI lemma has been generalised by removing the minimality assumption in Song, Lanzon, Patra, and Petersen (2012) and has been extended to descriptor systems in Xiong et al. (2016). These results have been well applied to the stability analysis of NI systems ((Liu & Xiong, 2015); Xiong et al., 2010) and to the controller synthesis of NI systems (Song, Lanzon, Patra, & Petersen, 2010; Song et al., 2012; Xiong, Lam, & Petersen, 2016). In addition, NI theory has been widely used in engineering, such as resonant controller design for piezoelectric tube scanner (Das, Pota, & Petersen, 2014), damping controller design for nanopositioners (Das, Pota, & Petersen, 2015) and feedback resonance compensator design for hard disk drive servo system (Rahman, Al Mamun, Yao, & Das, 2015).

Model reduction is recognised as one of the cornerstones of control theory and still a topic of active research. Efficient approaches have been developed for model reduction problems over past decades, such as the classical balanced truncation method (Moore, 1981), Hankel-norm approximation (Glover, 1984) and H_∞ model reduction (Kavranoglu & Bettayeb, 1993). Given an nth-order linear time-invariant system $G(s)$, the H_∞ model reduction problem is to find a lower-order system $G_r(s)$ satisfying $\|G(s) - G_r(s)\|_\infty < \gamma$. H_∞ model reduction method based on linear matrix inequalities (LMIs) has attracted much attention in recent years (Li, Yin, & Gao, 2014; Li, Yu, & Gao, 2015; Shen & Lam, 2015; Wei, Qiu, Karimi, & Wang, 2014, 2015). The passivity-preserving model reduction method has been presented in Li et al. (2014). H_∞ model reduction method has been applied to the frequency-limited model reduction for linear systems in Shen and Lam (2015). A more complicated situation to preserve positivity of the reduced-order system over a limited frequency interval has been handled in Li et al. (2015). In practical applications, many large-scale systems can be modelled as NI systems, such as the RLC network in Li et al. (2014), the tapered bar with force actuators and collocated position sensors in Geromel, Egas, and Kawaoka (2005). For these systems, it is often desirable to preserve the NI property when the order is reduced. Unfortunately, the existing model reduction methods are not applicable to NI systems, because they cannot guarantee the NI property of the reduced-order systems. Tu, Du, and Fan (2014) considered the model reduction problem for NI systems, where the balanced truncation method has been used. However, the method proposed in Tu et al. (2014) is not applicable to the NI systems with non-minimal realisations and the approximation error cannot be minimised. Hence, the model reduction problem for NI systems is still open and remains challenging. This motivates the research of this paper.

In this paper, we investigate the H_∞ model reduction problem for NI systems. For a given stable NI system $G(s)$, the objective of the paper is to find a stable reduced-order NI system $G_r(s)$

CONTACT Junlin Xiong junlin.xiong@gmail.com

so that $\|G(s) - G_r(s)\|_\infty < \gamma$, where γ is a prescribed positive number. Sufficient conditions in terms of matrix inequalities are derived for the existence of an H_∞ reduced-order NI system. In these new conditions, the reduced-order system matrices are decoupled with the matrix variables induced by the bounded real lemma (Gahinet & Apkarian, 1994) and NI lemma (Song et al., 2012). Moreover, the desired reduced-order system is constructed by using the feasible solutions. Iterative algorithms are provided to solve the matrix inequalities and to minimise the H_∞ approximation error. Finally, an RLC network example is given to demonstrate the effectiveness of the proposed model reduction method. The main challenge of this research is how to transform the H_∞ model reduction problem for NI systems into a convex optimisation problem. The conservatism of our results is that only a sub-optimal reduced-order NI system can be found. The contribution of this paper is that an H_∞ model reduction method for NI systems is developed.

Notation: All the matrices are assumed to be compatible dimensions and the symbol $\star$ within a square matrix represents the symmetric part. $\mathbb{R}^{m\times n}$ and $\mathcal{R}^{m\times n}$ denote all the $m \times n$ real matrices and real-rational proper transfer matrices, respectively. $0_{m\times n}$ denotes an $m \times n$ zero matrix and I_n represents identity matrix of order n. $\mathcal{H}$ denotes the set of matrices defined by $\mathcal{H} \triangleq \{U : U = [\,0_{(r+m)\times n}\ \ \Lambda\,]$, where $\Lambda \in \mathbb{R}^{(r+m)\times r}\}$. For a matrix A, A^{-1} and A^{T} stand for the inverse and transpose of A, respectively. $\Re[\,\cdot\,]$ denotes the real part of a complex number. The notation $P > 0$ (≥ 0) means that matrix P is positive definite (semi definite). For a matrix $X \in \mathbb{R}^{n\times n}$, sym$(X)$ indicates $X^{\mathrm{T}} + X$.

2. Problem formulation

Consider a stable linear time-invariant NI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$. The transfer function of (1) is given by $G(s) = C(sI - A)^{-1}B$.

A reduced-order system for (1) is

$$\begin{aligned} \dot{x}_r(t) &= A_r x_r(t) + B_r u(t), \\ y_r(t) &= C_r x_r(t), \end{aligned} \tag{2}$$

where $x_r(t) \in \mathbb{R}^r, y_r(t) \in \mathbb{R}^m, A_r \in \mathbb{R}^{r\times r}, B_r \in \mathbb{R}^{r\times m}, C_r \in \mathbb{R}^{m\times r}$ with $1 \leq r < n$. The transfer function of (2) is $G_r(s) = C_r(sI - A_r)^{-1}B_r$.

From (1) and (2), the approximation error system is given by

$$\begin{aligned} \dot{x}_e(t) &= A_e x_e(t) + B_e u(t), \\ e(t) &= C_e x_e(t), \end{aligned} \tag{3}$$

where $x_e(t) = [\,x^{\mathrm{T}}(t)\ x_r^{\mathrm{T}}(t)\,]^{\mathrm{T}}$ is the augmented state vector, $e(t) = y(t) - y_r(t)$ is the output error and

$$A_e = \begin{bmatrix} A & 0_{n\times r} \\ 0_{r\times n} & A_r \end{bmatrix}, \quad B_e = \begin{bmatrix} B \\ B_r \end{bmatrix}, \quad C_e = \begin{bmatrix} C & -C_r \end{bmatrix}.$$

The transfer function of the approximation error system (3) is $G_e(s) = C_e(sI - A_e)^{-1}B_e$.

Note that the stability of systems (1) and (2) is equivalent to that of system (3). Hence, the H_∞ model reduction problem for NI systems can be formulated as follows.

Problem 2.1: *Given $\gamma > 0$ and r $(1 \leq r < n)$. The H_∞ model reduction problem for NI system (1) is to find a reduced-order system (2), such that*

(1) *the reduced-order system $G_r(s)$ in (2) is stable NI*;
(2) *the approximation error system $G_e(s)$ in (3) satisfies $\|G_e(s)\|_\infty < \gamma$.*

Some preliminaries are presented. First, the definition of stable NI system is given.

Definition 2.1 (Lanzon & Petersen, 2008)**:** A square real-rational proper transfer function matrix $G(s) \in \mathcal{R}^{m\times m}$ is stable NI if

(1) $G(s)$ has no poles in $\Re[s] \geq 0$.
(2) $j[G(j\omega) - G^{*}(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$.

The following definition of NI systems is a generalisation of Definition 2.1.

Definition 2.2 (Xiong et al., 2010)**:** A square real-rational proper transfer function matrix $G(s) \in \mathcal{R}^{m\times m}$ is NI if

(1) $G(s)$ has no poles at the origin and in $\Re[s] > 0$.
(2) $j[G(j\omega) - G^{*}(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$ except values of ω where $j\omega$ is a pole of $G(s)$.
(3) If $j\omega_0$ is a pole of $G(s)$, it is at most a simple pole, and the residue matrix $K_0 \triangleq \lim_{s\to j\omega_0}(s - j\omega_0)jG(s)$ is positive semi-definite Hermitian.

The NI lemma without minimality assumption is given as follows.

Lemma 2.1 (Song et al., 2012)**:** *Let (A, B, C, D) be a state-space realisation of $G(s) \in \mathcal{R}^{m\times m}$, where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$, $m \leq n$. If $\det(A) \neq 0$, $D = D^{\mathrm{T}}$, and there exists a matrix $R \in \mathbb{R}^{n\times n}$, $R = R^{\mathrm{T}} > 0$, such that the following conditions*

$$AR + RA^{\mathrm{T}} \leq 0 \quad \text{and} \quad B + ARC^{\mathrm{T}} = 0$$

hold, then $G(s)$ is NI.

The following lemma is known as the bounded real lemma.

Lemma 2.2 (Gahinet & Apkarian, 1994)**:** *The error system in (3) is stable and satisfies $\|G(s)_e\|_\infty < \gamma$ if and only if there exists a matrix $Q \in \mathbb{R}^{(n+r)\times(n+r)}$, $Q = Q^{\mathrm{T}} > 0$, such that*

$$\Sigma \triangleq \begin{bmatrix} sym(QA_e) & QB_e & C_e^{\mathrm{T}} \\ B_e^{\mathrm{T}}Q & -\gamma I_m & 0_{m\times m} \\ C_e & 0_{m\times m} & -\gamma I_m \end{bmatrix} < 0. \tag{4}$$

To facilitate the presentation, some useful matrices are defined as

$$\bar{A} \triangleq \begin{bmatrix} A & 0_{n\times r} \\ 0_{r\times n} & 0_{r\times r} \end{bmatrix}, \quad \bar{F} \triangleq \begin{bmatrix} 0_{n\times r} & 0_{n\times m} \\ I_r & 0_{r\times m} \end{bmatrix},$$

$$\bar{B} \triangleq \begin{bmatrix} B \\ 0_{r\times m} \end{bmatrix}, \quad \bar{M} \triangleq \begin{bmatrix} 0_{n\times r} \\ I_r \end{bmatrix},$$

$$\bar{C} \triangleq \begin{bmatrix} C & 0_{m\times r} \end{bmatrix}, \quad \bar{H} \triangleq \begin{bmatrix} 0_{m\times r} & -I_m \end{bmatrix}, \quad \bar{N} \triangleq \begin{bmatrix} 0_{m\times r} & I_m \end{bmatrix}.$$

The system matrices of system (3) can be rewritten as

$$A_e = \bar{A} + \bar{F}U, \quad B_e = \bar{B} + \bar{F}V, \quad C_e = \bar{C} + \bar{H}U, \tag{5}$$

where

$$U = \begin{bmatrix} 0_{r\times n} & A_r \\ 0_{m\times n} & C_r \end{bmatrix} \in \mathcal{H}, \quad V = \begin{bmatrix} B_r \\ 0_{m\times m} \end{bmatrix}.$$

3. H_∞ negative imaginary model reduction

In this section, the main results of this paper are presented. Sufficient conditions in terms of matrix inequalities are derived for the existence of a solution to Problem 2.1. It is shown that the reduced-order NI system can be found by solving matrix inequalities. Iterative algorithms are provided to solve the matrix inequalities and to minimise the H_∞ approximation error.

First, a necessary and sufficient condition is proposed for the existence of an H_∞ reduced-order system.

Lemma 3.1: *Given $\gamma > 0$, r ($1 \le r < n$), and a stable system $G(s)$ in (1). There exists a reduced-order system (2) such that the approximation error system (3) is stable and satisfies $\|G_e(s)\|_\infty < \gamma$ if and only if there exist matrices $\hat{U} \in \mathcal{H}$, $L \in \mathcal{H}$, $V \in \mathbb{R}^{(r+m)\times m}$, $Q \in \mathbb{R}^{(n+r)\times(n+r)}$, $Q = Q^{\mathrm{T}} > 0$ and a diagonal matrix $X \in \mathbb{R}^{(r+m)\times(r+m)}$, $X > 0$, such that*

$$\begin{bmatrix} \mathrm{sym}(Q\bar{A} - \hat{U}^{\mathrm{T}}L) + \hat{U}^{\mathrm{T}}X\hat{U} & Q(\bar{B}+\bar{F}V) & \bar{C}^{\mathrm{T}} & Q\bar{F} + L^{\mathrm{T}} \\ \star & -\gamma I_m & 0_{m\times m} & 0_{m\times(m+r)} \\ \star & \star & -\gamma I_m & \bar{H} \\ \star & \star & \star & -X \end{bmatrix} < 0. \tag{6}$$

Proof: In view of Lemma 2.2, $G_e(s)$ is stable and satisfies $\|G_e(s)\|_\infty < \gamma$ if and only if there exists a matrix $Q \in \mathbb{R}^{(n+r)\times(n+r)}$, $Q = Q^{\mathrm{T}} > 0$, such that (4) holds. Now we prove that (4) is equivalent to (6).

($\Rightarrow$) Suppose that there exists a matrix $Q \in \mathbb{R}^{(n+r)\times(n+r)}$, $Q = Q^{\mathrm{T}} > 0$ such that (4) holds. There always exists a real diagonal matrix $X > 0$ such that $-X - W\Sigma^{-1}W^{\mathrm{T}} < 0$, where

$$W = \begin{bmatrix} \bar{F}^{\mathrm{T}}Q & 0_{(m+r)\times m} & \bar{H}^{\mathrm{T}} \end{bmatrix}.$$

Using the Schur complement, $-X - W\Sigma^{-1}W^{\mathrm{T}} < 0$ is equivalent to

$$\begin{bmatrix} \Sigma & W^{\mathrm{T}} \\ W & -X \end{bmatrix} = \begin{bmatrix} \mathrm{sym}(QA_e) & QB_e & C_e^{\mathrm{T}} & Q\bar{F} \\ \star & -\gamma I_m & 0_{m\times m} & 0_{m\times(m+r)} \\ \star & \star & -\gamma I_m & \bar{H} \\ \star & \star & \star & -X \end{bmatrix} < 0. \tag{7}$$

Let

$$T \triangleq \begin{bmatrix} I_{n+r} & 0_{(n+r)\times m} & 0_{(n+r)\times m} & 0_{(n+r)\times(m+r)} \\ 0_{m\times(n+r)} & I_m & 0_{m\times m} & 0_{m\times(m+r)} \\ 0_{m\times(n+r)} & 0_{m\times m} & I_m & 0_{m\times(m+r)} \\ -U & 0_{(m+r)\times m} & 0_{(m+r)\times m} & I_{m+r} \end{bmatrix}.$$

Multiplying (7) to the right by T and to the left by T^{T}, one obtains

$$\begin{bmatrix} \mathrm{sym}(QA_e - Q\bar{F}U) - U^{\mathrm{T}}XU & QB_e & C_e^{\mathrm{T}} - U^{\mathrm{T}}\bar{H}^{\mathrm{T}} & Q\bar{F} + U^{\mathrm{T}}X \\ \star & -\gamma I_m & 0_{m\times m} & 0_{m\times(m+r)} \\ \star & \star & -\gamma I_m & \bar{H} \\ \star & \star & \star & -X \end{bmatrix} < 0. \tag{8}$$

Substituting (5) into (8), we have that

$$\begin{bmatrix} \mathrm{sym}(Q\bar{A}) - U^{\mathrm{T}}XU & Q(\bar{B}+\bar{F}V) & \bar{C}^{\mathrm{T}} & Q\bar{F} + U^{\mathrm{T}}X \\ \star & -\gamma I_m & 0_{m\times m} & 0_{m\times(m+r)} \\ \star & \star & -\gamma I_m & \bar{H} \\ \star & \star & \star & -X \end{bmatrix} < 0. \tag{9}$$

Let $\hat{U} = U$ and $L = XU$, we arrive at (6).

($\Leftarrow$) Suppose that there exist matrices $\hat{U} \in \mathcal{H}$, $L \in \mathcal{H}$, $V \in \mathbb{R}^{(r+m)\times m}$, $Q \in \mathbb{R}^{(n+r)\times(n+r)}$, $Q = Q^{\mathrm{T}} > 0$ and diagonal matrix $X \in \mathbb{R}^{(r+m)\times(r+m)}$, $X > 0$, such that (6) holds. Using the Schur complement, the inequality (6) is equivalent to

$$\mathrm{sym}(Q\bar{A}) - \hat{U}^{\mathrm{T}}L - L^{\mathrm{T}}\hat{U} + \hat{U}^{\mathrm{T}}X\hat{U} - \Xi_{12}\Xi_{22}^{-1}\Xi_{12}^{\mathrm{T}} < 0, \tag{10}$$

where

$$\Xi_{12} = \begin{bmatrix} Q(\bar{B}+\bar{F}V) & \bar{C}^{\mathrm{T}} & Q\bar{F} + L^{\mathrm{T}} \end{bmatrix},$$

$$\Xi_{22} = \begin{bmatrix} -\gamma I_m & 0_{m\times m} & 0_{m\times(m+r)} \\ \star & -\gamma I_m & \bar{H} \\ \star & \star & -X \end{bmatrix}.$$

It follows from

$$(L - X\hat{U})^{\mathrm{T}}X^{-1}(L - X\hat{U}) \ge 0$$

that

$$-\hat{U}^{\mathrm{T}}L - L^{\mathrm{T}}\hat{U} + \hat{U}^{\mathrm{T}}X\hat{U} \ge -L^{\mathrm{T}}X^{-1}L. \tag{11}$$

Combing (10) with (11), we have that

$$\mathrm{sym}(Q\bar{A}) - L^{\mathrm{T}}X^{-1}L - \Xi_{12}\Xi_{22}^{-1}\Xi_{12}^{\mathrm{T}} < 0,$$

which is equivalent to

$$\begin{bmatrix} \mathrm{sym}(Q\bar{A}) - L^{\mathrm{T}}X^{-1}L & Q(\bar{B}+\bar{F}V) & \bar{C}^{\mathrm{T}} & Q\bar{F} + L^{\mathrm{T}} \\ \star & -\gamma I_m & 0_{m\times m} & 0_{m\times(m+r)} \\ \star & \star & -\gamma I_m & \bar{H} \\ \star & \star & \star & -X \end{bmatrix} < 0.$$

Let $U = X^{-1}L$, we arrive at (9).

Multiplying (9) to the right by T^{-1} and to the left by $T^{-\mathsf{T}}$, we obtain (7), which implies that the inequality (4) holds. The proof is completed. ■

Remark 3.1: Multiplier relaxation technique is widely used for H_∞ model reduction problems, such as Li, Lam, Wang, and Date (2011), Li et al. (2014), Shen and Lam (2015), Wei et al. (2014, 2015). It is also a commonly used technique for H_∞ control syntheses, such as Feng and Yagoubi (2013), Shen and Lam (2014), Qiu, Ding, Gao, and Yin (2016).

The following result provides sufficient conditions for the existence of a solution to Problem 2.1 and the construction of the desired reduced-order system.

Theorem 3.1: *Given $\gamma > 0$, r $(1 \le r < n)$, and consider the stable NI system (1). If there exist matrices $P \in \mathbb{R}^{r\times r}$, $P = P^\mathsf{T} > 0$, $Q \in \mathbb{R}^{(n+r)\times(n+r)}$, $Q = Q^\mathsf{T} > 0$, $\bar{\Omega} \in \mathbb{R}^{2r\times r}$, $\bar{A}_\mathrm{r} \in \mathbb{R}^{r\times r}$, $L \in \mathcal{H}$, $C_\mathrm{r} \in \mathbb{R}^{m\times r}$, nonsingular matrix $Y \in \mathbb{R}^{r\times r}$ and diagonal matrix $X \in \mathbb{R}^{(r+m)\times(r+m)}$, $X > 0$, such that the following matrix inequalities hold:*

$$\Phi + sym(\bar{\Omega}\bar{\Gamma}^\mathsf{T}) \le 0, \tag{12}$$

$$\begin{bmatrix} sym(Q\bar{A} - \hat{U}^\mathsf{T}L) + \hat{U}^\mathsf{T}X\hat{U} & Q(\bar{B} + \bar{F}V) & \bar{C}^\mathsf{T} & Q\bar{F} + L^\mathsf{T} \\ \star & -\gamma I_m & 0_{m\times m} & 0_{m\times(m+r)} \\ \star & \star & -\gamma I_m & \bar{H} \\ \star & \star & \star & -X \end{bmatrix} < 0, \tag{13}$$

where

$$\bar{\Gamma} = \begin{bmatrix} \bar{A}_\mathrm{r} \\ -Y \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0_{r\times r} & P \\ P & 0_{r\times r} \end{bmatrix}, \quad \hat{U} = \begin{bmatrix} 0_{r\times n} & \bar{A}_\mathrm{r}Y^{-1} \\ 0_{m\times n} & C_\mathrm{r} \end{bmatrix},$$
$$V = \begin{bmatrix} -\bar{A}_\mathrm{r}Y^{-1}PC_\mathrm{r}^\mathsf{T} \\ 0_{m\times m} \end{bmatrix}.$$

Then, Problem 2.1 is solvable and the desired reduced-order system matrices are given by $A_\mathrm{r} = \bar{A}_\mathrm{r}Y^{-1}$, $B_\mathrm{r} = -A_\mathrm{r}PC_\mathrm{r}^\mathsf{T}$, C_r.

Proof: Suppose that there exist matrices $P = P^\mathsf{T} > 0$, $Q = Q^\mathsf{T} > 0$, $L \in \mathcal{H}$, $\bar{\Omega}$, $\bar{A}_\mathrm{r}$, C_r, nonsingular matrix Y and diagonal matrix $X > 0$, such that the inequalities (12), (13) hold. The reduced-order system $G_\mathrm{r}(s)$ is

$$G_\mathrm{r}(s) = \left[\begin{array}{c|c} \bar{A}_\mathrm{r}Y^{-1} & -A_\mathrm{r}PC_\mathrm{r}^\mathsf{T} \\ \hline C_\mathrm{r} & 0_{m\times m} \end{array}\right].$$

Let $\Gamma = \bar{\Gamma}Y^{-1}$ and $\Omega = \bar{\Omega}Y^\mathsf{T}$, (12) becomes

$$\Phi + \mathrm{sym}(\Omega\Gamma^\mathsf{T}) \le 0. \tag{14}$$

Substituting $A_\mathrm{r} = \bar{A}_\mathrm{r}Y^{-1}$ and $\bar{\Gamma} = [\begin{smallmatrix} \bar{A}_\mathrm{r} \\ -Y \end{smallmatrix}]$ into $\Gamma = \bar{\Gamma}Y^{-1}$, we have $\Gamma = [\begin{smallmatrix} A_\mathrm{r} \\ -I_r \end{smallmatrix}]$. Selecting $\Psi \triangleq [\begin{smallmatrix} I_r \\ A_\mathrm{r}^\mathsf{T} \end{smallmatrix}]$, one has that $\Psi^\mathsf{T}\Gamma = 0$.

Pre- and post-multiplying (14) by Ψ^T and Ψ, we have

$$\Psi^\mathsf{T}(\Phi + \mathrm{sym}(\Omega\Gamma^\mathsf{T}))\Psi = \Psi^\mathsf{T}\Phi\Psi = A_\mathrm{r}P + PA_\mathrm{r}^\mathsf{T} \le 0.$$

The reduced-order system matrix B_r is $B_\mathrm{r} = -A_\mathrm{r}PC_\mathrm{r}^\mathsf{T}$. According to Lemma 2.1, $G_\mathrm{r}(s)$ is NI.

Based on Lemma 3.1, the inequality (13) implies that the approximation error system (3) is stable and satisfies $\|G_\mathrm{e}(s)\|_\infty < \gamma$. Hence, Problem 2.1 is solvable. The proof is completed. ■

Remark 3.2: Compared to the conditions in Lemmas 2.1 and 2.2, the matrices Q, P in Theorem 3.1 are not coupled with the reduced-order system matrices A_r, B_r, C_r. This characteristic is beneficial for solving Problem 2.1. The inequalities (12), (13) are not LMIs with respect to the parameters P, Q, $\bar{\Omega}$, $\bar{A}_\mathrm{r}$, L, C_r, Y, X. However, if the parameters $\bar{\Omega}$, C_r are fixed, these inequalities become LMIs. These LMIs can be solved efficiently by available numerical software.

An iterative LMIs algorithm is provided to solve the conditions in Theorem 3.1.

Algorithm 1. Iterative Algorithm for NI Model Reduction

1: **Input** $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times m}$, $\gamma > 0$, $1 \le r < n$, $\varepsilon > 0$.
2: **Output** $A_\mathrm{r} \in \mathbb{R}^{r\times r}$, $B_\mathrm{r} \in \mathbb{R}^{r\times m}$, $C_\mathrm{r} \in \mathbb{R}^{m\times r}$.
3: **Initilization**

Use the existing methods, such as balanced truncation method, to get a reduced-order system $\mathcal{K}_\mathrm{r} = \left[\begin{array}{c|c} \mathcal{A}_\mathrm{r} & \mathcal{B}_\mathrm{r} \\ \hline \mathcal{C}_\mathrm{r} & 0_{m\times m} \end{array}\right]$.

Fix $\bar{\Omega} = \begin{bmatrix} \mathcal{A}_\mathrm{r}^\mathsf{T} & -I_r \end{bmatrix}^\mathsf{T}$, solve (12) for P, Y, $\bar{A}_\mathrm{r}$. Let $A_\mathrm{r} = \bar{A}_\mathrm{r}Y^{-1}$.

Set $i = 0$. Choose $C_\mathrm{r}^{(i)} = \mathcal{C}_\mathrm{r}$, $B_\mathrm{r}^{(i)} = -A_\mathrm{r}P(\mathcal{C}_\mathrm{r})^\mathsf{T}$, $\mu^{(-1)} = -1$.

4: **repeat**
5: Let $\hat{U}^{(i)} = \begin{bmatrix} 0_{r\times n} & A_\mathrm{r} \\ 0_{m\times n} & C_\mathrm{r}^{(i)} \end{bmatrix}$ and $V^{(i)} = \begin{bmatrix} -B_\mathrm{r}^{(i)} \\ 0_{m\times m} \end{bmatrix}$, solve the following optimisation problem for Q, L, X, μ :

$$\begin{aligned} &\min \quad \mu \\ &\text{s.t.} \quad \begin{bmatrix} \begin{matrix} \mathrm{sym}(Q\bar{A} - \hat{U}^\mathsf{T}L) \\ +\hat{U}^\mathsf{T}X\hat{U} - \mu I_{n+r} \end{matrix} & Q(\bar{B} + \bar{F}V) & \bar{C}^\mathsf{T} & Q\bar{F} + L^\mathsf{T} \\ \star & -\gamma I_m & 0_{m\times m} & 0_{m\times(m+r)} \\ \star & \star & -\gamma I_m & \bar{H} \\ \star & \star & \star & -X \end{bmatrix} < 0. \end{aligned} \tag{15}$$

6: **if** $\mu \le 0$ **then**
7: Output A_r, $B_\mathrm{r}^{(i)}$ and $C_\mathrm{r}^{(i)}$.
8: **else**
9: Fix $\mu^{(i)} = \mu$, solve the following optimisation problem for Q, L, X:

$$\begin{aligned} &\min \quad \mathrm{trace}(X) \\ &\text{s.t.} \quad \begin{bmatrix} \begin{matrix} \mathrm{sym}(Q\bar{A} - \hat{U}^\mathsf{T}L) \\ +\hat{U}^\mathsf{T}X\hat{U} - \mu^{(i)} I_{n+r} \end{matrix} & Q(\bar{B} + \bar{F}V) & \bar{C}^\mathsf{T} & Q\bar{F} + L^\mathsf{T} \\ \star & -\gamma I_m & 0_{m\times m} & 0_{m\times(m+r)} \\ \star & \star & -\gamma I_m & \bar{H} \\ \star & \star & \star & -X \end{bmatrix} < 0. \end{aligned} \tag{16}$$

Update $C_\mathrm{r}^{(i+1)}$ and $B_\mathrm{r}^{(i+1)}$ according to

$$C_\mathrm{r}^{(i+1)} = \bar{N}X^{-1}L\bar{M}, \quad B_\mathrm{r}^{(i+1)} = -A_\mathrm{r}P(C_\mathrm{r}^{(i+1)})^\mathsf{T}.$$

Set $i = i + 1$.
end if
until $|\mu^{(i)} - \mu^{(i-1)}| \le \varepsilon$.

Remark 3.3 The optimisation problems (15), (16) are convex and can be solved effectively. If a solution $\mu \le 0$ to (15) is found, then the obtained reduced-order system is a solution to Problem 2.1. If a solution $\mu \le 0$ cannot be found, then we conclude that there may not exist a reduced-order NI system such that $\|G_\mathrm{e}(s)\|_\infty < \gamma$. For the H_∞ model reduction problem, it is desired that the approximation error is as small as possible. To minimise the H_∞ approximation error, the following optimisation algorithm is provided.

Algorithm 2. Minimisation of H_∞ approximation error

Input $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $1 \le r < n$, $\varepsilon > 0$, $\Delta_\gamma > 0$.
Output $A_\mathrm{r} \in \mathbb{R}^{r\times r}$, $B_\mathrm{r} \in \mathbb{R}^{r\times m}$, $C_\mathrm{r} \in \mathbb{R}^{m\times r}$, γ^*.
Initilization

Use the existing methods, such as balanced truncation method, to get a reduced-order system $\mathcal{K}_\mathrm{r} = \left[\begin{array}{c|c} \mathcal{A}_\mathrm{r} & \mathcal{B}_\mathrm{r} \\ \hline \mathcal{C}_\mathrm{r} & 0_{m\times m}\end{array}\right]$.
Fix $\bar{\Omega} = \begin{bmatrix} \mathcal{A}_\mathrm{r}^\mathsf{T} & -I_r \end{bmatrix}^\mathsf{T}$, solve (12) for P, Y, $\bar{A}_\mathrm{r}$. Let $A_\mathrm{r} = \bar{A}_\mathrm{r}Y^{-1}$.
Set $i, j = 0$ and $\gamma^{(j)}$ to be a sufficiently large number. Choose $C_\mathrm{r}^{(i)} = \mathcal{C}_\mathrm{r}$, $B_\mathrm{r}^{(i)} = -A_\mathrm{r}P(\mathcal{C}_\mathrm{r})^\mathsf{T}$, $\mu^{(-1)} = -1$.

repeat
print Let $\hat{U}^{(i)} = \begin{bmatrix} 0_{r\times n} & A_\mathrm{r} \\ 0_{m\times n} & C_\mathrm{r}^{(i)} \end{bmatrix}$ and $V^{(i)} = \begin{bmatrix} -B_\mathrm{r}^{(i)} \\ 0_{m\times m} \end{bmatrix}$, solve (15) for Q, L, X, μ.
if $\mu \le 0$ **then**
Denote the obtained $B_\mathrm{r}^{(i)}$ and $C_\mathrm{r}^{(i)}$ as $B_\mathrm{r}^{(j)}$ and $C_\mathrm{r}^{(j)}$. Set $\gamma^{(j+1)} = \gamma^{(j)} - \Delta_\gamma$, $j = j + 1$.
else
Fix $\mu^{(i)} = \mu$, solve (16) for Q, L, X.
Update $C_\mathrm{r}^{(i+1)}$ and $B_\mathrm{r}^{(i+1)}$ according to

$$C_\mathrm{r}^{(i+1)} = \bar{N}X^{-1}L\bar{M}, \quad B_\mathrm{r}^{(i+1)} = -A_\mathrm{r}P(C_\mathrm{r}^{(i+1)})^\mathsf{T}.$$

Set $i = i + 1$.
end if
until $|\mu^{(i)} - \mu^{(i-1)}| \le \varepsilon$, output A_r, $B_\mathrm{r}^{(j-1)}$, $C_\mathrm{r}^{(j-1)}$ and $\gamma^* = \gamma^{(j-1)}$.

4. Illustrative example

In this section, an example is provided to illustrate the effectiveness of the proposed method. The optimisation problems are solved by YALMIP (Löfberg, 2004). The performance of the proposed method is compared with the NI balanced truncation method (Tu et al., 2014).

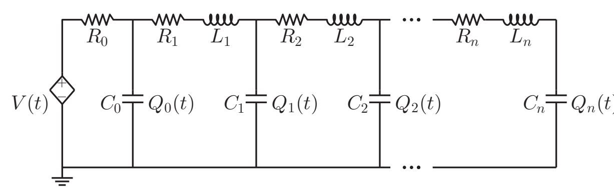

Figure 1. RLC network.

RLC networks as an illustrative example are widely used in numerous works, such as Li et al. (2014), Petersen (2015) and Xiong et al. (2016). In this paper, we choose the n-stage RLC network in Figure 1 as an illustrative example. The input is the voltage $V(t)$ and the output is the total charge on the capacitances

$$Q(t) = \sum_{k=0}^{n} Q_k(t).$$

The input–output relationship from $u(t) = V(t)$ to $y(t) = Q(t)$ is given by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned}$$

where $x(t) = \begin{bmatrix} u_0(t) & i_{L_1}(t) & u_1(t) & \cdots & i_{L_n}(t) & u_n(t) \end{bmatrix}^\mathsf{T}$, $u_k(t)$ is the voltage across capacitor C_k and $i_{L_k}(t)$ represent the current through inductor L_k,

$$A = \begin{bmatrix} -\frac{1}{C_0R_0} & -\frac{1}{C_0} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{L_1} & -\frac{R_1}{L_1} & -\frac{1}{L_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{C_1} & 0 & -\frac{1}{C_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{L_2} & -\frac{R_2}{L_2} & -\frac{1}{L_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{L_n} & -\frac{R_n}{L_n} & -\frac{1}{L_n} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{C_n} & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{1}{C_0R_0} & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^\mathsf{T},$$
$$C = \begin{bmatrix} C_0 & 0 & C_1 & 0 & C_2 & \cdots & C_n \end{bmatrix}.$$

It can be verified that the system is a stable NI system by either Definition 2.1 or Lemma 2.1. Here, we considered an 11-dimensional RLC network with $C_k = 1F$, $L_k = 1H$, $R_k = 0.5\Omega$. The goal of this example is to find the reduced-order NI systems with as small as possible H_∞ approximation error.

The reduced-order systems returned by Algorithm 2 are given in Table 1. For $r = 3$, the sub-optimal approximation error is $\gamma^* = 0.262$, which is smaller than 0.355 obtained by NI balanced truncation method in Tu et al. (2014). For $r = 2$, the sub-optimal approximation error is $\gamma^* = 0.494$, which is smaller than 0.975 obtained by NI balanced truncation method in Tu et al. (2014). For $r = 1$, the sub-optimal approximation error is $\gamma^* = 0.614$, which is smaller than 2.8412 obtained by NI balanced truncation method in Tu et al. (2014). The bode plot of the original and the reduced-order systems are shown in Figure 2. It can be seen from Figure 2 that the reduced-order system $G_\mathrm{r}(j\omega)$ satisfies $\angle G_\mathrm{r}(j\omega) \in (-\pi, 0)$ for all $\omega \in (0, \infty)$.

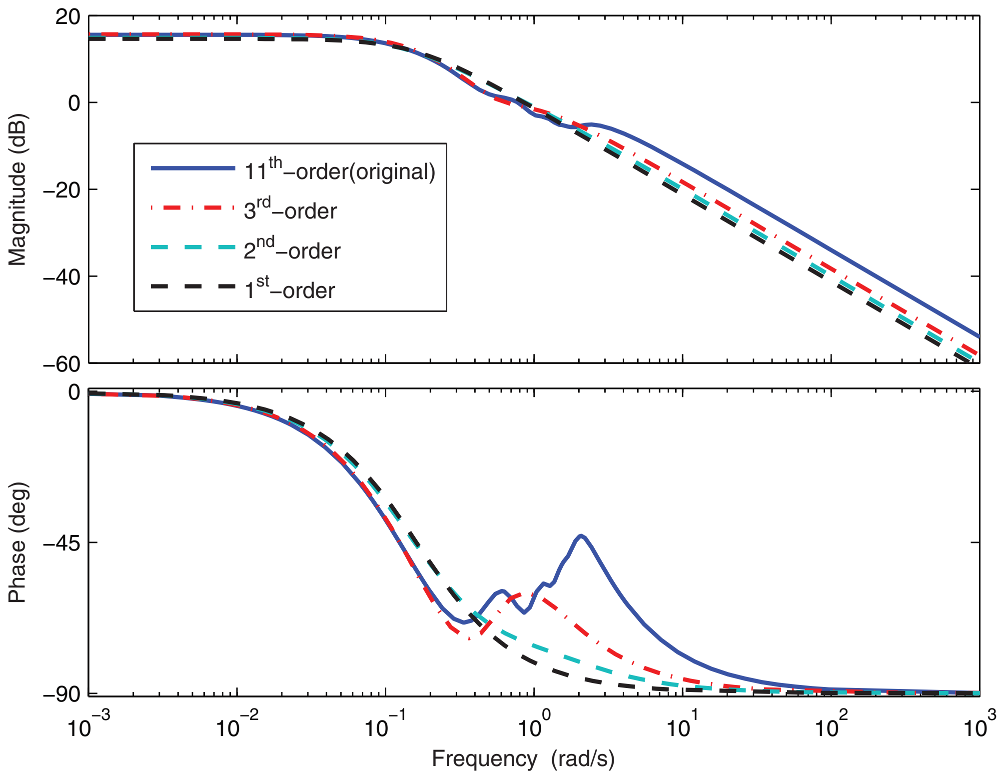

Figure 2. Bode plot of original and reduced-order systems.

Table 1. Reduced-order systems obtained by Algorithm 2.

Order	G_r	γ^*
Third	$\left[\begin{array}{ccc\|c} -0.5311 & 0.1309 & -0.5378 & -4.467 \\ 0.0353 & -0.5093 & 0.1103 & -6.665 \\ -0.1188 & -0.0366 & -0.3283 & -3.922 \\ \hline 0.368 & 1.037 & -2.492 & 0 \end{array}\right]$	0.262
Second	$\left[\begin{array}{cc\|c} -1.9671 & -1.0663 & -1.429 \\ 1.0314 & 0.4560 & 0.256 \\ \hline -0.9808 & -1.5377 & 0 \end{array}\right]$	0.494
First	$\left[\begin{array}{c\|c} -0.1612 & -0.432 \\ \hline -2.010 & 0 \end{array}\right]$	0.614

This means that $G_r(j\omega)$ has non-positive imaginary part, that is, $j[G_r(j\omega) - G_r^*(j\omega)] \geq 0$. In addition, the reduced-order system matrix A_r is stable. It follows from Definition 2.1 that the reduced-order system $G_r(s)$ is stable NI.

5. Conclusions

The H_∞ model reduction problem for NI systems has been studied in this paper. To preserve the NI property of the reduced-order systems, sufficient conditions have been established for the construction of an H_∞ reduced-order NI system. Iterative numerical algorithms have been provided to find the desired reduced-order system and to minimise the H_∞ approximation error. Finally, the efficiency of the proposed method has been illustrated by an RLC network example. The limitation of this paper is that the developed method can only find a sub-optimal reduced-order NI system. How to extend the results of this paper to find the optimal reduced-order NI system is worth future research.

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Notes on contributors

Lanlin Yu received her B.Eng. degree from Southwest University in 2013. She is currently pursuing her Ph.D. degree at the the University of Science and Technology of China. Her research interests include model reduction and negative imaginary systems.

Junlin Xiong received his B.Eng. and M.Sci. degrees from Northeastern University, China, and his Ph.D. degree from the University of Hong Kong, Hong Kong, in 2000, 2003 and 2007, respectively. From November 2007 to February 2010, he was a research associate at the University of New South Wales at the Australian Defence Force Academy, Australia. In March 2010, he joined the University of Science and Technology of China where he is currently a professor in the Department of Automation. Currently, he is an associate editor for the IET Control Theory and Application. His current research interests are in the fields of negative imaginary systems, large-scale systems and networked control systems.

ORCID

Junlin Xiong http://orcid.org/0000-0002-0128-4960

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