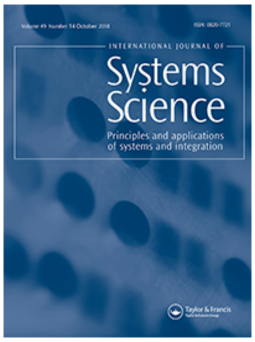

International Journal of Systems Science

ISSN: 0020-7721 (Print) 1464-5319 (Online) Journal homepage: http://www.tandfonline.com/loi/tsys20

Interval frequency negative imaginary systems

Junlin Xiong & Shengyu Zhai

To cite this article: Junlin Xiong & Shengyu Zhai (2018) Interval frequency negative imaginary systems, International Journal of Systems Science, 49:14, 2968-2980, DOI: 10.1080/00207721.2018.1529256

To link to this article: https://doi.org/10.1080/00207721.2018.1529256

Published online: 05 Oct 2018.

INTERNATIONAL JOURNAL OF SYSTEMS SCIENCE
2018, VOL. 49, NO. 14, 2968–2980
https://doi.org/10.1080/00207721.2018.1529256

Interval frequency negative imaginary systems

Junlin Xiong and Shengyu Zhai

Department of Automation, University of Science and Technology of China, Hefei, People's Republic of China

ABSTRACT

This paper studies interval frequency negative imaginary (IFNI) systems based on minimal state-space realisations. Firstly, the concept of the IFNI transfer functions is introduced, and the relationship between the IFNI transfer functions and the interval frequency positive real transfer functions is established. Secondly, based on the generalised KYP lemma, a necessary and sufficient condition is derived for IFNI transfer functions with minimal state-space realisation in terms of linear matrix inequalities. Our results coincide with the existing negative imaginary lemma when the interval frequency set becomes the positive frequency set. Also, a time domain interpretation of the IFNI property is provided in terms of the system input, output and state. Finally, two examples are used to illustrate the developed theory.

ARTICLE HISTORY
Received 15 December 2017
Accepted 16 September 2018

KEYWORDS
Generalised KYP lemma; interval frequency; negative imaginary systems; positive real systems

1. Introduction

Roughly speaking, negative imaginary (NI) systems are a class of dynamical systems satisfying negative imaginary property (Ferrante & Ntogramatzidis, 2013; Lanzon & Petersen, 2008; Liu & Xiong, 2016; Mabrok, Kallapur, Petersen, & Lanzon, 2014a; Xiong, Petersen, & Lanzon, 2010). The negative imaginary property arises in many practical engineering systems, for example, lightly damped structures (Lanzon & Petersen, 2008; Xiong et al., 2010), atomic force microscopes (Das, Pota, & Petersen, 2014; Mabrok, Kallapur, Petersen, & Lanzon, 2014b), robotic manipulator arms (Mabrok et al., 2014a), large vehicles platoons (Cai & Hagen, 2010), and hard disk drive servo systems (Rahman, Al Mamun, Yao, & Das, 2015). The negative imaginary property of dynamical systems was introduced and extended in Lanzon and Petersen (2008), Xiong et al. (2010), Mabrok et al. (2014a) where negative imaginary lemmas were established to determine the negative imaginary property based on minimal state-space realisations. Also, necessary and sufficient conditions were derived for the stability analysis of two interconnected negative imaginary systems, having potential applications in robust control (Xiong, Lam, & Petersen, 2016).

The concept of NI systems and the corresponding theory in Lanzon and Petersen (2008), Xiong et al. (2010), Mabrok et al. (2014a) have been extended in many directions. Firstly, the authors of Ferrante and Ntogramatzidis (2013) extended the concept of NI systems to the case where the transfer functions were not necessarily rational and proper. However, the NI transfer functions in Ferrante and Ntogramatzidis (2013) need to be symmetric. Such a symmetric restriction was removed in Liu and Xiong (2016). The concepts and results in Lanzon and Petersen (2008), Xiong et al. (2010) were also extended to the lossless negative imaginary case in Xiong, Petersen, and Lanzon (2012b). The concept of the NI systems and the negative imaginary lemma in Lanzon and Petersen (2008), Xiong et al. (2010), Mabrok et al. (2014a) were further extended to the discrete-time case in Ferrante, Lanzon, and Ntogramatzidis (2017), Liu and Xiong (2017). On the other hand, many efforts have been devoted to the extension of NI systems theory. Uncertain system control and disturbance attenuation (Yuan, Wang, & Guo, 2017, 2018) are important control problems, and Song designed state feedback negative imaginary controllers for robustly stabilising uncertain systems with strictly NI systems uncertainty (Song, Lanzon, Patra, & Petersen, 2012). The output feedback case was studied in Xiong et al. (2016) and structural constraints were allowed in the designed controllers. The absolute stability of NI systems interconnected to slope-restricted nonlinear uncertainties was studied in Dey, Patra, and Sen (2016). Model reduction problems for NI systems were discussed in Yu and Xiong (2017) where the H_∞ norm of the error system was minimised.

In practice, many dynamical systems do not satisfy the NI property over all the positive frequencies. For

CONTACT Junlin Xiong junlin.xiong@gmail.com Department of Automation, University of Science and Technology of China, Hefei,230026, People's Republic of China

example, it has been shown that the capacitance subsystem of piezoelectric tube scanners in Bhikkaji, Ratnam, Fleming, and Moheimani (2007) satisfies the NI property over a finite frequency range (Xiong, Petersen, & Lanzon, 2012a). In Patra and Lanzon (2011), the authors introduced a class of systems satisfying negative imaginary property in a interval frequency range, which need to be real-rational stable. In this paper, we further extend by allowing poles on imaginary axis. The introduction of IFNI systems is also motivated by practical examples. For example, the transfer function of the piezoelectric tube scanners (PTSs) model in Das et al. (2014) and Sallen-key low-pass filter (Pactitis, 2007; Patra & Lanzon, 2011) are actually IFNI systems (more details are given in the example section in this paper). Similar to the approach used in Xiong et al. (2012a), the generalised Kalman-Yakubovich-Popov lemma in Iwasaki and Hara (2005), Iwasaki, Hara, and Fradkov (2005) provided us a foundation to study IFNI systems.

The main contributions of this paper can be highlighted as follows: (1) the concept of IFNI systems is defined, and includes low frequency negative imaginary (LFNI) systems, middle frequency negative imaginary (MFNI) systems and high frequency negative imaginary (HFNI) systems as special cases; (2) the relationship between IFNI property and IFPR property of dynamical systems is established according to their definitions; (3) a necessary and sufficient condition is derived to test IFNI property of dynamical systems in terms of linear matrix inequalities, and either when the upper bound of the low frequency interval approaches to infinity, or when the lower bound of the high frequency interval approaches to zero, our result reduces to the generalised NI lemma in Mabrok et al. (2015); (4) a time domain interpretation of the IFNI property is provided in terms of the system input, output and state.

Notation: Let A^*, $\bar{A}$ and A^T denote the complex conjugate transpose, the complex conjugate and the transpose of a complex matrix A, respectively. $\mathbb{R}^{n\times n}$ denotes the set of $n \times n$ real matrices. The real part and imaginary part of complex number s, respectively, are denoted by $\Re[s]$ and $\Im[s]$. The set of square, real-rational, proper transfer functions is denoted by**G**.

2. Interval frequency negative imaginary transfer functions

The section introduces some useful definitions and lemmas, which can be used for streaming the main results of the paper.

Definition 2.1 (Mabrok et al., 2014a): Given a transfer function matrix $R(s) \in \mathbf{G}$. $R(s)$ is said to be *negative imaginary* if

(1) $R(s)$ has no poles in $\Re[s] > 0$;
(2) $j[R(j\omega) - R^*(j\omega)] \geq 0$ for all $\omega > 0$ where $j\omega$ is not a pole of $R(s)$;
(3) if $j\omega_0, \omega_0 > 0$, is a pole of $R(s)$, then it is a simple pole and the corresponding residue matrix of $jR(s)$ is positive semidefinite Hermitian;
(4) if $R(s)$ has a pole at $s = 0$, then $\lim_{s\to 0} s^2R(s)$ is positive semidefinite Hermitian, and $\lim_{s\to 0} s^kR(s) = 0$ for $k \geq 3$.

Before defining IFNI transfer functions, we first define the frequency interval set

$$\Omega \triangleq \Omega_L \bigcup \left(\bigcup_{l=1}^{N} \Omega_{M_l} \right) \bigcup \Omega_H, \tag{1}$$

where

$$\begin{aligned} \Omega_L &= \{\omega \in \mathbb{R} : 0 < \omega \leq \bar{\omega}_0\}, \\ \Omega_{M_l} &= \left\{\omega \in \mathbb{R} : \underline{\omega}_l \leq \omega \leq \bar{\omega}_l, \bar{\omega}_l > \underline{\omega}_l > \bar{\omega}_{l-1}\right\}, \\ &\quad l = 1, \ldots, N, \\ \Omega_H &= \left\{\omega \in \mathbb{R} : \underline{\omega}_h \leq \omega, \underline{\omega}_h > \bar{\omega}_N\right\}. \end{aligned}$$

Definition 2.2: Given a transfer function matrix $R(s) \in \mathbf{G}$. $R(s)$ is said to be *interval frequency negative imaginary* in the frequency interval set Ω if

(1) $R(s)$ has no poles in $\Re[s] > 0$;
(2) $j[R(j\omega) - R^*(j\omega)] \geq 0$ for all $\omega \in \Omega$ where $j\omega$ is not a pole of $R(s)$;
(3) if $j\omega_0, \omega_0 \in \Omega$, is a pole of $R(s)$, then it is a simple pole and the corresponding residue matrix of $jR(s)$ is positive semidefinite Hermitian;
(4) if $R(s)$ has a pole at $s = 0$, then $\lim_{s\to 0} s^2R(s)$ is positive semidefinite Hermitian and $\lim_{s\to 0} s^kR(s) = 0$ for $k \geq 3$;
(5) $R(\infty) = R^T(\infty)$.

Remark 2.3: In Definition 2.2, if $\Omega = \Omega_L$, then the interval frequency negative imaginary transfer function $R(s)$ is said to be low frequency negative imaginary (LFNI), which could be considered as an extension of the definition of finite frequency negative imaginary transfer functions (Xiong et al., 2012a) by allowing poles at the origin. If $\Omega = \Omega_{M_l}$, then $R(s)$ is said to be middle frequency negative imaginary (MFNI). If $\Omega = \Omega_H$, $R(s)$ is said to be high frequency negative imaginary (HFNI).

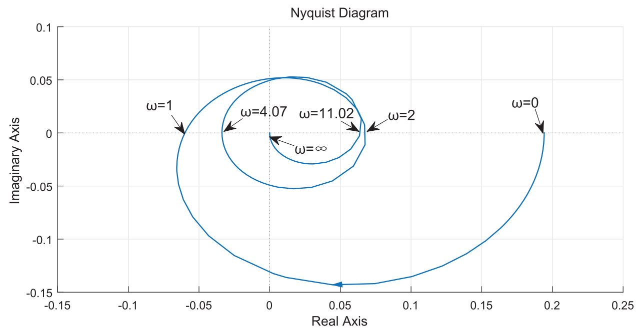

Figure 1. Positive-frequency Nyquist plot of the $G(s)$.

Example 2.4: Consider the stable transfer function

$$G(s) = \frac{s^4 - 2.6s^3 + 19s^2 - 14s + 23.3}{s^5 + 15s^4 + 85s^3 + 225s^2 + 274s + 120}.$$

By letting $\Im[G(j\omega)] = 0$, we found that $G(s)$ was IFNI over the frequency intervals $(0, 0.9983] \bigcup [2.0062, 4.0691] \bigcup [11.0191, \infty)$. The Nyquist plot for $G(s)$ is given in Figure 1.

Next, we introduce the definition of interval frequency positive real (IFPR) transfer functions. Let us define the frequency interval set

$$\bar{\Omega} \triangleq \bar{\Omega}_L \bigcup \left(\bigcup_{l=1}^{N} \bar{\Omega}_{M_l} \right) \bigcup \bar{\Omega}_H, \qquad (2)$$

where

$$\bar{\Omega}_L = \{\omega \in \mathbb{R} : |\omega| \le \bar{\omega}_0\},$$
$$\bar{\Omega}_{M_l} = \left\{\omega \in \mathbb{R} : \underline{\omega}_l \le |\omega| \le \bar{\omega}_l, \bar{\omega}_l > \underline{\omega}_l > \bar{\omega}_{l-1}\right\}, \quad l = 1, \ldots, N,$$
$$\bar{\Omega}_H = \left\{\omega \in \mathbb{R} : \underline{\omega}_h \le |\omega|, \underline{\omega}_h > \bar{\omega}_N\right\}.$$

The following facts deserve to be mentioned: $0 \notin \Omega$ while $0 \in \bar{\Omega}$, and hence $\bar{\Omega} = (-\Omega) \bigcup \{0\} \bigcup \Omega$.

Definition 2.5: Given a transfer function matrix $G(s) \in \mathbf{G}$. $G(s)$ is said to be *interval frequency positive real* if

(1) $G(s)$ has no poles in $\Re[s] > 0$;
(2) $G(j\omega) + G^*(j\omega) \ge 0$ for all $\omega \in \bar{\Omega}$ where $j\omega$ is not a pole of $G(s)$;
(3) if $j\omega, \omega \in \bar{\Omega} \bigcup \{0\}$, is pole of $G(s)$, then it is a simple pole and the corresponding residue matrix of $G(s)$ is positive semidefinite Hermitian.

Remark 2.6: In Definition 2.5, if $\bar{\Omega} = \bar{\Omega}_L$, then $G(s)$ is said to be low frequency positive real (LFPR), which is the same definition to that of finite frequency positive real (FFPR) transfer functions in Iwasaki, Hara, and Yamauchi (2003). If $\bar{\Omega} = \bar{\Omega}_{M_l}$, then $G(s)$ is said to be middle frequency positive real (MFPR). If $\bar{\Omega} = \bar{\Omega}_H$, $G(s)$ is said to be high frequency positive real (HFPR).

An useful lemma is as follows.

Lemma 2.7 (Xiong et al., 2010): *If* $A = A^* \ge 0$, *then* $\bar{A} = (\bar{A})^* \ge 0$.

Now, we are ready to establish the relationship between IFNI transfer functions and IFPR transfer functions. The proof follows the similar spirit as the proof for Lemma 2 in Xiong et al. (2012a).

Lemma 2.8: *Given a transfer function matrix* $R(s) \in \mathbf{G}$ *satisfying* $R(\infty) = R^T(\infty)$. *Then the following statements are equivalent:*

(1) *$R(s)$ is interval frequency negative imaginary.*
(2) *$\hat{R}(s) \triangleq R(s) - R(\infty)$ is interval frequency negative imaginary.*
(3) *$G(s) \triangleq s\hat{R}(s)$ is interval frequency positive real.*

Proof: ($1 \iff 2$) Let $\hat{R}(s) \triangleq R(s) - R(\infty)$. We prove that $R(s)$ is IFNI if and only if $\hat{R}(s)$ is IFNI according to Definition 2.2. Because $R(s)$ and $\hat{R}(s)$ have the same set of poles, one concludes that $R(s)$ has no poles in $\Re[s] > 0$ if and only if $\hat{R}(s)$ has no poles in $\Re[s] > 0$. When $j\omega$ is not a pole of $R(s)$, one has that $j[R(j\omega) - R^*(j\omega)] = j[\hat{R}(j\omega) - \hat{R}^*(j\omega)]$ for all $\omega \in \Omega$. If $j\omega_0, \omega_0 \in \Omega$, is a simple pole, then $\lim_{s\to j\omega_0}(s - j\omega_0)jR(s) = \lim_{s\to j\omega_0}(s - j\omega_0)j\hat{R}(s)$. Similarly, if $s = 0$ is a pole, then $\lim_{s\to 0} s^k R(s) = \lim_{s\to 0} s^k \hat{R}(s)$ for $k \ge 1$. Hence, $R(s)$ is IFNI if and only if $\hat{R}(s)$ is IFNI.

($2 \Rightarrow 3$) Note that $\hat{R}(s)$ and $G(s)$ have the same set of non-zero poles. Suppose $\hat{R}(s)$ is IFNI. Condition 1 in Definition 2.2 implies that Condition 1 in Definition 2.5 holds.

If $j\omega$, $\omega \in \Omega$, is not a pole of $\hat{R}(s)$, Condition 2 in Definition 2.2 implies that $G(j\omega) + G^*(j\omega) = j\omega[\hat{R}(j\omega) - \hat{R}^*(j\omega)] \ge 0$ for $\omega \in \Omega$. According to Lemma 2.7 and the continuity of $G(j\omega)$, we have $G(j\omega) + G^*(j\omega) \ge 0$ for all $\omega \in \bar{\Omega}$ such that $j\omega$ is not a pole of $G(s)$.

If $j\omega_0$, $\omega_0 \in \Omega$, is a pole of $\hat{R}(s)$, then $\pm j\omega_0$ are poles of both $\hat{R}(s)$ and $G(s)$. Note that $\hat{R}(s)$ can be factored as $(1/(s - j\omega_0)(s + j\omega_0))\hat{R}_1(s)$. According to Definition 2.2, we have that $\lim_{s\to j\omega_0}(s - j\omega_0)j\hat{R}(s) = (1/2\omega_0)\hat{R}_1(j\omega_0)$, which is positive semidefinite Hermitian; that is, $\hat{R}_1(j\omega_0) = \hat{R}_1^*(j\omega_0) \ge 0$. It follows from Lemma 2.7 that $\hat{R}_1(-j\omega_0) = \hat{R}_1^*(-j\omega_0) \ge 0$. The residue matrix of $G(s)$ at $j\omega_0$ is given by $\lim_{s\to j\omega_0}(s - j\omega_0)G(s) = \frac{1}{2}\hat{R}_1(j\omega_0)$, and hence is positive semidefinite Hermitian. Similarly, the residue matrix of $G(s)$ at $-j\omega_0$ is given by

$\frac{1}{2}\hat{R}_1(-j\omega_0)$, which is positive semidefinite Hermitian as well.

When $s=0$ is a pole of $\hat{R}(s)$, Condition 4 in Definition 2.2 implies that $s=0$ is either a simple or double pole. In the case of a simple pole, one has that $G(s)$ has no poles at the origin; furthermore, $G(0)+G^*(0)\geq 0$ holds due to the continuity of $G(j\omega)$. When zero is a double pole, one has that $G(s)$ has a simple zero pole. In this case, the residue matrix of $G(s)$ is given by $\lim_{s\to 0} sG(s)=\lim_{s\to 0} s^2\hat{R}(s)$, which is positive semidefinite Hermitian. Hence, according to Definition 2.5, $G(s)$ is IFPR.

($3\Rightarrow 2$) Suppose $G(s)$ is IFPR. Let $\hat{R}(s)=(1/s)G(s)$. Note that $\hat{R}(s)$ and $G(s)$ have the same non-zero pole set, which means that Condition 1 in Definition 2.2 holds. Because $j[\hat{R}(j\omega)-\hat{R}^*(j\omega)]=j[(1/j\omega)G(j\omega)+(1/j\omega)G^*(j\omega)]=(1/\omega)[G(j\omega)+G^*(j\omega)]\geq 0$ for $\omega\in\Omega$ where $j\omega$ is not a pole of $\hat{R}(s)$, Condition 2 in Definition 2.2 holds. If $j\omega_0$, $\omega_0\in\Omega$, is a pole of $\hat{R}(s)$, then it is a simple pole and $G(s)$ can be factored as $G(s)=(1/(s-j\omega_0)(s+j\omega_0))\hat{G}_1(s)$. The residue matrix of $G(s)$ at $s=j\omega_0$ can be calculated as $\lim_{s\to j\omega_0}(s-j\omega_0)(1/(s-j\omega_0)(s+j\omega_0))\hat{G}_1(s)=(1/j2\omega_0)\hat{G}_1(j\omega_0)\geq 0$. Then, the residue matrix of $j\hat{R}(s)$ at the same point is given by $\lim_{s\to j\omega_0}(s-j\omega_0)j(1/s)G(s)=(1/\omega_0)((1/j2\omega_0)\hat{G}_1(j\omega_0))\geq 0$. Condition 3 of Definition 2.2 holds. If $s=0$ is not a pole of $G(s)$, then $\hat{R}(s)$ has at most a simple pole at the origin. Hence $\lim_{s\to 0} s^k\hat{R}(s)=0$ for $k\geq 2$. If $s=0$ is a simple pole of $G(s)$, then it is a double pole of $\hat{R}(s)$ and $\lim_{s\to 0} s^2\hat{R}(s)=\lim_{s\to 0} sG(s)$, which is positive semidefinite Hermitian. This implies that Condition 4 in Definition 2.2 holds. According to Definition 2.2, $\hat{R}(s)$ is IFNI. ■

Let $\bar{\omega}_{cl}=(\underline{\omega}_l+\bar{\omega}_l)/2$ be the middle point of each frequency interval $\Omega_{M_l}/\bar{\Omega}_{M_l}$. We now define $\Phi=\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$, and Ψ_l, shown in Table 1, for each frequency interval. The matrices Φ and Ψ_l work together to characterise the frequency intervals in (1) and (2); the readers may refer to Iwasaki and Hara (2005) for more details.

The following lemma gives a version of IFPR lemma, which can be considered as an extension of Theorem 3 of Iwasaki et al. (2003).

Lemma 2.9: *Consider a transfer function matrix $G(s)\in$ **G** with state-space realisation (A,B,C,D). Suppose (A,B) is a controllable pair. Also, suppose $G(s)$ has no poles in the open right-half of the complex plane and the poles on the imaginary axis, if any, are simple. When A has eigenvalues $j\omega_i$ ($i\in\{1,\ldots,p\}$) such that $\omega_i\in\bar{\Omega}\bigcup\{0\}$, the residue matrix of $(sI-A)^{-1}$ at $s=j\omega_i$ is given by $\Phi_i\triangleq\lim_{s-j\omega_i}(s-j\omega_i)(sI-A)^{-1}$. Then the transfer function matrix $G(s)$ is IFPR if and only if*

(1) *there exist real symmetric matrices $P_0=P_0^T$, $Q_0=Q_0^T\geq 0$, $P_{N+1}=P_{N+1}^T$, $Q_{N+1}=Q_{N+1}^T\geq 0$ and Hermitian matrices $P_l=P_l^*$, $Q_l=Q_l^*\geq 0$, $l=1,\ldots,N$, such that*

$$\begin{bmatrix}A & B\\ I & 0\end{bmatrix}^T(\Phi\otimes P_l+\Psi_l\otimes Q_l)\begin{bmatrix}A & B\\ I & 0\end{bmatrix}\leq\begin{bmatrix}0 & C^T\\ C & D+D^T\end{bmatrix},\quad l=0,1,\ldots,N+1;\tag{3}$$

(2) *$C\Phi_iB=(C\Phi_iB)^*\geq 0$ for all $i\in\{1,\ldots,p\}$ if A has any eigenvalues on $j\bar{\Omega}\bigcup\{0\}$.*

Table 1. Ψ_l defined for each frequency interval.

	$\Omega_L/\bar{\Omega}_L$	$\Omega_{M_l}/\bar{\Omega}_{M_l}$	$\Omega_H/\bar{\Omega}_H$
l	0	$1,\ldots,N$	$N+1$
Ψ_l	$\begin{bmatrix}-1 & 0\\ 0 & \bar{\omega}_0^2\end{bmatrix}$	$\begin{bmatrix}-1 & j\bar{\omega}_{cl}\\ -j\bar{\omega}_{cl} & -\underline{\omega}_l\bar{\omega}_l\end{bmatrix}$	$\begin{bmatrix}1 & 0\\ 0 & -\underline{\omega}_h^2\end{bmatrix}$

Proof: Firstly, Condition 1 of Definition 2.5 is assumed to be true. Secondly, the poles on the imaginary axis are simple poles and the corresponding residue matrices are given by $C\Phi_iB$. Therefore, Condition 3 of Definition 2.5 is equivalent to Condition 2 of this lemma. Applying Theorem 4 in Iwasaki and Hara (2005) with $\Theta=-\left[\begin{smallmatrix}0 & C^T\\ C & D+D^T\end{smallmatrix}\right]$, one has that Condition 2 of Definition 2.5 holds if and only if there exist Hermitian matrices P_l and positive semidefinite Hermitian matrices Q_l such that (3) holds for $l=0,1,\ldots,N+1$. Finally, according to Lemma 2.7 and the fact that the matrices Ψ_0 and Ψ_{N+1} in (3) are real matrices, one has that the matrices P_0, Q_0, P_{N+1} and Q_{N+1} take real values. ■

Remark 2.10: A generalised KYP lemma was established in Iwasaki and Hara (2005), where the frequency parameter ω may take values on any line segment in the complex plane. However, the positive real property on any line segment of the imaginary axis has not been studied except the special case studied in Iwasaki et al. (2003).

3. Interval frequency negative imaginary lemma

In this section, an interval frequency negative imaginary lemma is first established based on Lemma 2.9. Then we study the relationship among different frequency negative imaginary lemmas and build up a connection with the existing negative imaginary lemma. Finally, a time-domain interpretation of IFNI property is presented in terms of the system input, output and state.

Theorem 3.1 (Interval Frequency Negative Imaginary Lemma): *Consider a transfer function matrix $R(s) \in \mathbf{G}$ with minimal state-pace realisation (A, B, C, D). Suppose that $R(s)$ has no poles in the open right-half of the complex plane and that the pure imaginary poles of $R(s)$, if any, are simple, and the zero pole, if any, are either a simple or double pole. When A has eigenvalues $j\omega_i$ $(i \in \{1, \ldots, p\})$ such that $\omega_i \in \Omega \bigcup \{0\}$, the residue matrix of $A(sI - A)^{-1}$ at $s = j\omega_i$ is given by $\Phi_i = \lim_{s\to j\omega_i}(s - j\omega_i)A(sI - A)^{-1}$. Then the following statements are equivalent:*

(1) *The transfer function matrix $R(s)$ is interval frequency negative imaginary.*
(2) *$D = D^T$, and the transfer function matrix $G(s)$ with state-space realisation (A, B, CA, CB) is interval frequency positive real.*
(3) *$D = D^T$, and $C\Phi_iB = (C\Phi_iB)^* \geq 0$ for all $i \in \{1, \ldots, p\}$ if $j\omega_i$ is an eigenvalue of A. Also, there exist real symmetric matrices $P_0 = P_0^T$, $Q_0 = Q_0^T \geq 0$, $P_{N+1} = P_{N+1}^T$, $Q_{N+1} = Q_{N+1}^T \geq 0$ and Hermitian matrices $P_l = P_l^*$, $Q_l = Q_l^* \geq 0$, $l = 1, \ldots, N$, such that*

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T (\Phi \otimes P_l + \Psi_l \otimes Q_l) \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} + \Theta \leq 0 \tag{4}$$

for all $l = 0, \ldots, N+1$, where $\Theta := -\left[\begin{smallmatrix} 0 & A^TC^T \\ CA & CB+B^TC^T \end{smallmatrix}\right]$.

Proof: $(1 \Leftrightarrow 2)$ Note that $G(s) = CA(sI - A)^{-1}B + CB = s[R(s) - R(\infty)]$. It follows from Lemma 2.8 that the two statements are equivalent.

$(2 \Leftrightarrow 3)$ The equivalence between Statement 2 and Statement 3 follows directly from Lemma 2.9. ∎

In terms of Theorem 3.1, we show that the LFNI lemma reduces to the generalised negative imaginary lemma (that is, Lemma 2 of Mabrok et al., 2015) when the upper bound of the low frequency interval approaches to infinity. The following results can be considered as a generalisation of Corollary 1 in Xiong et al. (2012a).

Corollary 3.2: *Suppose all the assumptions in Theorem 3.1 are satisfied. Consider the low frequency case $\Omega = \Omega_L$. When $\bar{\omega}_0 \to \infty$, the conditions in Theorem 3.1 are equivalent to the conditions in the generalised negative imaginary lemma in Mabrok et al. (2015).*

Proof: This result is readily established by applying Lemmas A.4 and A.9 in Appendix. ∎

Similarly for the HFNI case where $\Omega = \Omega_H$. When $\underline{\omega}_h \to 0$, the conditions in Theorem 3.1 reduce to that of the existing generalised negative imaginary lemma.

Corollary 3.3: *Suppose all the assumptions in Theorem 3.1 are satisfied, and consider the high frequency case $\Omega = \Omega_H$. If $\underline{\omega}_h \to 0$, then the conditions in Theorem 3.1 are equivalent to the conditions in the generalised negative imaginary lemma in Mabrok et al. (2015).*

Proof: This result can be obtained by applying Lemmas A.7 and A.9 in Appendix. ∎

Consider stable IFNI transfer function matrices. The following theorem provides a time-domain interpretation of the IFNI property with the system input, output and state.

Theorem 3.4: *Consider a stable transfer function matrix $R(s) \in \mathbf{G}$ with minimal state-space realisation (A, B, C, D) and $D = D^T$. Let the system input, output and state of $R(s)$ be denoted by $u(t)$, $y(t)$ and $x(t)$, respectively; suppose the system has zero initial conditions. Then, the following statements are equivalent:*

(1) *$R(s)$ is interval frequency negative imaginary in Ω.*
(2) *The inequality*

$$\int_0^\infty [\dot{y}(t) - D\dot{u}(t)]^T u(t)\,\mathrm{d}t \geq 0 \tag{5}$$

holds for all differentiable and square integrable inputs $u(t)$ such that

$$\int_0^\infty \begin{bmatrix} \dot{x}(t) & x(t) \end{bmatrix} \bar{\Psi}_l \begin{bmatrix} \dot{x}(t) & x(t) \end{bmatrix}^T \mathrm{d}t \geq 0,$$
$$l = 0, 1, \ldots, N+1, \tag{6}$$

where $\bar{\Psi}_l$ is the complex conjugate of Ψ_l.

Proof: Note that $R(s)$ is a stable transfer function. In view of Definition 2.2, $R(s)$ is IFNI if and only if $j[R(j\omega) - R^*(j\omega)] \geq 0$ holds for $\omega \in \Omega$. Let $G(s) = s[R(s) - R(\infty)] = CA(sI - A)^{-1}B + CB$. Then one has that $R(s)$ is IFNI if and only if

$$G(j\omega) + G(j\omega)^* = j\omega[R(j\omega) - R^*(j\omega)]$$
$$= -\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \Pi \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \geq 0 \tag{7}$$

holds for $\omega \in \Omega$, where

$$\Pi = -\begin{bmatrix} 0 & A^TC^T \\ CA & CB + B^TC^T \end{bmatrix}. \tag{8}$$

Next we will use Theorem 3 of Iwasaki et al. (2005) to complete the proof.

For $\omega \in \Omega_L$, it follows from Theorem 3 in Iwasaki et al. (2005) with the parameters $-\bar{\omega}_0$, $\bar{\omega}_0$, $\tau = 1$ and

Π in (8) that the inequality (7) holds, if and only if, the inequality (5) holds for all $u(t)$ such that

$$\int_0^{\infty} [-\bar{\omega}_0 x(t) + j\dot{x}(t)][\bar{\omega}_0 x(t) + j\dot{x}(t)]^* \, \mathrm{d}t \le 0. \quad (9)$$

Note that the left side of the above inequality is a matrix. Because the system state $x(t) \equiv 0$, $t \le 0$, it follows from Lemma A.10 that

$$\begin{aligned}\int_0^{\infty} &[\dot{x}(t)x^T(t) + x(t)\dot{x}^T(t)] \, \mathrm{d}t \\ &= \int_{-\infty}^{\infty} [\dot{x}(t)x^T(t) + x(t)\dot{x}^T(t)] \, \mathrm{d}t = 0. \quad (10)\end{aligned}$$

Hence, the inequality (9) becomes the inequality (6) with $l = 0$.

For $\omega \in \Omega_{M_l}$, $l = 1, \ldots, N$, it follows from Theorem 3 in Iwasaki et al. (2005) with the parameters $\underline{\omega}_l$, $\bar{\omega}_l$, $\tau = 1$ and Π in (8) that the inequality (7) holds, if and only if, the inequality (5) holds for all $u(t)$ such that

$$S_l = \int_0^{\infty} [\underline{\omega}_l x(t) + j\dot{x}(t)][\bar{\omega}_l x(t) + j\dot{x}(t)]^* \, \mathrm{d}t \le 0. \quad (11)$$

Due to the fact that $x(t) \equiv 0$, $t \le 0$, one has that

$$\begin{aligned}S_l &= \int_{-\infty}^{\infty} [\underline{\omega}_l x(t) + j\dot{x}(t)][\bar{\omega}_l x(t) + j\dot{x}(t)]^* \, \mathrm{d}t \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\underline{\omega}_l - \omega)(\bar{\omega}_l - \omega) X(j\omega) X^*(j\omega) \, \mathrm{d}\omega.\end{aligned}$$

That is, one has that $S_l = S_l^*$. Therefore, the inequality (11) can be written as

$$\begin{aligned}S_l &= \frac{1}{2}(S_l + S_l^*) \\ &= \int_0^{\infty} \frac{1}{2} \big\{ [\underline{\omega}_l x(t) + j\dot{x}(t)][\bar{\omega}_l x(t) + j\dot{x}(t)]^* \\ &\quad + [\bar{\omega}_l x(t) + j\dot{x}(t)][\underline{\omega}_l x(t) + j\dot{x}(t)]^* \big\} \, \mathrm{d}t \\ &= \int_0^{\infty} (\underline{\omega}_l \bar{\omega}_l x(t) x^T(t) - j\bar{\omega}_{cl} x(t) \dot{x}^T(t) \\ &\quad + j\bar{\omega}_{cl} \dot{x}(t) x^T(t) + \dot{x}(t)\dot{x}^T(t)) \, \mathrm{d}t \\ &= -\int_0^{\infty} \begin{bmatrix} \dot{x}(t) & x(t) \end{bmatrix} \bar{\Psi}_l \begin{bmatrix} \dot{x}(t) & x(t) \end{bmatrix}^T \, \mathrm{d}t \le 0,\end{aligned}$$

which is the inequality in (6) for $l = 1, \ldots, N$.

For $\omega \in \Omega_H$, it follows from Theorem 3 in Iwasaki et al. (2005) with the parameters $-\underline{\omega}_h$, $\underline{\omega}_h$, $\tau = -1$ and Π in (8) that the inequality (7) holds, if and only if, the inequality (5) holds for all $u(t)$ such that

$$\int_0^{\infty} \{-[-\underline{\omega}_h x(t) + j\dot{x}(t)][\underline{\omega}_h x(t) + j\dot{x}(t)]^*\} \, \mathrm{d}t \le 0. \quad (12)$$

In view of the equality (10), one has that the inequality (12) becomes the inequality

$$\int_0^{\infty} [\underline{\omega}_h^2 x(t) x^T(t) - \dot{x}(t)\dot{x}^T(t)] \, \mathrm{d}t \le 0,$$

which is the same inequality in (6) with $l = N+1$. This completes the proof. ■

Remark 3.5: Based on Theorem 3.1, the second statement in Theorem 3.4 can be directly derived from the first statement. The idea is the same as that in Remark 5 of Xiong et al. (2012a).

4. Illustrative examples

In this section, we consider two transfer functions from the piezoelectric tube scanner model in Das et al. (2014) and Sallen-key low pass filter in Pactitis (2007), respectively.

Example 4.1: In Das et al. (2014), the transfer function of the piezoelectric tube scanner (Das et al., 2014) from the voltage input V_{x+} to the displacement output d_x is given by

$$G(s) = \frac{-186.6s^2 + 1.348 \times 10^6 s - 2.412 \times 10^{10}}{s^3 + 1755s^2 + 3.452 \times 10^7 s + 4.459 \times 10^{10}},$$

which is shown to be MFNI based on Theorem 3.1. Note that the transfer function $G(s)$ has no poles on imaginary axis. By letting the imaginary part of $G(j\omega)$ be zero, we can find that $\Im[G(j\omega)] \le 0$ for $\omega \in (5784, 11958)$. The Nyquist plot of $G(s)$ is shown in Figure 2. Next, we use the developed result in Theorem 3.1 to verify this fact.

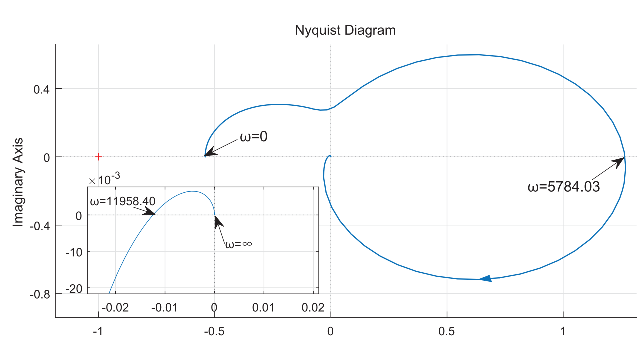

Figure 2. Nyquist plot of the transfer function ($\omega \ge 0$).

A minimal state-space realisation of $G(s)$ returned by MATLAB is given by

$$A=\begin{bmatrix}-1755.0 & -4213.9 & -2657.8\\ 8192.0 & 0 & 0\\ 0 & 2048.0 & 0\end{bmatrix},\quad B=\begin{bmatrix}32\\0\\0\end{bmatrix},$$
$$C=\begin{bmatrix}-5.8312 & 5.1422 & -44.9270\end{bmatrix},\quad D=0.$$

Let $\underline{\omega}_1=5784$ and $\bar{\omega}_1=11958$. To apply Theorem 3.1, we only need to check whether there exist Hermitian matrices P_1 and $Q_1\geq 0$ satisfying (4). Using the YALMIP toolbox, a solution to (4) can be found and given by

$$P=\begin{bmatrix}-1660 & -7898+6187i & 11381+9583i\\ -7898-6187i & -14187 & -26061+27622i\\ 11381-9583i & -26061-27622i & -56942\end{bmatrix},$$

$$Q=\begin{bmatrix}12.93 & 0.21-21.67i & -33.25-0.68i\\ 0.21+21.67i & 45.09 & 0.61-85.85i\\ -33.25+0.68i & 0.61+85.85i & 192.19\end{bmatrix}\geq 0.$$

If we set $\underline{\omega}_1$ to be a slightly smaller number or $\bar{\omega}_1$ to be a larger number, the LMI in (4) has no feasible solution. Hence, we can conclude that $G(s)$ is MFNI in the frequency set $(\underline{\omega}_1,\bar{\omega}_1)$.

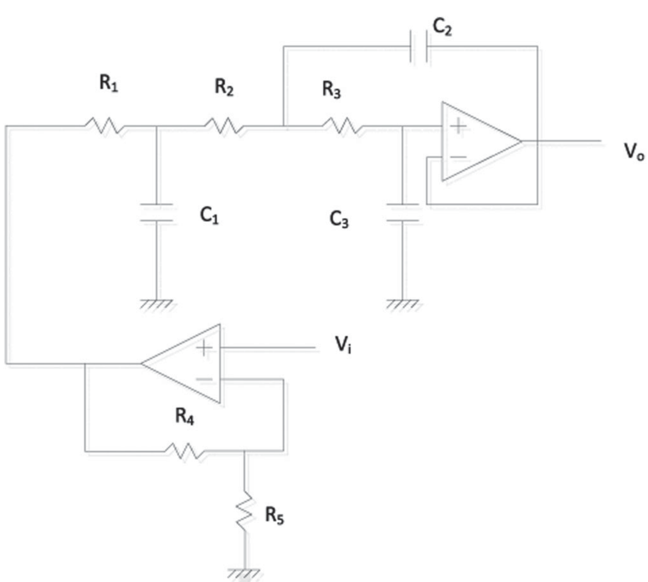

Figure 3. Sallen-key low-pass filter cascaded with a multiplier circuit.

Example 4.2: In Figure 3, a Sallen-key low pass filter (Pactitis, 2007) is cascaded with a gain multiplier circuit and the transfer function from $V_i(s)$ to $V_o(s)$ is given by

$$G(s)=\frac{(1+\frac{R_4}{R_5})\frac{1}{R_1R_2R_3C_1C_2C_3}}{s^3+(\frac{1}{R_1C_1}+\frac{1}{R_2C_1}+\frac{1}{R_2C_2}+\frac{1}{R_3C_2})s^2+(\frac{C_3R_3+R_1C_3+C_1R_1+C_3R_2}{R_1R_2R_3C_1C_2C_3})s+\frac{1}{R_1R_2R_3C_1C_2C_3}}$$

which is shown to be LFNI based on Theorem 3.1. Considering $R_1=11\text{K}\Omega$, $R_2=110\text{K}\Omega$, $R_3=33\text{K}\Omega$, $R_4=1\text{K}\Omega$, $R_5=1\text{K}\Omega$, $C_1=15\mu\text{F}$, $C_2=6.8\mu\text{F}$ and $C_3=1\mu\text{F}$, and the Nyquist plot of the above transfer function is shown in Figure 4. Setting the imaginary part of $G(j\omega)$ into zero, we can directly compute $\bar{\omega}_0=8.85$. One has that $\Im[G(j\omega)]\leq 0$ for $\omega\in(0,\bar{\omega}_0)$.

To verify the result in Theorem 3.1, we first found a minimal state realisation of $G(s)$ with

$$A=\begin{bmatrix}-12.4599 & -9.7904 & -7.6727\\ 8 & 0 & 0\\ 0 & 4 & 0\end{bmatrix},\quad B=\begin{bmatrix}4\\0\\0\end{bmatrix},$$

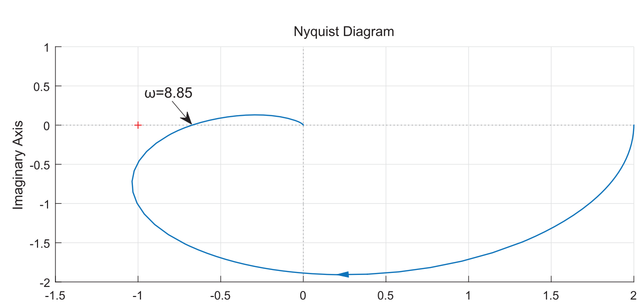

Figure 4. Nyquist plot of the transfer function $V_o(s)/V_i(s)$.

$$C = \begin{bmatrix} 0 & 0 & 3.8364 \end{bmatrix}, \quad D = 0.$$

Now solving the inequality (4), we have a feasible solution given by

$$P_0 = \begin{bmatrix} 0.0546 & 2.9333 & 0 \\ 2.9333 & 5.9083 & 0 \\ 0 & 0 & 7.3589 \end{bmatrix},$$

$$Q_0 = \begin{bmatrix} 0.0922 & 0.0068 & 0 \\ 0.0068 & 0.8462 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Also, we have Q_0. If we set the frequency $\bar{\omega}_0$ to be a slightly larger number, the LMI in (4) have no feasible solution. Hence, we can conclude that $G(s)$ is LFNI in the frequency set$(0, \bar{\omega}_0)$.

5. Conclusions

The paper studied the interval frequency negative imaginary property of dynamical linear systems. Firstly, we proposed the concept of interval frequency negative imaginary transfer functions and established a connection between interval frequency negative imaginary transfer functions and interval frequency positive real transfer functions. Base on this connection, a necessary and sufficient condition was proposed to determine the interval frequency negative imaginary property under the minimal state-space realisation assumption. Also, a time-domain interpretation of IFNI property was presented. When the interval frequency set becomes the whole positive frequency set, the developed IFNI lemma reduces to the normal negative imaginary lemma. Finally, two practical examples were provided to illustrate the interval frequency negative imaginary lemma. Future work might focus on the following two research problems. The first one is to generalise the interval frequency negative imaginary theory to discrete-time systems, the second is to develop a stability result for interconnected interval frequency negative imaginary systems.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

This work was supported by National Natural Science Foundation of China [grant number 61374026].

Notes on contributors

Junlin Xiong received his BEng. and MSci. degrees from Northeastern University, China, and his PhD degree from the University of Hong Kong, Hong Kong, in 2000, 2003 and 2007, respectively. From 2007 to 2010, he was a research associate at the University of New South Wales at the Australian Defence Force Academy, Australia. In March 2010, he joined the University of Science and Technology of China where he is currently a professor in the Department of Automation. Currently, he is an Associate Editor for the IET Control Theory and Application. His current research interests are in the fields of negative imaginary systems, large-scale systems and networked control systems.

Shengyu Zhai received his BEng. degree from HeFei University of Technology in 2015 and MEng. degree from University of Science and Technology of China in 2018, respectively. His main research interests are in negative imaginary theory.

ORCID

Junlin Xiong http://orcid.org/0000-0002-0128-4960

References

Bhikkaji, B., Ratnam, M., Fleming, A. J., & Moheimani, S. R. (2007). High-performance control of piezoelectric tube scanners. *IEEE Transactions on Control Systems Technology*, *15*(5), 853–866.

Brogliato, B., Lozano, R., Maschke, B., & Egeland, O. (2006). *Dissipative systems analysis and control: Theory and applications* (2nd ed.). New York: Springer-Verlag.

Cai, C., & Hagen, G. (2010). Stability analysis for a string of coupled stable subsystems with negative imaginary frequency response. *IEEE Transactions on Automatic Control*, *55*(8), 1958–1963.

Das, S. K., Pota, H. R., & Petersen, I. R. (2014). Resonant controller design for a piezoelectric tube scanner: A mixed negative-imaginary and small-gain approach. *IEEE Transactions on Control Systems Technology*, *22*(5), 1899–1906.

Dey, A., Patra, S., & Sen, S. (2016). Absolute stability analysis for negative-imaginary systems. *Automatica*, *67*, 107–113.

Ferrante, A., Lanzon, A., & Ntogramatzidis, L. (2017). Discrete-time negative imaginary systems. *Automatica*, *79*, 1–10.

Ferrante, A., & Ntogramatzidis, L. (2013). Some new results in the theory of negative imaginary systems with symmetric transfer matrix function. *Automatica*, *49*(7), 2138–2144.

Iwasaki, T., & Hara, S. (2005). Generalized KYP lemma: Unified frequency domain inequalities with design applications. *IEEE Transactions on Automatic Control*, *50*(1), 41–59.

Iwasaki, T., Hara, S., & Fradkov, A. L. (2005). Time domain interpretations of frequency domain inequalities on (semi) finite ranges. *Systems & control letters*, *54*(7), 681–691.

Iwasaki, T., Hara, S., & Yamauchi, H. (2003). Dynamical system design from a control perspective: Finite frequency positive-realness approach. *IEEE Transactions on Automatic Control*, *48*(8), 1337–1354.

Lanzon, A., & Petersen, I. R. (2008). Stability robustness of a feedback interconnection of systems with negative imaginary frequency response. *IEEE Transactions on Automatic Control*, *53*(4), 1042–1046.

Liu, M., & Xiong, J. (2016). On non-proper negative imaginary systems. *Systems & Control Letters*, *88*, 47–53.

Liu, M., & Xiong, J. (2017). Properties and stability analysis of discrete-time negative imaginary systems. *Automatica*, *83*, 58–64.

Mabrok, M., Kallapur, A. G., Petersen, I. R., & Lanzon, A. (2015). A generalized negative imaginary lemma and Riccati-based static state-feedback negative imaginary synthesis. *Systems & Control Letters*, *77*, 63–68.

Mabrok, M. A., Kallapur, A. G., Petersen, I. R., & Lanzon, A. (2014a). Generalizing negative imaginary systems theory to include free body dynamics: Control of highly resonant structures with free body motion. *IEEE Transactions on Automatic Control*, *59*(10), 2692–2707.

Mabrok, M. A., Kallapur, A. G., Petersen, I. R., & Lanzon, A. (2014b). Spectral conditions for negative imaginary systems with applications to nanopositioning. *IEEE/ASME Transactions on Mechatronics*, *19*(3), 895–903.

Pactitis, S. A. (2007). *Active filters: Theory and design*. Orlando, FL: CRC Press.

Patra, S., & Lanzon, A. (2011). Stability analysis of interconnected systems with 'mixed' negative-imaginary and small-gain properties. *IEEE Transactions on Automatic Control*, *56*(6), 1395–1400.

Rahman, M. A., Al Mamun, A., Yao, K., & Das, S. K. (2015). Design and implementation of feedback resonance compensator in hard disk drive servo system: A mixed passivity, negative-imaginary and small-gain approach in discrete time. *Journal of Control, Automation and Electrical Systems*, *26*(4), 390–402.

Song, Z., Lanzon, A., Patra, S., & Petersen, I. R. (2012). A negative-imaginary lemma without minimality assumptions and robust state-feedback synthesis for uncertain negative-imaginary systems. *Systems & Control Letters*, *61*(12), 1269–1276.

Xiong, J., Lam, J., & Petersen, I. R. (2016). Output feedback negative imaginary synthesis under structural constraints. *Automatica*, *71*, 222–228.

Xiong, J., Petersen, I. R., & Lanzon, A. (2010). A negative imaginary lemma and the stability of interconnections of linear negative imaginary systems. *IEEE Transactions on Automatic Control*, *55*(10), 2342–2347.

Xiong, J., Petersen, I. R., & Lanzon, A. (2012a). Finite frequency negative imaginary systems. *IEEE Transactions on Automatic Control*, *57*(11), 2917–2922.

Xiong, J., Petersen, I. R., & Lanzon, A. (2012b). On lossless negative imaginary systems. *Automatica*, *48*(6), 1213–1217.

Yu, L., & Xiong, J. (2017). H_∞ model reduction for negative imaginary systems. *International Journal of Systems Science*, *48*(7), 1515–1521.

Yuan, Y., Wang, Y., & Guo, L. (2018). Force reflecting control for bilateral teleoperation system under time-varying delays. *IEEE Transactions on Industrial Informatics*.

Yuan, Y., Wang, Z., & Guo, L. (2017). Event-triggered strategy design for discrete-time nonlinear quadratic games with disturbance compensations: The noncooperative case. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*.

Appendix

The following lemma extends the NI lemma in Xiong et al. (2010) to the case where the transfer functions allow poles at the origin.

Lemma A.1 (Lemma 2 of Mabrok et al., 2015): *Consider a transfer function matrix $R(s) \in \mathbf{G}$ with minimal state-space realisation (A, B, C, D). Then, $R(s)$ is negative imaginary if and only if $D = D^T$ and there exist matrices $P = P^T \geq 0$, L and W such that the following linear matrix inequality is satisfied:*

$$\begin{bmatrix} PA + A^TP & PB - A^TC^T \\ B^TP - CA & -CB - B^TC^T \end{bmatrix} = -\begin{bmatrix} L^TL & L^TW \\ W^TL & W^TW \end{bmatrix} \leq 0. \tag{A1}$$

Remark A.2: In Lemma A.1, if $s = 0$ is not a pole of $R(s)$, then $P = P^T \geq 0$ can be replaced by $P = P^T > 0$ (e.g. see.Lemma 7 in Xiong et al., 2010).

Given a square real-rational proper transfer function matrix $R(s)$ with at most double pole at the origin, one of its minimal state-space realisations can be of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{A2}$$

where

$$A = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}. \tag{A3}$$

Here $A_1 \in \mathbb{R}^{n_1\times n_1}$ is nonsingular, $A_2 = 0 \in \mathbb{R}^{n_2\times n_2}$, $A_3 = \left[\begin{smallmatrix} 0 & I_{k\times k} \\ 0_{k\times k} & 0 \end{smallmatrix}\right] \in \mathbb{R}^{2k\times 2k}$, $B_1 \in \mathbb{R}^{n_1\times m}$, $B_2 \in \mathbb{R}^{n_2\times m}$, $B_3 = \left[\begin{smallmatrix} B_{3a} \\ B_{3b} \end{smallmatrix}\right]$, $B_{3a} \in \mathbb{R}^{k\times m}$, $B_{3b} \in \mathbb{R}^{k\times m}$, $C_1 \in \mathbb{R}^{m\times n_1}$, $C_2 \in \mathbb{R}^{m\times n_2}$, $C_3 = [\, C_{3a} \; C_{3b} \,]$, $C_{3a} \in \mathbb{R}^{m\times k}$, $C_{3b} \in \mathbb{R}^{m\times k}$. Then $n_1 + n_2 + 2k = n$. The particular values of n_1, n_2 and k depend on the transfer function $R(s)$. Therefore, the transfer function matrix $R(s)$ can be expressed as

$$\begin{aligned} R(s) &= C_1(sI - A_1)^{-1}B_1 + C_2(sI - A_2)^{-1}B_2 \\ &\quad + C_3(sI - A_3)^{-1}B_3 + D \\ &= C_1(sI - A_1)^{-1}B_1 + \frac{C_2B_2 + C_3B_3}{s} + \frac{C_{3a}B_{3b}}{s^2} + D. \end{aligned}$$

Remark A.3: It follows from Lemma 2 of Mabrok et al. (2014a) that the matrix $[C_2 \;\; C_{3a}]$ is of full column rank, and the matrix $\left[\begin{smallmatrix} B_2 \\ B_{3b} \end{smallmatrix}\right]$ is of full row rank; also $m \geq k + n_2$ and (A_1, B_1, C_1) is a minimal state-space realisation.

Lemma A.4: *Suppose all the assumptions in Theorem 3.1 are satisfied, where a minimal state-pace realisation of $R(s)$ is given by (A2) and (A3). Consider the low frequency case $\Omega = \Omega_L$. If $\bar{\omega}_0 \to \infty$, then the equivalent conditions in Theorem 3.1 reduce to the conditions that $D = D^T$ and there exist symmetric matrices*

$P_1 \in \mathbb{R}^{n_1 \times n_1}$, $P_{3b} \in \mathbb{R}^{k \times k}$, $P_1 > 0$, $P_{3b} > 0$, *and matrices* $L_1 \in \mathbb{R}^{(n_1+k)\times n_1}$, $W \in \mathbb{R}^{(n_1+k)\times m}$ *such that*

$$P_1 A_1 + A_1^T P_1 = -L_1^T L_1, \tag{A4}$$

$$P_1 B_1 - A_1^T C_1^T = -L_1^T W, \tag{A5}$$

$$P_{3b} B_{3b} = C_{3a}^T, \tag{A6}$$

$$C_1 B_1 + B_1^T C_1^T + C_2 B_2 + B_2^T C_2^T + C_3 B_3 + B_3^T C_3^T = W^T W. \tag{A7}$$

Proof: Consider the low frequency case $\Omega = \Omega_L$ where the transfer function matrix $R(s)$ is LFNI. Condition 2 of Theorem 3.1 implies that $D = D^T$. In terms of Lemma 2.8, the transfer function matrix $G_r(s) = s[R(s) - D]$ is LFPR. A minimal state-space realisation of $G_r(s)$ is given by (see the proof to Lemma 3 of Mabrok et al., 2014a)

$$A_r = \begin{bmatrix} A_1 & 0 \\ 0 & 0_{k\times k} \end{bmatrix}, \quad B_r = \begin{bmatrix} B_1 \\ B_{3b} \end{bmatrix}, \quad C_r = \begin{bmatrix} C_1 A_1 & C_{3a} \end{bmatrix},$$

$$D_r = C_1 B_1 + C_2 B_2 + C_3 B_3.$$

It follows from Lemma 2.9 that there there exist symmetric matrix $P_r = P_r^T$ and $Q_r \geq 0$ such that

$$\begin{bmatrix} P_r A_r + A_r^T P_r - A_r^T Q_r A_r + \bar{\omega}_0^2 Q_r & P_r B_r - C_r^T - A_r^T Q_r B_r \\ B_r^T P_r - C_r - B_r^T Q_r A_r & -D_r - D_r^T - B_r^T Q_r B_r \end{bmatrix} \leq 0. \tag{A8}$$

and that $C_r \Phi_{ri} B_r = (C_r \Phi_{ri} B_r)^* \geq 0$ for all $i \in \{1, \ldots, p\}$ if A_r has eigenvalues on $j\bar{\Omega}_L$, where $\Phi_{ri} = \lim_{s\to j\omega_i}(s - j\omega_i)(sI - A_r)^{-1}$.

If $\bar{\omega}_0 \to \infty$, the $(1,1)$ block of the LMI (A8) implies that $Q_r \to 0$. Letting $Q_r = 0$ in (A8) results in

$$\begin{bmatrix} P_r A_r + A_r^T P_r & P_r B_r - C_r^T \\ B_r^T P_r - C_r & -D_r - D_r^T \end{bmatrix} = -\begin{bmatrix} L^T L & L^T W \\ W^T L & W^T W \end{bmatrix}. \tag{A9}$$

The $(2,2)$ block in (A9) is the equality in (A7). By considering the partitioned forms in the realisation (A_r, B_r, C_r, D_r), the matrices P_r and L are partitioned accordingly as

$$P_r = \begin{bmatrix} P_1 & P_{12} \\ P_{12}^T & P_{3b} \end{bmatrix}, \quad L = \begin{bmatrix} L_1 & L_2 \end{bmatrix}.$$

Then, the $(1,1)$ block of (A9) can be written as

$$\begin{bmatrix} P_1 A_1 + A_1^T P_1 & A_1^T P_{12} \\ P_{12}^T A_1 & 0 \end{bmatrix} = -\begin{bmatrix} L_1^T L_1 & L_1^T L_2 \\ L_2^T L_1 & L_2^T L_2 \end{bmatrix}. \tag{A10}$$

The $(1,1)$ block in (A10) is the equality in (A4). Because the $(2,2)$ block of the matrix on the left side of (A10) is zero, one has that $L_2 = 0$. Furthermore, $A_1^T P_{12} = 0$ holds. Because A_1 is nonsingular, one has that $P_{12} = 0$.

The $(1,2)$ block in (A9) can be written as

$$\begin{bmatrix} P_1 B_1 - A_1^T C_1^T \\ P_{3b} B_{3b} - C_{3a}^T \end{bmatrix} = -\begin{bmatrix} L_1^T W \\ 0 \end{bmatrix}.$$

Therefore, the equalities in (A5) and (A6) hold. To complete the proof, we only need to prove that $P_1 > 0$ and$P_{3b} > 0$.

It follows from Remark A.3 that $\text{rank}(C_{3a}) = k = \text{rank}(B_{3b})$. Hence, it follows from (A6) that P_{3b} is nonsingular. By directly computing the residue matrix of $G_r(s)$ at the origin, one has that

$$\lim_{s\to 0} sG_r(s) = \lim_{s\to 0} sC_r(sI - A_r)^{-1} B_r = C_{3a} B_{3b} = B_{3b}^T C_{3a}^T \geq 0.$$

Again, it follows from (A6) that $B_{3b}^T P_{3b} B_{3b} = B_{3b}^T C_{3a}^T \geq 0$. Because B_{3b} is of full row rank, one has that $P_{3b} \geq 0$. Hence, the matrix P_{3b} is positive definite.

Similarly, we will prove $P_1 > 0$ by showing both P_1 being nonsingular and $P_1 \geq 0$. Suppose P_1 is singular. Without loss of generality, we can assume that the matrices P_1, A_1, B_1, C_1, L_1 in (A4) and (A5) are of the following forms (see the proof to Corollary 1 of Xiong et al., 2012a)

$$P_1 = \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix},$$

$$C_1 = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}, \quad L_1 = \begin{bmatrix} L_{11} & L_{12} \end{bmatrix},$$

where $P_{11} = P_{11}^T$ is nonsingular. Hence, (A4) can be rewritten as

$$\begin{bmatrix} P_{11} A_{11} + A_{11}^T P_{11} & P_{11} A_{12} \\ A_{12}^T P_{11} & 0 \end{bmatrix} = -\begin{bmatrix} L_{11}^T L_{11} & L_{11}^T L_{12} \\ L_{12}^T L_{11} & L_{12}^T L_{12} \end{bmatrix},$$

which implies that $L_{12} = 0$ and $P_{11} A_{12} = 0$. Because P_{11} is nonsingular, we have $A_{12} = 0$. Hence, $A_1 = \begin{bmatrix} A_{11} & 0 \\ A_{13} & A_{14} \end{bmatrix}$ and $L_1 = [L_{11}\ 0]$. Also, the equality (A5) can be rewritten as

$$\begin{bmatrix} P_{11} B_{11} - A_{11}^T C_{11}^T - A_{13}^T C_{12}^T \\ -A_{14}^T C_{12}^T \end{bmatrix} = -\begin{bmatrix} L_{11}^T W \\ 0 \end{bmatrix},$$

which implies that $A_{14}^T C_{12}^T = 0$. Hence, $C_1 A_1 = [C_{11} A_{11} + C_{12} A_{13}\ 0]$. Because A_1 is nonsingular, one has that

$$\text{rank}\left(\begin{bmatrix} C_1 \\ C_1 A_1 \\ \vdots \\ C_1 A_1^{n_1-1} \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} C_1 A_1 \\ C_1 A_1^2 \\ \vdots \\ C_1 A_1^{n_1} \end{bmatrix}\right) < n_1,$$

which means that the pair (A_1, C_1) is not observable. This contradicts the observability of (A_1, C_1). Hence P_1 is nonsingular.

Next, we prove that P_1 is positive definite. The equalities in (A4) and (A5) can be rewritten as

$$A_1 Y_1 + Y_1 A_1^T = -\hat{L}_1^T \hat{L}_1, \tag{A11}$$

$$B_1 - Y_1 A_1^T C_1^T = -\hat{L}_1^T W, \tag{A12}$$

where $Y_1 = P_1^{-1}$ is nonsingular and $\hat{L}_1 = L_1 P_1^{-1}$. Note that $P_1 > 0$ if and only if $Y_1 > 0$.

Note that the purely imaginary poles of $G(s)$ are simple poles. The real Jordan canonical form of A_1 is of the form

$$A_1 \sim \begin{bmatrix} A_{11} & 0 \\ 0 & A_{12} \end{bmatrix},$$

where $A_{11} \in \mathbb{R}^{n_{11}\times n_{11}}$ has eigenvalues in $\Re[s] < 0$, $A_{12} = \text{diag}\{A_{121}, \ldots, A_{12i}, \ldots, A_{12g}\} \in \mathbb{R}^{n_{12}\times n_{12}}$, $A_{12i} = \begin{bmatrix} 0 & -\omega_i I_{q_i\times q_i} \\ \omega_i I_{q_i\times q_i} & 0 \end{bmatrix} \in \mathbb{R}^{2q_i\times 2q_i}$, $\omega_i > 0, i \in \{1, \ldots, g\}$ and $g \leq p$; also $n_{12} = 2q_1 + \cdots + 2q_i + \cdots + 2q_g$, and $n_{11} + n_{12} = n_1$. Let us assume that matrices Y_1, A_1, B_1, C_1 and $\hat{L}_1$ in (A11) and (A12) are of the following forms without loss of generality:

$$Y_1 = \begin{bmatrix} Y_{11} & Y_{13} \\ Y_{13}^T & Y_{12} \end{bmatrix}, \quad A_1 = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{12} \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix},$$

$$C_1 = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}, \quad \hat{L}_1 = \begin{bmatrix} \hat{L}_{11} & \hat{L}_{12} \end{bmatrix}.$$

Then, (A11) can be rewritten as

$$\begin{bmatrix} A_{11}Y_{11}+Y_{11}A_{11}^T & A_{11}Y_{13}+Y_{13}A_{12}^T \\ A_{12}Y_{13}^T+Y_{13}^TA_{11}^T & A_{12}Y_{12}+Y_{12}A_{12}^T \end{bmatrix} \le 0. \qquad \text{(A13)}$$

Because all eigenvalues of A_{11} are in the open left-half plane, it follows from the $(1,1)$ block of (A13) that $Y_{11} \ge 0$. In the following, we will prove that $Y_{13} = 0$ and $Y_{12} \ge 0$.

Let us partition the matrices Y_{12} and A_{12} as follows:

$$Y_{12} = \begin{bmatrix} \tilde{Y}_{121} & Y_{123} \\ Y_{123}^T & Y_{12g} \end{bmatrix}, \quad A_{12} = \begin{bmatrix} \tilde{A}_{121} & 0 \\ 0 & A_{12g} \end{bmatrix},$$

$\tilde{Y}_{121} \in \mathbb{R}^{(n_{12}-2q_g)\times(n_{12}-2q_g)}$, $Y_{12g} \in \mathbb{R}^{2q_g\times 2q_g}$, $\tilde{A}_{121} = \operatorname{diag}\{A_{121},\dots,A_{12i},\dots,A_{12(g-1)}\}$. The $(2,2)$ block of (A13) can be written as

$$\begin{aligned} &A_{12}Y_{12}+Y_{12}A_{12}^T \\ &= \begin{bmatrix} \tilde{A}_{121}\tilde{Y}_{121}+\tilde{Y}_{121}\tilde{A}_{121}^T & \tilde{A}_{121}Y_{123}+Y_{123}A_{12g}^T \\ A_{12g}Y_{123}^T+Y_{123}^T\tilde{A}_{121}^T & A_{12g}Y_{12g}+Y_{12g}A_{12g}^T \end{bmatrix} \le 0. \end{aligned} \qquad \text{(A14)}$$

Let $Y_{12g} = \begin{bmatrix} Y_{12g1} & Y_{12g3} \\ Y_{12g3}^T & Y_{12g2} \end{bmatrix}$, $Y_{12g1} \in \mathbb{R}^{q_g\times q_g}$, $Y_{12g2} \in \mathbb{R}^{q_g\times q_g}$. The $(2,2)$ block of the above inequality can be rewritten as

$$\begin{aligned} &A_{12g}Y_{12g}+Y_{12g}A_{12g}^T \\ &= \begin{bmatrix} -\omega_g(Y_{12g3}+Y_{12g3}^T) & \omega_g(Y_{12g1}-Y_{12g2}) \\ \omega_g(Y_{12g1}-Y_{12g2}) & \omega_g(Y_{12g3}+Y_{12g3}^T) \end{bmatrix} \le 0. \end{aligned}$$

By noting that the $(1,1)$ block and the $(2,2)$ block of the above inequality have opposite signs, we conclude that $Y_{12g3}+Y_{12g3}^T = 0$. Hence, the $(1,2)$ block must satisfy $Y_{12g1}-Y_{12g2} = 0$. In the other words, we have that $Y_{12g} = \begin{bmatrix} Y_{12g2} & Y_{12g3} \\ -Y_{12g3} & Y_{12g2} \end{bmatrix}$ and $A_{12g}Y_{12g}+Y_{12g}A_{12g}^T = 0$. Hence, the $(1,2)$ block of (A14) must satisfy

$$\tilde{A}_{121}Y_{123}+Y_{123}A_{12g}^T = 0. \qquad \text{(A15)}$$

Because the matrices $\tilde{A}_{121}$ and $-A_{12g}^T$ have no common eigenvalues, the Sylvester (A15) has a unique solution $Y_{123} = 0$. Hence, the matrix Y_{12} is of the form $Y_{12} = \begin{bmatrix} \tilde{Y}_{121} & 0 \\ 0 & Y_{12g} \end{bmatrix}$.

After applying the same techniques as above to $i = g-1,\dots,1$, we derive that the matrix Y_{12} is of the following form

$$Y_{12} = \operatorname{diag}(Y_{121}, Y_{122},\dots,Y_{12g}), \quad Y_{12i} = \begin{bmatrix} Y_{12i2} & Y_{12i3} \\ -Y_{12i3} & Y_{12i2} \end{bmatrix}, \qquad \text{(A16)}$$

and that

$$A_{12}Y_{12}^T+Y_{12}A_{12}^T = 0. \qquad \text{(A17)}$$

Hence, the $(1,2)$ block of (A13) satisfies

$$A_{11}Y_{13}+Y_{13}A_{12}^T = 0. \qquad \text{(A18)}$$

Again, because the matrices A_{11} and $-A_{12}^T$ have no common eigenvalues, the Sylvester (A18) has a unique solution $Y_{13} = 0$. Hence, the matrix Y_1 is of the form $Y_1 = \begin{bmatrix} Y_{11} & 0 \\ 0 & Y_{12} \end{bmatrix}$.

In the following, we will prove that $Y_{12} \ge 0$ by using the positive semidefinite property of the residue matrices; that is, $C_r\Phi_{ri}B_r = (C_r\Phi_{ri}B_r)^* \ge 0, i \in \{1,\dots,p\}$. Let us first consider the residue matrix at $s = j\omega_g$; that is, $C_r\Phi_{rg}B_r = (C_r\Phi_{rg}B_r)^* \ge 0$.

It follows from (A17) that the $(2,2)$ block of (A13) is zero. Hence the matrix $\hat{L}_1$ in (A11) is of the form

$$\hat{L}_1 = \begin{bmatrix} \hat{L}_{11} & 0 \end{bmatrix}.$$

The equality (A12) can be rewritten as

$$\begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} - \begin{bmatrix} Y_{11}A_{11}^TC_{11}^T \\ Y_{12}A_{12}^TC_{12}^T \end{bmatrix} = -\begin{bmatrix} \hat{L}_{11}^TW \\ 0 \end{bmatrix},$$

which implies that

$$B_{12} = Y_{12}A_{12}^TC_{12}^T. \qquad \text{(A19)}$$

On the other hand, it follows from direct calculations that

$$\begin{aligned} \Phi_{12g} &= \lim_{s\to j\omega_g}(s-j\omega_g)(sI-A_{12g})^{-1} = \frac{1}{2}\begin{bmatrix} I & jI \\ -jI & I \end{bmatrix}, \\ \Phi_{1g} &= \lim_{s\to j\omega_g}(s-j\omega_g)(sI-A_{12})^{-1} \\ &= \operatorname{diag}\left(0_{2q_1\times 2q_1},\dots,0_{2q_{g-1}\times 2q_{g-1}},\Phi_{12g}\right), \\ \Phi_g &= \lim_{s\to j\omega_g}(s-j\omega_g)(sI-A_1)^{-1} = \begin{bmatrix} 0_{n_{11}\times n_{11}} & 0 \\ 0 & \Phi_{1g} \end{bmatrix}, \\ \Phi_{rg} &= \lim_{s\to j\omega_g}(s-j\omega_g)(sI-A_r)^{-1} = \begin{bmatrix} \Phi_g & 0 \\ 0 & 0_{k\times k} \end{bmatrix}. \end{aligned} \qquad \text{(A20)}$$

Let $C_{12} = [\tilde{C}_{121}\ C_{12g}]$ and $C_{12g} = [C_{12g1}C_{12g2}]$. Now, we are ready to compute the residue matrix at $s = j\omega_g$; that is,

$$\begin{aligned} &\lim_{s\to j\omega_g}(s-j\omega_g)G(s) \\ &= C_r\Phi_{rg}B_r = \begin{bmatrix} C_1A_1 & C_{3a} \end{bmatrix}\begin{bmatrix} \Phi_g & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B_1 \\ B_{3b} \end{bmatrix} \\ &= C_1A_1\Phi_gB_1 = \begin{bmatrix} C_{11} & C_{12} \end{bmatrix}\begin{bmatrix} A_{11} & 0 \\ 0 & A_{12} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \Phi_{1g} \end{bmatrix}\begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} \\ &= C_{12}A_{12}\Phi_{1g}B_{12} = C_{12}A_{12}\Phi_{1g}Y_{12}A_{12}^TC_{12}^T \\ &= \begin{bmatrix} \tilde{C}_{121} & C_{12g} \end{bmatrix}\begin{bmatrix} \tilde{A}_{121} & 0 \\ 0 & A_{12g} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \Phi_{12g} \end{bmatrix}\begin{bmatrix} \tilde{Y}_{121} & 0 \\ 0 & Y_{12g} \end{bmatrix} \\ &\times \begin{bmatrix} \tilde{A}_{121}^T & 0 \\ 0 & A_{12g}^T \end{bmatrix}\begin{bmatrix} \tilde{C}_{121}^T \\ C_{12g}^T \end{bmatrix} \\ &= C_{12g}A_{12g}\Phi_{12g}Y_{12g}A_{12g}^TC_{12g}^T \\ &= \frac{1}{2}\begin{bmatrix} C_{12g1} & C_{12g2} \end{bmatrix}\begin{bmatrix} 0 & -\omega_gI \\ \omega_gI & 0 \end{bmatrix}\begin{bmatrix} I & jI \\ -jI & I \end{bmatrix} \\ &\times \begin{bmatrix} Y_{12g2} & Y_{12g3} \\ -Y_{12g3} & Y_{12g2} \end{bmatrix}\begin{bmatrix} 0 & \omega_gI \\ -\omega_gI & 0 \end{bmatrix}\begin{bmatrix} C_{12g1}^T \\ C_{12g2}^T \end{bmatrix} \\ &= \frac{\omega_g^2}{2}\begin{bmatrix} C_{12g1} & C_{12g2} \end{bmatrix} \\ &\times \begin{bmatrix} Y_{12g2}-jY_{12g3} & Y_{12g3}+jY_{12g2} \\ -Y_{12g3}-jY_{12g2} & Y_{12g2}-jY_{12g3} \end{bmatrix}\begin{bmatrix} C_{12g1}^T \\ C_{12g2}^T \end{bmatrix} \\ &= \frac{\omega_g^2}{2}(C_{12g1}-jC_{12g2})(Y_{12g2}-jY_{12g3})(C_{12g1}-jC_{12g2})^* \ge 0. \end{aligned} \qquad \text{(A21)}$$

It follows from Remark A.3 that the pair (A_1, C_1) is observable. Hence, the observability matrix

$$\begin{bmatrix} C_1 \\ C_1A_1 \\ \vdots \\ C_1A_1^{n_1-1} \end{bmatrix} = \begin{bmatrix} C_{11} & \tilde{C}_{121} & C_{12g} \\ C_{11}A_{11} & \tilde{C}_{121}\tilde{A}_{121} & C_{12g}A_{12g} \\ \vdots & \vdots & \vdots \\ C_{11}A_{11}^{n_1-1} & \tilde{C}_{121}\tilde{A}_{121}^{n_1-1} & C_{12g}A_{12g}^{n_1-1} \end{bmatrix}$$

is of full column rank, which means that

$$\begin{bmatrix} C_{12g} \\ C_{12g}A_{12g} \\ C_{12g}A_{12g}^2 \\ C_{12g}A_{12g}^3 \\ C_{12g}A_{12g}^4 \\ \vdots \end{bmatrix} = \begin{bmatrix} C_{12g1} & C_{12g2} \\ \omega_g C_{12g2} & -\omega_g C_{12g1} \\ -\omega_g^2 C_{12g1} & -\omega_g^2 C_{12g2} \\ -\omega_g^3 C_{12g2} & \omega_g^3 C_{12g1} \\ \omega_g^4 C_{12g1} & \omega_g^4 C_{12g2} \\ \vdots & \vdots \end{bmatrix}$$

is of full column rank. Right-multiplying the above matrix by $\left[\begin{smallmatrix} I & 0 \\ -jI & I \end{smallmatrix}\right]$, one has that the matrix

$$\begin{bmatrix} C_{12g1} - jC_{12g2} & C_{12g2} \\ \omega_g C_{12g2} + j\omega_g C_{12g1} & -\omega_g C_{12g1} \\ -\omega_g^2 C_{12g1} + j\omega_g^2 C_{12g2} & -\omega_g^2 C_{12g2} \\ -\omega_g^3 C_{12g2} - j\omega_g^3 C_{12g1} & \omega_g^3 C_{12g1} \\ \omega_g^4 C_{12g1} - j\omega_g^4 C_{12g2} & \omega_g^4 C_{12g2} \\ \vdots & \vdots \end{bmatrix}$$

is of full column rank. Therefore, the matrix

$$\begin{bmatrix} C_{12g1} - jC_{12g2} \\ \omega_g C_{12g2} + j\omega_g C_{12g1} \\ -\omega_g^2 C_{12g1} + j\omega_g^2 C_{12g2} \\ -\omega_g^3 C_{12g2} - j\omega_g^3 C_{12g1} \\ \omega_g^4 C_{12g1} - j\omega_g^4 C_{12g2} \\ \vdots \end{bmatrix}$$

is of full column rank. Left-multiplying the above matrix by

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 & \cdots \\ -j\omega_g I & I & 0 & 0 & 0 & \cdots \\ 0 & -j\omega_g I & I & 0 & 0 & \cdots \\ 0 & 0 & -j\omega_g I & I & 0 & \cdots \\ 0 & 0 & 0 & -j\omega_g I & I & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

one has that the matrix

$$\begin{bmatrix} C_{12g1} - jC_{12g2} \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

is of full column rank. Hence, the matrix $C_{12g1} - jC_{12g2}$ is of column full rank. It follows from (A21) that $Y_{12g2} - jY_{12g3} \geq 0$, which is equivalent to $Y_{12g} = \left[\begin{smallmatrix} Y_{12g2} & Y_{12g3} \\ -Y_{12g3} & Y_{12g2} \end{smallmatrix}\right] \geq 0$.

Similarly, the residue matrix $C_r\Phi_{ri}B_r = (C_r\Phi_{ri}B_r)^* \geq 0$ implies that $Y_{12i} \geq 0$ for $i = g-1, \ldots, 1$. Therefore, $Y_{12} = \mathrm{diag}(Y_{121}, Y_{122}, \ldots, Y_{12g}) \geq 0$. Noting that we have proved that Y_1 is nonsingular, we conclude that $Y_1 > 0$, which implies that $P_1 > 0$. This completed the proof. ■

Remark A.5: In the proof to Lemma A.4, we have shown that the positive semidefiniteness of the residue matrices guarantees the positive semidefiniteness of the matrices P_1 and P_{3b}. The inverse is true as well. That is, the positive semidefiniteness of the matricex P_1 and P_{3b} also implies the positive semidefiniteness of the residue matrices.

Remark A.6: When $s = 0$ is not a double pole of $R(s)$, a minimal state-space realisation of $G_r(s)$ is given by (A_1, B_1, C_1A_1, D_r). The conditions in Lemma A.4 reduce to that $D = D^T$ and there exist a symmetric matrix $P_1 > 0$ and matrices L_1, W satisfying (A4), (A5) and (A7).

Lemma A.7: *Suppose all the assumptions in Theorem* 3.1 *are satisfied, where a minimal state-pace realisation of $R(s)$ is given in* (A2) *and* (A3). *Consider the high frequency case $\Omega = \Omega_H$. If $\underline{\omega}_h \to 0$, then the conditions in Theorem* 3.1 *reduce to the conditions in Lemma* A.4.

Proof: The proof is similar to that of Lemma A.4. Let us define the same transfer function $G_r(s) = s[R(s) - D] = C_r(sI - A_r)^{-1}B_r + D_r$. It follows from Lemma 2.9 that there there exist symmetric matrices $P_r = P_r^T$ and $Q_r \geq 0$ such that

$$\begin{bmatrix} P_rA_r + A_r^TP_r + A_r^TQ_rA_r - \underline{\omega}_h^2 Q_r & P_rB_r - C_r^T + A_r^TQ_rB_r \\ B_r^TP_r - C_r + B_r^TQ_rA_r & -D_r - D_r^T + B_r^TQ_rB_r \end{bmatrix} \leq 0, \quad \text{(A22)}$$

and that $C_r\Phi_{ri}B_r = (C_r\Phi_{ri}B_r)^* \geq 0$ for all $i \in \{1, \ldots, p\}$ if A_r has eigenvalues on $j\bar{\Omega}_L$, where $\Phi_{ri} = \lim_{s \to j\omega_i}(s - j\omega_i)(sI - A_r)^{-1}$.

When $\underline{\omega}_h \to 0$, the inequality (A22) becomes

$$\begin{bmatrix} P_rA_r + A_r^TP_r & P_rB_r - C_r^T \\ B_r^TP_r - C_r & -D_r - D_r^T \end{bmatrix} + \begin{bmatrix} A_r^T \\ B_r^T \end{bmatrix} Q_r \begin{bmatrix} A_r & B_r \end{bmatrix} \leq 0. \quad \text{(A23)}$$

Therefore, the condition that there exist symmetric matrices P_r and $Q_r \geq 0$ satisfying (A23) is equivalent to the condition that there exists a symmetric matrix P_r satisfying the equality (A9). Then, the rest of proof follows the same lines as in that of Lemma A.4. ■

Remark A.8: Note that the derived conditions (A4)–(A7) in Lemma A.4 are the same as the conditions proposed in Lemma 3 of Mabrok et al. (2014a). However, Lemma 3 of Mabrok et al. (2014a) only states that these conditions are a necessary condition for the transfer function $R(s)$ in (A2) and (A3) being negative imaginary. In the following, we will further show that these conditions are further equivalent to the conditions given in Lemma A.1 (that is, Lemma 2 of Mabrok et al., 2015). Hence, these conditions are also a sufficient condition.

Lemma A.9: *Given a transfer function $R(s)$ with the minimal state-space realisation* (A2) *and* (A3) *and $D = D^T$. Then the conditions in Lemma* A.4 *are equivalent to the conditions in Lemma* A.1.

Proof: Note that any negative imaginary transfer function matrix $R(s)$ has at most a double pole at the origin. Hence one can assume that $R(s)$ has a minimal state-space realisation being the form in (A2) and (A3) without introducing any conservatism.

($\Rightarrow$) One can directly verify that the conditions in Lemma A.1 hold by letting

$$P = \begin{bmatrix} P_1 & 0 & 0 \\ 0 & 0_{n_2\times n_2} & 0 \\ 0 & 0 & P_3 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0_{k\times k} & 0 \\ 0 & P_{3b} \end{bmatrix},$$

$$L = \begin{bmatrix} L_1 & 0_{n\times n_2} & 0_{n\times 2k} \end{bmatrix}. \tag{A24}$$

Suppose that there exit matrices $P \geq 0$, L and W satisfying

$$\begin{bmatrix} PA + A^TP & PB - A^TC^T \\ B^TP - CA & -CB - B^TC^T \end{bmatrix} = -\begin{bmatrix} L^TL & L^TW \\ W^TL & W^TW \end{bmatrix} \leq 0. \tag{A25}$$

The $(2, 2)$ block of (A25) is actually (A7). By considering the partitioned form in (A3), the matrices P and L are partitioned accordingly as

$$P = \begin{bmatrix} P_1 & P_{12} & P_{13} \\ P_{12}^T & P_2 & P_{23} \\ P_{13}^T & P_{23}^T & P_3 \end{bmatrix}, \quad P_3 = \begin{bmatrix} P_{3a} & P_{3c} \\ P_{3c}^T & P_{3b} \end{bmatrix},$$

$$L = \begin{bmatrix} L_1 & L_2 & L_3 \end{bmatrix}.$$

To complete the proof, we firstly prove that the equalities in (A4)–(A6) hold, and that the matrices P and L are of the form given in (A24); then, we show that $P_1 > 0$ and $P_{3b} > 0$.

By noting the forms in (A3), the $(1, 1)$ block of (A25) can be written as

$$\begin{bmatrix} P_1A_1 + A_1^TP_1 & A_1^TP_{12} & A_1^TP_{13} + P_{13}A_3 \\ P_{12}^TA_1 & 0 & P_{23}A_3 \\ P_{13}^TA_1 + A_3^TP_{13}^T & A_3^TP_{23}^T & P_3A_3 + A_3^TP_3 \end{bmatrix}$$

$$= -\begin{bmatrix} L_1^TL_1 & L_1^TL_2 & L_1^TL_3 \\ L_2^TL_1 & L_2^TL_2 & L_2^TL_3 \\ L_3^TL_1 & L_3^TL_2 & L_3^TL_3 \end{bmatrix}. \tag{A26}$$

The $(1, 1)$ block of the above equality is (A4). Because the $(2, 2)$ block of the matrix on the left side of (A26) is zero, one has that $L_2 = 0$. Furthermore,

$$A_1^TP_{12} = 0, \quad P_{23}A_3 = 0.$$

Note that A_1 is nonsingular. Hence, the equality $A_1^TP_{12} = 0$ implies that $P_{12} = 0$. Let $P_{23} = [P_{23a}\ P_{23b}]$. Then

$$P_{23}A_3 = \begin{bmatrix} P_{23a} & P_{23b} \end{bmatrix}\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & P_{23a} \end{bmatrix} = 0,$$

which implies that $P_{23a} = 0$.

On the other hand, the $(3, 3)$ block of (A26) implies that

$$P_3A_3 + A_3^TP_3 = \begin{bmatrix} P_{3a} & P_{3c} \\ P_{3c}^T & P_{3b} \end{bmatrix}\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}\begin{bmatrix} P_{3a} & P_{3c} \\ P_{3c}^T & P_{3b} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & P_{3a} \\ P_{3a} & P_{3c}^T + P_{3c} \end{bmatrix} \leq 0,$$

which implies that $P_{3a} = 0$. Because $P_3 = \begin{bmatrix} 0 & P_{3c} \\ P_{3c}^T & P_{3b} \end{bmatrix} \geq 0$, one has that $P_{3c} = 0$, which implies that $P_3A_3 + A_3^TP_3 = 0$ and $L_3 = 0$. It follows from (A26) that

$$P_{13}A_3 + A_1^TP_{13} = 0.$$

Because the matrices A_3 and $-A_1^T$ has no common eigenvalues, the above Sylvester equation has a unique solution $P_{13} = 0$. In a summary, we have that

$$P = \begin{bmatrix} P_1 & 0 & 0 \\ 0 & P_2 & P_{23} \\ 0 & P_{23}^T & P_3 \end{bmatrix}, \quad P_{23} = \begin{bmatrix} 0 & P_{23b} \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0 & 0 \\ 0 & P_{3b} \end{bmatrix}, \quad L = \begin{bmatrix} L_1 & 0 & 0 \end{bmatrix}.$$

It follows from the $(1, 2)$ block of (A25) that

$$\begin{bmatrix} P_1B_1 - A_1^TC_1^T \\ P_2B_2 + P_{23}B_3 \\ P_{23}^TB_2 + P_3B_3 - A_3^TC_3^T \end{bmatrix} = -\begin{bmatrix} L_1^TW \\ 0 \\ 0 \end{bmatrix}. \tag{A27}$$

The $(1, 1)$ block in the above equality implies that (A5). The $(2, 1)$ block in (A27) implies that

$$P_2B_2 + P_{23}B_3 = P_2B_2 + \begin{bmatrix} 0 & P_{23b} \end{bmatrix}\begin{bmatrix} B_{3a} \\ B_{3b} \end{bmatrix}$$

$$= \begin{bmatrix} P_2 & P_{23b} \end{bmatrix}\begin{bmatrix} B_2 \\ B_{3b} \end{bmatrix} = 0. \tag{A28}$$

It follows from Remark A.3 that the matrix $\begin{bmatrix} B_2 \\ B_{3b} \end{bmatrix}$ is of full row rank. Therefore, it follows from (A28) that $[P_2\ P_{23b}] = 0$, which further implies that $P_{23} = 0$. Therefore, the matrix P has the form in (A24). Also, the $(3, 1)$ block in (A27) becomes

$$P_3B_3 - A_3^TC_3^T = \begin{bmatrix} 0 & 0 \\ 0 & P_{3b} \end{bmatrix}\begin{bmatrix} B_{3a} \\ B_{3b} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}\begin{bmatrix} C_{3a}^T \\ C_{3b}^T \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ P_{3b}B_{3b} - C_{3a}^T \end{bmatrix} = 0,$$

which implies that (A6) holds.

We already know that $P_1 \geq 0$ and $P_{3b} \geq 0$. By following the similar lines as in the proof to Lemma A.4, one has that the matricesP_1 and P_{3b} are nonsingular. Therefore, $P_1 > 0$ and $P_{3b} > 0$ hold. This completes the proof. ■

Lemma A.10: *Provided that the integrals exist, the following relation holds:*

$$\int_{-\infty}^{\infty} [\dot{x}(t)x(t)^T + x(t)\dot{x}(t)^T]\,\mathrm{d}t = 0. \tag{A29}$$

Proof: By the Parseval's theorem (Brogliato, Lozano, Maschke, & Egeland, 2006), we have

$$\int_{-\infty}^{\infty} [\dot{x}(t)x(t)^T + x(t)\dot{x}(t)^T]\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} [j\omega X(j\omega)X^*(j\omega) + X(j\omega)(j\omega X(j\omega))^*]\,\mathrm{d}w = 0.$$

The last equality holds because the integrand is always zero. ■