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Moment matching model reduction for negative imaginary systems

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Abstract

In this paper, moment matching model reduction problem for negative imaginary systems is considered. For a given negative imaginary system with poles at the origin, our goal is to find a reduced-order negative imaginary system such that a prescribed number of the moments and the poles at the origin are preserved. Firstly, the original negative imaginary system is split into an asymptotically stable subsystem, a lossless negative imaginary subsystem and an average subsystem. Then, moment matching model reduction is implemented on the asymptotically stable subsystem and the lossless negative imaginary subsystem. The resulting reduced-order system preserves the negative imaginary structure and the poles at the origin. Also, the proposed model reduction method is extended to the positive real systems. Numerical examples demonstrate the effectiveness of the proposed model reduction method.

1. Introduction

Negative imaginary (NI) systems theory has attracted much attention during the last ten years ([1–5] and references therein). It has been widely applied in different areas of control systems engineering, such as the feedback controller design for a DC machine [6], the robust cooperative control of multi-agent networked systems [7], the decentralized integral control of lightly damped flexible structures with collocated position sensors and force actuators [8]. An overview of NI systems theory and applications is referred to the survey paper [9]. Flexible

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structures with free body motion often lead to high order NI systems including poles at the origin, which arise in areas such as rotating flexible spacecraft [10], flexible link manipulators [11]. This pose serious difficulties in system analysis and synthesis. Therefore, an important issue concerns model reduction for high order NI systems to simplify further analysis and synthesis.

Recently, structure preserving model reduction problems for NI systems has drawn profound interest [12–15]. For the stable NI systems with minimal state-space realization, the use of balanced truncation method has been employed in [12]. H_∞ model reduction problem for stable NI system has been investigated in [13]. More complicated situations to preserve NI structure of the reduced-order system with a prescribed H_2 performance and the mixed H_2/H_∞ performance have been handled in [14,15]. However, no results on model reduction for NI systems with poles at the origin have been studied in the literature. When approximating a high order NI system, it is desired that the NI structure is preserved for reduced-order systems. Unfortunately, these existing model reduction methods are not applicable to NI systems with poles at the origin. Therefore, it is necessary to develop new approaches to solve the model reduction problem for NI systems with poles at the origin preserved. This motivates the research of this paper.

In the field of model reduction, moment matching model reduction method has been widely used to solve the structure preserving model reduction problems ([16–21] and references therein). For instance, structure preserving model reduction for second-order time-delay systems has been presented in [16,17], moment matching model reduction for port-Hamiltonian systems has been addressed in [18,19], bilinear structure preserving model reduction problem has been studied in [20,21]. For an overview of moment matching model reduction method, we refer to literature [22]. The key idea of the moment matching model reduction is to equalize the leading coefficients of the Laurent series expansion of the transfer functions of the original system and the reduced-order system at the selected points. The Partial realization problem is solved when the expansion is considered around infinity [23]. The Padé approximation is a problem of the moment matching at zero [24]. In the general case, the moment matching model reduction problem is known as the rational interpolation, which is of the interest of this paper.

For a given NI system with poles at the origin, we aim to find a reduced-order NI system that achieves moment matching. Firstly, the original system is split into an asymptotically stable subsystem, a lossless NI subsystem and an average subsystem. Then, sufficient conditions for the construction of the reduced-order asymptotically stable subsystem and lossless NI subsystem are derived. It shows that a reduced-order NI system can be obtained by direct computation of the projection matrix. It is a big advantage for high order systems because only matrix-vector multiplications are required. Moreover, the proposed model reduction method is also extended to the positive real systems with blocking zero at zero frequency. Finally, the effectiveness of the proposed moment matching model reduction method is illustrated by several numerical examples. The advantage of the proposed model reduction method is that the NI structure as well as the poles at the origin for the reduced-order systems can be guaranteed.

Notation: All the matrices are assumed to be compatible dimensions. $\mathbb{R}^{m\times n}$ and $\mathcal{R}^{m\times n}$ denote all the $m\times n$ real matrices and real rational proper transfer function matrices, respectively. $0_{m\times n}$ denotes an $m\times n$ zero matrix, I_n represents identity matrix of order n and e_i represents a column vector with the i-th entry equals to 1, other entries equal to 0. For a matrix A, A^{-1} and $A^\top$ stand for the inverse and the transpose, $\sigma(A)$ denotes the set of eigenvalues of A, respectively. ∅ denotes the empty set. $\Re[\cdot]$ denotes the real part of a complex number. The notation $P>0(\geq 0)$ means that matrix P is positive definite (semi-definite).

2. Problem statement

Consider an NI system $G(s)$ with the minimal state-space realization

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^n$, $C \in \mathbb{R}^{1\times n}$. The transfer function of system (1) is $G(s) = C(sI - A)^{-1}B$. The moment of system (1) is defined as follows.

Definition 1 [25]**.** The 0-moment of system (1) at $s_* \in \mathbb{C}$ is the complex number $\eta_0(s_*) = C(s_*I - A)^{-1}B$. The m-moment of system (1) at s_* is the complex number

$$\eta_m(s_*) = \frac{(-1)^m}{m!}\left[\frac{d^m}{ds^m}(C(sI-A)^{-1}B)\right]_{s=s_*}.$$

It should be noted that the moments determine the coefficients of the Laurent series expansion of the transfer function in the neighborhood of $s_* \in \mathbb{C}$, that is

$$\begin{aligned} G(s) &= G(s_*) + G^{(1)}(s_*)\frac{s-s_*}{1!} + \cdots + G^{(m)}(s_*)\frac{(s-s_*)^m}{m!} \\ &= \eta_0(s_*) + \eta_1(s_*)\frac{s-s_*}{1!} + \cdots + \eta_m(s_*)\frac{(s-s_*)^m}{m!}. \end{aligned}$$

In this paper, we only consider the case with $m = 0$.

A reduced-order system for system (1) is given by

$$\begin{aligned} \dot{x}_r(t) &= A_r x_r(t) + B_r u(t), \\ y_r(t) &= C_r x_r(t), \end{aligned} \tag{2}$$

where $x_r(t) \in \mathbb{R}^r$, $y_r(t) \in \mathbb{R}$, $A_r \in \mathbb{R}^{r\times r}$, $B_r \in \mathbb{R}^r$, $C_r \in \mathbb{R}^{1\times r}$, with $1 \leq r < n$. The transfer function of system (2) is $G_r(s) = C_r(sI - A_r)^{-1}B_r$.

Now, we formally state the moment matching model reduction problem for NI systems as follows.

Problem 1. Given an NI system (1) and $1 \leq r < n$. Let $\mathcal{S} = \{s_i | s_i \in \mathbb{C}, i = 1, \ldots, r\}$ be a set of interpolation points such that $\mathcal{S} \cap \{\sigma(A)\} = \emptyset$. The moment matching model reduction problem for NI system (1) is to find the reduced-order system (2) such that

(1) system (2) preserves the NI structure;
(2) $G_r(s_i) = G(s_i), \ s_i \in \mathcal{S}$.

Remark 1. Define the corresponding tangent directions as $\mathcal{Q} = \{b_i | b_i \in \mathbb{C}^q, i = 1, \ldots, r\}$. For the multiple-input multiple-output (MIMO) NI system, the goal of Problem 1 is to find the reduced-order NI system such that $G_r(s_i)b_i = G(s_i)b_i, \ s_i \in \mathcal{S}, \ b_i \in \mathcal{Q}$. Moment matching model reduction problems for single-input single-output (SISO) systems have been studied in numerous works, such as [18,25,26]. These results are easily extended to the case of MIMO systems [27,28]. Hence, we study and present results for the SISO case without loss of generality. Moreover, some necessary remarks about the extension to MIMO case are proposed in the paper, see Remarks 5 and 9.

Some preliminaries about NI systems are presented.

Definition 2 [29]**.** A square real-rational proper transfer function matrix $G(s) \in \mathcal{R}^{q\times q}$ is NI if

(1) $G(s)$ has no poles in $\Re[s] > 0$;
(2) $j[G(j\omega) - G^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$ except values of ω where $j\omega$ is a pole of $G(s)$;
(3) if $j\omega_0$, $\omega_0 > 0$ is a pole of $G(s)$, it is at most a simple pole, and the residue matrix $K_0 \triangleq \lim_{s\to j\omega_0}(s - j\omega_0)jG(s)$ is positive semi-definite Hermitian;
(4) if $s = 0$ is a pole of $G(s)$, then $\lim_{s\to 0} s^k G(s) = 0$ for all $k \geq 3$ and $\lim_{s\to 0} s^2 G(s)$ is Hermitian and positive definite.

Remark 2. Note that when the system $G(s)$ is asymptotically stable, Definition 2 coincides with the definition of stable NI systems in [1].

Definition 3 [30]**.** A square real-rational proper transfer function matrix $G(s) \in \mathcal{R}^{q\times q}$ is lossless NI if

(1) $G(s)$ is NI;
(2) $j[G(j\omega) - G^*(j\omega)] = 0$ for all $\omega \in (0, \infty)$ except values of ω where $j\omega$ is a pole of $G(s)$.

The following lemma provides a necessary and sufficient condition to test the NI structure.

Lemma 1 [31]**.** *Let (A, B, C, D) be a minimal state-space realization of $G(s) \in \mathcal{R}^{q\times q}$, where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times q}, C \in \mathbb{R}^{q\times n}, D \in \mathbb{R}^{q\times q}$, $q \leq n$. Then $G(s)$ is NI if and only if $D = D^\top$ and there exist matrices $X \in \mathbb{R}^{n\times n}$, $X = X^\top \geq 0$, L, W such that*

$$\begin{bmatrix} XA + A^\top X & XB - A^\top C^\top \\ B^\top X - CA & -(CB + B^\top C^\top) \end{bmatrix} = -\begin{bmatrix} L^\top L & L^\top W \\ W^\top L & W^\top W \end{bmatrix} \leq 0. \tag{3}$$

The following result gives a characterization for the NI structure of a system with no poles at the origin.

Lemma 2 [1]**.** *Let (A, B, C, D) be a minimal state-space realization of $G(s) \in \mathcal{R}^{q\times q}$, where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times q}, C \in \mathbb{R}^{q\times n}, D \in \mathbb{R}^{q\times q}$, $q \leq n$. Then $G(s)$ is NI if and only if* $\det(A) \neq 0$, *$D = D^\top$ and there exists a matrix $Y \in \mathbb{R}^{n\times n}$, $Y = Y^\top > 0$, such that*

$$AY + YA^\top \leq 0 \quad \text{and} \quad B + AYC^\top = 0. \tag{4}$$

Remark 3. By replacing the "$\leq$" sign with the "$=$" sign in the inequality of (4), we obtain the lossless NI Lemma, see Theorem 1 in [30]. Also, the strongly strict NI Lemma obtained by replacing the "$\leq$" sign with the "$<$" sign (Theorem 3.3 in [32]). Note that Eq. (3) is equivalent to Eq. (4) when $G(s)$ is asymptotically stable.

3. Moment matching with preservation of the NI structure

In this section, we first split the NI system into an asymptotically stable subsystem, a lossless NI subsystem and an average subsystem. Sufficient conditions are derived for the construction of the reduced-order asymptotically stable subsystem and the reduced-order lossless NI subsystem that matches prescribed moments of the original system.

Given an NI system (1), the minimal state-space realization can be transformed into the following block diagonal form [29]

$$\begin{aligned} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} &= \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} C_1 & C_2 & C_3 & C_4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}, \end{aligned} \tag{5}$$

where $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $A_2 \in \mathbb{R}^{2n_2 \times 2n_2}$, $B_1 \in \mathbb{R}^{n_1}$, $B_2 \in \mathbb{R}^{2n_2}$, $B_3 \in \mathbb{R}^{n_3}$, $B_{4a} \in \mathbb{R}^{n_4}$, $B_{4b} \in \mathbb{R}^{n_4}$, $C_1 \in \mathbb{R}^{1 \times n_1}$, $C_2 \in \mathbb{R}^{1 \times 2n_2}$, $C_3 \in \mathbb{R}^{1 \times n_3}$, $C_{4a} \in \mathbb{R}^{1 \times n_4}$, $C_{4b} \in \mathbb{R}^{1 \times n_4}$, $A_3 = 0_{n_3}$, $n_3 + n_4 = 1$, $n_1 + 2n_2 + n_3 + 2n_4 = n$,

$$A_4 = \begin{bmatrix} 0 & I_{n_4} \\ 0 & 0_{n_4} \end{bmatrix}, \quad B_4 = \begin{bmatrix} B_{4a} \\ B_{4b} \end{bmatrix}, \quad C_4 = \begin{bmatrix} C_{4a} & C_{4b} \end{bmatrix}.$$

The eigenvalues of A_1 have strictly negative real parts, A_2 is a diagonalizable matrix with nonzero purely imaginary eigenvalues. Then, the transfer function of system (1) can be rewritten as

$$G(s) = \sum_{i=1}^{2} G_i(s) + \frac{C_3 B_3 + C_4 B_4}{s} + \frac{C_{4a} B_{4b}}{s^2},$$

where $G_i(s) = C_i(sI - A_i)^{-1} B_i$, $(i = 1, 2)$. Let (A_i, B_i, C_i), $(i = 1, 2)$ denote the subsystems in Eq. (5). Here, the subsystem (A_1, B_1, C_1) is asymptotically stable, the subsystem (A_2, B_2, C_2) is lossless NI.

Remark 4. According to Theorem 2.1.2 in [33], for the given NI system with the minimal state-space realization (1), there always exist a nonsingular transformation T such that $(T^{-1}AT, T^{-1}B, CT)$ be the real Jordan canoncial form. Moreover, the transformation T can be obtained using some numerically algorithms such as the real Schur decomposition, see [33]. Similarly, we can choose this transformation such that the real Jordan blocks of $T^{-1}AT$ are ordered according to the eigenvalues of the matrix A. Following the proof of Lemma 7 in [29], the zero eigenvalues of A only have Jordon blocks of order one or two. Thus, we can obtain that the real Jordan canoncial form $(T^{-1}AT, T^{-1}B, CT)$ is of the form in Eq. (5). Moreover, it follows from Lemma 2 in [29] that the matrix $\begin{bmatrix} C_3 & C_{4a} \end{bmatrix}$ is of full column rank, the matrix $[\begin{smallmatrix} B_3 \\ B_{4b} \end{smallmatrix}]$ is of full row rank. In addition, the subsystems with realizations (A_i, B_i, C_i), $(i = 1, 2)$ are minimal realizations.

Let $\mathcal{S}_1 = \{s_k | s_k \in \mathbb{C}, k = 1, \ldots, r_1\}$ be a set of interpolation points such that $\mathcal{S}_1 \cap \{\sigma(A_1)\} = \emptyset$, $\mathcal{S}_2 = \{s_j | s_j \in \mathbb{C}, j = 1, \ldots, 2r_2\}$ be a set of interpolation points such that $\mathcal{S}_2 \cap \{\sigma(A_2)\} = \emptyset$. According to Definition 1, the moments of subsystems (A_i, B_i, C_i), $(i = 1, 2)$ in Eq. (5) at the selected interpolation points are given by

$$\begin{aligned} \eta_0(s_k) &= C_1(s_k I - A_1)^{-1} B_1, \quad s_k \in \mathcal{S}_1, \\ \eta_0(s_j) &= C_2(s_j I - A_2)^{-1} B_2, \quad s_j \in \mathcal{S}_2. \end{aligned}$$

A reduced-order system that matches the moments of system (5) at the selected interpolation points $s_k \in \mathcal{S}_1$, $s_j \in \mathcal{S}_2$ is given by

$$\begin{bmatrix} \dot{x}_{r_1}(t) \\ \dot{x}_{r_2}(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} A_{r_1} & 0 & 0 & 0 \\ 0 & A_{r_2} & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix} \begin{bmatrix} x_{r_1}(t) \\ x_{r_2}(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} B_{r_1} \\ B_{r_2} \\ B_3 \\ B_4 \end{bmatrix} u(t),$$
$$y_r(t) = \begin{bmatrix} C_{r_1} & C_{r_2} & C_3 & C_4 \end{bmatrix} \begin{bmatrix} x_{r_1}(t) \\ x_{r_2}(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}, \tag{6}$$

where $A_{r_1} \in \mathbb{R}^{n_1 \times n_1}$, $A_{r_2} \in \mathbb{R}^{2n_2 \times 2n_2}$, $B_{r_1} \in \mathbb{R}^{r_1}$, $B_{r_2} \in \mathbb{R}^{2r_2}$, $C_{r_1} \in \mathbb{R}^{1 \times r_1}$, $C_{r_2} \in \mathbb{R}^{1 \times 2r_2}$, A_3, A_4 are defined in Eq. (5), $r_1 + 2r_2 + n_3 + 2n_4 = r$. The eigenvalues of A_{r_1} have strictly negative real parts, A_{r_2} is a diagonalizable matrix with nonzero purely imaginary eigenvalues. The moments of the reduced-order system (6) satisfy the following equations

$$\begin{aligned} \tilde{\eta}_0(s_k) &= C_{r_1}(s_k I - A_{r_1})^{-1} B_{r_1} = \eta_0(s_k), \quad s_k \in \mathcal{S}_1, \\ \tilde{\eta}_0(s_j) &= C_{r_2}(s_j I - A_{r_2})^{-1} B_{r_2} = \eta_0(s_j), \quad s_j \in \mathcal{S}_2. \end{aligned}$$

In terms of the block diagonal form (5), a new condition is proposed to test the NI structure.

Lemma 3. *Let Eq. (5) be a minimal state-space realization of $G(s)$, where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^n$, $C \in \mathbb{R}^{1 \times n}$. Then $G(s)$ is NI if and only if there exist matrices $X_1 \in \mathbb{R}^{n_1 \times n_1}$, $X_1 = X_1^\top > 0$, $X_2 \in \mathbb{R}^{2n_2 \times 2n_2}$, $X_2 = X_2^\top > 0$, $X_4 = diag\{0_{n_4}, X_{4b}\}$, $X_{4b} \in \mathbb{R}^{n_4 \times n_4}$, $X_{4b} > 0$, L_1, W such that*

$$X_1 A_1 + A_1^\top X_1 = -L_1^\top L_1, \tag{7}$$

$$X_2 A_2 + A_2^\top X_2 = 0, \tag{8}$$

$$X_1 B_1 - A_1^\top C_1^\top = -L_1^\top W, \tag{9}$$

$$X_2 B_2 - A_2^\top C_2^\top = 0, \tag{10}$$

$$X_{4b} B_{4b} - C_{4a}^\top = 0, \tag{11}$$

$$\hat{D} + \hat{D}^\top = W^\top W, \tag{12}$$

where $\hat{D} = C_1 B_1 + C_2 B_2 + C_3 B_3 + C_4 B_4$.

Proof. The proof is similar to Lemma A.4 in [34], hence it is omitted here. □

3.1. Moment matching model reduction: NI systems with no poles at the origin

In this subsection, we show how to achieve moment matching and structure preserving for the NI systems with no poles at the origin.

Consider the case when the NI system (5) having no poles at the origin, that is, $n_3 = n_4 = 0$, n_1, $n_2 \neq 0$, $n_1 + n_2 = n$. Then, the minimal state-space realization of system (5) can be rewritten as

$$\hat{A} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \ \hat{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}. \tag{13}$$

In this case, the conditions in Lemma 3 reduced to the following result.

Lemma 4. *Let Eq. (13) be a minimal state-space realization of $G(s)$, where $\hat{A} \in \mathbb{R}^{n\times n}$, $\hat{B} \in \mathbb{R}^{n}$, $\hat{C} \in \mathbb{R}^{1\times n}$. Then $G(s)$ is NI if and only if there exists a block diagonal matrix $Y = diag\{Y_1, Y_2\} \in \mathbb{R}^{n\times n}$, $Y = Y^\top > 0$ such that*

$$A_1Y_1 + Y_1A_1^\top \leq 0 \quad \text{and} \quad B_1 + A_1Y_1C_1^\top = 0, \tag{14}$$

$$A_2Y_2 + Y_2A_2^\top = 0 \quad \text{and} \quad B_2 + A_2Y_2C_2^\top = 0. \tag{15}$$

Proof. The proof is presented in Appendix. □

Based on Lemma 4, the following theorem provides the conditions for the construction of the reduced-order NI systems.

Theorem 1. *Given an NI system (13), $1 \leq r_1 < n_1$, $1 \leq r_2 < n_2$, $r_1 + 2r_2 = r$, the interpolation points sets $\mathcal{S}_1$, $\mathcal{S}_2$. Let Y be a solution of (14)–(15). Construct matrix $U \in \mathbb{R}^{n\times r}$ as*

$$U = diag\{U_1, U_2\} \tag{16}$$

where

$$\begin{aligned} U_1 &= \begin{bmatrix} (s_1I - A_1)^{-1}B_1 & \dots & (s_{r_1}I - A_1)^{-1}B_1 \end{bmatrix}, \\ U_2 &= \begin{bmatrix} (s_1I - A_2)^{-1}B_2 & \dots & (s_{2r_2}I - A_2)^{-1}B_2 \end{bmatrix}, \end{aligned} \tag{17}$$

are the real basis matrices of the generalized reachability matrices. If the matrix $U^\top Y^{-1}\hat{A}^{-1}U$ is nonsingular, then the reduced-order NI system that matches the moments of system (13) is given by

$$\begin{aligned} \hat{A}_r &= (U^\top Y^{-1}\hat{A}^{-1}U)^{-1}(U^\top Y^{-1}U), \\ \hat{B}_r &= -(U^\top Y^{-1}\hat{A}^{-1}U)^{-1}U^\top\hat{C}^\top, \quad \hat{C}_r = \hat{C}U. \end{aligned} \tag{18}$$

Proof. Firstly, we prove that the reduced-order system (18) is NI. It follows from $Y > 0$ that $U^\top Y^{-1}U > 0$. This implies that the matrix $\hat{A}_r$ is nonsingular. Let $Y_r = (U^\top Y^{-1}U)^{-1}$. Then, we have that

$$Y_r = diag\{Y_{r_1}, Y_{r_2}\} = diag\{(U_1^\top Y_1^{-1}U_1)^{-1}, (U_2^\top Y_2^{-1}U_2)^{-1}\} > 0.$$

Moreover, the system matrices in Eq. (18) can be rewritten as

$$\hat{A}_r = \begin{bmatrix} A_{r_1} & 0 \\ 0 & A_{r_2} \end{bmatrix}, \ \hat{B}_r = -\begin{bmatrix} B_{r_1} \\ B_{r_2} \end{bmatrix}, \ \hat{C}_r = \begin{bmatrix} C_{r_1} & C_{r_2} \end{bmatrix},$$

where

$$\begin{aligned} A_{r_1} &= (U_1^\top Y_1^{-1}A_1^{-1}U_1)^{-1}Y_{r_1}^{-1}, \ A_{r_2} = (U_2^\top Y_2^{-1}A_2^{-1}U_2)^{-1}Y_{r_2}^{-1}, \\ B_{r_1} &= (U_1^T Y_1^{-1}A_1^{-1}U_1)^{-1}U_1^T C_1^T, \ B_{r_2} = (U_2^\top Y_2^{-1}A_2^{-1}U_2)^{-1}U_2^\top C_2^\top, \end{aligned}$$

$C_{r_1} = C_1 U_1, \ C_{r_2} = C_2 U_2.$

It follows from Lemma 4 that the following equivalent conditions hold

$$\begin{aligned} & A_1 Y_1 + Y_1 A_1^{\mathsf{T}} \leq 0 \Leftrightarrow Y_1^{-1}(A_1 Y_1 + Y_1 A_1^{\mathsf{T}}) Y_1^{-1} \leq 0, \\ \Leftrightarrow \ & Y_1^{-1} A_1 + A_1^{\mathsf{T}} Y_1^{-1} \leq 0 \Leftrightarrow A_1^{-\mathsf{T}} (Y_1^{-1} A_1 + A_1^{\mathsf{T}} Y_1^{-1}) A_1^{-1} \leq 0, \\ \Leftrightarrow \ & Y_1^{-1} A_1^{-1} + (Y_1^{-1} A_1^{-1})^{\mathsf{T}} \leq 0 \Leftrightarrow \ U_1^{\mathsf{T}} Y_1^{-1} A_1^{-1} U_1 + (U_1^{\mathsf{T}} Y_1^{-1} A_1^{-1} U_1)^{\mathsf{T}} \leq 0, \\ \Leftrightarrow \ & (U_1^{\mathsf{T}} Y_1^{-1} A_1^{-1} U_1)^{-1} + (U_1^{\mathsf{T}} Y_1^{-1} A_1^{-1} U_1)^{-\mathsf{T}} \leq 0, \\ \Leftrightarrow \ & A_{r_1} Y_{r_1} + Y_{r_1} A_{r_1}^{\mathsf{T}} \leq 0. \end{aligned}$$

Similarly, we can obtain that

$$A_2 Y_2 + Y_2 A_2^{\mathsf{T}} = 0 \Leftrightarrow \ A_{r_2} Y_{r_2} + Y_{r_2} A_{r_2}^{\mathsf{T}} = 0.$$

Moreover, $B_{r_1} = -A_{r_1} Y_{r_1} C_{r_1}^{\mathsf{T}}$, $B_{r_2} = -A_{r_2} Y_{r_2} C_{r_2}^{\mathsf{T}}$. Thus, according to Lemma 4, the reduced-order system (18) is NI.

Now we prove that the transfer functions of the reduced-order system (18) and the original system (13) are equal at the selected interpolation points $s_k \in \mathcal{S}_1, \ s_j \in \mathcal{S}_2$. Note that,

$$\begin{aligned} G_{r_1}(s_k) &= C_{r_1}(s_k I - A_{r_1})^{-1} B_{r_1} = -C_{r_1}(s_k I - A_{r_1})^{-1} A_{r_1} Y_{r_1} C_{r_1}^{\mathsf{T}} \\ &= -C_1 U_1 \{ s_k U_1^{\mathsf{T}} Y_1^{-1} A_1^{-1} U_1 - (U_1^{\mathsf{T}} Y_1^{-1} U_1) \}^{-1} U_1^{\mathsf{T}} C_1^{\mathsf{T}} \\ &= -C_1 U_1 \{ U_1^{\mathsf{T}} (s_k Y_1^{-1} A_1^{-1} - Y_1^{-1}) U_1 \}^{-1} U_1^{\mathsf{T}} C_1^{\mathsf{T}}, \\ G_{r_2}(s_j) &= C_{r_2}(s_j I - A_{r_2})^{-1} B_{r_2} = -C_{r_2}(s_j I - A_{r_2})^{-1} A_{r_2} Y_{r_2} C_{r_2}^{\mathsf{T}} \\ &= -C_2 U_2 \{ s_j U_2^{\mathsf{T}} Y_2^{-1} A_2^{-1} U_2 - (U_2^{\mathsf{T}} Y_2^{-1} U_2) \}^{-1} U_2^{\mathsf{T}} C_2^{\mathsf{T}} \\ &= -C_2 U_2 \{ U_2^{\mathsf{T}} (s_j Y_2^{-1} A_2^{-1} - Y_2^{-1}) U_2 \}^{-1} U_2^{\mathsf{T}} C_2^{\mathsf{T}}. \end{aligned} \tag{19}$$

According to Lemma 4, we have that $B_1 = -A_1 Y_1 C_1^{\mathsf{T}}, \ B_2 = -A_2 Y_2 C_2^{\mathsf{T}}$. Thus, $U_1, \ U_2$ in Eq. (17) can be rewritten as

$$\begin{aligned} U_1 &= \begin{bmatrix} -(s_1 I - A_1)^{-1} A_1 Y_1 C_1^{\mathsf{T}} & \dots & -(s_{r_1} I - A_1)^{-1} A_1 Y_1 C_1^{\mathsf{T}} \end{bmatrix} \\ &= \begin{bmatrix} -(s_1 Y_1^{-1} A_1^{-1} - Y_1^{-1})^{-1} C_1^{\mathsf{T}} & \dots & -(s_{r_1} Y_1^{-1} A_1^{-1} - Y_1^{-1})^{-1} C_1^{\mathsf{T}} \end{bmatrix}, \\ U_2 &= \begin{bmatrix} -(s_1 I - A_2)^{-1} A_2 Y_2 C_2^{\mathsf{T}} & \dots & -(s_{2r_2} I - A_2)^{-1} A_2 Y_2 C_2^{\mathsf{T}} \end{bmatrix} \\ &= \begin{bmatrix} -(s_1 Y_2^{-1} A_2^{-1} - Y_2^{-1})^{-1} C_2^{\mathsf{T}} & \dots & -(s_{2r_2} Y_2^{-1} A_2^{-1} - Y_2^{-1})^{-1} C_2^{\mathsf{T}} \end{bmatrix}. \end{aligned} \tag{20}$$

Combining Eq. (20) with Eq. (19), one obtains that

$$\begin{aligned} G_{r_1}(s_k) &= -C_1 U_1 \begin{bmatrix} -\tilde{U}_1 & \cdots & -U_1^{\mathsf{T}} C_1^{\mathsf{T}} & \cdots & -\tilde{U}_{r_1} \end{bmatrix}^{-1} U_1^{\mathsf{T}} C_1^{\mathsf{T}} \\ &= C_1 U_1 \begin{bmatrix} \tilde{U}_1 & \cdots & U_1^{\mathsf{T}} C_1^{\mathsf{T}} & \cdots & \tilde{U}_{r_1} \end{bmatrix}^{-1} \begin{bmatrix} \tilde{U}_1 & \cdots & U_1^{\mathsf{T}} C_1^{\mathsf{T}} & \cdots & \tilde{U}_{r_1} \end{bmatrix} e_k \\ &= C_1 U_1 e_k = C_1 (s_k I - A_1)^{-1} B_1 = G_1(s_k), \end{aligned}$$

where

$$\begin{aligned} \tilde{U}_1 &= U_1^{\mathsf{T}} (s_k Y_1^{-1} A_1^{-1} - Y_1^{-1})(s_1 Y_1^{-1} A_1^{-1} - Y_1^{-1})^{-1} C_1^{\mathsf{T}}, \\ \tilde{U}_{r_1} &= U_1^{\mathsf{T}} (s_k Y_1^{-1} A_1^{-1} - Y_1^{-1})(s_{r_1} Y_1^{-1} A_1^{-1} - Y_1^{-1})^{-1} C_1^{\mathsf{T}}. \end{aligned}$$

Similarly, we can also obtain that $G_{r_2}(s_j) = G_2(s_j)$. $\square$

Remark 5. Define the corresponding tangent directions as $\mathcal{Q}_1 = \{b_k | b_k \in \mathbb{C}^q, k = 1, \ldots, r_1\}$, $\mathcal{Q}_2 = \{l_j | l_j \in \mathbb{C}^q, j = 1, \ldots, 2r_2\}$. Theorem 1 can be extended to the MIMO case by replacing the matrices U_1, U_2 in Eq. (17) as

$$U_1 = \left[(s_1 I - A_1)^{-1} B_1 b_1 \quad \ldots \quad (s_{r_1} I - A_1)^{-1} B_1 b_{r_1}\right],$$
$$U_2 = \left[(s_1 I - A_2)^{-1} B_2 l_1 \quad \ldots \quad (s_{2r_2} I - A_2)^{-1} B_2 l_{2r_2}\right]. \tag{21}$$

Following the proof of $G_{\mathrm{r}}(s_k) = G(s_k)$ and $G_{\mathrm{r}}(s_j) = G(s_j)$ in Theorem 1, we have that the transfer function of the obtained reduced-order NI system (18) satisfies

$$G_{\mathrm{r}}(s_k) b_k = G(s_k) b_k, \;\; s_k \in \mathcal{S}_1, \;\; b_k \in \mathcal{Q}_1,$$
$$G_{\mathrm{r}}(s_j) l_j = G(s_j) l_j, \;\; s_j \in \mathcal{S}_2, \;\; l_j \in \mathcal{Q}_2.$$

Thus, the results in Theorem 1 are also applicable for the MIMO NI systems.

Remark 6. In [35], the guidelines to choose the interpolation points s_k, s_j have been established, which can be concluded as follows.

(1) A purely imaginary interpolation point leads to very good local approximation and to a very slow convergence at all frequencies away from the interpolation point.
(2) A real interpolation point offer a good approximation in a large neighborhood around the interpolation point, except around some lightly damped eigenvalues on the imaginary axis.
(3) The combination of real, imaginary, and complex interpolation points is generally preferred over a single interpolation point, however the choice and number of these points is not straightforward.

In [36], a rational Krylov algorithm with interpolation points selected as spectral zeros of the original transfer function has been presented for passive linear systems. However, the choice of the interpolation points for NI systems is still an active field of research and remains challenging. Our future research will focus on how to choose interpolation points to obtain the reduced-order NI system with small approximation error.

Remark 7. For strongly strict NI systems, the system matrix $\hat{A} = A_1$ satisfies $A_1 Y_1 + Y_1 A_1^{\mathrm{T}} < 0$, which implies that

$$A_{\mathrm{r}_1} Y_{\mathrm{r}_1} + Y_{\mathrm{r}_1} A_{\mathrm{r}_1}^{\mathrm{T}} = (U_1^{\mathrm{T}} Y_1^{-1} A_1^{-1} U_1)^{-1} + (U_1^{\mathrm{T}} Y_1^{-1} A_1^{-1} U_1)^{-\mathrm{T}} < 0.$$

Thus, the obtained reduced-order system matrix A_{r_1} is stable for the arbitrary interpolation points $s_k \in \mathcal{S}_1$.

Remark 8. For lossless NI systems, the system matrix $\hat{A} = A_2$ satisfies

$$A_2 Y_2 + Y_2 A_2^{\mathrm{T}} = 0,$$

which implies that

$$Y_2^{-1} A_2^{-1} = -A_2^{-\mathrm{T}} Y_2^{-1} \Rightarrow Y_2^{-1} A_2^{-1} A_2^{-\mathrm{T}} Y_2^{-1} = A_2^{-\mathrm{T}} Y_2^{-1} Y_2^{-1} A_2^{-1},$$

that is, the matrix $Y_2^{-1} A_2^{-1}$ is normal. According to Antoulas [37], one obtains that the matrix $U_2^{\mathrm{T}} Y_2^{-1} A_2^{-1} U_2$ can never be singular. Thus, the obtained reduced-order system matrix A_{r_2} is nonsingular for the arbitrary interpolation points $s_j \in \mathcal{S}_2$.

3.2. Moment matching model reduction: NI systems with poles at the origin

In this subsection, by using the transformation between NI systems and positive real systems, the moment matching model reduction problems for NI systems with poles at the origin is transformed equivalently into that for the positive real systems with blocking zero at zero frequency. Then, the moment matching model reduction method is extended to the positive real systems with blocking zero at zero frequency.

Consider the case when the NI system (5) having poles at the origin, that is, $n_3+n_4=1$, $n_1+2n_2+n_3+n_4=n$. Then, according to Lemma 3 in [29], the transfer function

$$\begin{aligned}R(s)=sG(s)&=C_1A_1(sI-A_1)^{-1}B_1+C_2A_2(sI-A_2)^{-1}B_2\\&+\frac{C_{4a}B_{4b}}{s}+C_1B_1+C_2B_2+C_3B_3+C_4B_4,\end{aligned}\tag{22}$$

is positive real. A minimal state-space realization of $R(s)$ is given by

$$\begin{aligned}&A_{\rm PR}=\begin{bmatrix}A_1&0&0\\0&A_2&0\\0&0&0_{n_4}\end{bmatrix},\ B_{\rm PR}=\begin{bmatrix}B_1\\B_2\\B_{4b}\end{bmatrix},\ C_{\rm PR}=\begin{bmatrix}C_1A_1&C_2A_2&C_{4a}\end{bmatrix},\\&\hat{D}=C_1B_1+C_2B_2+C_3B_3+C_4B_4.\end{aligned}\tag{23}$$

In terms of the block diagonal form (23), a new condition is derived to test the positive real structure.

Lemma 5. *Let Eq. (23) be a minimal state-space realization of $R(s)$, where $A_{\rm PR}\in\mathbb{R}^{(n_1+2n_2+n_4)\times(n_1+2n_2+n_4)}$, $B_{\rm PR}\in\mathbb{R}^{n_1+2n_2+n_4}$, $C_{\rm PR}\in\mathbb{R}^{1\times(n_1+2n_2+n_4)}$, $\hat{D}\in\mathbb{R}$, $\hat{D}=\hat{D}^{\mathsf T}$. Then $R(s)$ is positive real if and only if there exists a block diagonal matrix $P_{\rm PR}=diag\{\hat{P}_1,\hat{P}_2,\hat{P}_3\}$, $\hat{P}_1\in\mathbb{R}^{n_1\times n_1}$, $\hat{P}_1=\hat{P}_1^{\mathsf T}>0$, $\hat{P}_2\in\mathbb{R}^{2n_2\times 2n_2}$, $\hat{P}_2=\hat{P}_2^{\mathsf T}>0$, $\hat{P}_3\in\mathbb{R}^{n_4\times n_4}$, $\hat{P}_3=\hat{P}_3^{\mathsf T}>0$ such that*

$$\begin{bmatrix}P_{\rm PR}A_{\rm PR}+A_{\rm PR}^{\mathsf T}P_{\rm PR}&P_{\rm PR}B_{\rm PR}-C_{\rm PR}^{\mathsf T}\\B_{\rm PR}^{\mathsf T}P_{\rm PR}-C_{\rm PR}&-\hat{D}-\hat{D}^{\mathsf T}\end{bmatrix}\le 0.\tag{24}$$

Proof. According to the GKYP Lemma [38], $R(s)$ is positive real if and only if there exist matrices $P_{\rm PR}\in\mathbb{R}^{(n_1+2n_2+n_4)\times(n_1+2n_2+n_4)}$, $P_{\rm PR}=P_{\rm PR}^{\mathsf T}>0$, $\tilde{L}$, $\tilde{W}$ such that

$$\begin{bmatrix}P_{\rm PR}A_{\rm PR}+A_{\rm PR}^{\mathsf T}P_{\rm PR}&P_{\rm PR}B_{\rm PR}-C_{\rm PR}^{\mathsf T}\\B_{\rm PR}^{\mathsf T}P_{\rm PR}-C_{\rm PR}&-\hat{D}-\hat{D}^{\mathsf T}\end{bmatrix}=-\begin{bmatrix}\tilde{L}\tilde{L}^{\mathsf T}&\tilde{L}\tilde{W}\\\tilde{W}^{\mathsf T}\tilde{L}^{\mathsf T}&\tilde{W}^{\mathsf T}\tilde{W}\end{bmatrix}\le 0.$$

By following the similar lines as in the proof of Lemma A.4 [34], we conclude that the above inequality and Eq. (24) are equivalent. □

Let (A_i,B_i,C_iA_i,C_iB_i) $(i=1,2)$ denote the subsystems in Eq. (23). Here, the subsystems $(A_i,\ B_i,\ C_iA_i,\ C_iB_i)$ always have a blocking zero at zero frequency since the matrices A_1, A_2 are nonsingular. This implies that the positive real system without a blocking zero at zero frequency is always transformed into an NI system with poles at the origin. Hence, we perform moment matching model reduction on subsystems $(A_i,\ B_i,\ C_iA_i,\ C_iB_i)$ to preserve the blocking zero and the positive real structure for the reduced-order systems.

The following theorem provides the conditions for the construction of the reduced-order positive real systems that preserve the blocking zero and match the moments of the original system (23).

Theorem 2. *Given the positive real system (23), $1\le r_1<n_1$, $1\le r_2<n_2$, $r_1+2r_2=r$. Let $P_{\rm PR}$ be a solution of Eq. (24), $s_1=0$, $\{s_k\}_{k=2}^{r_1}\subset\mathbb{C}$ be a set of interpolation points such*

that $s_k \cap \sigma(A_1) = \emptyset$, $\{s_j\}_{j=2}^{2r_2} \subset \mathbb{C}$ *be a set of interpolation points such that* $s_j \cap \sigma(A_2) = \emptyset$. *Construct a matrix* $U \in \mathbb{R}^{(n_1+2n_2+n_4)\times(r_1+2r_2+n_4)}$ *as*

$$U = diag\{U_1, U_2, I_{n_4}\}, \tag{25}$$

where the matrices U_1, U_2 *are defined in Eq. (17). A reduced-order positive real system that matches the moments of the original system (23) is given by*

$$\begin{aligned}\tilde{A}_{\mathrm{PR}} &= (U^\top P_{\mathrm{PR}} U)^{-1} U^\top P_{\mathrm{PR}} A_{\mathrm{PR}} U, \quad \tilde{B}_{\mathrm{PR}} = (U^\top P_{\mathrm{PR}} U)^{-1} U^\top P_{\mathrm{PR}} B_{\mathrm{PR}}, \\ \tilde{C}_{\mathrm{PR}} &= C_{\mathrm{PR}} U, \quad \hat{D} = C_1 B_1 + C_2 B_2 + C_3 B_3 + C_4 B_4.\end{aligned} \tag{26}$$

Proof. Firstly, we prove that the reduced-order system $R_{\mathrm{r}}(s)$ in Eq. (26) is positive real. Let $\tilde{P}_{\mathrm{PR}} = U^\top P_{\mathrm{PR}} U$. Then, we have that $\tilde{P}_{\mathrm{PR}} = \tilde{P}_{\mathrm{PR}}^\top > 0$. Moreover,

$$\begin{aligned}\Xi &= \begin{bmatrix} \tilde{P}_{\mathrm{PR}} \tilde{A}_{\mathrm{PR}} + \tilde{A}_{\mathrm{PR}}^\top \tilde{P}_{\mathrm{PR}} & \tilde{P}_{\mathrm{PR}} \tilde{B}_{\mathrm{PR}} - \tilde{C}_{\mathrm{PR}}^\top \\ \tilde{B}_{\mathrm{PR}}^\top \tilde{P}_{\mathrm{PR}} - \tilde{C}_{\mathrm{PR}} & -\hat{D} - \hat{D}^\top \end{bmatrix} \\ &= \begin{bmatrix} U^\top (P_{\mathrm{PR}} A_{\mathrm{PR}} + A_{\mathrm{PR}}^\top P_{\mathrm{PR}}) U & U^\top (P_{\mathrm{PR}} B_{\mathrm{PR}} - C_{\mathrm{PR}}^\top) \\ (B_{\mathrm{PR}}^\top P_{\mathrm{PR}} - C_{\mathrm{PR}}) U & -\hat{D} - \hat{D}^\top \end{bmatrix} \\ &= \Theta^\top \begin{bmatrix} P_{\mathrm{PR}} A_{\mathrm{PR}} + A_{\mathrm{PR}}^\top P_{\mathrm{PR}} & P_{\mathrm{PR}} B_{\mathrm{PR}} - C_{\mathrm{PR}}^\top \\ B_{\mathrm{PR}}^\top P_{\mathrm{PR}} - C_{\mathrm{PR}} & -\hat{D} - \hat{D}^\top \end{bmatrix} \Theta \le 0,\end{aligned} \tag{27}$$

where $\Theta = diag\{U, 1\}$. Thus, according to Lemma 5, the reduced-order system (26) is positive real.

Now we prove that the transfer function of the reduced-order system (26) and the original system (23) are equal at the selected interpolation points $s_1 = 0$. The following string of equalities lead to the desired result,

$$\begin{aligned}R_{\mathrm{r}_1}(s_k) &= \tilde{C}_{\mathrm{PR}_1} (s_k I - \tilde{A}_{\mathrm{PR}_1})^{-1} \tilde{B}_{\mathrm{PR}_1} + C_1 B_1 \\ &= C_1 A_1 U_1 \{ s_k I - (U_1^\top \hat{P}_1 U_1)^{-1} U_1^\top \hat{P}_1 A_1 U_1 \}^{-1} (U_1^\top \hat{P}_1 U_1)^{-1} U_1^\top \hat{P}_1 B_1 + C_1 B_1 \\ &= C_1 A_1 U_1 \{ U_1^\top \hat{P}_1 (s_k I - A_1) U_1 \}^{-1} U_1^\top \hat{P}_1 B_1 + C_1 B_1 \\ &= C_1 A_1 U_1 \begin{bmatrix} \hat{U}_1 & \cdots & U_1^\top \hat{P}_1 B_1 & \cdots & \hat{U}_{r_1} \end{bmatrix}^{-1} U_1^\top \hat{P}_1 B_1 + C_1 B_1 \\ &= C_1 A_1 U_1 \begin{bmatrix} \hat{U}_1 & \cdots & U_1^\top \hat{P}_1 B_1 & \cdots & \hat{U}_{r_1} \end{bmatrix}^{-1} \\ &\quad \times \begin{bmatrix} \hat{U}_1 & \cdots & U_1^\top \hat{P}_1 B_1 & \cdots & \hat{U}_{r_1} \end{bmatrix} e_k + C_1 B_1 \\ &= C_1 A_1 U_1 e_k + C_1 B_1 = C_1 A_1 (s_k I - A_1)^{-1} B_1 + C_1 B_1 = R_1(s_k),\end{aligned}$$

where

$$\begin{aligned}\hat{U}_1 &= -U_1^\top \hat{P}_1 (s_k I - A_1) A_1^{-1} B_1, \\ \hat{U}_{r_1} &= U_1^\top \hat{P}_1 (s_k I - A_1)(s_{r_1} I - A_1)^{-1} B_1.\end{aligned}$$

Similarly, we also obtain that $R_{\mathrm{r}_2}(s_j) = R_2(s_j)$. Moreover, we have that

$$R_{\mathrm{r}_1}(s_1) = R_1(s_1) = 0, \quad R_{\mathrm{r}_2}(s_1) = R_2(s_1) = 0.$$

That is, the blocking zero at zero frequency is also preserved for the reduced-order system. □

Remark 9. Similar to Theorem 1, when the matrices U_1, U_2 in Eq. (25) are defined as in Eq. (21), then the obtained results in Theorem 2 are also applicable for the MIMO NI systems.

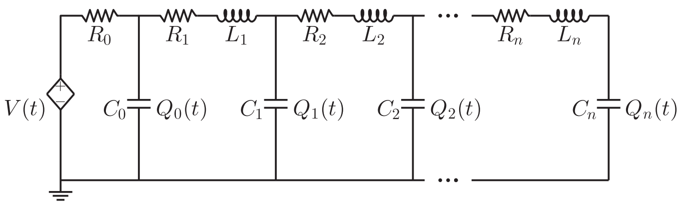

Fig. 1. Ladder RLC network.

Based on Theorem 2, the following result provides the condition for the construction of the reduced-order NI system with poles at the origin.

Lemma 6. *Given an NI system with poles at the origin in Eq. (5), $1 \leq r_1 < n_1$, $1 \leq r_2 < n_2$, $r_1 + 2r_2 = r$. Let $s_1 = 0$, $\{s_k\}_{i=2}^{r_1} \subset \mathbb{C}$ be a set of interpolation points such that $s_k \cap \sigma(A_1) = \emptyset$, $\{s_j\}_{j=2}^{2r_2} \subset \mathbb{C}$ be a set of interpolation points such that $s_j \cap \sigma(A_2) = \emptyset$. Construct a matrix $U \in \mathbb{R}^{(n_1+2n_2+n_4)\times(r_1+2r_2+n_4)}$ as in Eq. (25). A reduced-order NI system with poles at the origin that matches the moments of the origin system (5) is given by*

$$G_r(s) = \frac{1}{s}(\tilde{C}_{PR}(sI - \tilde{A}_{PR})^{-1}\tilde{B}_{PR} + \hat{D}), \tag{28}$$

where $\tilde{A}_{PR}$, $\tilde{B}_{PR}$, $\tilde{C}_{PR}$, $\hat{D}$ are defined in Eq. (26).

Proof. According to Theorem 2, the obtained reduced-order subsystems $R_{r_1}(s)$ and $R_{r_2}(s)$ are both positive real systems with blocking zero at zero frequency. Thus, the stability of the transformed reduced-order subsystems $G_{r_1}(s) = \frac{1}{s}R_{r_1}(s)$ and $G_{r_2}(s) = \frac{1}{s}R_{r_2}(s)$ can be guaranteed. That is, the transformed reduced-order subsystem $G_{r_1}(s) = \frac{1}{s}R_{r_1}(s)$ is asymptotically stable, the transformed reduced-order subsystem $G_{r_2}(s) = \frac{1}{s}R_{r_2}(s)$ is lossless NI. This implies that the transformed reduced-order system $G_r(s)$ in Eq. (28) preserves the NI structure of the original system (5). □

4. Illustrative examples

In this section, we provide three examples to demonstrate the effectiveness of the proposed moment matching model reduction methods.

4.1. Example 1: Stable NI systems

Consider the n-stage RLC network as shown in Fig. 1, which is borrowed from [13]. Defining the state as

$$x(t) = \begin{bmatrix} u_0(t) & i_{L_1}(t) & u_1(t) & \cdots & i_{L_n}(t) & u_n(t) \end{bmatrix}^{\mathrm{T}},$$

where $u_s(t)$ is the voltage across capacitor C_s, $i_{L_s}(t)$ represent the current through inductor L_s. A stable NI system can be obtained by considering the input–output relationship from

Table 1
Comparison of reduced-order systems in Example 4.1.

Method	Reduced-order system	$\frac{\|G_e(s)\|_\infty}{\|G(s)\|_\infty}$	NI
Moment matching [37]	$\frac{1.033s^2-0.374s+0.2807}{s^3+1.823s^2+0.9711s+0.1065}$	0.5607	No
NI moment matching	$\frac{1.661s^2+2.329s+1.508}{s^3+3.539s^2+2.585s+0.3064}$	0.1796	Yes

$u(t) = V(t)$ to $y(t) = \sum_{k=0}^{n} Q_k(t)$ with

$$A = \begin{bmatrix} -\frac{1}{C_0R_0} & -\frac{1}{C_0} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{L_1} & -\frac{R_1}{L_1} & -\frac{1}{L_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{C_1} & 0 & -\frac{1}{C_1} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{L_2} & -\frac{R_2}{L_2} & -\frac{1}{L_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{L_n} & -\frac{R_n}{L_n} & -\frac{1}{L_n} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{C_n} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{C_0R_0} \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{29}$$

$$C = \begin{bmatrix} C_0 & 0 & C_1 & 0 & C_2 & \cdots & C_n \end{bmatrix}.$$

Here, we consider an 11th-order RLC network with $C_s = 1F$, $L_s = 1H$, $R_s = 0.5\Omega$. Let $\mathcal{S}_1 = \{-0.8, -0.15, j\}$. For comparison, we reduce the system order to 3rd-order by two different methods, the moment matching model reduction for linear systems in [37] and the NI moment matching model reduction method proposed in the paper. Moreover, the relative H_∞ approximation errors with respect to the original NI system and the reduced-order system are computed. The obtained reduced-order systems and the relative H_∞ approximation errors are listed in Table 1. A reduced-order stable NI system can be obtained by the proposed model reduction method with transfer function given in Table 1, while the moment matching model reduction method can not. Moreover, the achieved relative H_∞ approximation error is smaller than 0.5607 obtained by the moment matching model reduction method [37].

Fig. 2 shows the bode plots of the original and the reduced-order systems. It can be seen from Fig. 2 that the reduced-order system $G_r(j\omega)$ obtained by the proposed model reduction method satisfies $\angle G_r(j\omega) \in (-\pi, 0)$ for all $\omega \in (0, \infty)$. This means that $G_r(j\omega)$ has non-positive imaginary part, that is, $j[G_r(j\omega) - G_r^*(j\omega)] \geq 0$. In addition, the reduced-order system is stable. Thus, as shown in Fig. 2, the reduced-order system obtained by the proposed model reduction method is stable NI and approximate the original system well. However, the reduced-order system obtained by the moment matching model reduction method [37] does not satisfy the NI structure.

To further illustrate the effectiveness of the proposed model reduction method, we consider the internal stability of a positive feedback interconnection of the high order NI system (29) and a positive feedback controller, as shown in Fig. 3. The positive feedback controller is given by

$$K(s) = \frac{k}{s^2 + 2\xi\omega s + \omega^2}, \tag{30}$$

where $k > 0$, $\xi > 0$, $\omega > 0$. By using Nyquist arguments, it is clear that $K(s)$ in Eq. (30) is strict negative imaginary.

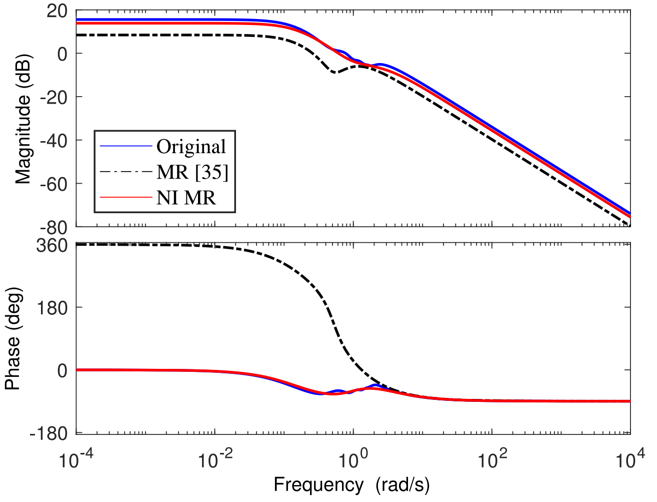

Fig. 2. Bode plots of the original and reduced-order systems.

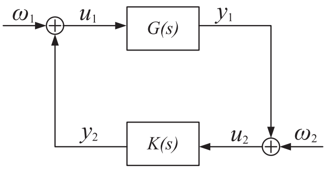

Fig. 3. Positive feedback interconnection.

Note that the transfer function of the original system (29) satisfies $G(\infty)=0,\ G(0)=6$, the transfer function $K(s)$ in Eq. (30) satisfies $K(\infty)=0,\ K(0)=\frac{k}{\omega^2}$, and the transfer function of the reduced-order NI system obtained by the proposed model reduction satisfies $G_r(\infty)=0,\ G_r(0)=4.96$.

Let $\lambda_{max}(\cdot)$ denotes the the maximum eigenvalue of a matrix with only real eigenvalues. According to Theorem 5 in [1], the closed loop system as shown in Fig. 3 is internally stable equivalent to that the dc gain condition $\lambda_{max}(G(0)K(0))<1$ holds. Thus, we have that $\frac{k}{\omega^2}<0.167$. It follows that $\frac{k}{\omega^2}<0.202$, which is equivalent to that the dc gain condition $\lambda_{max}(G_r(0)K(0))<1$ holds. This implies that the positive feedback system consisting of the reduced-order NI system and the given controller in Eq. (30) is internally stable. Thus, a given positive feedback controller (30) for the high order NI system (29) can perform satisfactorily for the reduced-order NI system obtained by the proposed model reduction method.

4.2. Example 2: NI systems with poles at the origin

Consider a train system as shown in Fig. 4. The forces acting on the engine m_1 consist of the forces due to the spring with spring constant k, the friction force with fraction coefficient μ, and the force generated by the engine F. From Newton's second law of motion, the

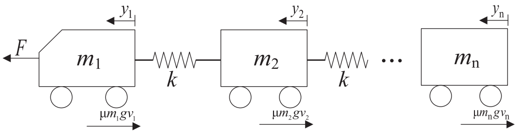

Fig. 4. A train system consisting of an engine m_1 and cars m_s.

dynamics of the system can be described by the following equations

$$\begin{aligned} m_1\ddot{y}_1 &= F - k(y_1 - y_2) - \mu m_1 g\dot{y}_1, \\ m_2\ddot{y}_2 &= k(y_1 - y_2) - k(y_2 - y_3) - \mu m_2 g\dot{y}_2, \\ &\vdots \\ m_n\ddot{y}_n &= k(y_{n-1} - y_n) - \mu m_n g\dot{y}_n, \end{aligned}$$

where F is the force generated by the engine, $g = 9.8\,\mathrm{m/s^2}$ is the gravitation constant, y_1 represents the position of the engine m_1 and y_s represents the position of the cars m_s. Choose the state as $x = \begin{bmatrix} y_1 & \dot{y}_1 & y_2 & \dot{y}_2 & \cdots & y_n & \dot{y}_n \end{bmatrix}^{\mathrm{T}}$, the input-output relationship from $u = F$ to $y = y_1$ is given by Eq. (1) with

$$B = \begin{bmatrix} 0 & \frac{1}{m_1} & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}},\ C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The main diagonal of the system matrix A has 0 in the $(2s-1, 2s-1)$ position and $-\mu g$ in the $(2s, 2s)$ position. The super diagonal of A contains $\frac{k_{s-1}}{m_{s-1}}$ in the $(2s-2, 2s-1)$ position and 1 in the $(2s-1, 2s)$ position. The sub-diagonal of A has 0 in the $(2s-1, 2s-2)$ position and $-\frac{k_{s-1}}{m_s}$ in the $(2s, 2s-1)$ position. Additionally, the system matrix A has $\frac{k_{s-1}}{m_s}$ in the $(2s, 2s-3)$ position. For the case of $s = 3$, the system matrix A is given by

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_1}{m_1} & -\mu g & \frac{k_1}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{k_1}{m_2} & 0 & -\frac{k_1+k_2}{m_2} & -\mu g & \frac{k_2}{m_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k_2}{m_3} & 0 & -\frac{k_2}{m_3} & -\mu g \end{bmatrix}.$$

The transfer function of the train system satisfies the NI property and has a simple pole at the origin. Here, we consider a 10th-order train system with $\mu = 0.001$, $m_s = 2\,\mathrm{kg}$, $k_s = 2\,\mathrm{N/m}$. The goal of this example is to find a 4th-order NI system with a simple pole at the origin.

Let $s_1 = 0$, $\{s_k\}_{k=2}^{4} = \{-0.15, -0.009, -0.0055\}$. A reduced-order NI system that matches the moments of the original system at s_1, $\{s_k\}_{k=2}^{4}$ is given by

$$\begin{aligned} \begin{bmatrix} \dot{x}_{\mathrm{r}_1}(t) \\ \dot{x}_3(t) \end{bmatrix} &= \begin{bmatrix} A_{\mathrm{r}_1} & 0 \\ 0 & A_3 \end{bmatrix} \begin{bmatrix} x_{\mathrm{r}_1}(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} B_{\mathrm{r}_1} \\ B_3 \end{bmatrix} u(t), \\ y_{\mathrm{r}}(t) &= \begin{bmatrix} C_{\mathrm{r}_1} & C_3 \end{bmatrix} \begin{bmatrix} x_{\mathrm{r}_1}(t) \\ x_3(t) \end{bmatrix}, \end{aligned}$$

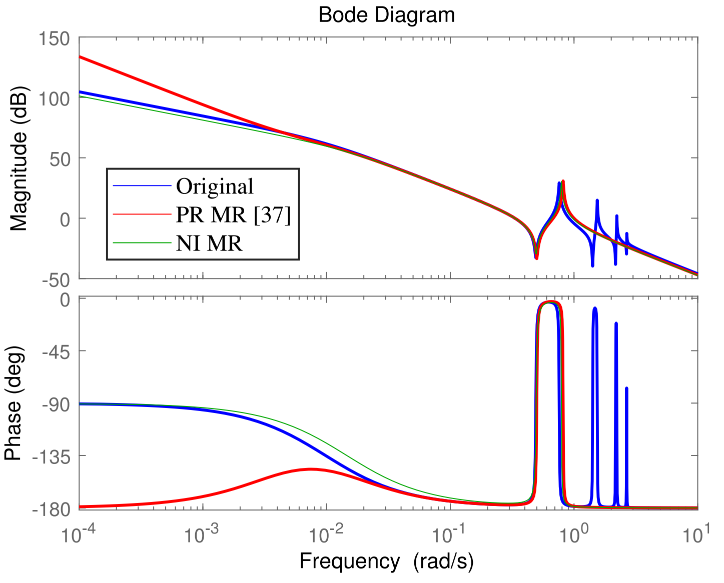

Fig. 5. Bode plots of the original and reduced-order systems.

where

$$A_{r_1}=\begin{bmatrix}-0.0146 & 0 & 0\\ 0 & -0.01 & -1.5826\\ 0 & 1.5826 & -0.01\end{bmatrix},\quad B_{r_1}=\begin{bmatrix}-1.0972\\ -0.002\\ 0.0202\end{bmatrix},\quad C_{r_1}=\begin{bmatrix}0.1044\\ -0.3268\\ -0.0032\end{bmatrix}^{\mathrm{T}}$$

The transfer function of the reduced-order system is

$$G_r(s)=\frac{0.4276s^2+0.006246s+0.1044}{s^4+0.02452s^3+0.6264s^2+0.009113s}. \tag{31}$$

It can be verified that the reduced-order system is an NI system with a simple pole at the origin.

Let $\tilde{s}_1=1,\ \{s_k\}_{k=2}^4=\{-0.15,-0.009,-0.0055\}$. The reduced-order positive real system obtained by the passive preserving moment matching model reduction method [39] is given by

$$R_r(s)=\frac{0.444s^3+0.006208s^2+0.1111s+0.000449}{s^4+0.02359s^3+0.6669s^2+0.009228s}. \tag{32}$$

It has no blocking zero at zero frequency. Thus, the transformed reduced-order system $G_r(s)=\frac{1}{s}R_r(s)$ has a double pole at the origin.

Fig. 5 shows the bode plots of the original and the reduced-order systems. It can be seen from Fig. 5 that the reduced-order system obtained by the proposed model reduction method is NI system with a simple pole at the origin. It approximate the original system well. However, the transformed reduced-order system obtained by the passive preserving model reduction method [39] has a relatively large approximation error. Compared with the passive preserving model reduction method [39], the proposed model reduction method guarantees the blocking zero at zero frequency for the reduced-order positive real system. Thus, it guarantees the same poles at the origin for the original and the transformed reduced-order systems.

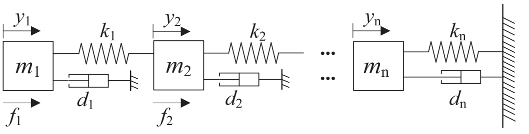

Fig. 6. Mass-spring-damper system.

4.3. Example 3: MIMO NI systems

Consider a mass-spring-damper system as shown in Fig. 6, which is borrowed from [15]. The masses, spring constants, damping constants are denoted by m_s, k_s, d_s, $s = 1, \dots, n$. y_s is the displacement of the mass m_s. $u = \begin{bmatrix} f_1 & f_2 \end{bmatrix}^\mathsf{T}$ is the applied input force and the output $y = \begin{bmatrix} y_1 & y_2 \end{bmatrix}$ is the displacement of the mass. A minimal realization corresponding to four masses is given by Eq. (1) with

$$A = \begin{bmatrix} 0 & I_4 \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \quad B^\mathsf{T} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{m_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{m_2} & 0 & 0 \end{bmatrix}^\mathsf{T},$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{33}$$

where

$$K = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2+k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3+k_4 \end{bmatrix}, \quad F = \begin{bmatrix} f_1 \\ f_2 \\ 0 \\ 0 \end{bmatrix}, \quad q = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \quad x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix},$$
$$M = diag\{m_1, m_2, m_3, m_4\}, \quad D = diag\{d_1, d_2, d_3, d_4\}.$$

Let $m_s = 1\,\text{kg}$, $k_s = 2\,\text{N/m}$, $d_s = 3.5\,\text{Ns/m}$. The goal of this example is to find a 2nd-order NI system.

Let $\mathcal{S}_1 = \{-0.19, -0.8\}$. A reduced-order system that matches the moments of the original system (31) at $\mathcal{S}_1$ is given by Eq. (18) with

$$\hat{A}_\mathrm{r} = \begin{bmatrix} -0.1794 & 0.01242 \\ -0.4344 & -1.429 \end{bmatrix}, \quad \hat{B}_\mathrm{r} = \begin{bmatrix} -0.01058 & -0.01242 \\ 0.4345 & 0.6293 \end{bmatrix}, \quad \hat{C}_\mathrm{r} = \begin{bmatrix} -55.96 & -1.269 \\ -49.49 & -0.9138 \end{bmatrix}.$$

It can be verified that the obtained reduced-order system is NI by Lemma 2. Moreover, the matrix $Y_\mathrm{r} = \begin{bmatrix} 2008.9 & 59.2 \\ 59.2 & 3 \end{bmatrix}$ is found to satisfy Eq. (4).

Fig. 7 shows the time domain simulations for the original and the reduced-order systems. It can be seen that the reduced-order system follow the original output accurately over the low frequency, while there exists a relatively large error over high frequency. Thus, it can be concluded that the proposed model reduction method is also applicable for MIMO case. However, the choice of the interpolation points s_i and the tangent directions b_i is still an active field of research and remains challenging. The future work will be focused on how to choose s_i and b_i to optimize the approximation error over the full positive frequency range.

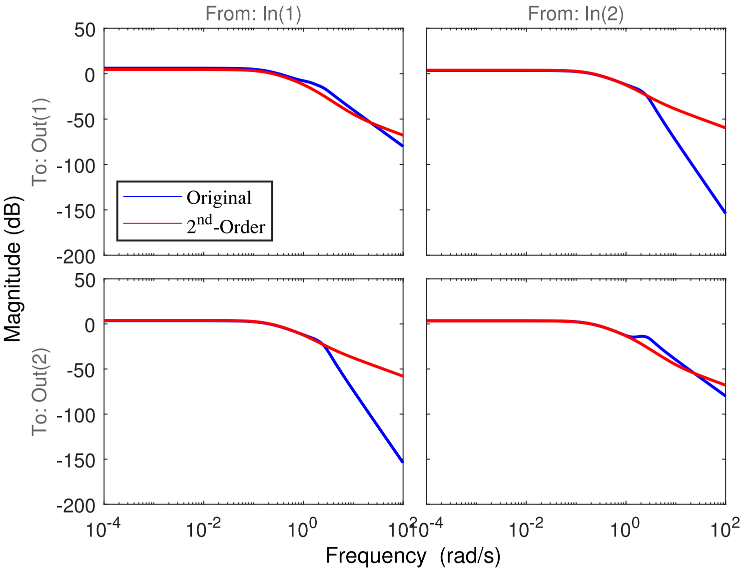

Fig. 7. Time domain simulations of the original and reduced-order systems.

5. Conclusions

The moment matching model reduction problem for negative imaginary systems with poles at the origin has been studied in this paper. Sufficient conditions have been established for the construction of the reduced-order negative imaginary system. It has shown that a desired reduced-order system can be obtained by direct computation of the projection matrix. The resulting reduced-order system preserves the negative imaginary structure and matches the prescribed moments of the original system. Moreover, the proposed model reduction method has been extended to the positive real case. Finally, several numerical examples have shown the effectiveness of the proposed model reduction method. The limitation of the proposed model reduction method is that a prior approximation error bound can not be guaranteed. How to select the interpolation points to obtain the optimal reduced-order system is worth future research.

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Appendix

In this appendix, the proof of Lemma 4 is given as follows.

According to Lemma 3, system (13) is NI if and only if there exist matrices $X_1 \in \mathbb{R}^{n_1 \times n_1}$, $X_1 = X_1^{\mathsf{T}} > 0,\ X_2 \in \mathbb{R}^{2n_2 \times 2n_2},\ X_2 = X_2^{\mathsf{T}} > 0,\ L_1,\ W$ such that

$$
\begin{aligned}
&X_1 A_1 + A_1^{\mathsf{T}} X_1 = -L_1^{\mathsf{T}} L_1, \\
&X_2 A_2 + A_2^{\mathsf{T}} X_2 = 0, \\
&X_1 B_1 - A_1^{\mathsf{T}} C_1^{\mathsf{T}} = -L_1^{\mathsf{T}} W,
\end{aligned}
$$

$$X_2 B_2 - A_2^\top C_2^\top = 0,$$
$$C_1 B_1 + C_2 B_2 + B_1^\top C_1^\top + B_2^\top C_2^\top = W^\top W,$$

which are equivalent to

$$\begin{aligned} A_1 Y_1 + Y_1 A_1^\top &= -\hat{L}_1^\top \hat{L}_1, \\ A_2 Y_2 + Y_2 A_2^\top &= 0, \\ B_1 - Y_1 A_1^\top C_1^\top &= -\hat{L}_1^\top W, \\ B_2 - Y_2 A_2^\top C_2^\top &= 0, \\ C_1 B_1 + C_2 B_2 + B_1^\top C_1^\top + B_2^\top C_2^\top &= W^\top W, \end{aligned}$$

where $Y_1 = X_1^{-1}$, $Y_2 = X_2^{-1}$, $\hat{L}_1 = L_1 X_1^{-1}$. Note that the above equalities can be rewritten as

$$\begin{aligned} A_1 Y_1 + Y_1 A_1^\top &= -\hat{L}_1^\top \hat{L}_1, \\ A_2 Y_2 + Y_2 A_2^\top &= 0, \\ B_1 - Y_1 A_1^\top C_1^\top &= -\hat{L}_1^\top W, \\ B_2 + A_2 Y_2 C_2^\top &= 0, \\ C_1 (A_1 Y_1 + Y_1 A_1^\top) &= W^\top W + C_1 \hat{L}_1^\top W + W^\top \hat{L}_1 C_1^\top. \end{aligned}$$

Thus, we have that (15) in Lemma 4 hold. Combining with the first, the third and the last equations, one obtains that

$$\begin{aligned} A_1 Y_1 + Y_1 A_1^\top &= -\hat{L}_1^\top \hat{L}_1, \\ B_1 - Y_1 A_1^\top C_1^\top &= -\hat{L}_1^\top W, \\ (W + \hat{L}_1 C_1^\top)^\top (W + \hat{L}_1 C_1^\top) &= 0. \end{aligned}$$

Hence, $W = -\hat{L}_1 C_1^\top$. Then, the second equation can be rewritten as

$$B_1 = Y_1 A_1^\top C_1^\top + \hat{L}_1^\top \hat{L}_1 C_1^\top = -A_1 Y_1 C_1^\top.$$

That is, Eq. (14) in Lemma 4 holds. Therefore, the conditions in Lemma 3 are equivalent to the conditions in Lemma 4 for system (13). This complete the proof.

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