SIAM J. CONTROL OPTIM.
Vol. 57, No. 2, pp. 1043–1067




# KRASOVSKII AND RAZUMIKHIN STABILITY THEOREMS FOR STOCHASTIC SWITCHED NONLINEAR TIME-DELAY SYSTEMS*

WEI REN† AND JUNLIN XIONG‡

**Abstract.** This paper studies stability properties of stochastic switched nonlinear time-delay systems. The stability analysis is based on two extensions of the Lyapunov-based method: the Krasovskii approach and the Razumikhin approach. In terms of the Krasovskii approach, Krasovskii-type stability conditions are derived based on Lyapunov–Krasovskii functions and average dwell-time condition. In terms of the Razumikhin approach, Razumikhin-type stability conditions are obtained via Lyapunov–Razumikhin functions, the small gain condition, and the fixed dwell-time condition. Furthermore, as a widespread phenomenon in switched systems, the asynchronous switching case is studied. Both Krasovskii-type and Razumikhin-type stability conditions are established for the asynchronous switching case. Finally, the developed results are illustrated via two examples from the mechanical rotational cutting process and networked switched control systems.

**Key words.** Lyapunov–Krasovskii function, Lyapunov–Razumikhin function, time delay, stochastic switched systems, asynchronous switching, stochastic stability

**AMS subject classifications.** 34A38, 34K20, 34K34, 34K50, 37C75, 93C10, 93D09, 93D30, 93E03, 93E15

**DOI.** 10.1137/17M1111711

## 1. Introduction.

As a special class of stochastic hybrid systems [1, 2], stochastic switched systems are dynamical systems consisting of a family of continuous-time subsystems and a switching rule that orchestrates switching between them, where either the continuous-time subsystems or the switching rule is subject to stochastic perturbations. Stochastic switched systems can be used to model many physical and manmade systems and have numerous applications in diverse fields like networked control systems [3, 4, 5], power systems [6, 7], and server systems [2, Chapter 7]. Besides stochastic perturbations, time delays are another class of practical imperfections affecting system performance. For stochastic switched systems, time delays may lead to asynchronous switching, which in turn deteriorates system stability and performance; see [4, 5, 8, 9]. In this paper, we focus on stability analysis of stochastic switched nonlinear time-delay systems with/without asynchronous switching.

To analyze stability of control systems, the Lyapunov-based method is commonly used and effective; see [10, 11, 13]. However, classic Lyapunov theory cannot be applied directly to time-delay systems as time delays cause a violation of monotonic decrease conditions [9, 14]. As a result, there are generally two ways to extend the Lyapunov-based method. The first one is the Krasovskii approach, which is based on Lyapunov–Krasovskii functions (LKFs) [15, 16, 17]. An LKF is a positive definite function with a negative definite derivative along the system solution. The second one is the Razumikhin approach, the essence of which is Lyapunov–Razumikhin functions

*Received by the editors January 17, 2017; accepted for publication (in revised form) February 4, 2019; published electronically March 19, 2019.
http://www.siam.org/journals/sicon/57-2/M111171.html
**Funding:** This work was supported by the National Natural Science Foundation of China through grants 61374026 and 61773357.
†Department of Automatic Control, KTH Royal Institute of Technology, SE-10044, Stockholm, Sweden (gtppwe@gmail.com).
‡Department of Automation, University of Science and Technology of China, Hefei 230026, China (xiong77@ustc.edu.cn).

(LRFs) [8, 18, 19, 20, 21]. An LRF is a positive definite function whose derivative is negative definite under the Razumikhin condition [18, 21]. These two approaches have been used successfully in stability analysis and controller design of time-delay systems, such as impulsive time-delay systems [19, 22], functional differential systems [23], time-delay logistics networks [24], and time-delay fuzzy systems [25].

In this paper, we would like to continue this research line and aim to establish both Krasovskii-type and Razumikhin-type conditions for stability of stochastic switched nonlinear time-delay systems with/without asynchronous switching. To our knowledge, most available results in the literature are on linear deterministic switched systems [5, 10, 26, 27] or are based on exponential Lyapunov functions [20, 28] and the average dwell-time (ADT) condition [8, 28, 29, 30]. These results cannot be easily and directly applied to stochastic switched nonlinear time-delay systems. For instance, the obtained linear matrix inequalities [5, 26, 27] for linear switched systems are not available for stochastic switched nonlinear systems. Therefore, as the first contribution of this paper, we employ Krasovskii-type and Razumikhin-type stability conditions for stochastic switched nonlinear time-delay systems with synchronous switching. The Krasovskii-type conditions are based on LKFs and the ADT condition, whereas the Razumikhin-type conditions depend on LRFs, the small gain condition, and the fixed dwell-time (FDT) condition. Consequently, the applicability ranges of both the Krasovskii and Razumikhin approaches are expanded.

Compared with the previous works [5, 28, 31, 32, 33, 34] based on LKFs, the derived Krasovskii-type conditions have the following advantages. First, the derivatives of the LKFs in this paper are allowed to depend on the delayed state trajectory, which is not the case in [5, 28, 32]. Second, stochastic switched nonlinear time-delay systems are studied in this paper, whereas linear switched time-delay systems were considered in [5, 33, 34] and external disturbances have not been addressed in [31]. Thus, the obtained Krasovskii-type conditions are more general. On the other hand, the obtained Razumikhin-type stability conditions have advantages over those in previous works. First, in contrast with [20] using the comparison principle, we apply the Razumikhin approach in this paper. Hence, we avoid constructing the comparison system. Second, instead of exponential Lyapunov functions [26, 30, 35, 36], general LRFs are implemented with the FDT condition and the small gain condition in this paper. As a result, the obtained Razumikhin-type stability conditions can be applied to study those that cannot be analyzed via exponential Lyapunov functions.

As the second contribution of this paper, we study the asynchronous switching case, which is widespread in switched systems [8, 26, 28, 30]. The asynchronous switching arises from time delays or external disturbances and causes instability of switched systems in the switching intervals. For the asynchronous switching case, both Krasovskii-type and Razumikhin-type stability conditions are established in this paper. In what follows, Krasovskii-type and Razumikhin-type stability conditions are extended from the synchronous switching case to the asynchronous switching case. Furthermore, to obtain the Krasovskii-type conditions for the asynchronous switching case, a comparison principle is proposed for impulsive switched time-delay systems.

The rest of this paper is organized as follows. In section 2, the considered problem is formulated and some necessary preliminaries are introduced. In sections 3 and 4, both Krasovskii-type and Razumikhin-type stability conditions are established for stochastic switched nonlinear time-delay systems. The synchronous and asynchronous switching cases are studied in sections 3 and 4, respectively. Finally, the developed results are applied in section 5 to mechanical rotational cutting process and networked switched control systems.

**Notation**. $\mathbb{R} := (-\infty, +\infty)$; $\mathbb{R}^+_t := [t, +\infty)$ for a given $t \in \mathbb{R}$; $\mathbb{R}^+ := (0, +\infty)$; $\mathbb{N} := \{0, 1, 2, \ldots\}$; $\mathbb{N}^+ := \{1, 2, \ldots\}$. $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space. For two vectors $x, y \in \mathbb{R}^n$, $x \prec y$ ($x \preceq y$) if $x_i < y_i$ ($x_i \leq y_i$) for all $i \in \{1, \ldots, n\}$. For a given vector or matrix $A$, $A^\top$ denotes its transpose. For a matrix $A = A^\top \in \mathbb{R}^{n\times n}$, $\mathrm{tr}[A]$, $\lambda_{\max}$, and $\lambda_{\min}$ denote the trace, the largest, and the smallest eigenvalues of $A$, respectively. $|\cdot|$ represents the Euclidean vector norm; $\mathbb{P}\{\cdot\}$ denotes the probability measure; $\mathbb{E}[\cdot]$ denotes the mathematical expectation. For a given function $f : \mathbb{R}^+_{t_0} \to \mathbb{R}^n$ and the initial time $t_0 \geq \tau > 0$, $\|f\|_\tau := \sup_{t\in[t_0-\tau,t_0]} |f(t)|$; $\|f\|_{[t_0,t)} := \sup_{t\in[t_0,t]} |f(t)|$; $\|f\|$ denotes the supremum norm on $[t_0, \infty)$. $\mathcal{C}^{1,2}$ stands for the class of the nonnegative functions on $\mathbb{R}^+_{t_0} \times \mathbb{R}^n$, which are continuously differentiable on the first augment and continuously twice differentiable on the second augment. Let $\mathbb{PC}([a, b]; \mathbb{R}^n)$ denote the class of piecewise continuous functions mapping $[a, b]$ to $\mathbb{R}^n$ and having finite right-hand continuous jumps on $[a, b]$. Given a function $f : \mathbb{R} \to \mathbb{R}$, denote $f(t^-) := \limsup_{s\to 0^-} f(t + s)$. A function $\alpha : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is of class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing; $\alpha(t)$ is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded; $\alpha(t)$ is of class $\mathcal{L}$ if it is continuous and strictly decreasing to zero as $t \to \infty$; $\alpha(t)$ is of class $\mathcal{VK}$ (or $\mathcal{VK}_\infty$) if it is of class $\mathcal{K}$ (or $\mathcal{K}_\infty$) and convex; $\alpha(t)$ is of class $\mathcal{CK}$ (or $\mathcal{CK}_\infty$) if it is of class $\mathcal{K}$ (or $\mathcal{K}_\infty$) and concave. A function $\beta(s, t) : \mathbb{R}^+_0 \times \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is of class $\mathcal{KL}$ if $\beta(s, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and of class $\mathcal{L}$ for each fixed $s \geq 0$.

## 2. Problem formulation and preliminaries.

In this section, we introduce the system model of stochastic switched nonlinear time-delay systems and some necessary preliminaries. As a mathematical tool, a comparison principle is proposed for impulsive switched time-delay systems.

Consider the following stochastic switched nonlinear time-delay system:

$$(2.1)\qquad \begin{cases} dx(t) = f_{\sigma(t)}(t, x_t, u)dt + g_{\sigma(t)}(t, x_t, u)dB(t), \quad t \in \mathbb{R}^+_{t_0}, \\ x(t) = \xi(t), \quad t \in [t_0 - \tau, t_0], \quad t_0 \geq \tau, \end{cases}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the system state, $u(t) \in \mathbb{PC}(\mathbb{R}^+_{t_0}; \mathbb{R}^{n_u})$ is the external input, and $B(t) \in \mathbb{R}^{n_w}$ is an $n_w$-dimensional $\mathfrak{F}_t$-adapted Brownian motion defined on a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P}, \{\mathfrak{F}_t\}_{t\geq t_0})$. The delayed state is denoted by $x_t := x(t-\tau(t))$, where the time-delay function $\tau(t) : \mathbb{R}^+_{t_0} \to [0, \tau]$ is continuous and upper bounded by a constant $\tau \geq 0$. The initial function $\xi : [t_0 - \tau, t_0] \to \mathbb{R}^{n_x}$ is an $\mathfrak{F}_{t_0}$-adapted random variable with finite $\mathbb{E}[\|\xi\|_\tau]$. The switching signal $\sigma : \mathbb{R}^+_{t_0} \to \mathcal{M} := \{1, \ldots, M\}$ is piecewise right-continuous. Denote by $\mathcal{T} = \{t_0, t_1, \ldots\}$ the switching time sequence. For each $i \in \mathcal{M}$, the functions $f_i : \mathbb{R}^+_{t_0} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x}$ and $g_i : \mathbb{R}^+_{t_0} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x\times n_w}$ are assumed to be Lipschitz and Borel-measurable with $f_i(t, 0, 0) \equiv 0$ and $g_i(t, 0, 0) \equiv 0$ for all $t \in \mathbb{R}^+_{t_0}$, which thus implies that $x(t) \equiv 0$ is a trivial solution to the system (2.1). Assume that the system (2.1) has a unique solution process; see [37] for more details.

Definition 2.1. *The system* (2.1) *is* stochastically input-to-state stable (SISS) *if for any $\varepsilon \in (0, 1)$, there exist $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$ such that for all $\xi \in \mathbb{R}^{n_x}, u \in \mathbb{R}^{n_u}$,*

$$(2.2)\qquad \mathbb{P}\{|x(t)| \leq \beta(\mathbb{E}[\|\xi\|_\tau], t - t_0) + \gamma(\|u\|)\} \geq 1 - \varepsilon \quad \forall t \in \mathbb{R}^+_{t_0}.$$

*If, in addition, the function $\gamma(\|u\|)$ in* (2.2) *is replaced by the form $\int_{t_0}^{t} \gamma(|u(s)|)ds$, then the system* (2.1) *is* stochastically integral input-to-state stable (SiISS).

*Remark* 2.1. Definition 2.1 on SISS and SiISS is parallel to those in [9, 19] for deterministic time-delay systems, the one in [22] for stochastic impulsive time-delay systems, and those in [32, 38, 39] for deterministic impulsive time-delay systems. In addition, Definition 2.1 can be extended further to other stability properties like ISS in mean square and weighted SISS; see [22].

DEFINITION 2.2 (see [22, 29]). *For a switching signal $\sigma$ and any $T_2 \geq T_1 \geq t_0$, let $N(T_2, T_1)$ be the switching number of $\sigma$ over the interval $(T_1, T_2]$. If there exist $N_0 \in \mathbb{N}^+$ and $\tau_a > 0$ such that*

$$-N_0 + \frac{T_2 - T_1}{\tau_a} \leq N(T_2, T_1) \leq N_0 + \frac{T_2 - T_1}{\tau_a},$$

*then $N_0$ and $\tau_a$ are called the* chatter bound *and the* average dwell-time*, or ADT, respectively.*

DEFINITION 2.3 (see [8, 37]). *Given any $\mathcal{C}^{1,2}$ functions $V_i : \mathbb{R}^+_{t_0-\tau} \times \mathbb{R}^{n_x} \to \mathbb{R}^+_0$, $i \in \mathcal{M}$,* the infinitesimal operator $\mathscr{L}$*, associated with the continuous dynamics of the system* (2.1)*, is defined as*

$$\mathscr{L}V_i(t, x_t) = \frac{\partial V_i(t,x)}{\partial t} + \frac{\partial V_i(t,x)}{\partial x} f_i(t, x_t, u) + \frac{1}{2}\mathrm{tr}\left[g_i^T(t, x_t, u)\frac{\partial^2 V_i(t,x)}{\partial x^2} g_i(t, x_t, u)\right].$$

Note that for a function $W : \mathbb{R}^+_{t_0-\tau} \to \mathbb{R}^+_0$, the upper Dini derivative of $W(t)$ is defined as $D^+W(t) = \limsup_{s\to 0^+}(W(t+s) - W(t))/s$; see [22, 38]. In addition, the upper Dini derivative is a class of the Clark generalized directional derivative and thus is an upper bound of the usual directional derivative; see [40, section 2].

From Itô's differential formula in [41, Chapter IV.3], we have

$$dV_i(t, x) = \mathscr{L}V_i(t, x_t)dt + \frac{\partial V_i(t,x)}{\partial x} g_i(t, x_t, u)dB(t) \quad \forall t \notin \mathcal{T}.$$

Taking expectation, and from the proofs of Lemma 1 and Theorem 1 in [36], one has

$$d\mathbb{E}[V_i(t,x)] = \mathbb{E}[\mathscr{L}V_i(t, x_t)]dt \quad \forall t \notin \mathcal{T}.$$

In addition, it follows from [22] that $\mathbb{E}[\mathscr{L}V_i(t, x_t)]$ is continuous in $[t_k, t_{k+1})$ and that $D^+\mathbb{E}[V_i(t, x(t))] = \mathbb{E}[\mathscr{L}V_i(t, x_t)]$ for all $t \in [t_k, t_{k+1})$.

Before presenting the main results of this paper, the following comparison principle is proposed for impulsive switched time-delay systems.

PROPOSITION 2.4. *Assume that $\mathfrak{X}_i(t), \mathfrak{U}_i(t) \in \mathbb{PC}(\mathbb{R}^+_{t_0-\tau}; \mathbb{R}^{n_x})$ for all $i \in \mathcal{M}$ and that $\chi_3, \phi_2 \in \mathbb{PC}(\mathbb{R}^+_{t_0}; \mathbb{R}^{n_x})$ is continuous in $[t_k, t_{k+1})$, where $k \in \mathbb{N}$. Suppose there exist continuously nondecreasing functions $\chi_1 : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}, \chi_2 : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ and a continuously increasing function $\phi_1 : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ such that for all $k \in \mathbb{N}$,*

$$\begin{cases} D^+\mathfrak{X}_{\sigma(t)}(t) \preceq \chi_1(\mathfrak{X}_{\sigma(t)}(t)) + \chi_2(\mathfrak{X}_{\sigma(t-\tau(t))}(t-\tau(t))) + \chi_3(t), & t \in (t_k, t_{k+1}), \\ \mathfrak{X}_{\sigma(t_k)}(t_k) \preceq \phi_1(\mathfrak{X}_{\sigma(t_k^-)}(t_k^-)) + \phi_2(t_k^-), \end{cases} \tag{2.3}$$

$$\begin{cases} D^+\mathfrak{U}_{\sigma(t)}(t) \succ \chi_1(\mathfrak{U}_{\sigma(t)}(t)) + \chi_2(\mathfrak{U}_{\sigma(t-\tau(t))}(t-\tau(t))) + \chi_3(t), & t \in (t_k, t_{k+1}), \\ \mathfrak{U}_{\sigma(t_k)}(t_k) \succeq \phi_1(\mathfrak{U}_{\sigma(t_k^-)}(t_k^-)) + \phi_2(t_k^-). \end{cases} \tag{2.4}$$

*If $\mathfrak{X}_i(t) \preceq \mathfrak{U}_i(t)$ for all $t \in [t_0 - \tau, t_0]$ and all $i \in \mathcal{M}$, then $\mathfrak{X}_{\sigma(t)}(t) \preceq \mathfrak{U}_{\sigma(t)}(t)$ for all $t \in [t_0, \infty)$ and all $i \in \mathcal{M}$.*

*Proof.* Since $\mathfrak{X}_i(t) \preceq \mathfrak{U}_i(t)$ for all $t \in [t_0-\tau, t_0]$, we first prove that $\mathfrak{X}_{\sigma(t)}(t) \preceq \mathfrak{U}_{\sigma(t)}(t)$ for $t \in [t_0, t_1)$ via *reductio ad absurdum.* Then using the mathematical induction approach, we prove that $\mathfrak{X}_{\sigma(t)}(t) \preceq \mathfrak{U}_{\sigma(t)}(t)$ for all $t \in [t_0, \infty)$.

Suppose that $\mathfrak{X}_{\sigma(t)}(t) \preceq \mathfrak{U}_{\sigma(t)}(t)$ holds for all $t \in (t_0, t_1)$. If not, then there exists a $t \in (t_0, t_1)$ such that $\mathfrak{X}_{\sigma(t)}(t) \preceq \mathfrak{U}_{\sigma(t)}(t)$ does not hold. At such a time instant $t$, there exists at least an $l \in \{1, \dots, n_x\}$ such that the $l$th component of $\mathfrak{X}_{\sigma(t)}(t)$, i.e., $\mathfrak{X}^l_{\sigma(t)}(t)$, satisfies $\mathfrak{X}^l_{\sigma(t)}(t) > \mathfrak{U}^l_{\sigma(t)}(t)$. Define $\bar{t} := \inf\{t \in (t_0, t_1) | \exists l \in \{1, \dots, n_x\} \text{ s.t. } \mathfrak{X}^l_{\sigma(t)}(t) > \mathfrak{U}^l_{\sigma(t)}(t)\}$ and $j := \min\{l \in \{1, \dots, n_x\} | \mathfrak{X}^l_{\sigma(\bar{t})}(\bar{t}) \geq \mathfrak{U}^l_{\sigma(\bar{t})}(\bar{t})\}$. Hence, we have that

$$\mathfrak{X}_{\sigma(t_0)}(t) \prec \mathfrak{U}_{\sigma(t_0)}(t), \quad t \in (t_0, \bar{t}), \tag{2.5}$$

$$\mathfrak{X}^j_{\sigma(t_0)}(\bar{t}) = \mathfrak{U}^j_{\sigma(t_0)}(\bar{t}), \tag{2.6}$$

$$\mathfrak{X}^j_{\sigma(t_0)}(t) > \mathfrak{U}^j_{\sigma(t_0)}(t), \quad t \in (\bar{t}, \bar{t}+\Delta t), \tag{2.7}$$

where $\Delta t > 0$ is arbitrarily small. Moreover, we obtain from (2.6)–(2.7) that

$$\frac{\mathfrak{X}^j_{\sigma(t_0)}(t) - \mathfrak{X}^j_{\sigma(t_0)}(\bar{t})}{t-\bar{t}} > \frac{\mathfrak{U}^j_{\sigma(t_0)}(t) - \mathfrak{U}^j_{\sigma(t_0)}(\bar{t})}{t-\bar{t}}, \quad t \in (\bar{t}, \bar{t}+\Delta t).$$

In what follows, we have that $D^+\mathfrak{X}^j_{\sigma(t_0)}(\bar{t}) \geq D^+\mathfrak{U}^j_{\sigma(t_0)}(\bar{t})$.

On the other hand, from the first inequalities in (2.3)–(2.4), we have that at the time instant $t = \bar{t}$, $\sigma(\bar{t}) = \sigma(t_0)$ and

$$D^+\mathfrak{X}_{\sigma(\bar{t})}(\bar{t}) \preceq \chi_1(\mathfrak{X}_{\sigma(\bar{t})}(\bar{t})) + \chi_2(\mathfrak{X}_{\sigma(\bar{t}-\tau(\bar{t}))}(\bar{t}-\tau(\bar{t}))) + \chi_3(\bar{t}), \tag{2.8}$$

$$D^+\mathfrak{U}_{\sigma(\bar{t})}(\bar{t}) \succ \chi_1(\mathfrak{U}_{\sigma(\bar{t})}(\bar{t})) + \chi_2(\mathfrak{U}_{\sigma(\bar{t}-\tau(\bar{t}))}(\bar{t}-\tau(\bar{t}))) + \chi_3(\bar{t}). \tag{2.9}$$

If $\bar{t}-\tau(\bar{t}) \in [t_0-\tau, t_0]$, then it follows from the assumption that $\mathfrak{X}_i(\bar{t}-\tau(\bar{t})) \preceq \mathfrak{U}_i(\bar{t}-\tau(\bar{t}))$ for all $i \in \mathcal{M}$. Otherwise, $\bar{t}-\tau(\bar{t}) \in (t_0, \bar{t}]$, which implies that $\sigma(\bar{t}-\tau(\bar{t})) = \sigma(t_0)$. In this case, $\mathfrak{X}_{\sigma(\bar{t}-\tau(\bar{t}))}(\bar{t}-\tau(\bar{t})) \preceq \mathfrak{U}_{\sigma(\bar{t}-\tau(\bar{t}))}(\bar{t}-\tau(\bar{t}))$ holds the definition of $\bar{t}$ and (2.5). Since $\chi_1, \chi_2$ are nondecreasing and $\chi_3 \in \mathbb{PC}([t_0, \infty); \mathbb{R}^{n_x})$, we obtain from (2.8)–(2.9) that $D^+\mathfrak{X}_{\sigma(t_0)}(\bar{t}) \prec D^+\mathfrak{U}_{\sigma(t_0)}(\bar{t})$, which implies that $D^+\mathfrak{X}^j_{\sigma(t_0)}(\bar{t}) < D^+\mathfrak{U}^j_{\sigma(t_0)}(\bar{t})$. This is a contradiction. As a result, $\mathfrak{X}_{\sigma(t)}(t) \preceq \mathfrak{U}_{\sigma(t)}(t)$ for all $t \in (t_0, t_1)$.

Furthermore, suppose that $\mathfrak{X}_{\sigma(t)}(t) \preceq \mathfrak{U}_{\sigma(t)}(t)$ holds for all $t \in [t_0, t_k)$. As a result, it follows that $\mathfrak{X}_{\sigma(t)}(t) \preceq \mathfrak{U}_{\sigma(t)}(t)$ holds for $t \in [t_k-\tau, t_k)$. In addition, due to the second inequalities in (2.3)–(2.4) and because $\phi_1$ is continuously increasing and $\phi_2 \in \mathbb{PC}([t_0, \infty); \mathbb{R}^{n_x})$, one has that

$$\begin{aligned}\mathfrak{X}_{\sigma(t_k)}(t_k) &\preceq \phi_1(\mathfrak{X}_{\sigma(t_k^-)}(t_k^-)) + \phi_2(t_k^-) \\ &\preceq \phi_1(\mathfrak{U}_{\sigma(t_k^-)}(t_k^-)) + \phi_2(t_k^-) \preceq \mathfrak{U}_{\sigma(t_k)}(t_k).\end{aligned}$$

That is, $\mathfrak{X}_{\sigma(t_k)}(t_k) \preceq \mathfrak{U}_{\sigma(t_k)}(t_k)$.

If $\mathfrak{X}_{\sigma(t)}(t) \preceq \mathfrak{U}_{\sigma(t)}(t)$ is not valid for all $t \in (t_k, t_{k+1})$, then there exist $\bar{t} \in (t_k, t_{k+1})$ and $j \in \{1, \dots, n_x\}$ such that $\mathfrak{X}^j_{\sigma(t_k)}(\bar{t}) > \mathfrak{U}^j_{\sigma(t_k)}(\bar{t})$. Along the same lines as the proof for the case of $(t_0, t_1)$, we have that $\mathfrak{X}_{\sigma(t)}(t) \preceq \mathfrak{U}_{\sigma(t)}(t)$ for all $t \in [t_k, t_{k+1})$. According to the mathematical induction, we get that $\mathfrak{X}_{\sigma(t)}(t) \preceq \mathfrak{U}_{\sigma(t)}(t)$ for all $t \in [t_0, \infty)$. ◻

*Remark* 2.2. Proposition 2.4 provides a comparison principle for vector-valued functions. If $\mathfrak{X}_{\sigma(t)}(t)$ and $\mathfrak{U}_{\sigma(t)}(t)$ are scalar, then the result is still valid. Different versions of the comparison principle in the previous works are included as the special cases of Proposition 2.4, such as Lemma C.1 in [42] for hybrid systems, Lemma 1 in [22] for impulsive delayed systems, and Lemma 2.7 in [35] for switched systems.

**3. Stochastic switched time-delay systems with synchronous switching.** In this section, SISS and SiISS are studied for the system (2.1) in the synchronous switching case. Both Krasovskii-type and Razumikhin-type stability conditions are established.

**3.1. Krasovskii approach based stability analysis.** In the following, Krasovskii-type conditions are established for both SISS and SiISS of the system (2.1).

THEOREM 3.1. *Consider the system* (2.1). *Assume that there exist LKFs* $V_i : \mathbb{R}^+_{t_0-\tau} \times \mathbb{R}^{n_x} \to \mathbb{R}^+_0$, $i \in \mathcal{M}$, $\alpha_1, \varphi_1, \varphi_2 \in \mathcal{K}_\infty, \alpha_2 \in \mathcal{CK}_\infty$ *and constants* $\lambda_1 > \lambda_2 \geq 0, \mu > 1$ *such that*

(A.1) *for all* $t \in \mathbb{R}^+_{t_0-\tau}$, $\alpha_1(|x(t)|) \leq V_i(t, x(t)) \leq \alpha_2(|x(t)|)$;

(A.2) *for all* $t \in \mathbb{R}^+_{t_0} \backslash \mathcal{T}$, $k \in \mathbb{N}^+$, $\mathscr{L} V_{\sigma(t)}(t, x_t) \leq -\lambda_1 V_{\sigma(t)}(t, x(t)) + \lambda_2 V_{\sigma(t-\tau(t))} (t - \tau(t), x_t) + \varphi_1(|u(t)|)$;

(A.3) *for all* $t \in \mathcal{T}$, $V_{\sigma(t)}(t, x(t)) \leq \mu V_{\sigma(t^-)}(t^-, x(t^-)) + \varphi_2(|u(t^-)|)$;

(A.4) $\tau_a > \frac{\ln \mu}{\lambda_0}$, *where* $\lambda_0 \in (0, \bar{\lambda})$ *and* $\bar{\lambda}$ *is the unique solution to* $\lambda - \lambda_1 + \lambda_2 e^{\lambda\tau} = 0$,

*then the system* (2.1) *is both SISS and SiISS.*

*Proof.* We prove the theorem via three steps. First, the existence of $\bar{\lambda}$ in (A.4) is established. Second, we prove the boundedness of the LKFs via *reductio ad absurdum*. Finally, based on the bounds of the LKFs and (A.4), we prove the convergence of the system state, which in turn guarantees SISS and SiISS of the system (2.1).

*Step* 1. Define $\Gamma(\lambda) := \lambda - \lambda_1 + \lambda_2 e^{\lambda\tau}$. Observe that $\Gamma(0) = -\lambda_1 + \lambda_2 < 0$ and that $\Gamma(\lambda) \to \infty$ as $\lambda \to \infty$. In addition, $\Gamma'(\lambda) := 1 + \lambda_2 \tau e^{\lambda\tau} \geq 1$. Thus, there exists a unique $\bar{\lambda} > 0$ such that $\Gamma(\lambda) = 0$ and $\Gamma(\lambda_0) < 0$ for all $\lambda_0 \in (0, \bar{\lambda})$.

*Step* 2. Since $\alpha_1 \in \mathcal{K}_\infty$ and $\alpha_2 \in \mathcal{CK}_\infty$, we obtain from (A.1) and Jensen's inequality in [41, Chapter II, 18.3] that for all $t \in \mathbb{R}^+_{t_0-\tau}$ and $i \in \mathcal{M}$,

$$\mathbb{E}[V_i(t, x(t))] \leq \alpha_2(\mathbb{E}[|x(t)|]). \tag{3.1}$$

Because $u(t)$ is defined on $[t_0, \infty)$, we have from (A.3) and (3.1) that $\mathbb{E}[V_i(t, x(t))] \leq \mu\alpha_2(\mathbb{E}[\|\xi\|_\tau])$ for all $t \in [t_0 - \tau, t_0]$ and $i \in \mathcal{M}$.

Define $W_{\sigma(t)}(t) := e^{\lambda_0(t-t_0)} V_{\sigma(t)}(t, x(t))$ for $t \in \mathbb{R}^+_{t_0-\tau}$, where $\lambda_0$ is from (A.4). Obviously, $\mathbb{E}[W_i(t)] < \mathbb{E}[V_i(t, x(t))] \leq \mu\alpha_2(\mathbb{E}[\|\xi\|_\tau])$ holds for all $t \in [t_0 - \tau, t_0]$. In the following, we prove that for all $t \in [t_0, \infty)$,

$$\mathbb{E}[W_{\sigma(t)}(t)] \leq \mu^{N(t,t_0)} H_1(t, N(t, t_0)) + H_2(t, N(t, t_0)), \tag{3.2}$$

where $H_1(t, N(t, t_0)) := M_1 + M_2(t, N(t, t_0))$, $M_1 := \mu\alpha_2(\mathbb{E}[\|\xi\|_\tau])$ and

$$\begin{aligned} M_2(t, N(t, t_0)) := & \sum_{i=0}^{N(t,t_0)-1} \mu^{-i} \int_{t_i}^{t_{i+1}} e^{\lambda_0(s-t_0)} \varphi_1(|u(s)|) ds \\ & + \mu^{-N(t,t_0)} \int_{t_{N(t,t_0)}}^{t} e^{\lambda_0(s-t_0)} \varphi_1(|u(s)|) ds, \\ H_2(t, N(t, t_0)) := & \sum_{i=1}^{N(t,t_0)} \mu^{N(t,t_i)} e^{\lambda_0(t_i-t_0)} \varphi_2(|u(t_i^-)|). \end{aligned}$$

Observe that $H_1(t, N(t, t_0)) \equiv M_1$ for $t \in [t_0 - \tau, t_0]$. For $t \in [t_0, t_1)$, $H_1(t, 0) = M_1 + \int_{t_0}^t e^{\lambda_0(s-t_0)} \varphi_1(|u(s)|) ds, H_2(t, 0) \equiv 0, \sigma(t) = \sigma(t_0)$, and $N(t, t_0) = 0$. Suppose that $\mathbb{E}[W_{\sigma(t)}(t)] \le H_1(t,0)$ holds for $t \in [t_0, t_1)$. If not, define $t^* := \inf\{t \in (t_0, t_1) | \mathbb{E}[W_{\sigma(t)}(t)] > H_1(t, 0)\}$. Therefore, from the continuity of $W_{\sigma(t)}(t)$ and $H_1(t,0)$ in $(t_0, t_1)$, we have that

$$\mathbb{E}[W_{\sigma(t^*)}(t^*)] = H_1(t^*, 0); \tag{3.3}$$

$$\mathbb{E}[W_{\sigma(t)}(t)] > H_1(t, 0), \quad t \in (t^*, t^* + \Delta t), \tag{3.4}$$

where $\Delta t > 0$ is arbitrarily small. We get from (3.2)–(3.4) that

$$\begin{aligned} D^+\mathbb{E}[W_{\sigma(t^*)}(t^*)] &\ge \limsup_{\Delta t \to 0^+} \frac{1}{\Delta t} \int_{t^*}^{t^*+\Delta t} e^{\lambda_0 (s-t_0)} \varphi_1(|u(s)|) ds \\ &= e^{\lambda_0 (t^*-t_0)} \varphi_1(|u(t^*)|). \end{aligned} \tag{3.5}$$

On the other hand, if $t^* - \tau(t^*) \in [t_0 - \tau, t_0]$, it follows from (3.1) and the definition of $W_{\sigma(t)}(t)$ that $\mathbb{E}[W_i(t^* - \tau(t^*))] \le \mathbb{E}[V_i(t^* - \tau(t^*), x(t^* - \tau(t^*)))] \le M_1$ for all $i \in \mathcal{M}$. Otherwise, $\mathbb{E}[W_{\sigma(t^* - \tau(t^*))}(t^* - \tau(t^*))] \le H_1(t^* - \tau(t^*), 0)$ holds from the definition of $t^*$. Since $H_1(t, N(t, t_0))$ is constant in $[t_0 - \tau, t_0]$ and increases in $[t_0, t^*]$, we have that

$$\begin{aligned} \mathbb{E}[W_{\sigma(t^*-\tau(t^*))}(t^* - \tau(t^*))] &\le H_1(t^* - \tau(t^*), N(t^* - \tau(t^*), t_0)) \\ &\le H_1(t^*, 0) = \mathbb{E}[W_{\sigma(t^*)}(t^*)], \end{aligned}$$

which implies that

$$\begin{aligned} \mathbb{E}[V_{\sigma(t^*-\tau(t^*))}(t^* - \tau(t^*), x_{t^*})] &= e^{-\lambda_0 (t^* - \tau(t^*) - t_0)} \mathbb{E}[W_{\sigma(t^*-\tau(t^*))}(t^* - \tau(t^*))] \\ &\le e^{-\lambda_0 (t^* - \tau(t^*) - t_0)} \mathbb{E}[W_{\sigma(t^*)}(t^*)] \\ &\le e^{\lambda_0 \tau} \mathbb{E}[V_{\sigma(t^*)}(t^*, x(t^*))]. \end{aligned} \tag{3.6}$$

In what follows, from (A.2), (A.4), and (3.6), we yield that

$$\begin{aligned} D^+\mathbb{E}[W_{\sigma(t^*)}(t^*)] &\le e^{\lambda_0 (t^*-t_0)} (\lambda_0 \mathbb{E}[V_{\sigma(t^*)}(t^*, x(t^*))] - \lambda_1 \mathbb{E}[V_{\sigma(t^*)}(t^*, x(t^*))] \\ &\quad + \lambda_2 \mathbb{E}[V_{\sigma(t^*-\tau(t^*))}(t^* - \tau(t^*), x_{t^*})] + \varphi_1(|u(t^*)|)) \\ &\le e^{\lambda_0 (t^*-t_0)} (\lambda_0 \mathbb{E}[V_{\sigma(t^*)}(t^*, x(t^*))] - \lambda_1 \mathbb{E}[V_{\sigma(t^*)}(t^*, x(t^*))] \\ &\quad + \lambda_2 e^{\lambda_0 \tau} \mathbb{E}[V_{\sigma(t^*)}(t^*, x(t^*))] + \varphi_1(|u(t^*)|)) \\ &< e^{\lambda_0 (t^*-t_0)} \varphi_1(|u(t^*)|), \end{aligned} \tag{3.7}$$

which contradicts (3.5). Therefore, $\mathbb{E}[W_{\sigma(t)}(t)] \le H_1(t, 0)$ for all $t \in [t_0, t_1)$.

In the following, suppose that (3.2) holds for all $t \in [t_0, t_k)$, $k \in \mathbb{N}$. Thus, at the switching time instance $t_k$, we get from (A.3) and (3.2) that

$$\begin{aligned} \mathbb{E}[W_{\sigma(t_k)}(t_k)] &\le \mu \mathbb{E}[W_{\sigma(t_k^-)}(t_k^-)] + e^{\lambda_0 (t_k - t_0)} \varphi_2(|u(t_k^-)|) \\ &\le \mu^{N(t_k^-, t_0)+1} H_1(t_k^-, N(t_k^-, t_0)) + \mu H_2(t_k^-, N(t_k^-, t_0)) \\ &\quad + e^{\lambda_0 (t_k - t_0)} \varphi_2(|u(t_k^-)|). \end{aligned} \tag{3.8}$$

Since $N(t_k^-, t_0) + 1 = N(t_k, t_0)$ and $N(t, t_0) - i = N(t, t_i)$ for all $t \in \mathbb{R}^+_{t_0}$, one has that

$$\begin{aligned}
&\mu^{N(t_k^-,t_0)+1} H_1(t_k^-, N(t_k^-, t_0)) \\
&= \mu^{N(t_k,t_0)} M_1 + \mu^{N(t_k,t_0)} \sum_{i=0}^{N(t_k^-,t_0)-1} \mu^{-i} \int_{t_i}^{t_{i+1}} e^{\lambda_0(s-t_0)} \varphi_1(|u(s)|) ds \\
&\quad + \mu^{N(t_k,t_0)} \mu^{-N(t_k^-,t_0)} \int_{t_{N(t_k^-,t_0)}}^{t_k^-} e^{\lambda_0(s-t_0)} \varphi_1(|u(s)|) ds, \\
&\le \mu^{N(t_k,t_0)} M_1 + \mu^{N(t_k,t_0)} \sum_{i=0}^{N(t_k,t_0)-1} \mu^{-i} \int_{t_i}^{t_{i+1}} e^{\lambda_0(s-t_0)} \varphi_1(|u(s)|) ds \\
&= \mu^{N(t_k,t_0)} H_1(t_k, N(t_k, t_0))
\end{aligned} \tag{3.9}$$

and that

$$\begin{aligned}
&\mu H_2(t_k^-, N(t_k^-, t_0)) + e^{\lambda_0(t_k-t_0)} \varphi_2(|u(t_k)|) \\
&= \sum_{i=1}^{N(t_k^-,t_0)} \mu^{N(t_k^-,t_i)+1} e^{\lambda_0(t_i-t_0)} \varphi_2(|u(t_i^-)|) + e^{\lambda_0(t_k-t_0)} \varphi_2(|u(t_k^-)|) \\
&\le \sum_{i=1}^{N(t_k^-,t_0)} \mu^{N(t_k,t_i)} e^{\lambda_0(t_i-t_0)} \varphi_2(|u(t_i^-)|) + \mu e^{\lambda_0(t_k-t_0)} \varphi_2(|u(t_k^-)|) \\
&= H_2(t_k, N(t_k, t_0)).
\end{aligned} \tag{3.10}$$

Combining (3.8)–(3.10) gives that

$$\mathbb{E}[W_{\sigma(t_k)}(t_k)] \le \mu^{N(t_k,t_0)} H_1(t_k, N(t_k, t_0)) + H_2(t_k, N(t_k, t_0)).$$

That is, (3.2) is valid for $t = t_k$.

Suppose that (3.2) holds for all $t \in (t_k, t_{k+1})$. If not, define $t^\star := \inf\{t \in (t_k, t_{k+1}) | \mathbb{E}[W_{\sigma(t)}(t)] > \mu^{N(t,t_0)} H_1(t, N(t, t_0)) + H_2(t, N(t, t_0))\}$. Thus, it follows from the continuity of $W_{\sigma(t)}(t)$ in $(t_k, t_{k+1})$ that

$$\mathbb{E}[W_{\sigma(t^\star)}(t^\star)] = \mu^{N(t^\star,t_0)} H_1(t^\star, N(t^\star, t_0)) + H_2(t^\star, N(t^\star, t_0)), \tag{3.11}$$

$$\mathbb{E}[W_{\sigma(t)}(t)] > \mu^{N(t,t_0)} H_1(t, N(t, t_0)) + H_2(t, N(t, t_0)), \quad t \in (t^\star, t^\star + \Delta t), \tag{3.12}$$

where $\Delta t > 0$ is arbitrarily small. Similar to (3.5), we get that

$$D^+ \mathbb{E}[W_{\sigma(t^\star)}(t^\star)] \ge e^{\lambda_0(t^\star - t_0)} \varphi_1(|u(t^\star)|). \tag{3.13}$$

Assume there exists certain $\jmath \in \{1, \dots, k\}$ such that $t^\star - \tau(t^\star) \in [t_\jmath, t_{\jmath+1})$. Therefore, we obtain from (3.2) and (3.11) that

$$\begin{aligned}
&\mathbb{E}[W_{\sigma(t^\star-\tau(t^\star))}(t^\star - \tau(t^\star))] \\
&\le \mu^{N(t_\jmath,t_0)} H_1(t^\star - \tau(t^\star), N(t_\jmath, t_0)) + H_2(t^\star - \tau(t^\star), N(t_\jmath, t_0)) \\
&\le \mu^{N(t^\star,t_0)} H_1(t^\star, N(t^\star, t_0)) + H_2(t^\star, N(t^\star, t_0)) \\
&= \mathbb{E}[W_{\sigma(t^\star)}(t^\star)],
\end{aligned} \tag{3.14}$$

where the second "$\le$" holds due to the fact that $\mu^{N(t,t_0)}H_1(t,N(t,t_0))$ and $H_2(t,N(t,t_0))$ are nondecreasing with respect to $t$. Therefore, from (A.2), (A.4), (3.14) and along a similar fashion as (3.6)–(3.7), we have that

$$\begin{aligned}
D^+\mathbb{E}[W_{\sigma(t^\star)}(t^\star)] &\le e^{\lambda_0(t^\star-t_0)}(\lambda_0\mathbb{E}[V_{\sigma(t^\star)}(t^\star,x(t^\star))]-\lambda_1\mathbb{E}[V_{\sigma(t^\star)}(t^\star,x(t^\star))]\\
&\quad+\lambda_2\mathbb{E}[V_{\sigma(t^\star-\tau(t^\star))}(t^\star-\tau(t^\star),x_{t^\star})]+\varphi_1(|u(t^\star)|))\\
&\le e^{\lambda_0(t^\star-t_0)}(\lambda_0\mathbb{E}[V_{\sigma(t^\star)}(t^\star,x(t^\star))]-\lambda_1\mathbb{E}[V_{\sigma(t^\star)}(t^\star,x(t^\star))]\\
&\quad+\lambda_2e^{\lambda_0\tau}\mathbb{E}[V_{\sigma(t^\star)}(t^\star,x(t^\star))]+\varphi_1(|u(t^\star)|))\\
&< e^{\lambda_0(t^\star-t_0)}\varphi_1(|u(t^\star)|),
\end{aligned}$$

which contradicts (3.13). In what follows, (3.2) holds for all $t\in(t_k,t_{k+1})$.

According to the mathematical induction, we obtain from the preceding analysis that (3.2) holds for all $t\in[t_0,\infty)$. Furthermore, it follows from (3.2) that

$$\begin{aligned}
\mathbb{E}[V_{\sigma(t)}(t,x(t))] &\le e^{-\lambda_0(t-t_0)}\mu^{N(t,t_0)}H_1(t,N(t,t_0))\\
&\quad+e^{-\lambda_0(t-t_0)}H_2(t,N(t,t_0)) \quad \forall t\in[t_0,\infty).
\end{aligned} \tag{3.15}$$

*Step* 3. Based on Definition 2.2 and (A.4), there exists an $\omega>0$ such that

$$N(t,s)\le N_0+\frac{t-s}{\tau_a}\le N_0+\frac{(\lambda_0-\omega)(t-s)}{\ln\mu},$$

which in turn implies that

$$\mu^{N(t,t_0)}e^{-\lambda_0(t-t_0)}\le\mu^{N_0+\frac{(\lambda_0-\omega)(t-t_0)}{\ln\mu}}e^{-\lambda_0(t-t_0)}=\mu^{N_0}e^{-\omega(t-t_0)}. \tag{3.16}$$

According to (3.16) and the fact that $\mu^{N(t,t_0)-i}=\mu^{N(t,t_{i+1})+1}$, we obtain that

$$\begin{aligned}
&\mu^{N(t,t_0)}e^{-\lambda_0(t-t_0)}M_2(t,N(t,t_0))\\
&\le\sum_{i=0}^{N(t,t_0)-1}e^{-\lambda_0(t-t_0)}\mu^{N(t,t_{i+1})+1}e^{(\lambda_0-\omega)(t_{i+1}-t_0)}\int_{t_i}^{t_{i+1}}e^{\omega(s-t_0)}\varphi_1(|u(s)|)ds\\
&\quad+e^{-\lambda_0(t-t_0)}e^{(\lambda_0-\omega)(t-t_0)}\int_{t_{N(t,t_0)}}^{t}e^{\omega(s-t_0)}\varphi_1(|u(s)|)ds\\
&\le\mu^{1+N_0}\sum_{i=0}^{N(t,t_0)-1}e^{-\omega(t-t_0)}\int_{t_i}^{t_{i+1}}e^{\omega(s-t_0)}\varphi_1(|u(s)|)ds\\
&\quad+e^{-\omega(t-t_0)}\int_{t_{N(t,t_0)}}^{t}e^{\omega(s-t_0)}\varphi_1(|u(s)|)ds\\
&\le\mu^{1+N_0}e^{-\omega(t-t_0)}\int_{t_0}^{t}e^{\omega(s-t_0)}\varphi_1(|u(s)|)ds.
\end{aligned} \tag{3.17}$$

Similarly,

$$\begin{aligned}
e^{-\lambda_0(t-t_0)}H_2(t,N(t,t_0)) &= \sum_{i=1}^{N(t,t_0)}\mu^{N(t,t_i)}e^{-\lambda_0(t-t_i)}\varphi_2(|u(t_i)|)\\
&\le\mu^{N_0}\sum_{i=1}^{N(t,t_0)}e^{-\omega(t-t_i)}\varphi_2(|u(t_i)|).
\end{aligned} \tag{3.18}$$

Therefore, it follows from (3.15)–(3.18) that

$$\mathbb{E}[V_{\sigma(t)}(t,x(t))] \leq \mu^{N_0} e^{-\omega(t-t_0)} M_1 + \mu^{1+N_0} e^{-\omega(t-t_0)} \int_{t_0}^{t} e^{\omega(s-t_0)} \varphi_1(|u(s)|)ds$$

$$+ \mu^{N_0} \sum_{i=1}^{N(t,t_0)} e^{-\omega(t-t_i)} \varphi_2(|u(t_i)|) \tag{3.19}$$

$$\leq \mu^{N_0} e^{-\omega(t-t_0)} M_1 + \Lambda\varphi(\|u\|), \tag{3.20}$$

where $\Lambda := (\omega^{-1}\mu + (1-e^{-\omega\theta_1})^{-1})\mu^{N_0}$, $\varphi(v) := \max\{\varphi_1(v), \varphi_2(v)\}$, and $0 < \theta_1 \leq \inf\{t_{k+1} - t_k | k \in \mathbb{N}\}$. Using Markov's inequality in [41, Chapter II, 18.1] to (3.20) yields that for all $t \in \mathbb{R}^+_{t_0}$,

$$\mathbb{P}\{V_{\sigma(t)}(t,x(t)) \leq \bar{\beta}(\mathbb{E}[\|\xi\|_\tau], t-t_0) + \bar{\varphi}(\|u\|)\}$$

$$\geq 1 - \frac{\mathbb{E}[V_{\sigma(t)}(t,x(t))]}{\bar{\beta}(\mathbb{E}[\|\xi\|_\tau], t-t_0) + \bar{\varphi}(\|u\|)} \geq 1 - \varepsilon_1, \tag{3.21}$$

where $\bar{\beta}(v,t) := \mu^{N_0+1} e^{-\omega t} \alpha_2(v)/\varepsilon_1$ and $\bar{\varphi}(v) := \Lambda\varphi(v)/\varepsilon_1$. As a result, it follows from (A.1) and (3.21) that the system (2.1) is SISS with $\beta(v,t) := \alpha_1^{-1}(2\bar{\beta}(v,t))$ and $\gamma(v) := \alpha_1^{-1}(2\bar{\varphi}(v))$.

On the other hand, (3.19) can be rewritten as

$$\mathbb{E}[V_{\sigma(t)}(t,x(t))] \leq \mu^{N_0} e^{-\omega(t-t_0)} M_1 + (1+\mu)\mu^{N_0} \int_{t_0}^{t} (\varphi_1(|u(s)|) + \theta_1^{-1}\varphi_2(|u(s)|))ds.$$

Just like the SISS analysis, we also conclude that the system (2.1) is SiISS. ∎

*Remark* 3.1. Let us examine the statement of Theorem 3.1 in some detail.

(i) The condition (A.1) is a fairly standard assumption and ensures that each $V_i(t,x(t))$ is positive definite and radially unbounded; see also [11, section 4].

(ii) The condition (A.2) is used to estimate the derivatives of the LKFs along the vector field of each subsystem. Both $\lambda_1$ and $\lambda_2$ offer the quantities of such estimate and the assumption that $\lambda_1 > \lambda_2 \geq 0$ ensures that each subsystem is stable in the switching intervals. Condition (A.2) allows the derivatives of the LKFs to be related to the current state, the delayed state trajectory, and the external disturbance. Condition (A.3) restricts the jumps of the LKFs by the switches and implies that the jumps of the LKFs depend on both the current state and the external disturbance. In previous works [28, 30, 33, 34], the derivatives of the Lyapunov functions (both the LKFs and the LRFs) do not depend on the delayed state trajectory, and the jumps of the Lyapunov functions only depend on the current state. As a result, (A.2)–(A.3) in Theorem 3.1 are more general and include those in [28, 34] as the special cases; see also Remark 3.2. In addition, similar conditions can be found in [22], which studied SISS of impulsive time-delay systems. If the considered system is an impulsive time-delay system, then (A.3) is thought of as the condition for the impulsive times, and Theorem 3.1 is reduced to Theorem 1 in [22].

(iii) The condition (A.4) is the ADT condition, which balances the coefficients in (A.2)–(A.3) and constrains the frequency of the switching on average to establish SISS and SiISS of the system (2.1). If $\mu \equiv 1$ and $\varphi_2(v) \equiv 0$ in (A.3), then the switching is neutral for system stability [28, 30], thereby having no constraints on the ADT. In previous works like [28], the small gain condition was used to study switched nonlinear time-delay systems. The relationship between the small gain condition and the ADT condition has been studied in [43, section 4.2].

*Remark* 3.2. Based on Krasovskii-type stability conditions in Theorem 3.1, some special cases and extensions can be derived. If $\lambda_2 \equiv 0$, then (A.2) is reduced to the classic decreasing condition for the LKFs; see [2, 34, 36]. In this case, the obtained result is valid with $\lambda_0 \in (0, \lambda_1)$ in (A.4). Besides SISS and SiISS, the stability properties like ISS in mean square and weighted-ISS can also be established along the lines of a similar stability analysis as in the proof of Theorem 3.1. For instance, if $\alpha_1(v) := a_1 v^2$ and $\alpha_2(v) := a_2 v^2$, where $a_2 > a_1 > 0$, then it follows from (3.20) that

$$\mathbb{E}[|x(t)|^2] \leq \beta_2(\mathbb{E}[\|\xi\|_\tau], t - t_0) + \varphi_2(|u(t)|) \quad \forall t \in \mathbb{R}^+_{t_0},$$

where $\beta_2(v,t) := a_1^{-1} a_2 \mu^{N_0+1} e^{-\omega t} v$ and $\varphi_2(v) := a_1^{-1}\Lambda\varphi(v)$, which implies that the system (2.1) is exponentially ISS in mean square; see [22]. Furthermore, Krasovskii-type conditions in Theorem 3.1 can also be applied for stochastic impulsive switched nonlinear time-delay systems. In this case, (A.2) is for impulsive switching intervals and (A.3) is for impulsive switching time instances; see [12, 32]. Hence, the previous works on both switched time-delay systems [4, 5, 28] and impulsive time-delay systems [22, 34, 39] are recovered as the special cases of Theorem 3.1.

**3.2. Razumikhin approach based stability analysis.** In this subsection, the Razumikhin-type conditions are derived for SISS of the system (2.1) with synchronous switching. Instead of the ADT condition, the FDT condition and the small gain condition are applied. To begin with, assume there exists $\boldsymbol{\theta} > 0$ such that $\boldsymbol{\theta} \leq \inf\{t_{k+1} - t_k | k \in \mathbb{N}\}$.

THEOREM 3.2. *Consider the system* (2.1). *Assume that there exist LRFs* $V_i : \mathbb{R}^+_{t_0-\tau} \times \mathbb{R}^{n_x} \to \mathbb{R}^+_0$, $i \in \mathcal{M}$, $\alpha_1, \rho_1, \rho_2 \in \mathcal{K}_\infty, \psi \in \mathcal{VK}_\infty, \alpha_2, \phi \in \mathcal{CK}_\infty$ *and constants* $\boldsymbol{\theta} > \delta > 0$ *such that* (A.1) *holds and*

(B.1) *for all* $t \in \mathbb{R}^+_{t_0} \backslash \mathcal{T}$, $V_{\sigma(t)}(t, x(t)) > \max\{\rho_1(|V_{\sigma(t)}(t,x(t))|^\tau), \rho_2(|u(t)|)\}$ *implies that* $\mathscr{L}V_{\sigma(t)}(t, x_t) \leq -\psi(V_{\sigma(t)}(t, x(t)))$;

(B.2) *for all* $t \in \mathcal{T}$, $V_{\sigma(t)}(t, x(t)) \leq \phi(V_{\sigma(t^-)}(t^-, x(t^-)))$;

(B.3) *the FDT condition holds, that is,* $\int_a^{\phi(a)} \frac{ds}{\psi(s)} \leq \boldsymbol{\theta} - \delta$ *for all* $a > 0$;

(B.4) *the small gain condition holds, that is,* $\rho_1(v) < v$ *for all* $v > 0$,

*where* $|V_{\sigma(t)}(t,x(t))|^\tau := \sup_{s\in[t-\tau,t]} |V_{\sigma(t)}(s, x(s))|$, *then the system* (2.1) *is SISS.*

*Proof.* Since $\alpha_1, \psi \in \mathcal{VK}_\infty$ and $\alpha_2, \phi \in \mathcal{CK}_\infty$, along similar lines as the proof of Theorem 3.1, (3.1) holds and $\mathbb{E}[V_i(t, x(t))] \leq \max\{\alpha_2(\mathbb{E}[\|\xi\|_\tau]), \phi(\alpha_2(\mathbb{E}[\|\xi\|_\tau]))\}$ for all $t \in [t_0 - \tau, t_0]$ and all $i \in \mathcal{M}$. In the following, based on the relation between $\max\{\rho_1(|V_{\sigma(t)}(t,x(t))|^\tau), \rho_2(|u(t)|)\}$ and $V_{\sigma(t)}(t, x(t))$, the proceeding proof is partitioned into two cases.

*Case* 1. $V_{\sigma(t)}(t, x(t)) > \max\{\rho_1(|V_{\sigma(t)}(t,x(t))|^\tau), \rho_2(|u(t)|)\}$ for all $t \in \mathbb{R}^+_{t_0}$. In this case, from Jensen's inequality in [41, Chapter II, 18.3], (B.1)–(B.2) can be written directly as follows:

$$\mathbb{E}[\mathscr{L}V_{\sigma(t)}(t, x_t)] \leq -\psi(\mathbb{E}[V_{\sigma(t)}(t, x(t))]), \quad t \notin \mathcal{T}; \tag{3.22}$$

$$\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))] \leq \phi(\mathbb{E}[V_{\sigma(t_{k-1})}(t_k^-, x(t_k^-))]), \quad t_k \in \mathcal{T}. \tag{3.23}$$

For any $k \in \mathbb{N}$, integrating (3.22) from $t_k$ to any $t \in [t_k, t_{k+1})$ and letting $t \to t_{k+1}^-$ give that

$$\int_{t_k}^{t_{k+1}^-} \frac{\mathbb{E}[\mathscr{L}V_{\sigma(t)}(t, x_t)]dt}{\psi(\mathbb{E}[V_{\sigma(t)}(t, x(t))])} \leq -(t_{k+1}^- - t_k) \leq -\boldsymbol{\theta}. \tag{3.24}$$

According to (3.24), define the function

$$F(\varrho) := \int_{\varsigma}^{\varrho} \frac{ds}{\psi(s)},$$

where $\varsigma > 0$ is fixed and $\varrho > 0$. Obviously, $F : \mathbb{R}_0^+ \to \mathbb{R}$ is continuous and strictly increasing, and so is its inverse $F^{-1} : \mathbb{R} \to \mathbb{R}_0^+$. Thus, using Itô's formula in [41, Chapter IV, 3] and Fubini's theorem in [41, Chapter II, 12.2], (3.24) is rewritten as

$$F(\mathbb{E}[V_{\sigma(t_k)}(t_{k+1}^-, x(t_{k+1}^-))]) - F(\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))]) \leq -\boldsymbol{\theta}. \tag{3.25}$$

It follows from (B.3), (3.23), and (3.25) that for all $k \in \mathbb{N}$,

$$\begin{aligned}
&F(\mathbb{E}[V_{\sigma(t_{k+1})}(t_{k+1}, x(t_{k+1}))]) - F(\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))]) \\
&\leq F(\phi(\mathbb{E}[V_{\sigma(t_k)}(t_{k+1}^-, x(t_{k+1}^-))])) - F(\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))]) \\
&\leq F(\phi(\mathbb{E}[V_{\sigma(t_k)}(t_{k+1}^-, x(t_{k+1}^-))])) - F(\mathbb{E}[V_{\sigma(t_k)}(t_{k+1}^-, x(t_{k+1}^-))]) \\
&\quad + F(\mathbb{E}[V_{\sigma(t_k)}(t_{k+1}^-, x(t_{k+1}^-))]) - F(\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))]) \\
&\leq \boldsymbol{\theta} - \delta - \boldsymbol{\theta} = -\delta.
\end{aligned}$$

That is,

$$\mathbb{E}[V_{\sigma(t_{k+1})}(t_{k+1}, x(t_{k+1}))] \leq F^{-1}(F(\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))]) - \delta). \tag{3.26}$$

In what follows, iterating the above analysis from $t_0$ to $t_k \in \mathcal{T}$ gives that

$$\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))] \leq F^{-1}(F(\mathbb{E}[V_{\sigma(t_0)}(t_0, x(t_0))]) - k\delta), \tag{3.27}$$

which is valid for all $k \in \mathfrak{K} := \{k \in \mathbb{N} | F(\mathbb{E}[V_{\sigma(t_0)}(t_0, x(t_0))]) - k\delta \geq \lim_{\varrho \downarrow 0} F(\varrho)\}$.

Denote $r := \mathbb{E}[V_{\sigma(t_0)}(t_0, x(t_0))]$ and $k_1 := \max_{k \in \mathfrak{K}} k$ (if it does not exist, denote $k_1 := \infty$). Based on (3.27), a class $\mathcal{KL}$ function is constructed as follows:

$$\beta_1(r, 0) := \max\{r, \phi(r)\}, \tag{3.28}$$

$$\beta_1(r, t_k - t_0) := F^{-1}(F(\beta_1(r, 0) - k\delta)), \quad k \in \{1, \ldots, k_1\}. \tag{3.29}$$

In $(t_k - t_0, t_{k+1} - t_0)$, $k \in \{1, \ldots, k_1\}$, $\beta_1(r, t)$ is required to decrease continuously and to lie above every solution of (3.22). If $k_1 < \infty$, then $\beta_1(r, t)$ in $[t_{k_1} - t_0, \infty)$ is defined to be continuous and decreasing to zero as $t \to \infty$.

From the construction of $\beta_1(r, t)$ in (3.28)–(3.29), we have that

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \leq \beta_1(\mathbb{E}[V_{\sigma(t_0)}(t_0, x(t_0))], t - t_0) \quad \forall t \geq t_0,$$

where $\beta_1(r, t)$ is continuous and decreases with the time line. If $k_1 = \infty$, $\beta_1(r, t) \to 0$ as $t \to \infty$ by the construction. If not, we need to prove that $\beta_1(r, t) \to 0$ as $t \to \infty$.

CLAIM 1. *If $\beta_1(r, t_k - t_0) \to 0$ as $k \to \infty$, then $\beta_1(r, t) \to 0$ as $t \to \infty$.*

*Proof.* If the claim is not true, then there exists an $\epsilon > 0$ (which depends on the choice of $r$) that $\lim_{k \to \infty} \beta_1(r, t_k - t_0) = \epsilon$. Define $\vartheta := \min_{\epsilon \leq q \leq \beta_1(r,0)} \varphi(q)$ and using the mean value theorem, we have

$$\delta \leq F(\beta_1(r, t_k - t_0)) - F(\beta_1(r, t_{k+1} - t_0)) \leq \frac{\beta_1(r, t_k - t_0) - \beta_1(r, t_{k+1} - t_0)}{\vartheta},$$

which implies that $\beta_1(r, t_k - t_0) - \beta_1(r, t_{k+1} - t_0) \geq \delta\vartheta > 0$. Consequently, $\beta_1(r, t_k - t_0)$ decreases to zero as $k \to \infty$, which contradicts $\lim_{k \to \infty} \beta_1(r, t_k - t_0) = \epsilon > 0$. Therefore, the claim is true. ∎

Define $\beta_2(r,t) := \sup_{0\le b\le r}\beta_1(b,t)$ and $\beta_3(r,t) := \frac{1}{r}\int_r^{2r}\beta_2(s,t)ds + re^{-t}$. Observe that $\beta_3(r,t) \ge \beta_2(r,t) \ge \beta_1(r,t)$ for all $r,t>0$ and that $\beta_3 \in \mathcal{KL}$. Thus, we get that

$$\mathbb{E}[V_{\sigma(t)}(t,x(t))] \le \beta_3(\mathbb{E}[V_{\sigma(t_0)}(t_0,x(t_0))], t-t_0) \quad \forall t \ge t_0. \tag{3.30}$$

Applying Markov's inequality in [41, Chapter II, 18.1] to (3.30) yields that for arbitrary $\varepsilon_1 \in (0,1)$, there exists $\beta_4(v,t) := \beta_3(v,t)/\varepsilon_1$ such that for all $t \in \mathbb{R}^+_{t_0}$,

$$\begin{aligned}&\mathbb{P}\left\{V_{\sigma(t)}(t,x(t)) > \beta_4(\mathbb{E}[V_{\sigma(t_0)}(t_0,x(t_0))], t-t_0)\right\}\\ &\quad\le \frac{\mathbb{E}[V_{\sigma(t)}(t,x(t))]}{\beta_4(\mathbb{E}[V_{\sigma(t_0)}(t_0,x(t_0))], t-t_0)} \le \varepsilon_1,\end{aligned}$$

which combined with (A.1) gives that for arbitrary $\varepsilon_1 \in (0,1)$, there exists $\beta_5(v,t) := \alpha^{-1}(\beta_4(\alpha_2(v),t)) \in \mathcal{KL}$ such that

$$\mathbb{P}\{|x(t)| > \beta_5(\mathbb{E}[\|\xi\|_\tau], t-t_0)\} \le \varepsilon_1 \quad \forall t \in \mathbb{R}^+_{t_0}. \tag{3.31}$$

*Case* 2. $V_{\sigma(t)}(t,x(t)) \ge \max\{\rho_1(|V_{\sigma(t)}(t,x(t))|^\tau), \rho_2(|u(t)|)\}$ does not hold for all $t \in \mathbb{R}^+_{t_0}$. If $V_{\sigma(t)}(t,x(t)) \ge \max\{\rho_1(|V_{\sigma(t)}(t,x(t))|^\tau), \rho_2(|u(t)|)\}$ for some $t \in \mathbb{R}^+_{t_0}$, then it follows from Case 1 that (3.30) holds. Otherwise, by taking expectation, we have that $\mathbb{E}[V_{\sigma(t)}(t,x(t))] \le \max\{\rho_1(\mathbb{E}[|V_{\sigma(t)}(t,x(t))|^\tau]), \rho_2(\|u\|)\}$. Combining these two cases yields that (see also [23]), for all $t \in \mathbb{R}^+_{t_0}$,

$$\begin{aligned}\mathbb{E}[V_{\sigma(t)}(t,x(t))] \le \max\{&\rho_1(\mathbb{E}[\|V_{\sigma(t)}(t,x(t))\|^\tau]), \rho_2(\|u\|),\\ &\beta_3(\mathbb{E}[V_{\sigma(t_0)}(t_0,x(t_0))], t-t_0)\},\end{aligned} \tag{3.32}$$

$$\mathbb{E}[|V_{\sigma(t)}(t,x(t))|^\tau] \le \max\{\Upsilon(t-t_0)\mathbb{E}[|V_{\sigma(t)}(t_0,x(t_0))|^\tau], \mathbb{E}[\|V_{\sigma(t)}(t,x(t))\|^\tau]\}, \tag{3.33}$$

where $\|V_{\sigma(t)}(t,x(t))\|^\tau := \sup_{t\ge t_0}\{|V_{\sigma(t)}(t,x(t))|^\tau\}$, $\Upsilon(t) := 0.5[1-\mathrm{sgn}(t-\tau)]$, and sgn is a sign function with $\mathrm{sgn}(v) = 1$ for $v \ge 0$ and $\mathrm{sgn}(v) = -1$ for $v < 0$.

Using (B.4) and similar to the proof of Theorem 1 in [23], we have from (3.32)–(3.33) that there exist $\bar\beta(v,t) := \alpha_1^{-1}(\beta_3(\alpha_2(v),t))$ and $\bar\gamma := \alpha_1^{-1}(\rho_2(v))$ such that

$$\mathbb{E}[\|x(t)\|^\tau] \le \bar\beta(\mathbb{E}[\|\xi\|_\tau], t-t_0) + \bar\gamma(\|u\|) \quad \forall t \in \mathbb{R}^+_{t_0}. \tag{3.34}$$

Because $\mathbb{E}[\|x(t)\|^\tau] \ge \mathbb{E}[|x(t)|]$ for all $t \in \mathbb{R}^+_{t_0}$, it follows from (3.34) and Markov's inequality that for any $\varepsilon \in (0,1)$, there exist $\beta(v,t) := \max\{\beta_5(v,t), \bar\beta(v,t)/\varepsilon\}$ and $\gamma(v) := \bar\gamma(v)/\varepsilon$ such that

$$\mathbb{P}\{|x(t)| \le \beta(\mathbb{E}[\|\xi\|_\tau], t-t_0) + \gamma(\|u\|)\} \ge 1-\varepsilon \quad \forall t \in \mathbb{R}^+_{t_0}.$$

That is, the system (2.1) is SISS and the proof is completed. □

*Remark* 3.3. Compared with Krasovskii-type stability conditions in Theorem 3.1, Razumikhin-type stability conditions in Theorem 3.2 imply that the LRFs decrease if the Razumikhin condition, that is, $V_{\sigma(t)}(t,x(t)) > \max\{\rho_1(|V_{\sigma(t)}(t,x(t))|^\tau), \rho_2(|u(t)|)\}$, holds for all $t \in \mathbb{R}^+_{t_0}$. In addition, the jumps of the LRFs only depend on the current state. Although the Razumikhin approach is more retarded than the Krasovskii approach [14], Theorem 3.2 can still be applicable to some more general systems. The main reason lies in that the obtained Razumikhin-type conditions are more general from two perspectives:

(i) The functions $\psi, \phi$, and $\rho_1$ are not required to be linear. That is, the LRFs are not necessarily exponential, which is the essential difference from the Lyapunov functions in [5, 13, 22, 38] and the LKFs in Theorem 3.1. As a result, the Razumikhin-type stability conditions can be applied to those that cannot be analyzed by exponential Lyapunov functions.

(ii) The FDT condition is applied in Theorem 3.2 and implies that the switching intervals are lower bounded by a certain limit. In some physical switched system models like networked control systems [3, 4], the switching intervals are generally bounded in some given intervals. In these cases, the FDT condition is more practical than the ADT condition. The relationship between the FDT condition and the ADT condition has been studied in [43, section 3.2].

However, the Razumikhin-type stability conditions in Theorem 3.2 cannot guarantee other stability properties of the system (2.1), such as SiISS, ISS in mean square, and weighted ISS. In addition, $\rho_1(v)$ is required to satisfy the small gain condition, which also restricts the application range of Theorem 3.2 to some extent.

*Remark* 3.4. If there is no time delay, then Theorem 3.2 is similar to Theorem 1 in [8]. In Theorem 3.2, the Razumikhin condition $V_{\sigma(t)}(t,x(t))>\max\{\rho_1(|V_{\sigma(t)}(t,x(t))|^\tau),$ $\rho_2(|u(t)|)\}$ can be written as $V_{\sigma(t)}(t,x(t)) > \max\{\rho_1(V_{\sigma(t)}(t+s,x(t+s))),\rho_2(|u(t)|)\}$ for all $s\in[t-\tau,t]$; see [9, 14].

In the following, we present an alternative Razumikhin-type stability criterion for the systems (2.1) with synchronous switching. Different from the techniques used in the proof of Theorem 3.2, the Halanay-like inequality [46] is applied, which provides a new perspective on stability analysis of the system (2.1).

PROPOSITION 3.3. *Assume that all the conditions in Theorem* 3.2 *are satisfied with* $\rho_1(v) := \bar\rho v$*, where* $\bar\rho\in(0,1)$*; then the system* (2.1) *is SISS.*

*Proof.* Similar to the proof of Theorem 3.2, the stability analysis is divided into two cases. The proof for Case 1 is the same as that of Theorem 3.2 and (3.30) holds.

For Case 2, if $V_\sigma(t,x(t))\geq\max\{\bar\rho|V_\sigma(t,x(t))|^\tau,\rho_2(|u(t)|)\}$ for some $t\in\mathbb{R}^+_{t_0}$, then it follows from Case 1 that (3.30) holds. Otherwise, we have that

$$\mathbb{E}[V_{\sigma(t)}(t,x(t))]\leq\bar\rho\mathbb{E}[|V_{\sigma(t)}(t,x(t))|^\tau]+\rho_2(|u(t)|),\quad t\in\mathbb{R}^+_{t_0}. \tag{3.35}$$

Using Lemma 1 in [46], it can be obtained from (3.36) that

$$\mathbb{E}[V_{\sigma(t)}(t,x(t))]\leq e^{\frac{\ln\bar\rho}{\tau}(t-t_0)}\alpha_2(\mathbb{E}[\|\xi\|_\tau])+(1-\bar\rho)^{-2}\rho_2(\|u\|),\quad t\in\mathbb{R}^+_{t_0}. \tag{3.36}$$

Since $\bar\rho\in(0,1)$, we get that $\ln\bar\rho<0$.

Combining (3.30) in Case 1 and (3.36) in Case 2 yields that

$$\mathbb{E}[V_{\sigma(t)}(t,x(t))]\leq\beta(\mathbb{E}[\|\xi\|_\tau],t-t_0)+\gamma(\|u\|)\quad\forall t\in\mathbb{R}^+_{t_0},$$

where $\beta(v,t):=\beta_3(\alpha_2(v),t)+e^{\frac{\ln\bar\rho}{\tau}t}\alpha_2(v)$ and $\gamma(v):=(1-\bar\rho)^{-2}\rho_2(v)$. In the following, the preceding analysis is the same as the proof of Theorem 3.2 and then SISS of the system (2.1) is established. Hence, the proof is completed. ∎

*Remark* 3.5. In Theorem 3.2 and Proposition 3.3, $\rho_1(v)$ is required to satisfy the small gain condition. However, the analysis techniques in Theorem 3.2 and Proposition 3.3 are different and cannot be transformed mutually. The reason is that $\rho_1(v)$ in Proposition 3.3 is required to be linear; see [46, Remark 1] for more details.

**4. Stochastic switched time-delay systems with asynchronous switching.** In this section, the asynchronous switching case is studied. For this case, both Krasovskii-type and Razumikhin-type conditions are established for SISS and SiISS of the systems (2.1). First, some notation is introduced; see [8, 26]. For each $k \in \mathbb{N}$, denote by $\mathcal{T}_{\downarrow}(t_k, t_{k+1})$ and $\mathcal{T}_{\uparrow}(t_k, t_{k+1})$ separately the unions of dispersed intervals in $[t_k, t_{k+1})$ where LRFs decrease and increase. That is, $[t_k, t_{k+1}) = \mathcal{T}_{\downarrow}(t_k, t_{k+1}) \cup \mathcal{T}_{\uparrow}(t_k, t_{k+1})$. Moreover, $\mathcal{T}_{\downarrow}(t_{k+1} - t_k)$ and $\mathcal{T}_{\uparrow}(t_{k+1} - t_k)$ denote the lengths of $\mathcal{T}_{\downarrow}(t_k, t_{k+1})$ and $\mathcal{T}_{\uparrow}(t_k, t_{k+1})$, respectively.

**4.1. Krasovskii approach based stability analysis.** The Krasovskii-type stability conditions are given in the following theorem, which is not just an extension of Theorem 3.1 but also involves some additional technical difficulties.

THEOREM 4.1. *Consider the system* (2.1). *Assume that there exist LKFs* $V_i : \mathbb{R}^+_{t_0-\tau} \times \mathbb{R}^{n_x} \to \mathbb{R}^+_0$, $i \in \mathcal{M}$. *If there exist functions* $\alpha_1, \varphi_1, \varphi_2, \varphi_3 \in \mathcal{K}_\infty, \alpha_2 \in \mathcal{CK}_\infty$ *and constants* $\lambda_1 > \lambda_2 \geq 0, \lambda_3, \lambda_4 > 0, \rho, \mathcal{T}_0 \geq 0, \mu > 1$ *such that* (A.1) *and* (A.3) *hold with* $\mu$ *and* $\varphi_3$, *and*

(C.1) *for all* $t \in \mathcal{T}_{\downarrow}(t_k, t_{k+1})$, $k \in \mathbb{N}$, $\mathscr{L}V_{\sigma(t)}(t, x(t)) \leq -\lambda_1 V_{\sigma(t)}(t, x(t)) + \lambda_2 V_{\sigma(t-\tau(t))}(t - \tau(t), x_t) + \varphi_1(|u(t)|)$;

(C.2) *for all* $t \in \mathcal{T}_{\uparrow}(t_k, t_{k+1}), k \in \mathbb{N}$, $\mathscr{L}V_{\sigma(t)}(t, x(t)) \leq \lambda_3 V_{\sigma(t)}(t, x(t)) + \lambda_4 V_{\sigma(t-\tau(t))}(t - \tau(t), x_t) + \varphi_2(|u(t)|)$;

(C.3) $\rho < \frac{\pi}{\pi + \lambda_3}$, *where* $\pi$ *is the solution to* $\lambda - \lambda_1 + \lambda_2 e^{\lambda\tau} = 0$;

(C.4) *for arbitrary* $t \geq s \geq t_0$, $\mathcal{T}_{\uparrow}(t - s) \leq \mathcal{T}_0 + \rho(t - s)$;

(C.5) $\rho$ *and* $\mathcal{T}_0$ *satisfy* $-\pi + \rho(\lambda_3 + \pi) + \lambda_4 e^{(\lambda_3+\pi)\mathcal{T}_0} < 0$;

(C.6) $\tau_a > \frac{(\lambda_3+\pi)\mathcal{T}_0 + \ln\mu}{\lambda_0}$, *where* $\lambda_0$ *is the unique solution to* $-\pi + \rho(\lambda_3 + \pi) + \lambda + \lambda_4 e^{(\lambda_3+\pi)\mathcal{T}_0} e^{\lambda\tau} = 0$,

*then the system* (2.1) *is both SISS and SiISS under the asynchronous switching case.*

*Proof.* Because of the asynchronous switching, the LKFs decrease in $\mathcal{T}_{\downarrow}(t_k, t_{k+1})$ and increase in $\mathcal{T}_{\uparrow}(t_k, t_{k+1})$. Hence, the following proof is partitioned into three parts. The first part transforms the derivatives of the LKFs in $[t_k, t_{k+1})$ into an impulsive switched time-delay system. The second part bounds the LKFs in $[t_k, t_{k+1})$ using Proposition 2.4. The third part analyzes SISS and SiISS of the system (2.1) via the bounds of the LKFs and the ADT condition (C.6).

*Part* 1. Based on (C.1)–(C.2), for all $k \in \mathbb{N}$, the interval $[t_k, t_{k+1})$ is divided into finite subintervals and the number of the subintervals is assumed to be even without loss of generality. That is, $[t_k, t_{k+1}) = \cup_{0 \leq j \leq 2N-1} [t_{k_j}, t_{k_{j+1}})$, where $N, k \in \mathbb{N}$, $t_{k_0} = t_k$ and $t_{k_{2N}} = t_{k+1}$. $V_{\sigma(t)}(t, x(t))$ is right-hand continuous at $t_k$ and continuous at $t_{k_j}$, where $j \in \{1, \ldots, 2N - 1\}$. In addition, from (C.1)–(C.2), assume that

(a) for all $j \in \{0, \ldots, N - 1\}$, $t \in [t_{k_{2j+1}}, t_{k_{2j+2}})$, $\mathbb{E}[\mathscr{L}V_{\sigma(t)}(t, x(t))] \leq -\lambda_1 \mathbb{E}[V_{\sigma(t)}(t, x(t))] + \lambda_2 \mathbb{E}[V_{\sigma(t-\tau(t))}(t - \tau(t), x_t)] + \varphi_1(|u(t)|)$;

(b) for all $j \in \{0, \ldots, N-1\}$, $t \in [t_{k_{2j}}, t_{k_{2j+1}})$, $\mathbb{E}[\mathscr{L}V_{\sigma(t)}(t, x(t))] \leq \lambda_3 \mathbb{E}[V_{\sigma(t)}(t, x(t))] + \lambda_4 \mathbb{E}[V_{\sigma(t-\tau(t))}(t - \tau(t), x_t)] + \varphi_2(|u(t)|)$.

Similar to the proof of Theorem 3.1, we have that $\mathbb{E}[V_i(t, x(t))] \leq \alpha_2(\mathbb{E}[\|\xi\|_\tau])$ for all $t \in [t_0 - \tau, t_0]$ and $i \in \mathcal{M}$. Using Proposition 2.6 in [42] and Lemma 2 in [31], it can be obtained from item (a) that there exist $\pi > 0$ and $\bar{\varphi}_1 \in \mathcal{K}_\infty$ (maybe depending on $\varphi_1$) such that for all $t \in [t_{k_{2j+1}}, t_{k_{2j+2}})$, $j \in \{0, \ldots, N - 1\}$,

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \leq e^{-\pi(t - t_{k_{2j+1}})} \mathbb{E}[V_{\sigma(t)}(t_{k_{2j+1}}, x(t_{k_{2j+1}}))] + \bar{\varphi}_1(|u(t)|),$$

where $\pi > 0$ is the unique solution to $\pi - \lambda_1 + \lambda_2 e^{\pi\tau} = 0$. Due to the constance of $\sigma(t)$ on $[t_k, t_{k+1})$ and the continuity of $\mathbb{E}[V_{\sigma(t)}(t, x(t))]$ at the time instant $t_{k_{2j+2}}$, $j \in \{0, \ldots, N-2\}$, one has

$$\mathbb{E}[V_{\sigma(t)}(t_{k_{2j+2}}, x(t_{k_{2j+2}}))] \leq e^{-\pi\Delta_{2j+1}}\mathbb{E}[V_{\sigma(t)}(t_{k_{2j+1}}, x(t_{k_{2j+1}}))] + \bar{\varphi}_1(|u(t^-_{k_{2j+2}})|),$$

where $\Delta_{2j+1} := t_{k_{2j+2}} - t_{k_{2j+1}}$.

Therefore, following the above analysis, we obtain from (A.3), (C.1)–(C.2) that

$$\text{(4.1)}\quad \begin{cases} D^+\mathbb{E}[V_{\sigma(t)}(t, x(t))] \leq \lambda_3 V_{\sigma(t)}(t, x(t)) + \lambda_4 V_{\sigma(t-\tau(t))}(t-\tau(t), x_t) \\ \qquad\qquad + \varphi_2(|u(t)|), \quad t \in (t_{k_{2j}}, t_{k_{2j+1}}),\ j \in \{0, \ldots, N-1\}, \\ \mathbb{E}[V_{\sigma(t_k)}(t_{k_{2j+1}}, x(t_{k_{2j+1}}))] \leq e^{-\pi\Delta_{2j+1}}\mathbb{E}[V_{\sigma(t)}(t^-_{k_{2j+1}}, x(t^-_{k_{2j+1}}))] \\ \qquad\qquad + \bar{\varphi}_1(|u(t^-_{k_{2j+2}})|), \quad j \in \{0, \ldots, N-1\}, \\ \mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))] \leq \mu\mathbb{E}[V_{\sigma(t^-_k)}(t^-_k, x(t^-_k))] + \varphi_3(|u(t^-_k)|), \quad k \in \mathbb{N}, \\ \mathbb{E}[V_i(t, x(t))] \leq \alpha_2(\mathbb{E}[\|\xi\|_\tau]), \quad t \in [t_0 - \tau, t_0], \quad i \in \mathcal{M}. \end{cases}$$

*Part* 2. Let $\mathfrak{U}_{\sigma(t)}(t) \in \mathbb{R}$ be the solution to the following equation:

$$\text{(4.2)}\quad \begin{cases} D^+\mathfrak{U}_{\sigma(t)}(t) = \lambda_3\mathfrak{U}_{\sigma(t)}(t) + \lambda_4\mathfrak{U}_{\sigma(t-\tau(t))}(t-\tau(t)) + \varphi_2(|u(t)|) + \epsilon, \\ \qquad\qquad t \in (t_{k_{2j}}, t_{k_{2j+1}}), \quad j \in \{0, \ldots, N\}, \\ \mathfrak{U}_{\sigma(t_k)}(t_{k_{2j+1}}) = e^{-\pi\Delta_{2j+1}}\mathfrak{U}_{\sigma(t)}(t^-_{k_{2j+1}}) + \bar{\varphi}_1(|u(t^-_{k_{2j+2}})|), \quad j \in \{0, \ldots, N\}, \\ \mathfrak{U}_{\sigma(t_k)}(t_k) = \mu\mathfrak{U}_{\sigma(t^-_k)}(t^-_k) + \varphi_3(|u(t^-_k)|), \quad k \in \mathbb{N}, \\ \mathfrak{U}_i(t) = \alpha_2(\mathbb{E}[\|\xi\|_\tau]), \quad t \in [t_0 - \tau, t_0], \quad i \in \mathcal{M}, \end{cases}$$

where $\epsilon > 0$ is arbitrarily small. Since $\mathbb{E}[V_i(t, x(t))] \leq \mathfrak{U}_i(t)$ for all $t \in [t_0 - \tau, t_0]$ and $i \in \mathcal{M}$, it can be obtained from Proposition 2.4 that $\mathbb{E}[V_{\sigma(t)}(t, x(t))] \leq \mathfrak{U}_{\sigma(t)}(t)$ for all $t \in [t_0, t_1)$. Furthermore, it follows from the third inequalities in (4.1)–(4.2) that $\mathbb{E}[V_{\sigma(t_1)}(t_1, x(t_1))] \leq \mathfrak{U}_{\sigma(t_1)}(t_1, x(t_1))$. Repeating such a mechanism, we have that $\mathbb{E}[V_{\sigma(t)}(t, x(t))] \leq \mathfrak{U}_{\sigma(t)}(t)$ for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$. In the following, we derive the bound of $\mathbb{E}[V_{\sigma(t)}(t, x(t))]$ for $t \in [t_k, t_{k+1})$.

We have from (C.3) that $\pi - \rho(\lambda_3 + \pi) > 0$ and from (C.4) that

$$\text{(4.3)}\quad \begin{aligned} e^{-\pi\mathcal{T}_\downarrow(t-s)}e^{\lambda_3\mathcal{T}_\uparrow(t-s)} &\leq e^{-\pi(t-s)}e^{(\lambda_3+\pi)(\mathcal{T}_0+\rho(t-s))} \\ &= e^{(\lambda_3+\pi)\mathcal{T}_0}e^{-(\pi-\rho(\lambda_3+\pi))(t-s)}. \end{aligned}$$

Define $\Lambda_1 := \pi - \rho(\lambda_3 + \pi)$ and $\Lambda_2 := (\lambda_3 + \pi)\mathcal{T}_0$. For the systems (4.1)–(4.2), using (4.3) and similar to the proof strategy of Theorem 2 in [22], we get that for all $k \in \mathbb{N}$ and $t \in [t_k, t_{k+1})$,

$$\begin{aligned} \mathbb{E}[V_{\sigma(t)}(t, x(t))] &\leq e^{\Lambda_2}e^{-\lambda(t-t_k)}\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))] \\ &\quad + \int_{t_k}^t J_1 e^{-\lambda_5(t-s)}\varphi_2(|u(s)|)ds + \frac{J_2}{1 - e^{-\Lambda_1\theta_2}}\sup_{t\in[t_k,t)}\bar{\varphi}_1(|u(s)|) \qquad (4.4) \\ &\leq e^{\Lambda_2}e^{-\lambda(t-t_k)}\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))] \\ &\quad + \left(\frac{J_1}{\lambda_5} + \frac{J_2}{1 - e^{-\Lambda_1\theta_2}}\right)\sup_{s\in[t_k,t)}\{\bar{\varphi}_1(|u(s)|), \varphi_2(|u(s)|)\}, \qquad (4.5) \end{aligned}$$

where $J_1 \geq \frac{(\Lambda_1-\lambda_5)e^{\Lambda_2}}{-\Psi(\lambda_5)}$, $J_2 \geq \frac{\Lambda_1 e^{\Lambda_2}}{\Lambda_1 - \lambda_4 e^{\Lambda_2}}$, $0 < \theta_2 \leq \inf\{t_{k_{j+1}} - t_{k_j} | j \in \{0, \ldots, 2N-1\}\}$, $\lambda_5 \in (0, \lambda_0)$ and $\lambda_0$ is the solution to $\Psi(\lambda) := -\Lambda_1 + \lambda + \lambda_4 e^{\Lambda_2}e^{\lambda\tau} = 0$. Because

$\Psi(0) < 0$ holds from (C.5), $\Psi(\lambda) \to \infty$ as $\lambda \to \infty$, and $\Psi'(\lambda) = 1 + \tau\lambda_4 e^{\Lambda_2} e^{\lambda\tau} > 0$, there exists a unique $\lambda_0 > 0$ such that $\Psi(\lambda_0) = 0$. Moreover, $\Psi(\lambda) < 0$ for all $\lambda \in (0, \lambda_0)$.

*Part* 3. Define $\Lambda := \lambda_5^{-1} J_1 + \frac{J_2}{1-e^{-\Lambda_1\theta_1}}$ and $\hat{\varphi}(|u(t_k)|) := \sup_{s\in[t_k,t_{k+1})}\{\bar{\varphi}_1(|u(s)|), \varphi_2(|u(s)|)\}$ for each $k \in \mathbb{N}$. Using (A.3) and repeating (4.5) in Part 2 from $t_0$ to any $t \geq t_0$ yield that, for all $t \in \mathbb{R}^+_{t_0}$,

$$\begin{aligned}\mathbb{E}[V_{\sigma(t)}(t, x(t))] &\leq \mu^{N(t,t_0)} e^{\Lambda_2 N(t,t_0) - \lambda(t-t_0)} \mathbb{E}[V_{\sigma(t_0)}(t_0, x(t_0))] \\ &\quad + \sum_{i=0}^{N(t,t_0)} \mu^{N(t_i,t_0)} e^{\Lambda_2 N(t_i,t_0) - \lambda(t_i - t_0)} [\Lambda\hat{\varphi}(|u(t_i)|) + \varphi_3(|u(t_i^-)|)]. \end{aligned} \tag{4.6}$$

According to Definition 2.2, we have that for all $t \in \mathbb{R}^+_{t_0}$,

$$\begin{aligned}\mu^{N(t,t_0)} e^{\Lambda_2 N(t,t_0) - \lambda(t-t_0)} &= e^{-\lambda(t-t_0)} e^{N(t,t_0)(\ln\mu + \Lambda_2)} \\ &\leq e^{-\lambda(t-t_0)} e^{N_0(\ln\mu+\Lambda_2)} e^{(\frac{\ln\mu+\Lambda_2}{\tau_a})(t-t_0)} \\ &= e^{N_0(\ln\mu+\Lambda_2)} e^{(t-t_0)(-\lambda + \frac{\ln\mu+\Lambda_2}{\tau_a})}. \end{aligned} \tag{4.7}$$

Define $\varpi := -\lambda + \frac{\ln\mu+\Lambda_2}{\tau_a}$. It follows from (C.6) that $\varpi < 0$.

Combining (4.6) and (4.7) yields that

$$\begin{aligned}\mathbb{E}[V_{\sigma(t)}(t, x(t))] &\leq e^{N_0(\ln\mu+\Lambda_2)} e^{\varpi(t-t_0)} \alpha_2(\mathbb{E}[\|\xi\|_\tau]) \\ &\quad + e^{N_0(\ln\mu+\Lambda_2)} \sum_{i=0}^{N(t,t_0)} e^{\varpi(t_i-t_0)} [\Lambda\hat{\varphi}(|u(t_i)|) + \varphi_3(|u(t_i^-)|)] \\ &\leq e^{N_0(\ln\mu+\Lambda_2)} e^{\varpi(t-t_0)} \alpha_2(\mathbb{E}[\|\xi\|_\tau]) + e^{N_0(\ln\mu+\Lambda_2)} (1-e^{\varpi})^{-1} \varphi(|u(t)|) \\ &=: \beta(\mathbb{E}[\|\xi\|_\tau], t - t_0) + \gamma(\|u\|), \end{aligned} \tag{4.8}$$

where $\beta(v, t) := e^{N_0(\ln\mu+\Lambda_2)} e^{\varpi t} \alpha_2(v)$, $\gamma(v) := e^{N_0(\ln\mu+\Lambda_2)}(1-e^{\varpi})^{-1}\varphi(v)$ and

$$\varphi(|u(t)|) := (1+\Lambda) \sup_{s\in[t_0,t)} \{\bar{\varphi}_1(|u(s)|), \varphi_2(|u(s)|), \varphi_3(|u(s)|)\}.$$

In what follows, it follows from (4.8) and the Markov inequality that for any $\varepsilon > 0$,

$$\mathbb{P}\left\{|x(t)| \leq \bar{\beta}(\mathbb{E}[\|\xi\|_\tau], t - t_0) + \bar{\gamma}(\|u\|)\right\} \geq 1 - \varepsilon \quad \forall t \in \mathbb{R}^+_{t_0},$$

where $\bar{\beta}(v, t) := \alpha_1^{-1}(2\beta(v, t))/\varepsilon$ and $\bar{\gamma}(v) := \alpha_1^{-1}(2\gamma(v))/\varepsilon$. As a result, the system (2.1) is SISS.

On the other hand, (4.4) is rewritten as follows: for all $t \in [t_k, t_{k+1})$, $k \in \mathbb{N}$,

$$\begin{aligned}\mathbb{E}[V_{\sigma(t)}(t, x(t))] &\leq e^{\Lambda_2} e^{-\lambda(t-t_k)} \mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))] \\ &\quad + \int_{t_k}^{t} (J_1\varphi_2(|u(s)|) + J_2\theta_2^{-1}\bar{\varphi}_1(|u(s)|))ds. \end{aligned}$$

Along similar lines as the SISS analysis, the SiISS of the system (2.1) is established. ∎

*Remark* 4.1. In Theorem 4.1, if the LKFs increase in $(t_{k_{2j}}, t_{k_{2j+1}})$ and decrease in $(t_{k_{2j+1}}, t_{k_{2j+2}})$, then the similar equivalent impulsive switched time-delay system is also obtained and the stability result is derived. If the number of the subintervals is odd, then the stability analysis proceeds in the similar fashion but is more complex. In the proof of Theorem 4.1, the essential technique is how to transform the evolution of the LKF in the switching intervals into an impulsive switched time-delay system (4.3). A similar transformation technique appeared in previous works like [22, 27]. Furthermore, the estimate of the system state of the impulsive switched time-delay systems is involved; see [22, 38] for more details.

**4.2. Razumikhin approach based stability analysis.** As the counterpart of Theorem 3.2, Razumikhin-type conditions are established in the following theorem for SISS of the system (2.1) with asynchronous switching.

THEOREM 4.2. *Consider the system* (2.1). *There exist LRFs* $V_i : \mathbb{R}^+_{t_0-\tau} \times \mathbb{R}^{n_x} \to \mathbb{R}^+_0$, $i \in \mathcal{M}$, $\alpha_1, \rho_1, \rho_2 \in \mathcal{K}_\infty, \psi_1 \in \mathcal{VK}_\infty, \alpha_2, \psi_2, \phi \in \mathcal{CK}_\infty$ *and constants* $\boldsymbol{\theta}, \delta > 0$ *such that* (A.1), (B.2), *and* (B.4) *hold and*

(E.1) *for all* $t \in \mathcal{T}_\downarrow(t_k, t_{k+1})$, $k \in \mathbb{N}$, $V_{\sigma(t)}(t, x(t)) \geq \max\{\rho_1(|V_{\sigma(t)}(t, x(t))|^\tau), \rho_2(|u(t)|)\}$ *implies* $\mathscr{L}V_{\sigma(t)}(t, x_t) \leq -\psi_1(V_{\sigma(t)}(t, x(t)))$;

(E.2) *for all* $t \in \mathcal{T}_\uparrow(t_k, t_{k+1})$, $k \in \mathbb{N}$, $V_{\sigma(t)}(t, x(t)) \geq \max\{\rho_1(|V_{\sigma(t)}(t, x(t))|^\tau), \rho_2(|u(t)|)\}$ *implies* $\mathscr{L}V_{\sigma(t)}(t, x_t) \leq \psi_2(V_{\sigma(t)}(t, x(t)))$;

(E.3) *for an arbitrary* $a > 0$, $\max\{\int_a^{\phi(a)} \psi_1^{-1}(s)ds, \int_a^{\phi(a)} \psi_2^{-1}(s)ds\} \leq \boldsymbol{\theta} - \delta - 2\mathcal{T}_{\max}$, *where* $\mathcal{T}_{\max} := \sup\{\mathcal{T}_\uparrow(t_{k+1} - t_k)|k \in \mathbb{N}\}$,

*then the system* (2.1) *is SISS in the asynchronous switching case.*

*Proof.* Similar to the proof of Theorem 3.2, the proof is divided into two cases: the case that $V_{\sigma(t)}(t, x(t)) > \max\{\rho_1(|V_{\sigma(t)}(t, x(t))|^\tau), \rho_2(|u(t)|)\}$ for all $t \in \mathbb{R}^+_{t_0}$ and the case that $V_{\sigma(t)}(t, x(t)) \leq \max\{\rho_1(|V_{\sigma(t)}(t, x(t))|^\tau), \rho_2(|u(t)|)\}$ holds for all $t \in \mathbb{R}^+_{t_0}$. The stability analysis for the second case is the same as that in Theorem 3.2. In the following, we prove that for the first case, (3.26) still holds in the asynchronous switching case.

Since $\psi_1 \in \mathcal{VK}, \psi_2 \in \mathcal{CK}$, using Jensen's inequality, (E.1)–(E.2) are rewritten as

$$\mathbb{E}[\mathscr{L}V_{\sigma(t)}(t, x_t)] \leq -\psi_1(\mathbb{E}[V_{\sigma(t)}(t, x(t))]), \quad t \in \mathcal{T}_\downarrow(t_k, t_{k+1}), \tag{4.9}$$

$$\mathbb{E}[\mathscr{L}V_{\sigma(t)}(t, x_t)] \leq \psi_2(\mathbb{E}[V_{\sigma(t)}(t, x(t))]), \quad t \in \mathcal{T}_\uparrow(t_k, t_{k+1}). \tag{4.10}$$

Moreover, the interval $[t_k, t_{k+1})$ is divided into finite subintervals, that is, $[t_k, t_{k+1}) = \cup_{0\leq j\leq 2N-1}[t_{k_j}, t_{k_{j+1}})$, where $t_{k_0} = t_k$ and $t_{k_{2N}} = t_{k+1}$. Assume for all $j \in \{0, \ldots, N–1\}$, (4.9) holds for $t \in [t_{2j}, t_{2j+1})$ and (4.10) holds for $t \in [t_{2j+1}, j_{2i+2})$.

Similar to the technique applied in the proof of Theorem 3.2, define the function

$$F(\varrho) := \int_\varsigma^\varrho \frac{ds}{\ell(s)\psi_1(s) + (1 - \ell(s))\psi_2(s)},$$

where $\varsigma > 0$ is fixed, $\varrho > 0$, and $\ell : \mathbb{R}^+_0 \to \{0, 1\}$ is a logical function. For each $k \in \mathbb{N}$, $\ell(t) = 1$ if $t \in \mathcal{T}_\downarrow(t_k, t_{k+1})$ and $\ell(t) = 0$ if $t \in \mathcal{T}_\uparrow(t_k, t_{k+1})$. Observe that the function $F$ and its inverse $F^{-1}$ are continuous and strictly increasing. It follows from (E.3) that for any $t \in [t_{k_{2j}}, t_{k_{2j+1}})$, $j \in \{0, \ldots, N - 1\}$,

$$F(\mathbb{E}[V_{\sigma(t)}(t, x(t))]) - F(\mathbb{E}[V_{\sigma(t)}(t_{k_{2j}}, x(t_{k_{2j}}))]) \leq -(t - t_{k_{2j}}), \tag{4.11}$$

and for any $t \in [t_{k_{2j+1}}, t_{k_{2j+2}})$, $j \in \{0, \ldots, N - 1\}$,

$$F(\mathbb{E}[V_{\sigma(t)}(t, x(t))]) - F(\mathbb{E}[V_{\sigma(t)}(t_{k_{2j+1}}, x(t_{k_{2j+1}}))]) \leq t - t_{k_{2j+1}}. \tag{4.12}$$

As a result, it follows from (B.2) (E.3), (4.11), and (4.12) that, for $t_k, t_{k+1} \in \mathcal{T}$,

$$\begin{aligned}
&F(\mathbb{E}[V_{\sigma(t_{k+1})}(x(t_{k+1}))]) - F(\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))]) \\
&\leq F(\phi(\mathbb{E}[V_{\sigma(t_k)}(t_{k+1}^-, x(t_{k+1}^-))])) - F(\mathbb{E}[V_{\sigma(t_k)}(t_{k+1}^-, x(t_{k+1}^-))]) \\
&\quad + F(\mathbb{E}[V_{\sigma(t_k)}(t_{k+1}^-, x(t_{k+1}^-))]) - F(\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))]) \\
&\leq F(\phi(\mathbb{E}[V_{\sigma(t_k)}(t_{k+1}^-, x(t_{k+1}^-))])) - F(\mathbb{E}[V_{\sigma(t_k)}(t_{k+1}^-, x(t_{k+1}^-))]) \\
&\quad + F(\mathbb{E}[V_{\sigma(t_k)}(t_{k+1}^-, x(t_{k+1}^-))]) - F(\mathbb{E}[V_{\sigma(t_k)}(t_{k_{2N-1}}, x(t_{k_{2N-1}}))]) \\
&\quad + \sum_{j=0}^{2N-2} \big(F(\mathbb{E}[V_{\sigma(t_k)}(t_{k_{j+1}}, x(t_{k_{j+1}}))]) - F(\mathbb{E}[V_{\sigma(t_k)}(t_{k_j}, x(t_{k_j}))])\big) \\
&\leq \boldsymbol{\theta} - \delta - 2\mathcal{T}_{\max} - \mathcal{T}_{\downarrow}(t_{k+1} - t_k) + \mathcal{T}_{\uparrow}(t_{k+1} - t_k) \leq -\delta. \qquad \square
\end{aligned}$$

That is, it holds that $\mathbb{E}[V_{\sigma(t_{k+1})}(t_{k+1}, x(t_{k+1}))] \leq F^{-1}(F(\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))]) - \delta)$. The proceeding analysis is similar to the proof of Theorem 3.2 and is omitted here. Hence, SISS of the system (2.1) is established in the asynchronous switching case.

The counterpart of Proposition 3.3 is given as follows. The proof is a combination of the proof strategies of Theorem 4.2 and Proposition 3.3 and hence is omitted here.

PROPOSITION 4.3. *If all the conditions in Theorem* 4.2 *are satisfied with* $\rho_1(v) := \bar{\rho}v$ *and* $\bar{\rho} \in (0, 1)$, *then the system* (2.1) *is SISS in the asynchronous switching case.*

## 5. Applications.

In this section, the obtained results in the previous sections are illustrated via two numerical examples from the mechanical rotational cutting process and networked switched control systems.

### 5.1. Mechanical rotational cutting process.

Consider the mechanical rotational cutting process [48, 49], which can be modeled as the following classic form:

$$\ddot{x}(t) + 2\xi_0\omega_0\dot{x}(t) + \omega_0^2 x(t) = F/m, \tag{5.1}$$

where $x(t) \in \mathbb{R}$ is the displacement of the tool in the feed direction, $\xi_0 \in \mathbb{R}$ is the damping ratio, $\omega_0 \in \mathbb{R}$ is the natural angular frequency of the tool, $m > 0$ is the modal mass, and $F > 0$ is the cutting force.

During the cutting process, the cutting force needs to be adjusted according to variations in the workpiece material. Moreover, there are fluctuations in the cutting process, thereby affecting the cutting force. In this case, the fluctuations can be modeled as a random disturbance. As a result, the cutting process can be modeled as the following stochastic switched time-delay system:

$$\begin{aligned}
&\ddot{x}(t) + \gamma_1\dot{x}(t) + \gamma_2(x(t) + x^3(t)) \\
&\quad = -\gamma_{3\sigma(t)}x(t-\tau) + [\delta_{1\sigma(t)}x(t) + \delta_{2\sigma(t)}x(t-\tau)]\frac{dB(t)}{dt},
\end{aligned} \tag{5.2}$$

where $\gamma_1 \in \mathbb{R}$ is the term proportional to the product of natural frequency, the damping ratio $\gamma_2 \in \mathbb{R}$ represents the tool stiffness, and $\gamma_{3\sigma(t)} \in \mathbb{R}$ is the delay term proportional to effective cutting stiffness of the workpiece per unit of chip width. $\sigma : \mathbb{R}_0^+ \to \mathcal{M} := \{1, \ldots, M\}$ is the switching signal, $B(t)$ is standard Brownian motion, and $\delta_{1\sigma(t)}, \delta_{2\sigma(t)} \in \mathbb{R}$. Note that in (5.2), the cutting force is related to the delayed displacement of the tool; see [48, 49] for the details.

Define $y_1(t) := x(t), y_2(t) := \dot{x}(t)$ and $y(t) = (y_1(t), y_2(t))^\top$. Hence, the system (5.2) is written as a general stochastic switched nonlinear time-delay system model:

$$
\begin{aligned}
dy(t) &= f_{\sigma(t)}(t, y_t, u)dt + g_{\sigma(t)}(t, y_t, u)dB(t) \\
&= \begin{bmatrix} y_2(t) + U_{\sigma(t)}(t) \\ -\gamma_1 y_1(t) - \gamma_2(y_1(t) + y_1^3(t)) - \gamma_{3\sigma(t)} y_1(t-\tau) \end{bmatrix} dt \\
&\quad + \begin{bmatrix} 0 \\ \delta_{1\sigma(t)} y_1(t) + \delta_{2\sigma(t)} y_1(t-\tau) \end{bmatrix} dB(t),
\end{aligned} \tag{5.3}
$$

where the control input is $U_{\sigma(t)}(t) := \eta_{1\sigma(t)} y_1^3(t) y_2(t) + \eta_{2\sigma(t)} y_1(t) + \eta_{3\sigma(t)} y_2(t) + u(t)$ with $\eta_{1\sigma(t)}, \eta_{2\sigma(t)}, \eta_{3\sigma(t)} \in \mathbb{R}$ and the disturbance $u(t) \in \mathbb{R}$.

To study SISS of the system (5.3), let the LKFs and LRFs be $V_1(t, y(t)) = V_2(t, y(t)) = y^\top(t) y(t)$. For all $t \notin \mathcal{T}$, the derivatives of Lyapunov functions satisfy

$$
\begin{aligned}
\mathscr{L}V_{\sigma(t)}(t, y_t) &\leq (2\eta_{1\sigma(t)} - 2\gamma_{3\sigma(t)}) y_1^3(t) y_2(t) + (\varepsilon + 2\eta_{2\sigma(t)} + 2\delta_{1\sigma(t)}^2) y_1^2(t) \\
&\quad - 2\gamma_1 y_2^2(t) + (2 + 2\eta_{3\sigma(t)} - 2\gamma_2) y_1(t) y_2(t) + 2\gamma_2 y_2(t) y_1(t-\tau) \\
&\quad + 2\delta_{2\sigma(t)}^2 y_1^2(t-\tau) + \varepsilon^{-1} u^2(t),
\end{aligned} \tag{5.4}
$$

and for the switching time instants, there exists $\mu \geq 1$ such that

$$
V_{\sigma(t_k)}(t_k, y(t_k)) \leq \mu V_{\sigma(t_k^-)}(t_k^-, y(t_k^-)), \quad k \in \mathbb{N}.
$$

Let $M = 2, \gamma_1 = 0.5, \gamma_2 = 1.5$, and $\mu = 1.1$. Pick $\gamma_{31} = 0.5, \eta_{11} = 1.5, \eta_{21} = -2, \eta_{31} = 0, \delta_{11} = 0.5, \delta_{21} = 0.5$ and $\gamma_{32} = 0.9, \eta_{12} = 1, \eta_{22} = -4, \eta_{32} = 1, \delta_{12} = 1, \delta_{22} = 1$. First, for the LKFs, we have from (5.4) that

$$
\begin{aligned}
\mathscr{L}V_1(t, y_t) &\leq -0.8 V_1(t, y(t)) + 0.7 V_1(t-\tau, y_t) + 2u^2(t), \\
\mathscr{L}V_2(t, y_t) &\leq -0.9 V_2(t, y(t)) + 0.6 V_2(t-\tau, y_t) + 2u^2(t).
\end{aligned}
$$

From Theorem 3.1, if $\tau_a > 1.2929$, then the system (5.3) is SISS. Second, consider the LRFs. If $V_{\sigma(t)}(t, y(t)) \geq \max\{0.5|V_{\sigma(t)}(t, y(t))|^\tau, u^2(t)\}$ for all $\sigma(t) \in \{1, 2\}$, then

$$
\mathscr{L}V_1(t, y_t) \leq -0.5 V_1(t, y(t)), \quad \mathscr{L}V_2(t, y_t) \leq -0.9 V_2(t, y(t)).
$$

From Theorem 3.2, the system (5.3) is SISS if $\boldsymbol{\theta} - \delta > 0.1906$.

Set $y(t) = [2, -3]^\top$ for $t \in [0, 0.5]$, $\tau = 0.5$, $u(t) = 0.1\sin(t)$ and the stochastic perturbation is the Gaussian white noise with zero-mean and variance of 30. Under the periodic switching time sequence, the state responses of the system (5.3) are given in Figures 5.1–5.2. Figure 5.1 gives the state response of the system (5.3) under Krasovskii-type conditions with the ADT $\tau_a = 1.5$, whereas Figure 5.2 presents the state response of the system (5.3) under Razumikhin-type conditions with $\boldsymbol{\theta} = 0.3$.

**5.2. Networked switched control systems.** Consider networked stochastic switched control systems subject to network-induced delays in both the state measurement and the switching signal. The system model is given by [5, 8]

$$
\begin{aligned}
dx(t) &= (A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t) + C_{\sigma(t)} v(t)) dt \\
&\quad + (D_{\sigma(t)} x(t) + E_{\sigma(t)} u(t) + F_{\sigma(t)} v(t)) dB(t),
\end{aligned} \tag{5.5}
$$

where $x(t) \in \mathbb{R}^{n_x}$ is the system state, $u(t) \in \mathbb{R}^{n_u}$ is the controlled input, $v(t) \in \mathbb{R}^{n_v}$ is external disturbance, and $B(t)$ is a one-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P}, \{\mathfrak{F}_t\}_{t \geq t_0})$ and satisfies $\mathbb{E}[dB(t)] = 0$ and $\mathbb{E}[dB^2(t)] = dt$. The switching signal $\sigma : \mathbb{R}_0^+ \to \mathcal{M} := \{1, \ldots, M\}$ is piecewise continuous. Assume that the matrices in (5.5) have the appropriate dimensions.

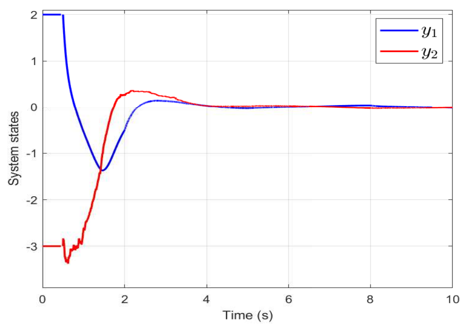


FIG. 5.1. *State response of the system* (5.3)*. The LKFs are applied with a periodic switching time sequence and the period* 1.5*.*

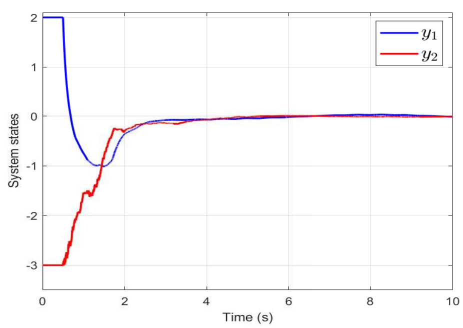


FIG. 5.2. *State response of the system* (5.3)*. The LRFs are applied with a periodic switching time sequence and* $\boldsymbol{\theta} = 0.3$*.*

Because of the network, the information to be transmitted needs to be sampled. The sampling times and the switching times are assumed to be the same. However, the network induces the delays at both the sensor and actuator sides, which in turn leads the switching to be delayed [5]. The total delays of the state measurement and the control actuation are augmented as $\tau(t)$, which is bounded by $\tau \leq h_k := t_{k+1} - t_k$, $k \in \mathbb{N}$; see [3, 5]. As a result, the switching of the plant and the controller is asynchronous in $[t_k, t_k + \tau(t_k))$ and synchronous in $[t_k + \tau(t_k), t_{k+1})$. That is, if the $i$th subsystem is active in $[t_k, t_{k+1})$, then the $j$th controller is active in $[t_k, t_k + \tau(t_k))$ and the $i$th controller is active in $[t_k + \tau(t_k), t_{k+1})$, where $i, j \in \mathcal{M}, i \neq j$. Assume that the transmission information is compressed as one single packet, the sensors are time-driven, and the controllers and the actuators are event-driven. Therefore, the networked switched control system (5.5) is modeled as

$$\begin{aligned} dx(t) = {} & (A_{\sigma(t)}x(t) + B_{\sigma(t)}K_{\sigma(t+\tau(t))}x_t + C_{\sigma(t)}v(t))dt \\ & + (D_{\sigma(t)}x(t) + E_{\sigma(t)}K_{\sigma(t+\tau(t))}x_t + F_{\sigma(t)}v(t))dB(t). \end{aligned} \tag{5.6}$$

In the following theorem, based on LKFs, sufficient conditions are derived to guarantee SISS of the system (5.6) and the stabilizing controller gains are given.

THEOREM 5.1. *Consider the system* (5.6) *and suppose that the constants* $\lambda_1 > \lambda_2 \geq 0$, $\lambda_3, \lambda_4 \geq 0$, $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2 \geq 0$, $\tau \in [0, \lambda_1^{-1})$, *and* $\mu > 1$ *are given. If there exist matrices* $X_i = X_i^\top > 0, Q_i = Q_i^\top > 0, R_i = R_i^\top > 0, Y_i$ *with appropriate dimensions such that for all* $(i,j) \in \mathcal{M} \times \mathcal{M}, i \neq j, \mathcal{H}_i, \mathcal{H}_j \leq 0$ *and*

$$Q_j - \mu Q_i \geq 0, \quad R_j - \mu R_i \geq 0, \quad X_j - \mu X_i \leq 0, \tag{5.7}$$

$$R_i - \tau\lambda_1 R_i - \lambda_1 Q_i \leq 0, \quad R_j - \tau\lambda_1 R_j - \lambda_1 Q_j \leq 0, \tag{5.8}$$

$$\mathcal{H}_i = \begin{bmatrix} \Sigma_i & B_iY_i & C_i & X_iD_i^\top \\ * & Q_i - \lambda_2 X_i & 0 & Y_i^\top E_i^\top \\ * & * & -\boldsymbol{\rho}_1 I & F_i^\top \\ * & * & * & -X_i \end{bmatrix}, \mathcal{H}_j = \begin{bmatrix} \Sigma_j & B_iY_j & C_i & X_jD_i^\top \\ * & -Q_j - \lambda_4 X_j & 0 & Y_j^\top E_i^\top \\ * & * & -\boldsymbol{\rho}_2 I & F_i^\top \\ * & * & * & -X_j \end{bmatrix}.$$

*where* $\Sigma_i = A_iX_i + X_iA_i^\top + \lambda_1 X_i + Q_i + \tau R_i$ *and* $\Sigma_j = A_iX_j + X_jA_i^\top - \lambda_3 X_j + Q_j + \tau R_j$, *then there exist stabilizing switched controllers such that the system* (5.6) *is SISS for the switching signal satisfying the ADT condition* (C.6) *in Theorem* 4.1, *and the controller gains are* $K_i = Y_iX_i^{-1}$, $i \in \mathcal{M}$.

*Proof.* To prove the theorem, we consider the worst case: $\tau(t) \equiv \tau$. As a result, in the switching interval $[t_k, t_{k+1})$, $k \in \mathbb{N}$, the closed-loop system is divided into two parts: for $t \in [t_k, t_k + \tau)$,

$$dx(t) = (A_ix(t) + B_iK_jx_t + C_iv(t))dt + (D_ix(t) + E_iK_jx_t + F_iv(t))dB(t); \tag{5.9}$$

for $t \in [t_k + \tau, t_{k+1})$,

$$dx(t) = (A_ix(t) + B_iK_ix_t + C_iv(t))dt + (D_ix(t) + E_iK_ix_t + F_iv(t))dB(t). \tag{5.10}$$

For all $\sigma(t - \tau) = i \in \mathcal{M}$, the LKFs are defined as

$$V_i(t, x(t)) := x^\top(t)P_ix(t) + \int_{t-\tau}^t x^\top(s)\bar{Q}_ix(s)ds + \int_{-\tau}^0 \int_{t+\theta}^t x^\top(s)\bar{R}_ix(s)dsd\theta,$$

where $\bar{Q}_i := P_iQ_iP_i$, $\bar{R}_i := P_iR_iP_i$, and $P_i, Q_i, R_i \in \mathbb{R}^{n_x \times n_x}$ are positive definite. Thus, it follows from Theorem 4.1 that if for all $(i, j) \in \mathcal{M} \times \mathcal{M}$ and $i \neq j$,

$$\alpha_1(|x(t)|) \leq V_i(t, x(t)) \leq \alpha_2(|x(t)|), \quad x(t) \in \mathbb{R}^{n_x}, \tag{5.11}$$

$$\mathscr{L}V_i(t, x(t)) \leq -\lambda_1 V_i(t, x(t)) + \lambda_2 V_i(t - \tau, x_t) + \boldsymbol{\rho}_1 |v(t)|^2, \ t \in [t_k + \tau, t_{k+1}), \tag{5.12}$$

$$\mathscr{L}V_i(t, x(t)) \leq \lambda_3 V_i(t, x(t)) + \lambda_4 V_i(t - \tau, x_t) + \boldsymbol{\rho}_2 |v(t)|^2, \ t \in [t_k, t_k + \tau), \tag{5.13}$$

$$V_i(t_k, x(t_k)) \leq \mu V_j(t_k, x(t_k)) + \boldsymbol{\rho}_3 |v(t_k)|^2, \tag{5.14}$$

where $\boldsymbol{\rho}_\imath \geq 0$ and $\imath = 1, 2, 3$, then the system (5.6) is SISS under the ADT condition (C.6).

Define $\alpha_1(v) := \min_{i\in\mathcal{M}}\{\lambda_{\min}(P_i)\}v^2$ and $\alpha_2(v) := \max_{i\in\mathcal{M}}\{\lambda_{\max}(P_i) + \tau(\lambda_{\max}(\bar{Q}_i) + \lambda_{\max}(\bar{R}_i))\}v^2$; then (A.1) holds. For all $t \in [t_k + \tau, t_{k+1})$, we have from (5.10) and (5.12) that

$$\begin{aligned}
&\mathscr{L}V_i(t, x(t)) + \lambda_1 V_i(t, x(t)) - \lambda_2 V_i(t - \tau, x_t) - \boldsymbol{\rho}_1 v^\top(t)v(t) \\
&\leq 2x^\top(t)P_i(A_ix(t) + B_iK_ix_t + C_iv(t)) + (D_ix(t) + E_iK_ix_t + F_iv(t))^\top P_i \\
&\quad \times (D_ix(t) + E_iK_ix_t + F_iv(t)) + x^\top(t)(\bar{Q}_i + \tau\bar{R}_i + \lambda_1 P_i)x(t) \\
&\quad - x_t^\top(\bar{Q}_i + \lambda_2 P_i)x_t - \int_{t-\tau}^t x^\top(s)(\bar{R}_i - \lambda_1\bar{Q}_i - \tau\lambda_1\bar{R}_i)x(s)ds \\
&\quad - \lambda_2 \int_{t-2\tau}^{t-\tau} x_s^\top(\bar{Q}_i + \tau\bar{R}_i)x_sds - \boldsymbol{\rho}_1 v^\top(t)v(t) \\
&\leq 2x^\top(t)P_i(A_ix(t) + B_iK_ix_t + C_iv(t)) + (D_ix(t) + E_iK_ix_t + F_iv(t))^\top P_i \\
&\quad \times (D_ix(t) + E_iK_ix_t + F_iv(t)) + x^\top(t)(\bar{Q}_i + \tau\bar{R}_i + \lambda_1 P_i)x(t) \\
&\quad - x_t^\top(\bar{Q}_i + \lambda_2 P_i)x_t - \boldsymbol{\rho}_1 v^\top(t)v(t),
\end{aligned} \tag{5.15}$$

where the first "$\leq$" holds due to Jensen's inequality and the second "$\leq$" holds because of (5.8) and positive definite matrices $P_i, \bar{Q}_i, \bar{R}_i$. As a result, the right-hand side of (5.15) can be rewritten as $\begin{bmatrix} x^\top(t) & x_t^\top & v^\top(t)\end{bmatrix} \Phi_{1i} \begin{bmatrix} x^\top(t) & x_t^\top & v^\top(t)\end{bmatrix}^\top$, where

$$\Phi_{1i} := \begin{bmatrix} \Xi_{1i} & P_iB_iK_i + \bar{D}_i^\top P_iE_iK_i & P_iC_i + \bar{D}_i^\top P_iF_i \\ * & K_i^\top E_i^\top P_iE_iK_i - \lambda_2 P_i - \bar{Q}_i & K_i^\top E_i^\top P_iF_i \\ * & * & F_i^\top F_i - \boldsymbol{\rho}_1 I \end{bmatrix},$$

$\Xi_{1i} := A_i^\top P_i + P_iA_i + D_i^\top P_iD_i + \lambda_1 P_i + \bar{Q}_i + \tau\bar{R}_i$, $i \in \mathcal{M}$, and $*$ denotes symmetric terms in block matrix. From the Schur complement lemma, $\Phi_{1i} \leq 0$ equals to

$$\Phi_{2i} := \begin{bmatrix} \Xi_{2i} & P_iB_iK_i & P_iC_i & D_i^\top P_i \\ * & -\bar{Q}_i - \lambda_2 P_i & 0 & K_i^\top E_i^\top P_i \\ * & * & -\boldsymbol{\rho}_1 I & F_i^\top P_i \\ * & * & * & -P_i \end{bmatrix} \leq 0, \tag{5.16}$$

where $\Xi_{2i} := \bar{A}_i^\top P_i + P_i\bar{A}_i + \lambda_1 P_i + \bar{Q}_i + \tau\bar{R}_i$ and $i \in \mathcal{M}$. Pre- and postmultiplying $\Phi_{2i}$ by $\mathrm{diag}\{P_i^{-1}, P_i^{-1}, I, P_i^{-1}\}$, respectively, yield that

$$\Phi_{3i} := \begin{bmatrix} \Xi_{3i} & B_iK_iP_i^{-1} & C_i & P_i^{-1}D_i^\top \\ * & -Q_i - \lambda_2 P_i^{-1} & 0 & P_i^{-1}K_i^\top E_i^\top \\ * & * & -\boldsymbol{\rho}_1 I & F_i^\top \\ * & * & * & -P_i^{-1} \end{bmatrix} \leq 0, \tag{5.17}$$

where $\Xi_{3i} := P_i^{-1}\bar{A}_i^\top + \bar{A}_iP_i^{-1} + \lambda_1 P_i^{-1} + Q_i + \tau R_i$, and $i \in \mathcal{M}$. Define $X_i := P_i^{-1}$ and $Y_i := K_iP_i^{-1}$. Thus, $\mathcal{H}_i \leq 0$ implies (5.17).

Similarly, for all $t \in [t_k, t_k + \tau)$, we obtain from (5.9) that (5.13) equals to

$$\begin{aligned} &\mathscr{L}V_j(t, x(t)) - \lambda_3 V_i(t, x(t)) - \lambda_4 V_i(t - \tau, x_t) - \boldsymbol{\rho}_2 v^\top(t)v(t) \\ &\leq x^\top(t)P_j(A_ix(t) + B_iK_ix_t + C_iv(t)) + (D_ix(t) + E_iK_ix_t + F_iv(t))^\top P_j \\ &\quad \times (D_ix(t) + E_iK_ix_t + F_iv(t)) + x^\top(t)\bar{Q}_jx(t) - x_t^\top\bar{Q}_jx_t + \tau x^\top(t)\bar{R}_jx(t) \\ &\quad - \lambda_3 x^\top(t)P_jx(t) - \lambda_4 x_t^\top P_jx_t - \boldsymbol{\rho}_2 v^\top(t)v(t). \end{aligned}$$

Just like the case for $t \in [t_k + \tau, t_{k+1})$, we obtain that $\mathcal{H}_j \leq 0$ implies (5.13).

For all $k \in \mathbb{N}$, $V_i(t_k, x(t_k)) \leq \mu V_j(t_k, x(t_k)) + \boldsymbol{\rho}_3 v^\top(t_k)v(t_k)$ holds if $P_i \leq \mu P_j, \bar{Q}_i \leq \mu\bar{Q}_j, \bar{R}_i \leq \mu\bar{R}_j$. From the Schur complement lemma and the proof of Theorem 3 in [8], $P_i - \mu P_j \leq 0$ holds if and only if $-X_i + \mu^{-1}X_j \leq 0$. That is, if (5.7) holds, then $P_i - \mu P_j \leq 0$, which implies $\bar{Q}_i - \mu\bar{Q}_j \leq P_j(\mu^2 Q_i - \mu Q_j)P_j \leq 0$ and $\bar{R}_i - \mu\bar{R}_j \leq P_j(\mu^2 R_i - \mu R_j)P_j \leq 0$. In addition, if there is a feasible solution satisfying $\mathcal{H}_i, \mathcal{H}_j \leq 0$ and (5.7)–(5.8), then the controller gains are $K_i = Y_iX_i^{-1}$, where $i \in \mathcal{M}$. ∎

If the switching is synchronous, then the conditions in Theorem 5.1 can be reduced according to Theorem 3.1. Similarly, if there exist $\rho_1, \rho_2 \in \mathcal{K}_\infty$ satisfying the small gain condition such that $V_{\sigma(t)}(t, x(t)) \geq \max\{\rho_1(|V_{\sigma(t)}(t, x(t))|^\tau), \rho_2(|u(t)|)\}$ holds for all $t \in \mathbb{R}^+_{t_0}$, then according to Theorem 4.2 and in the same fashion as the proof of Theorem 5.1, similar conditions can be derived to establish SISS of the system (5.6).

**6. Conclusions.** In this paper, stochastic stability was studied for stochastic switched nonlinear time-delay systems without/with asynchronous switching. Based on two extended Lyapunov approaches, both Krasovskii-type and Razumikhin-type stability conditions were established for both the synchronous and the asynchronous switching case. Finally, mechanical systems and networked switched control systems were used to illustrate the obtained theory. A direct extension of this paper is to analyze stochastic stability of stochastic impulsive switched time-delay systems.

## REFERENCES


[1] A. R. Teel, A. Subbaraman, and A. Sferlazza, *Stability analysis for stochastic hybrid systems: A survey*, Automatica, 50 (2014), pp. 2435–2456.

[2] Z. Li, Y. Soh, and C. Wen, *Switched and Impulsive Systems: Analysis, Design and Applications*, Springer, New York, 2005.

[3] M. Donkers, W. Heemels, N. Van De Wouw, and L. Hetel, *Stability analysis of networked control systems using a switched linear systems approach*, IEEE Trans. Automat. Control, 56 (2011), pp. 2101–2115.

[4] K. Lee and R. Bhattacharya, *Stability analysis of large-scale distributed networked control systems with random communication delays: A switched system approach*, Systems Control Lett., 85 (2015), pp. 77–83.

[5] D. Ma and J. Zhao, *Stabilization of networked switched linear systems: An asynchronous switching delay system approach*, Systems Control Lett., 77 (2015), pp. 46–54.

[6] V. Ugrinovskii and H. R. Pota, *Decentralized control of power systems via robust control of uncertain Markov jump parameter systems*, Internat. J. Control, 78 (2005), pp. 662–677.

[7] S. V. Dhople, Y. C. Chen, L. DeVille, and A. D. Domínguez-García, *Analysis of power system dynamics subject to stochastic power injections*, IEEE Trans. Circuits Syst., 60 (2013), pp. 3341–3353.

[8] W. Ren and J. Xiong, *Stability and stabilization of switched stochastic systems under asynchronous switching*, Systems Control Lett., 97 (2015), pp. 184–192.

[9] B. Zhou and A. V. Egorov, *Razumikhin and Krasovskii stability theorems for time-varying time-delay systems*, Automatica, 71 (2015), pp. 281–291.

[10] G. Yang and D. Liberzon, *A Lyapunov-based small-gain theorem for interconnected switched systems*, Systems Control Lett., 78 (2015), pp. 47–54.

[11] H. K. Khalil, *Nonlinear Systems*, Prentice-Hall, Upper Saddle River, NJ, 2002.

[12] W. Ren and J. Xiong, *Lyapunov conditions for stability of stochastic impulsive switched systems*, IEEE Trans. Circuits Syst., 65 (2018), pp. 1994–2004.

[13] J. P. Hespanha, D. Liberzon, and A. R. Teel, *Lyapunov conditions for input-to-state stability of impulsive systems*, Automatica, 44 (2008), pp. 2735–2744.

[14] I. V. Medvedeva and A. P. Zhabko, *Synthesis of Razumikhin and Lyapunov-Krasovskii approaches to stability analysis of time-delay systems*, Automatica, 51 (2015), pp. 372–377.

[15] N. N. Krasovsky, *Stability of Motion: Applications of Lyapunov's Second Method to Differential Systems and Equations with Delay*, Stanford University Press, Stanford, CA, 1963.

[16] P. Pepe and Z.-P. Jiang, *A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems*, Systems Control Lett., 55 (2005), pp. 1006–1014.

[17] F. Mazenc, S.-I. Niculescu, and M. Krstic, *Lyapunov-Krasovskii functionals and application to input delay compensation for linear time-invariant systems*, Automatica, 48 (2012), pp. 1317–1323.

[18] J. K. Hale, *Theory of Functional Differential Equations*, Springer, New York, 1977.

[19] C. Ning, Y. He, M. Wu, and J. She, *Improved Razumikhin-type theorem for input-to-state stability of nonlinear time-delay systems*, IEEE Trans. Automat. Control, 59 (2014), pp. 1983–1988.

[20] Y. Liu and W. Feng, *Razumikhin-Lyapunov functional method for the stability of impulsive switched systems with time delay*, Math. Comput. Model., 49 (2009), pp. 249–264.

[21] X. Mao, *Razumikhin-type theorems on exponential stability of neutral stochastic differential equations*, SIAM J. Math. Anal., 28 (1997), pp. 389–401.

[22] X. Wu, Y. Tang, and W. Zhang, *Input-to-state stability of impulsive stochastic time-delay systems under linear assumptions*, Automatica, 66 (2015), pp. 195–204.

[23] A. R. Teel, *Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem*, IEEE Trans. Automat. Control, 43 (1998), pp. 960–964.

[24] S. Dashkovskiy, H. R. Karimi, and M. Kosmykov, *A Lyapunov-Razumikhin approach for stability analysis of logistics networks with time-delays*, Internat. J. Systems Sci., 43 (2012), pp. 845–853.

[25] S. Zhou and T. Li, *Robust stabilization for time-delay discrete-time fuzzy systems via basis-dependent Lyapunov-Krasovskii function*, Fuzzy Sets and Systems, 151 (2005), pp. 139–153.

[26] L. Zhang and H. Gao, *Asynchronously switched control of switched linear systems with average dwell time*, Automatica, 46 (2010), pp. 953–958.

[27] J. Xiong, J. Lam, Z. Shu, and X. Mao, *Stability analysis of continuous-time switched systems with a random switching signal*, IEEE Trans. Automat. Control, 59 (2014), pp. 180–186.

[28] Y.-E. Wang, X.-M. Sun, P. Shi, and J. Zhao, *Input-to-state stability of switched nonlinear systems with time delays under asynchronous switching*, IEEE Trans. Cybernetics, 43 (2013), pp. 2261–2265.

[29] J. P. Hespanha and A. S. Morse, *Stability of switched systems with average dwell-time*, in Proceedings of the IEEE Conference on Decision and Control, 1999, pp. 2655–2660.

[30] M. A. Müller and D. Liberzon, *Input/output-to-state stability and state-norm estimators for switched nonlinear systems*, Automatica, 48 (2012), pp. 2029–2039.

[31] C. Yang and W. Zhu, *Stability analysis of impulsive switched systems with time delays*, Math. Comput. Model., 50 (2009), pp. 1188–1194.

[32] J. Liu, X. Liu, and W.-C. Xie, *Input-to-state stability of impulsive and switching hybrid systems with time-delay*, Automatica, 47 (2011), pp. 899–908.

[33] X.-M. Sun, J. Zhao, and D. J. Hill, *Stability and $\mathscr{L}_2$-gain analysis for switched delay systems: A delay-dependent method*, Automatica, 42 (2005), pp. 1769–1774.

[34] Y. Chen and W. X. Zheng, *Stability analysis and control for switched stochastic time-delay systems*, Internat. J. Robust Nonlinear Control, 26 (2015), pp. 303–328.

[35] D. Chatterjee and D. Liberzon, *Stability analysis of deterministic and stochastic switched systems via a comparison principle and multiple Lyapunov functions*, SIAM J. Control Optim., 45 (2006), pp. 174–206.

[36] P. Zhao, W. Feng, and Y. Kang, *Stochastic input-to-state stability of switched stochastic nonlinear systems*, Automatica, 48 (2012), pp. 2569–2576.

[37] X. Mao, *Stochastic Differential Equations and Applications*, 2nd ed., Horwood, Chichester, UK, 2007.

[38] J. Liu, X. Liu, and W.-C. Xie, *Class-$\mathcal{KL}$ estimates and input-to-state stability analysis of impulsive switched systems*, Systems Control Lett., 61 (2012), pp. 738–746.

[39] W.-H. Chen and W. X. Zheng, *Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays*, Automatica, 45 (2009), pp. 1481–1488.

[40] A. R. Teel and L. Praly, *On assigning the derivative of a disturbance attenuation control Lyapunov function*, Math. Control Signals Systems, 13 (2000), pp. 95–124.

[41] L. Rogers and D. Williams, *Diffusions, Markov Processes, and Martingales: Volume* 1*, Foundations*, Cambridge University Press, Cambridge, UK, 2000.

[42] C. Cai and A. R. Teel, *Characterizations of input-to-state stability for hybrid systems*, Systems Control Lett., 58 (2009), pp. 47–53.

[43] S. Dashkovskiy and A. Mironchenko, *Input-to-state stability of nonlinear impulsive systems*, SIAM J. Control Optim., 51 (2013), pp. 1962–1987.

[44] W. Ren and J. Xiong, *Stability analysis of impulsive stochastic nonlinear systems*, IEEE Trans. Automat. Control, 62 (2017), pp. 4791–4797.

[45] E. Fridman, *Tutorial on Lyapunov-based methods for time-delay systems*, Eur. J. Control, 20 (2014), pp. 271–283.

[46] F. Mazenc, M. Malisoff, and S. I. Niculescu, *Stability and control design for time-varying systems with time-varying delays using a trajectory-based approach*, SIAM J. Control Optim., 55 (2017), pp. 533–556.

[47] J. Zhang, Y. Lin, and P. Shi, *Output tracking control of networked control systems via delay compensation controllers*, Automatica, 57 (2015), pp. 85–92.

[48] S. Dashkovskiy and L. Naujok, *Nonlinear techniques to characterize prechatter and chatter vibrations in the machining of metals*, Internat. J. Bifur. Chaos, 11 (2010), pp. 449–467.

[49] F. A. Khasawneh and E. Munch, *Chatter detection in turning using persistent homology*, Mech. Systems Signal Process., 70 (2016), pp. 527–541.