linear matrix inequality algorithm is proposed to compute the filtering matrices, which can be parameterized by a positive definite matrix independent of the Lyapunov matrix. The effectiveness of the derived condition has been demonstrated by an illustrative example.

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A Negative Imaginary Lemma and the Stability of Interconnections of Linear Negative Imaginary Systems

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***Abstract*—The note is concerned with linear negative imaginary systems. First, a previously established Negative Imaginary Lemma is shown to remain true even if the system transfer function matrix has poles on the imaginary axis. This result is achieved by suitably extending the definition of negative imaginary transfer function matrices. Secondly, a necessary and sufficient condition is established for the internal stability of the positive feedback interconnections of negative imaginary systems. Meanwhile, some properties of linear negative imaginary systems are developed. Finally, an undamped flexible structure example is presented to illustrate the theory.**

***Index Terms*—Linear systems, negative imaginary systems, positive real systems, stability.**

I. INTRODUCTION

Systems which dissipate energy often lead to positive real systems. The study of positive real systems has achieved great successes both in theory and in practice [1], [2]. The positive real property may be seen as a generalization of the positive definite property of matrices to the case of dynamic systems [2], where only the real part of the transfer function matrix is considered. Positive real systems have many uses in practice. For instance, they can be realized with electrical circuits using only resistors, inductors and capacitors [1]. For mechanical positive real systems, velocity sensors and force actuators can be used to implement a control system with a guarantee of closed-loop stability.

One major limitation of positive real systems is that their relative degree must be zero or one [2]. This limits the application of positive real theory. For example, a lightly damped flexible structure with a collocated velocity sensor and force actuator can typically be modeled by a sum of second-order transfer functions as $G(s) = \sum_{i=0}^{\infty}(\psi_i^2 s/(s^2 + 2\zeta_i\omega_i s + \omega_i^2))$, where ω_i is the mode frequency, $\zeta_i > 0$ is the damping coefficient associated of the i-th mode, and ψ_i is determined by the boundary condition on the underlying partial differential equation. In some cases (for example, when using piezoelectric sensors), the sensor output is proportional to position rather than velocity. In this case, the transfer function $G(s)$ given above is the transfer function from the actuator input to the derivative of the sensor output. In the case of a lightly damped flexible structure with a collocated position sensor and force actuator, the transfer function will be of the form $G(s) = \sum_{i=0}^{\infty}(\psi_i^2/(s^2 + 2\zeta_i\omega_i s + \omega_i^2))$. It can be seen that in this case, the relative degree of the system is more than unity. Hence, the standard positive real theory will not be helpful in establishing closed-loop stability. However, such a transfer function would satisfy the following negative imaginary condition $j[G(j\omega) - G^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$. Such systems are called "systems with negative imaginary frequency

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response" in [3] or "negative imaginary systems" in this note. Compared to positive real systems, negative imaginary systems must have a relative degree between zero and two. Note also that systems with relative degree two can have the finite frequency positive real property [4], but no corresponding stability result is available in [4].

The negative imaginary property of negative imaginary systems may be seen as a generalization to the case of dynamic systems, of the negative definite property of matrices, where the imaginary part of the system transfer function matrix is considered. A complete state-space characterization of linear negative imaginary systems has been established in [3]. A necessary and sufficient condition has also been derived to guarantee the internal stability of a positive feedback interconnection of linear time-invariant MIMO negative imaginary systems.

However, all the results in [3] were built upon the requirement that the systems under consideration are asymptotically stable (that is, the poles of the systems lie in the open left half plane). Inspired by the Positive Real Lemma (see, e.g., [2, Theor. 3.1] or [5, Theor. 3]), which holds for dynamic systems that are Lyapunov stable (that is, the poles of the system are in the closed left half plane), we would like to extend the results in [3] to the case where the system poles may be on the imaginary axis.

To achieve this, negative imaginary concepts for square real-rational proper transfer function matrices, which extend corresponding ones in [3], are introduced to allow purely imaginary poles. The relationship between the negative imaginary property and the positive real property for transfer function matrices is also established in this note. Then, we show that the Negative Imaginary Lemma (that is, [3, Lemma 1]) remains true when the transfer function matrix has poles on the imaginary axis. In addition, a Strict Negative Imaginary Lemma is established and some properties of negative imaginary systems are presented. A necessary and sufficient condition for the internal stability of a positive feedback interconnection of two negative imaginary systems is proposed in terms of the dc loop gain (i.e., the loop gain at zero frequency) of the feedback system. This result extends the main result of [3] to allow for negative imaginary systems with purely imaginary poles. Note that this result is quantitatively different from the corresponding stability result (e.g., [2, Lemma 3.37]) for the feedback interconnections of positive real systems which requires no such dc gain condition.

Notation: $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrices. $\Re[\cdot]$ is the real part of complex numbers. $\lambda_{\max}(A)$ is the maximum eigenvalue for a square matrix A that has only real eigenvalues. $\bar{A}$, A^T and A^* denote the complex conjugate, the transpose and the complex conjugate transpose of a complex matrix A, respectively. Also $R^{\sim}(s)$ denotes the adjoint of a transfer function matrix $R(s)$ given by $R^T(-s)$. When $s = j\omega$, we have $R^{\sim}(j\omega) = R^T(-j\omega) = R^*(j\omega)$.

II. Negative Imaginary Transfer Function Matrices

In this section, we propose new definitions of negative imaginary transfer function matrices. These new definitions generalize the corresponding ones in [3] to the case where the transfer function matrices are allowed to have poles on the imaginary axis.

Definition 1: A square real-rational proper transfer function matrix $R(s)$ is termed *negative imaginary* if:

1) $R(s)$ has no poles at the origin and in $\Re[s] > 0$;
2) $j[R(j\omega) - R^*(j\omega)] \geq 0$ for all $\omega \in (0,\infty)$ except values of ω where $j\omega$ is a pole of $R(s)$;
3) if $j\omega_0$, $\omega_0 \in (0,\infty)$, is a pole of $R(s)$, it is at most a simple pole, and the residue matrix $K_0 \triangleq \lim_{s\to j\omega_0}(s - j\omega_0)jR(s)$ is positive semidefinite Hermitian.

Remark 1: In the above definition, the requirement of $R(s)$ having no poles at $s = 0$ ensures that the dc gain of the transfer function matrix exists. The dc gain plays an essential role in the internal stability analysis of a positive feedback interconnection of negative imaginary systems.

Remark 2: When $R(s)$ is asymptotically stable, Definition 1 coincides with the one given in [3]. Hence, the definition here can be considered as a generalization of that given in [3].

Remark 3: If a scalar transfer function $R(s)$ is negative imaginary, then $R(s)$ will have nonpositive imaginary part when $s = j\omega$, $\omega \in (0,\infty)$. However, the converse of this statement is not necessarily true. For example, $R(s) = 1/(s + a)$ has negative imaginary part for all real a when $s = j\omega$, $\omega \in (0,\infty)$. However, it is negative imaginary according to Definition 1 only if $a > 0$.

Definition 2 (See [3]): A square real-rational proper transfer function matrix $R(s)$ is termed *strictly negative imaginary* if:

1) $R(s)$ has no poles in $\Re[s] \geq 0$;
2) $j[R(j\omega) - R^*(j\omega)] > 0$ for $\omega \in (0,\infty)$.

Definition 3 (See [1]): A square real-rational transfer function matrix $F(s)$ is *positive real* if:

1) No element of $F(s)$ has a pole in $\Re[s] > 0$;
2) $F(s) + F^*(s) \geq 0$ for $\Re[s] > 0$.

The above negative imaginary concepts are closely related to the positive real concept. This point can be seen from the following lemma, which provides a necessary and sufficient condition for a real-rational transfer function matrix to be positive real.

Lemma ([1, Theor. 2.7.2]): Let $F(s)$ be a square real-rational transfer function matrix. Then $F(s)$ is positive real if and only if:

1) no element of $F(s)$ has a pole in $\Re[s] > 0$;
2) $F(j\omega) + F^*(j\omega) \geq 0$ for all real ω except values of ω where $j\omega$ is a pole of $F(s)$;
3) If $j\omega_0$ is a pole of any element of $F(s)$, it is at most a simple pole, and the residue matrix, $K_0 \triangleq \lim_{s\to j\omega_0}(s - j\omega_0)F(s)$ in case ω_0 is finite, and $K_\infty \triangleq \lim_{\omega\to\infty}(F(j\omega)/(j\omega))$ in case ω_0 is infinite, is positive semidefinite Hermitian.

Another useful lemma is as follows.

Lemma 2: If $A = A^* \geq 0$, then $\bar{A} = (\bar{A})^* \geq 0$.

Proof: The lemma follows immediately from $x^*\bar{A}x = (\bar{\bar{x}})^*\bar{A}(\bar{\bar{x}}) = \overline{\bar{x}^*A\bar{x}} = \bar{x}^*A\bar{x} \geq 0$. ■

Now, we are ready to give a description of the relationship between negative imaginary transfer function matrices and positive real transfer function matrices.

Lemma 3: Given a square real-rational strictly proper transfer function matrix $R(s)$. Then $R(s)$ is negative imaginary if and only if:

1) $R(s)$ has no poles at the origin;
2) $F(s) \triangleq sR(s)$ is positive real.

Proof: The proof is based on Definition 1 and Lemma 1. Note that $F(s)$ and $R(s)$ have the same set of poles.

(Necessity) Suppose $R(s)$ is negative imaginary. Condition 1 in Definition 1 implies that $R(s)$ has no poles at the origin and $F(s)$ has no poles in $\Re[s] > 0$.

When $j\omega$, $\omega > 0$, is not a pole of $F(s)$, Condition 2 in Definition 1 implies that $F(j\omega) + F^*(j\omega) = j\omega[R(j\omega) - R^*(j\omega)] \geq 0$. Since $s = 0$ is not a pole of $R(s)$, we have that $F(0) + F^*(0) = 0$. So $F(j\omega) + F^*(j\omega) \geq 0$ for $\omega \in [0,\infty)$ with $j\omega$ not a pole of $F(s)$. In view of Lemma 2, we have $\overline{F(j\omega) + F^*(j\omega)} \geq 0$ for $\omega \geq 0$. That is, $F(-j\omega) + F^*(-j\omega) \geq 0$ for $\omega \geq 0$. So $F(j\omega) + F^*(j\omega) \geq 0$ for $\omega \leq 0$. Thus, we have that $F(j\omega) + F^*(j\omega) \geq 0$ for any $\omega \in (-\infty,\infty)$ with $j\omega$ not a pole of $F(s)$.

If $j\omega_0$, $\omega_0 > 0$, is a pole of $F(s)$, then $\pm j\omega_0$ are poles of $R(s)$ and $F(s)$. In this case, $R(s)$ can be factored as $R(s) = (1/(s^2 + \omega_0^2))R_1(s)$. Condition 3 in Definition 1 implies that $j\omega_0$ is at most a simple pole, and that $\lim_{s\to j\omega_0}(s - j\omega_0)jR(s) = (1/(2\omega_0))R_1(j\omega_0)$ is positive semidefinite Hermitian. That is, $R_1(j\omega_0) = R_1^*(j\omega_0) \geq 0$.

In view of Lemma 2, we have $R_1(-j\omega_0) = R_1^*(-j\omega_0) \geq 0$. Now the residue matrix of $F(s)$ at $j\omega_0$ given by $\lim_{s\to j\omega_0}(s - j\omega_0)F(s) = \lim_{s\to j\omega_0}(s - j\omega_0)sR(s) = (1/2)R_1(j\omega_0)$ is positive semidefinite Hermitian. Also, the residue matrix of $F(s)$ at $-j\omega_0$ is given by $(1/2)R_1(-j\omega_0)$, which is positive semidefinite Hermitian. Moreover, $F(s)$ has no infinite poles since $R(s)$ is strictly proper. Hence, according to Lemma 1, $F(s)$ is positive real.

(Sufficiency) Suppose $F(s)$ is positive real and has no poles at the origin. According to Lemma 1, Conditions 1 and 3 of Definition 1 hold. Also, $j\omega[R(j\omega) - R^*(j\omega)] \geq 0$ for $\omega \in (-\infty, \infty)$ with $j\omega$ not a pole of $R(s)$. So $j[R(j\omega) - R^*(j\omega)] \geq 0$ for $\omega \in (0, \infty)$ with $j\omega$ not a pole of $R(s)$. Therefore, Condition 2 of Definition 1 holds and $R(s)$ is negative imaginary. ■

Example 1: As an application of Lemma 3, we can say that $R(s) = 1/(s^2 + 1)$ is negative imaginary if and only if $F(s) = s/(s^2 + 1)$ is positive real. This can be verified by directly using their definitions.

The following lemma states a useful property of negative imaginary transfer function matrices.

Lemma 4: A square real-rational proper transfer function matrix $R(s)$ is negative imaginary if and only if $R(\infty) = R^T(\infty)$ and $\hat{R}(s) \triangleq R(s) - R(\infty)$ is negative imaginary.

Proof: (Necessity) It follows from Condition 2 in Definition 1 that $\lim_{\omega\to\infty} j[R(j\omega) - R^*(j\omega)] \geq 0$, that is, $j[R(\infty) - R^T(\infty)] \geq 0$. In view of Lemma 2, $\overline{j[R(\infty) - R^T(\infty)]} \geq 0$, that is, $j[R(\infty) - R^T(\infty)] \leq 0$. Therefore, we must have that $R(\infty) = R^T(\infty)$.

Since $R(\infty) = R^T(\infty)$, we have that $j[\hat{R}(j\omega) - \hat{R}^*(j\omega)] = j[R(j\omega) - R^*(j\omega)]$ and $\lim_{s\to j\omega_0}(s - j\omega_0)j\hat{R}(s) = \lim_{s\to j\omega_0}(s - j\omega_0)jR(s)$. So $\hat{R}(s)$ is negative imaginary according to Definition 1.

(Sufficiency) The sufficient part follows as in a similar fashion to the necessity part under the condition that $R(\infty) = R^T(\infty)$. ■

Remark 4: Suppose $R(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is a negative imaginary transfer function matrix. Then $\hat{R}(s) = R(s) - R(\infty) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$, and $F(s) = s\hat{R}(s) \sim \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right]$.

The following lemma gives a property of these state-space realizations.

Lemma 5: Suppose $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, and $\det(A) \neq 0$. Then the following statements are equivalent.

1) (A, B, C, D) is a minimal realization.
2) $(A, B, C, 0)$ is a minimal realization.
3) (A, B, CA, CB) is a minimal realization.

Proof: The equivalence between Statement 1 and Statement 2 is obvious. The equivalence between Statement 2 and Statement 3 follows from the fact that A is invertible. ■

The following lemma characterizes the negative imaginary properties of the sum of negative imaginary transfer function matrices.

Lemma 6: Given two negative imaginary transfer function matrices $R(s)$, $\Delta(s)$, and a strictly negative imaginary transfer function matrix $R_s(s)$. Then:

1) $R(s) + \Delta(s)$ is negative imaginary;
2) If $\Delta(s)$ has no poles on the imaginary axis, then $R_s(s) + \Delta(s)$ is strictly negative imaginary.

Proof: To prove Part 1, let $H(s) \triangleq R(s) + \Delta(s)$. Then, the set of poles of $H(s)$ is the union of the sets of poles of $R(s)$ and $\Delta(s)$. So $H(s)$ has no poles at the origin and in $\Re[s] > 0$. For any given $\omega \in (0, \infty)$, if $j\omega$ is not a pole of $H(s)$, then $j[H(j\omega) - H^*(j\omega)] = j[R(j\omega) - R^*(j\omega)] + j[\Delta(j\omega) - \Delta^*(j\omega)] \geq 0$. If $j\omega_0$ is a pole of $H(s)$, we have three cases.

1) $j\omega_0$ is a pole of $R(s)$ but not a pole of $\Delta(s)$. Then $K_H \triangleq \lim_{s\to j\omega_0}(s - j\omega_0)jH(s) = \lim_{s\to j\omega_0}(s - j\omega_0)jR(s) + \lim_{s\to j\omega_0}(s - j\omega_0)j\Delta(s) = K_R + 0 \geq 0$, where $K_R = \lim_{s\to j\omega_0}(s - j\omega_0)jR(s)$.
2) $j\omega_0$ is not a pole of $R(s)$ but a pole of $\Delta(s)$, Then $K_H = K_\Delta \geq 0$, where $K_\Delta = \lim_{s\to j\omega_0}(s - j\omega_0)j\Delta(s)$.
3) $j\omega_0$ is a pole of both $R(s)$ and $\Delta(s)$, Then $K_H = K_R + K_\Delta \geq 0$.

Therefore, we have $0 \leq K_H < \infty$. Also $j\omega$ must be a simple pole (otherwise $K_H = \infty$). This proves that $H(s)$ is negative imaginary.

The proof for Part 2 is trivial as both $R_s(s)$ and $\Delta(s)$ have no poles on the imaginary axis. ■

III. Negative Imaginary Lemma

The Negative Imaginary Lemma proposed in this section extends the Negative Imaginary Lemma in [3] to the case where the transfer function matrices may have poles on the imaginary axis. Also, a Strict Negative Imaginary Lemma is established for strictly negative imaginary transfer function matrices.

The following lemma is analogous to the Positive Real Lemma (e.g., see [2, Lemma 3.1] or [5, Theor. 3]), where the systems may have purely imaginary poles.

Lemma 7 (Negative Imaginary Lemma): Let (A, B, C, D) be a minimal state-space realization of an $m \times m$ real-rational proper transfer function matrix $R(s)$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$. Then $R(s)$ is negative imaginary if and only if:

1) $\det(A) \neq 0$, $D = D^T$;
2) there exists a matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, such that

$$AY + YA^T \leq 0, \quad \text{and} \quad B + AYC^T = 0.$$

Proof: The equivalence follows from the following sequence of equivalent reformulations.

$R(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is negative imaginary.

$\Leftrightarrow$ $D = D^T$, and $\hat{R}(s) \sim \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$ is negative imaginary (see Lemma 4).

$\Leftrightarrow$ $\det(A) \neq 0$, $D = D^T$, and $F(s) \sim \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right]$ is positive real (according to Lemma 3 and Lemma 5).

$\Leftrightarrow$ $\det(A) \neq 0$, $D = D^T$, and there exist $X = X^T > 0$, Q, W such that

$$\begin{aligned} XA + A^TX &= -Q^TQ \\ B^TX + W^TQ &= CA \\ CB + B^TC^T &= W^TW. \end{aligned}$$

This equivalence is via the Positive Real Lemma (see, e.g., [2, Lemma 3.1]). The rest of the proof follows along the lines of the proof of [3, Lemma 1]. ■

Remark 5: It follows from the equation $B + AYC^T = 0$ that $\text{rank}(B) = \text{rank}(C)$ since both A and Y are invertible.

A useful property of negative imaginary systems is stated in the following corollary.

Corollary 1: If $R(s)$ is negative imaginary and has the minimal state-space realization (A, B, C, D), then there exists a real-rational strictly proper transfer function matrix $M(s) \sim \left[\begin{array}{c|c} A & B \\ \hline LY^{-1}A^{-1} & 0 \end{array}\right]$ such that

$$j[R(j\omega) - R^*(j\omega)] = \omega M^*(j\omega)M(j\omega)$$

when $j\omega$ is not a pole of $R(s)$. Here, $Y = Y^T > 0$ and L are the solutions of $L^TL = -AY - YA^T$ and $B + AYC^T = 0$.

Proof: Define

$$W(s) \triangleq sM(s) = \left[\begin{array}{c|c} A & B \\ \hline LY^{-1} & LY^{-1}A^{-1}B \end{array}\right]$$

$$\hat{R}(s) \triangleq R(s) - R(\infty) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

$$F(s) \triangleq s\hat{R}(s) = \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right].$$

Then we get the equation shown at the bottom of the page. That is

$$s\left[R(s) - R^{\sim}(s)\right] = -s^2 M^{\sim}(s)M(s).$$

When $s \neq 0$, we have $R(s) - R^{\sim}(s) = -sM^{\sim}(s)M(s)$. When $s = 0$, we have $R(0) = -CA^{-1}B = CYC^T = R^T(0)$ and hence $R(0) - R^{\sim}(0) = R(0) - R^T(0) = 0$. So

$$R(s) - R^{\sim}(s) = -sM^{\sim}(s)M(s)$$

holds for any s with s not a pole of $R(s)$. Let $s = j\omega$ with $j\omega$ not a pole of $R(s)$. Then, we have $R(j\omega) - R^*(j\omega) = -j\omega M^*(j\omega)M(j\omega)$. That is

$$j\left[R(j\omega) - R^*(j\omega)\right] = \omega M^*(j\omega)M(j\omega).$$

This completes the proof. ■

Remark 6: It follows from the above proof that $R(s) - R^{\sim}(s) = -sM^{\sim}(s)M(s)$ for all s with s not a pole of $R(s)$.

For strictly negative imaginary transfer function matrices, a strict version of negative imaginary lemma is also derived. The result is analogous to the Weak Strict Positive Real Lemma (see e.g., Lemma 3.18 of [2] or Theorem 1 of [6]).

Lemma 8 (Strict Negative Imaginary Lemma): Let (A, B, C, D) be a minimal state-space realization of an $m \times m$ real-rational proper transfer function matrix $R(s)$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$. Then $R(s)$ is strictly negative imaginary if and only if:

1) A is Hurwitz, $D = D^T$, $\mathrm{rank}(B) = \mathrm{rank}(C) = m$;
2) There exists a matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, such that

$$AY + YA^T \leq 0, \quad \text{and} \quad B + AYC^T = 0;$$

3) the transfer function matrix $M(s) \sim \left[\begin{array}{c|c} A & B \\ \hline LY^{-1}A^{-1} & 0 \end{array}\right]$ has full column rank at $s = j\omega$ for any $\omega \in (0, \infty)$. Here $L^T L = -AY - YA^T$. That is, $\mathrm{rank}(M(j\omega)) = m$ for any $\omega \in (0, \infty)$.

Proof: (Necessity) Suppose $R(s)$ is strictly negative imaginary. Then $R(s)$ has no poles in $\Re[s] \geq 0$, which implies that A is Hurwitz. Also $R(s)$ is negative imaginary. Hence, according to the Negative Imaginary Lemma, we have $D = D^T$ and there exists a matrix $Y = Y^T > 0$ such that $AY + YA^T \leq 0$ and $B + AYC^T = 0$.

To prove $\mathrm{rank}(B) = m$, suppose that, on the contrary, $\mathrm{rank}(B) < m$. Then there exists a nonzero vector $x \in \mathbb{R}^m$ such that $Bx = 0$. Therefore

$$\begin{aligned} x^T\{j[R(j\omega) - R^*(j\omega)]\}x &= jx^T C(j\omega I - A)^{-1}Bx \\ &\quad - jx^T B^T(-j\omega I - A^T)^{-1}C^T x \\ &= 0 \end{aligned}$$

for any $\omega \in (0, \infty)$. This contradicts $j[R(j\omega) - R^*(j\omega)] > 0$ for all $\omega \in (0, \infty)$. Thus, we conclude that $\mathrm{rank}(B) = m$. That is, B has full column rank. Similarly, we have $\mathrm{rank}(C) = m$.

Next, we prove Condition 3. It follows from Corollary 1 that

$$\omega M^*(j\omega)M(j\omega) = j\left[R(j\omega) - R^*(j\omega)\right] > 0$$

for any $\omega \in (0, \infty)$. This implies that $M(j\omega)x \neq 0$ for any $\omega \in (0, \infty)$ and any nonzero complex vector x. Therefore, $\mathrm{rank}(M(j\omega)) = m$ for any $\omega \in (0, \infty)$.

$$\begin{aligned}
W^{\sim}(s)W(s) &= \left[\begin{array}{c|c} -A^T & -Y^{-1}L^T \\ \hline B^T & B^TA^{-T}Y^{-1}L^T \end{array}\right]\left[\begin{array}{c|c} A & B \\ \hline LY^{-1} & LY^{-1}A^{-1}B \end{array}\right] \\
&= \left[\begin{array}{cc|c} A & 0 & B \\ -Y^{-1}L^TLY^{-1} & -A^T & -Y^{-1}L^TLY^{-1}A^{-1}B \\ \hline B^TA^{-T}Y^{-1}L^TLY^{-1} & B^T & B^TA^{-T}Y^{-1}L^TLY^{-1}A^{-1}B \end{array}\right] \\
&= \left[\begin{array}{cc|c} A & 0 & B \\ Y^{-1}(AY + YA^T)Y^{-1} & -A^T & Y^{-1}(AY + YA^T)Y^{-1}A^{-1}B \\ \hline B^TA^{-T}Y^{-1}(-AY - YA^T)Y^{-1} & B^T & B^TA^{-T}Y^{-1}(-AY - YA^T)Y^{-1}A^{-1}B \end{array}\right] \\
&= \left[\begin{array}{cc|c} A & 0 & B \\ Y^{-1}A + A^TY^{-1} & -A^T & Y^{-1}B - A^TC^T \\ \hline CA - B^TY^{-1} & B^T & CB + B^TC^T \end{array}\right] \\
&= \left[\begin{array}{cc|c} A & 0 & B \\ 0 & -A^T & -A^TC^T \\ \hline CA & B^T & CB + B^TC^T \end{array}\right] \\
&= \left[\begin{array}{c|c} A & B \\ \hline CA & CB \end{array}\right] + \left[\begin{array}{c|c} -A^T & -A^TC^T \\ \hline B^T & B^TC^T \end{array}\right] \\
&= F(s) + F^{\sim}(s) \\
&= s\hat{R}(s) - s\hat{R}^{\sim}(s) \\
&= s\left[\hat{R}(s) - \hat{R}^{\sim}(s)\right] \\
&= s\left[R(s) - R^{\sim}(s)\right].
\end{aligned}$$

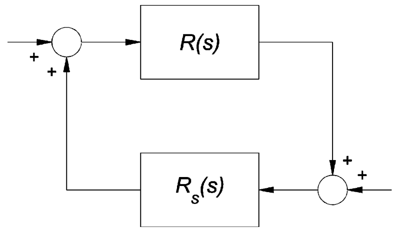

Fig. 1. Positive feedback interconnection.

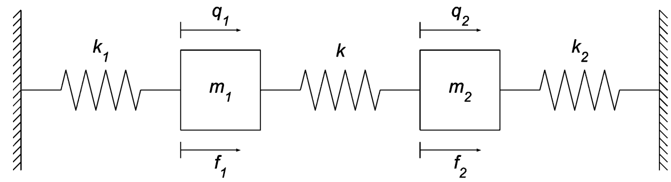

Fig. 2. Undamped flexible structure.

(Sufficiency) Condition 1 and Condition 2 imply that $R(s)$ is negative imaginary. According to Corollary 1, we have

$$j\left[R(j\omega) - R^*(j\omega)\right] = \omega M^*(j\omega)M(j\omega)$$

for any $\omega \in (0,\infty)$. Condition 3 implies that $M(j\omega)x \neq 0$ for any nonzero complex vector x. Therefore,

$$jx^*\left[R(j\omega) - R^*(j\omega)\right]x^* = \omega x^* M^*(j\omega)M(j\omega)x > 0$$

for any $\omega \in (0,\infty)$ and any nonzero complex vector x. That is

$$j\left[R(j\omega) - R^*(j\omega)\right] > 0$$

for $\omega \in (0,\infty)$. This completes the proof. ■

Remark 7: Condition 3 in Lemma 8 can be equivalently replaced by the condition that $\begin{bmatrix} A - j\omega I & B \\ LY^{-1}A^{-1} & 0 \end{bmatrix}$ has full column rank for all $\omega \in (0,\infty)$. The equivalence follows from [7, Lemma 3.33].

Similar to Corollary 1 for negative imaginary transfer function matrices, we have the following corollary for strictly negative imaginary transfer function matrices.

Corollary 2: A real-rational proper transfer function matrix $R(s)$ is strictly negative imaginary if and only if there exists a stable real-rational strictly proper transfer function matrix $M(s)$ with full column rank at $j\omega$ for any $\omega \in (0,\infty)$ such that

$$j\left[R(j\omega) - R^*(j\omega)\right] = \omega M^*(j\omega)M(j\omega)$$

for all $\omega \in (0,\infty)$.

Proof: The result follows from the proofs of Corollary 1 and Lemma 8. ■

At frequencies of zero and infinity, negative imaginary transfer function matrices have the following properties.

Corollary 3:

1) Given a negative imaginary transfer function matrix $R(s)$, then $R(0) - R(\infty) \geq 0$.
2) Given a strictly negative imaginary transfer function matrix $R(s)$, then $R(0) - R(\infty) > 0$.

Proof: The proof is the same as the proof of Lemma 2 of [3]. ■

IV. Stability of Interconnections of Negative Imaginary Systems

In this section, we consider the internal stability of a positive feedback interconnection of two negative imaginary systems, denoted by $[R(s), R_s(s)]$, as shown in Fig. 1.

A necessary and sufficient condition is provided for the stability of the system given in Fig. 1 in terms of the dc loop gain (i.e., the loop gain at zero frequency).

Theorem 1: Given a negative imaginary transfer function matrix $R(s)$ and a strictly negative imaginary transfer function matrix $R_s(s)$ that also satisfy $R(\infty)R_s(\infty) = 0$ and $R_s(\infty) \geq 0$. Then the positive feedback interconnection $[R(s), R_s(s)]$ is internally stable if and only if $\lambda_{\max}(R(0)R_s(0)) < 1$.

Proof: The proof follows along the same lines as the proof of [3, Theor. 5] except that our Lemma 7 is used instead of [3, Lemma 1] and [7, Theor. 5.7] is used instead of [7, Coroll. 5.6]. ■

The following corollaries are a restatement of the above theorem, written in the same form as the small gain theorem (see [7, Theor. 9.1]).

Corollary 4: Given $\gamma > 0$ and a strictly negative imaginary transfer function matrix $R(s)$ with $R(\infty) \geq 0$. Then, the positive feedback interconnection $[\Delta(s), R(s)]$ is internally stable for all negative imaginary transfer function matrices $\Delta(s)$ satisfying $\Delta(\infty)R(\infty) = 0$ and $\lambda_{\max}(\Delta(0)) < \gamma$ (respectively $\leq \gamma$) if and only if $\lambda_{\max}(R(0)) \leq 1/\gamma$ (respectively $< 1/\gamma$).

Proof: The proof is the same as the proof of Corollary 6 of [3].■

Corollary 5: Given $\gamma > 0$ and a negative imaginary transfer function matrix $R(s)$. Then the positive feedback interconnection $[\Delta(s), R(s)]$ is internally stable for all strictly negative imaginary transfer function matrices $\Delta(s)$ satisfying $\Delta(\infty)R(\infty) = 0$, $\Delta(\infty) \geq 0$ and $\lambda_{\max}(\Delta(0)) < \gamma$ (respectively, $\leq \gamma$) if and only if $\lambda_{\max}(R(0)) \leq (1/\gamma)$ (respectively, $< 1/\gamma$).

Proof: The proof is the same as the proof of [3, Coroll. 6]. ■

Remark 8: The results in this section are simply "restatements" of the results in [3] with the new definitions in this note. However, the results here allow one of the interconnected systems to have purely imaginary poles. Also, these results cannot be obtained from the integral quadratic constraint (IQC) theory [8] because the strictly negative imaginary systems satisfy the frequency domain condition on a punctured imaginary axis excluding the origin while the IQC theory requires that the systems satisfy a frequency domain condition on the whole imaginary axis; see [3] for more details.

V. Flexible Structure Example

To illustrate the main results of this note, let us consider an undamped flexible structure as depicted in Fig. 2. This is a two-degree-of-freedom spring mass system which is composed of two masses and three springs. The masses are attached to fixed walls via two springs and coupled together via a third one. The control inputs to the structure are the forces applied to the masses, and the outputs are the displacement of the masses. The parameters $m_1 > 0$, $m_2 > 0$, $k_1 > 0$, and $k_2 > 0$ are assumed to be known while the parameter $k > 0$ is uncertain.

Using Newton's law, the dynamics of the flexible structure can be described by

$$M\ddot{q} + Kq = f$$

where

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$$
$$K = \begin{bmatrix} k_1 + k & -k \\ -k & k_2 + k \end{bmatrix}.$$

Note that the matrices M and K are positive definite. Let $x_1 = q$ and $x_2 = \dot{q}$. Then the flexible structure can also be represented by the state-space equation

$$\begin{cases} \dot{x} = Ax + Bf \\ q = Cx + Df \end{cases}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the system state, f is the control input, and q is the measurement output. The system matrices are given by

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}, \quad C = [I \quad 0], \quad D = 0.$$

Note that this is a minimal state-space realization.

Clearly, the first condition in Lemma 7 holds. Solving the LMI in the second condition in Lemma 7 leads to a positive definite solution given by

$$Y = \begin{bmatrix} K^{-1} & 0 \\ 0 & M^{-1} \end{bmatrix}.$$

Therefore, we can conclude that the transfer function matrix $R(s) = C(sI - A)^{-1}B + D$ is negative imaginary. In other words, the flexible structure in Fig. 2 is a negative imaginary system.

In view of Theorem 1, the flexible structure $R(s)$ can be stabilized by any controller $C(s)$ which is strictly negative imaginary and satisfies $\lambda_{\max}(C(0)R(0)) < 1$ and $C(\infty) \geq 0$. For example, if the parameters in the structure are given by $m_1 = m_2 = 1$ kg and $k_1 = k_2 = 1$ N/m, then $\lambda_{\max}(R(0)) = 1$ for all $k > 0$. Note that the maximum eigenvalue of the dc gain is independent of the uncertain parameter k in this case. Hence, a simple choice for the stabilizing controller could be $C(s) = (1/(s + a))I$ with $a > 1$. A minimal state-space realization for this controller is given by $A_c = -aI$, $B_c = I$, $C_c = I$, and $D_c = 0$. Then, the stability of the closed-loop system is determined by the stability of the matrix

$$A_{cl} = \begin{bmatrix} A & BC_c \\ B_c C & A_c \end{bmatrix}.$$

It can be verified through numerical computations that the matrix A_{cl} is Hurwitz for all $k > 0$ if and only if $a > 1$.

VI. Conclusions

This note has studied the negative imaginary properties of square real-rational proper transfer function matrices. The Negative Imaginary Lemma was derived for transfer function matrices that may have poles on the imaginary axis. Also, a necessary and sufficient condition was established for the internal stability of positive feedback interconnections of negative imaginary systems. These results extend corresponding results of a previous paper in which no poles on the imaginary axis were allowed. The developed theory has applications in the feedback control of flexible structures with collocated position sensors and force actuators. For flexible structures that have rigid-body modes, however, the stability result in this note is not directly applicable because the dc gains of such systems are infinite. The question of how to extend the results of this note to deal with such systems may be an interesting area for future research.

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Dynamical Analysis of Neural Networks of Subgradient System

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***Abstract*—In this technical note, we consider a class of neural network, which is a generalization of neural network models considered in the optimization context. Under some mild assumptions, this neural network can be translated into a negative subgradient dynamical system. At first, we study the existence and uniqueness of solution of this neural network. Then, by nonsmooth Łojasiewicz inequality, we prove the convergence of the trajectories of this neural network. In the end, a constrained minimization problem is studied, which can be associated with this neural network. It is proved that the local constrained minimum of the cost function coincides with the stable equilibria point of this neural network.**

***Index Terms*—Constrained minima problem, nonsmooth Łojasiewicz inequality, subgradient system.**

I. Introduction

This technical note is concerned with dynamical analysis of neural network as follows:

$$\begin{cases} \dot{x}(t) \in -\partial E(x(t)) - N_K(x(t)) \\ x(0) = x_0 \end{cases} \tag{1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in R^n$ is the state vector, $E : R^n \rightarrow R$ is a cost function, $N_K(x)$ is the normal cone to K at

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