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Finite Frequency Negative Imaginary Systems

Junlin Xiong, Ian R. Petersen, and Alexander Lanzon

***Abstract*—This technical note is concerned with finite frequency negative imaginary (FFNI) systems. Firstly, the concept of FFNI transfer function matrices is introduced, and the relationship between the FFNI property and the finite frequency positive real property of transfer function matrices is studied. Then the technical note establishes an FFNI lemma which gives a necessary and sufficient condition on matrices appearing in a minimal state-space realization for a transfer function to be FFNI. Also, a time-domain interpretation of the FFNI property is provided in terms of the system input, output and state. Finally, an example is presented to illustrate the FFNI concept and the FFNI lemma.**

***Index Terms*—Lightly damped systems, negative imaginary systems, positive real systems.**

I. Introduction

Loosely speaking, negative imaginary linear systems are Lyapunov stable dynamical systems whose transfer function matrices satisfy the negative imaginary condition: $j[R(j\omega) - R^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$ [1]–[4]. In the SISO case, a negative imaginary transfer function $R(s)$ has non-positive imaginary part when $s = j\omega$ where $\omega \in (0, \infty)$. In other words, the phase of the transfer function satisfies $\angle R(j\omega) \in (-\pi, 0)$ where $\omega \in (0, \infty)$. Negative imaginary systems can model many practical physical systems. For example, a lightly damped flexible structure with collocated position sensors and force actuators can be modeled by a class of negative imaginary systems with transfer function given by $R(s) = \sum_{i=0}^{\infty} \psi_i^2/(s^2 + 2\zeta_i\omega_i s + \omega_i^2)$, where $\omega_i > 0$ is the mode frequency associated with the i-th mode, $\zeta_i > 0$ is the damping coefficient, and ψ_i is determined by the boundary condition on the underlying partial differential equation [1]–[3]. Also, a transfer function of the form $R(s) = k_1/(s^2 + 2\sigma_1\omega_1 s + \omega_1^2) + d_1$, which was used to model the voltage subsystem in a piezoelectric tube scanner system in [5], is negative imaginary. The negative imaginary theory is closely related to the positive real theory [6]–[8]. The concept of systems with counterclockwise input-output dynamics [9] is also related to the concept of negative imaginary systems.

In [2]–[4], a complete state-space characterization of negative imaginary linear systems was established in terms of the solvability of a linear matrix inequality and a linear matrix equation. A necessary and sufficient condition was also derived to guarantee the internal stability of a positive feedback interconnection of negative imaginary linear systems in terms of their DC loop gains. The stability result in [1]–[4]

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has been used in the case of a string of arbitrarily many coupled systems [10], where a sufficient stability condition is given in terms of a continued fraction of the subsystem DC gains. The control synthesis problem for negative imaginary systems has been explored in the full state feedback case in [11] and in the output feedback case by reformulating negative imaginary systems into systems that have bounded gain as in [12]. Moreover, a lossless negative imaginary theory has been developed to model negative imaginary systems whose poles are purely imaginary [13].

In this technical note, a new concept—finite frequency negative imaginary (FFNI) transfer function matrices—will be introduced. Roughly speaking, an FFNI transfer function matrix is a square real-rational proper transfer function, which is not only stable in the Lyapunov sense but also possesses the negative imaginary property for a finite frequency range. This concept can be considered as a generalization of the concept of negative imaginary transfer function matrices. The study of FFNI transfer function matrices is mainly motivated by the fact that many such transfer functions arise in practical control problems. For example, the capacitance subsystem of the piezoelectric tube scanner studied in [5] is modeled by $R(s) = (c_1 s^2 + c_2 s + c_3)/(s^2 + 2\sigma_1\omega_1 s + \omega_1^2)$; this transfer function is FFNI with the parameter values obtained through experiment. This example is also used in this technical note to illustrate the theory to be developed. For some lightly damped flexible structures, taking non-collocated position sensors and force actuators often leads to FFNI transfer functions. The study of FFNI transfer function matrices is also inspired by the finite frequency positive real (FFPR) theory developed in [14], where the FFPR theory was successfully applied to design dynamical systems with the FFPR property.

The organization of the technical note is as follows. Section II introduces the FFNI concept for square real-rational proper transfer function matrices. Several properties of such matrices are studied. The relationship between the FFNI property and the FFPR property of transfer function matrices is established. In Section III, the FFNI lemma—the main result of the technical note—is provided in terms of a linear matrix inequality and two linear matrix equations. The FFNI lemma gives a complete state-space characterization for systems to be FFNI in terms of their minimal realizations. When the limit frequency of an FFNI transfer function matrix approaches infinity, the FFNI lemma is shown to reduce to the negative imaginary lemma developed in [2]–[4]. Moreover, a time-domain interpretation of the FFNI property is presented in terms of the system input, output and state. Such an interpretation opens a door to develop the negative imaginary theory for nonlinear systems. An illustrative example is provided in Sections IV. Section V concludes the technical note.

Notation: $\bar{A}$, A^T and A^* denote the complex conjugate, the transpose and the complex conjugate transpose of a complex matrix A, respectively.

II. Finite Frequency Negative Imaginary Transfer Function Matrices

The idea behind the definition of FFNI transfer function matrices is that the negative imaginary conditions, which are used in [4] to define negative imaginary systems, are only required to hold on a finite frequency range.

Definition 1: A square real-rational proper transfer function matrix $R(s)$ is said to be finite frequency negative imaginary with limit frequency $\bar{\omega}$ if it satisfies the following conditions:

1) $R(s)$ has no poles at the origin and in the open right-half of the complex plane;
2) $j[R(j\omega) - R^*(j\omega)] \geq 0$ for all $\omega \in \Omega$, where $\Omega = \{\omega \in \mathbb{R} : 0 < \omega \leq \bar{\omega}, j\omega \text{ is not a pole of } R(s)\}$;
3) Every pole of $R(s)$ on $j\bar{\Omega}$, if any, is simple and the corresponding residue matrix of $jR(s)$ is positive semidefinite Hermitian, where $\bar{\Omega} = \{\omega \in \mathbb{R} : 0 < \omega \leq \bar{\omega}\}$;
4) $R(\infty) = R^T(\infty)$.

Remark 1: A complex number p is called a pole of order q of a transfer function matrix $R(s)$, if some element of $R(s)$ has a pole of order q at p and no element has a pole of order larger than q at p[15]. A simple pole is a pole of order one. Let (A, B, C, D) be a minimal state-space realization of $R(s)$, the poles of $R(s)$ are the eigenvalues of A[16].

Lemma 1: If $R(s)$ is an FFNI transfer function matrix with limit frequency $\bar{\omega}$, then the following properties hold:

1) $j[R(j\omega) - R^*(j\omega)] \leq 0$ for all $\omega \in -\Omega$.
2) Every pole of $R(s)$ in $-j\bar{\Omega}$, if any, is simple and the corresponding residue matrix of $jR(s)$ is negative semidefinite Hermitian.

Proof: (Property 1) For any $0 < \omega \leq \bar{\omega}$ such that $j\omega$ is not a pole of $R(s)$, we know that $j[R(j\omega) - R^*(j\omega)] \geq 0$. Following along the similar lines as in the proof of the necessity part of Lemma 3 in [4], we have $j[R(j\omega) - R^*(j\omega)] \leq 0$ for $-\bar{\omega} \leq \omega < 0$ when $j\omega$ is not a pole of $R(s)$.

(Property 2) Firstly, note that $R(s)$ can be factored into $R(s) = 1/[(s - j\omega)(s + j\omega)]R_1(s)$ whenever $j\omega$ is a pole of $R(s)$. Actually, $R_1(s)$ may be simply defined as $R_1(s) \triangleq (s^2 + \omega^2)R(s)$ so that the factorization is obtained. In addition, $R_1(s)$ needs not to be a proper transfer function matrix. Suppose $j\omega_0 (0 < \omega_0 \leq \bar{\omega})$ is a pole of $R(s)$. Then following along the similar lines as in the proof of Lemma 3 in [4], we have that the corresponding residue matrix of $jR(s)$ at $s = -j\omega_0$ is given by $1/(-2\omega_0)R_1(-j\omega_0) \leq 0$. ■

The FFNI concept is closely related to the FFPR concept developed in [14]. Before formally establishing the relationship between these concepts, let us recall the concept of FFPR transfer function matrices.

Definition 2: [14, Def. 4]: A square real-rational proper transfer function matrix $G(s)$ is said to be finite frequency positive real with limit frequency $\bar{\omega}$ if it satisfies the following conditions:

1) $G(s)$ has no poles in the open right-half of the complex plane;
2) $G(j\omega) + G^*(j\omega) \geq 0$, for all $\omega \in \Omega$, where $\Omega = \{\omega \in \mathbb{R} : |\omega| \leq \bar{\omega},\ j\omega \text{ is not a pole of } G(s)\}$;
3) Every pole of $G(s)$ in $j\bar{\Omega}$, if any, is simple and the corresponding residue matrix is positive semidefinite Hermitian, where $\bar{\Omega} = \{\omega \in \mathbb{R} : |\omega| \leq \bar{\omega}\}$.

In the above definition, the expression "limit frequency" is used instead of the term "bandwidth", which was used in [14]. Now, we are ready to state the relationship between FFNI transfer function matrices and FFPR transfer function matrices based on their definitions.

Lemma 2: Given a square real-rational proper transfer function matrix $R(s)$, suppose $R(s)$ has no poles at the origin, and $R(\infty) = R^T(\infty)$. Then the following statements are equivalent:

1) $R(s)$ is FFNI with limit frequency $\bar{\omega}$.
2) $\hat{R}(s) \triangleq R(s) - R(\infty)$ is FFNI with limit frequency $\bar{\omega}$.
3) $F(s) \triangleq s\hat{R}(s)$ is FFPR with limit frequency $\bar{\omega}$.

Proof: ($1 \iff 2$) The proof is similar to that of Lemma 4 in [4], and hence omitted.

($2 \iff 3$) Note that $F(s)$ and $\hat{R}(s)$ have the same set of poles. When $j\omega$ is not a pole of $\hat{R}(s)$, we have $F(j\omega) + F^*(j\omega) = j\omega[\hat{R}(j\omega) - \hat{R}^*(j\omega)]$. When $j\omega_0$ is a pole of $\hat{R}(s)$, we have $\lim_{s\to j\omega_0}(s - j\omega_0)F(s) = \omega_0 \lim_{s\to j\omega_0}(s - j\omega_0)j\hat{R}(s)$. Then the equivalence follows from the definitions and the properties in Lemma 1. ■

Lemma 2 allows us to translate an FFNI problem to an FFPR problem. In the next section, the FFNI lemma will be developed in this way.

III. Finite Frequency Negative Imaginary Lemma

The FFNI Lemma to be developed in this section gives a necessary and sufficient condition for a transfer function matrix to be FFNI in terms of the matrices appearing in a minimal state-space realization of the transfer function. This lemma could be considered as a generalization the Negative Imaginary Lemma [2]–[4] and is analogous to the FFPR Lemma [14].

Theorem 1 (Finite Frequency Negative Imaginary Lemma): Consider a real-rational proper transfer function matrix $R(s)$ with a minimal state-space realization (A, B, C, D). Suppose all poles of $R(s)$ are in the closed left-half of the complex plane, and the poles on the imaginary axis, if any, are simple. Let a positive scalar $\bar{\omega}$ be given. Also, suppose that if A has eigenvalues $j\omega_i$ $(i \in \{1, \ldots, q\})$ such that $0 < \omega_i \leq \bar{\omega}$, the residue of $(sI - A)^{-1}$ at $s = jw_i$ is given by $\Phi_i \triangleq \lim_{s \to j\omega_i}(s - j\omega_i)(sI - A)^{-1}$. Then the following statements are equivalent:

1) The transfer function matrix $R(s)$ is FFNI with limit frequency $\bar{\omega}$.
2) $\det(A) \neq 0$, $D = D^T$, and the transfer function matrix $F(s)$ with a minimal state-space realization (A, B, CA, CB) is FFPR with limit frequency $\bar{\omega}$.
3) $\det(A) \neq 0$, $D = D^T$, and $CA\Phi_i B = (CA\Phi_i B)^* \geq 0$ for all $i \in \{1, \ldots, q\}$ if A has any eigenvalues on $j\bar{\Omega}$ where $\bar{\Omega} = \{\omega \in \mathbb{R} : 0 < \omega \leq \bar{\omega}\}$. Also, there exist real symmetric matrices $P = P^T$ and $Q = Q^T \geq 0$ such that

$$PA + A^T P - A^T QA + \bar{\omega}^2 Q \leq 0 \quad (1)$$
$$C + B^T A^{-T} P = 0 \quad (2)$$
$$QA^{-1}B = 0. \quad (3)$$

4) $\det(A) \neq 0$, $D = D^T$, and $CA\Phi_i B = (CA\Phi_i B)^* \geq 0$ for all $i \in \{1, \ldots, q\}$ if A has any eigenvalues on $j\bar{\Omega}$ where $\bar{\Omega} = \{\omega \in \mathbb{R} : 0 < \omega \leq \bar{\omega}\}$. Also, there exist real symmetric matrices $Y = Y^T$ and $X = X^T \geq 0$ such that

$$AY + YA^T - AXA^T + \bar{\omega}^2 X \leq 0 \quad (4)$$
$$B + AYC^T = 0 \quad (5)$$
$$CX = 0. \quad (6)$$

Proof: $(1 \iff 2)$ Let $\hat{R}(s) = C(sI - A)^{-1}B = R(s) - R(\infty)$. We have $F(s) = CA(sI - A)^{-1}B + CB = s\hat{R}(s)$. Hence, this equivalence follows from the definitions and Lemma 2.

$(2 \iff 3)$ In view of the FFPR Lemma (that is, Theorem 3 of [14]), the statement in 2) is true if and only if the following conditions hold:

a) $CA\Phi_i B = (CA\Phi_i B)^* \geq 0$ for all $i \in \{1, \ldots, q\}$ if A has any eigenvalues on $j\bar{\Omega}$;
b) There exist real symmetric matrices $P = P^T$ and $Q = Q^T \geq 0$ such that

$$\begin{bmatrix} A & B \\ I & 0 \end{bmatrix}^T \begin{bmatrix} -Q & P \\ P & \bar{\omega}^2 Q \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \end{bmatrix} \leq \begin{bmatrix} 0 & A^T C^T \\ CA & CB + B^T C^T \end{bmatrix}. \quad (7)$$

Hence, we only need to prove that the inequality in (7) is equivalent to the inequality in (1) and the equations in (2), (3).

Note that the inequality in (7) can be rewritten as

$$\begin{bmatrix} -A^T QA + PA + A^T P + \bar{\omega}^2 Q & -A^T QB + PB - A^T C^T \\ -B^T QA + B^T P - CA & -B^T QB - CB - B^T C^T \end{bmatrix} \leq 0.$$

Pre- and post-multiplying this inequality by $\begin{bmatrix} I & 0 \\ -B^T A^{-T} & I \end{bmatrix}$ and its transpose, respectively, we obtain

$$\begin{bmatrix} -A^T QA + PA + A^T P + \bar{\omega}^2 Q & \Xi \\ \Xi^T & \bar{\omega}^2 B^T A^{-T} QA^{-1}B \end{bmatrix} \leq 0$$

where $\Xi = -A^T PA^{-1}B - \bar{\omega}^2 QA^{-1}B - A^T C^T$. Therefore, we must have $\bar{\omega}^2 B^T A^{-T} QA^{-1}B = 0$, which is equivalent to (3). Furthermore, the above inequality becomes

$$\begin{bmatrix} -A^T QA + PA + A^T P + \bar{\omega}^2 Q & -A^T PA^{-1}B - A^T C^T \\ -B^T A^{-T} PA - CA & 0 \end{bmatrix} \leq 0$$

which is equivalent to (1), (2) as the matrix A is nonsingular. Now we can conclude that the inequality in (7) is equivalent to the inequality in (1) and the equations in (2) and (3).

$(2 \iff 4)$ The proof is similar to the proof for $(2 \iff 3)$ by invoking duality. ■

Remark 2: In Theorem 1, an alternative method to compute the residue matrix Φ_i is to use the formula $\Phi_i = r_i l_i^*$ where r_i and l_i are column vectors such that $Ar_i = j\omega_i r_i$, $l_i^* A = j\omega_i l_i^*$, and $l_i^* r_i = 1$ (see Theorem 3 and Lemma 6 of [14] for more details).

Remark 3: Theorem 1 can also be derived from the generalized KYP lemma [8] by following the approach used here and paying attention to the case where the system has poles in the frequency range of interest. Furthermore, some versions of middle and high frequency negative imaginary lemmas could be derived similarly.

It follows from the definitions that when the limit frequency $\bar{\omega} \to \infty$, an FFNI transfer function matrix $R(s)$ reduces to a normal negative imaginary transfer function matrix. In the next result, we show that the conditions in the Finite Frequency Negative Imaginary Lemma will reduce to the conditions in the Negative Imaginary Lemma as $\bar{\omega} \to \infty$.

Corollary 1: Under the same assumptions as in Theorem 1, let the limit frequency $\bar{\omega} \to \infty$. Then the necessary and sufficient conditions in the finite frequency negative imaginary lemma lead to the necessary and sufficient conditions in the negative imaginary lemma.

Proof: To complete the proof, we need to show, under the assumptions of Theorem 1, that

a) the inequality in (4) and the equations in (5) and (6) are reduced to

$$AY + YA^T \leq 0, \quad \text{and} \quad B + AYC^T = 0; \quad (8)$$

b) the real symmetric matrix Y is positive definite;
c) the matrix $CA\Phi_i B$ is positive semidefinite Hermitian.

The proof is accordingly divided into three steps.

Step 1: Using similar techniques to [14], [17], the parameter X in (4) must approach zero as the limit frequency $\bar{\omega}$ approaches infinity. Hence, we have $X = 0$. Then the inequality in (4) and the equations in (5) and (6) reduce to (8).

Step 2: Under the condition that all the eigenvalues of A are in the closed left-half of the complex plane, we will prove that the real symmetric matrix Y must be positive definite.

Because all poles of $R(s)$ are assumed to be in the closed left-half plane, it follows from the inequality in (8) that $Y = Y^T \geq 0$. Next, we prove that Y is nonsingular by contradiction.

Suppose Y is singular. Then a unitary congruence transformation can be used to give

$$U^*YU = \begin{bmatrix} Y_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad U^*AU = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$
$$U^*B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CU = [\, C_1 \quad C_2 \,]$$

where $Y_1 = Y_1^T > 0$ is nonsingular and U is a unitary matrix. Hence, we can assume that the matrices Y, A, B and C in (8) are of the following forms without loss of generality:

$$Y = \begin{bmatrix} Y_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Y_1 = Y_1^T > 0, \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$
$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [\,C_1 \quad C_2\,].$$

Now, the inequality in (8) can be re-written as

$$\begin{bmatrix} A_{11}Y_1 + Y_1 A_{11}^T & Y_1 A_{21}^T \\ A_{21}Y_1 & 0 \end{bmatrix} \le 0.$$

Because the $(2,2)$ block of the above LMI is zero, we must have $A_{21}Y_1 = 0$. Furthermore, the non-singularity of Y_1 leads to $A_{21} = 0$. Therefore, the matrix A is of the form $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$. Also, the equation in (8) can be re-written as $\begin{bmatrix} B_1 + A_{11}Y_1C_1^T \\ B_2 + A_{21}Y_1C_1^T \end{bmatrix} = 0$. Because $A_{21} = 0$, we have $B_2 = 0$. Therefore, the matrix B is of the form $B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$. It follows from the matrix forms obtained above that the matrix pair (A, B) is not controllable. This contradicts the controllability of (A, B). Hence Y must be nonsingular.

In summary, we have both that $Y = Y^T \ge 0$ and that Y is nonsingular. Hence, $Y = Y^T > 0$.

Step 3: Under the assumption that the purely imaginary poles of $R(s)$, if any, are simple, we will prove that the matrix $CA\Phi_iB$ is positive semidefinite Hermitian. Firstly, in view of the equation in (8), we have that $CA\Phi_iB = -CA\Phi_iAYC^T$. In the sequel, it suffices to show that $A\Phi_iAY$ is negative semidefinite Hermitian.

Suppose that $R(s)$ has a purely imaginary pole pair at $\pm j\omega_i, \omega_i > 0$. Then there exists a nonsingular real matrix T (e.g., considering the real Jordan canonical form of the matrix A) such that $TAT^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$, where $A_1 \in \mathbb{R}^{(n-2)\times(n-2)}$ has no eigenvalues at $\pm j\omega_i$, and $A_2 \in \mathbb{R}^{2\times 2}$ is of the form $A_2 = \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix}$. Hence, we can assume that the matrices Y, A, B and C in (8) are of the following forms without loss of generality:

$$Y = \begin{bmatrix} Y_1 & Y_3 \\ Y_3^T & Y_2 \end{bmatrix} = Y^T > 0, \quad A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$
$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = [\,C_1 \quad C_2\,]$$

where A_1 is nonsingular and has no eigenvalues at $\pm j\omega_i$, $A_2 = \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix}$.

Now through direct calculation, we obtain

$$\Phi_{2i} \triangleq \lim_{s\to j\omega_i} (s - j\omega_i)(sI - A_2)^{-1} = \frac{1}{2}\begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix}$$
and
$$\Phi_i = \lim_{s\to j\omega_i} (s - j\omega_i)(sI - A)^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & \Phi_{2i} \end{bmatrix}.$$

On the other hand, the inequality in (8) can be re-written as

$$\begin{bmatrix} A_1Y_1 + Y_1A_1^T & A_1Y_3 + Y_3A_2^T \\ Y_3^TA_1^T + A_2Y_3^T & A_2Y_2 + Y_2A_2^T \end{bmatrix} \le 0. \tag{9}$$

Let $Y_2 = \begin{bmatrix} y_1 & y_3 \\ y_3 & y_2 \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Then it follows from (9) that

$$A_2Y_2 + Y_2A_2^T = \begin{bmatrix} -2\omega_i y_3 & \omega_i(y_1 - y_2) \\ \omega_i(y_1 - y_2) & 2\omega_i y_3 \end{bmatrix} \le 0.$$

Therefore, we must have $y_3 = 0$ and $y_1 = y_2 > 0$. That is, the matrix Y_2 is of the form $Y_2 = \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix}$ where $y > 0$. In this case, $A_2Y_2 + Y_2A_2^T = 0$. That is, the $(2,2)$ block of (9) is zero. Hence

$$A_1Y_3 + Y_3A_2^T = 0. \tag{10}$$

Because the matrices A_1 and $-A_2^T$ have no common eigenvalues, the Sylvester (10) has a unique solution which is given by $Y_3 = 0$. Therefore, the matrix Y is of the form $Y = \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix}$.

Now, we can calculate

$$A_2\Phi_{2i}A_2Y_2 = \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -\omega_i \\ \omega_i & 0 \end{bmatrix}\begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix} = -\frac{\omega_i^2 y}{2}\begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix}.$$

Therefore, $A_2\Phi_{2i}A_2Y_2 = (A_2\Phi_{2i}A_2Y_2)^* \le 0$. Hence

$$A\Phi_iAY = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \Phi_{2i} \end{bmatrix}\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & A_2\Phi_{2i}A_2Y_2 \end{bmatrix} \le 0.$$

Therefore, we have $-CA\Phi_iAYC^T = (-CA\Phi_iAYC^T)^* \ge 0$. This completes the proof. ■

The following theorem provides a time-domain interpretation of the FFNI properties in terms of the system input, output and state. It is expected that this result may give us a deeper understanding of FFNI systems.

Theorem 2: Consider a proper stable transfer function matrix $R(s)$ with $R(\infty) = R^T(\infty)$. Let u, y and x be the input, the output and the state of a minimal realization of $R(s)$. Then, the following statements are equivalent:

1) $R(s)$ is FFNI with limit frequency $\bar{\omega}$.
2) The inequality

$$\int_{-\infty}^{\infty} [\dot{y}(t) - D\dot{u}(t)]^T u(t)dt \ge 0 \tag{11}$$

holds for all square integrable and differentiable inputs u such that

$$\int_{-\infty}^{\infty} \dot{x}(t)\dot{x}^T(t)dt \le \bar{\omega}^2 \int_{-\infty}^{\infty} x(t)x^T(t)dt. \tag{12}$$

Proof: Let (A, B, C, D) be a minimal state-space realization of $R(s)$. Then the linear system whose transfer function is given by $R(s)$ can be represented as

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t). \end{cases} \tag{13}$$

Let us consider a new transfer function matrix $F(s) \triangleq s[R(s) - R(\infty)]$. Then (A, B, CA, CB) is a minimal state-space realization of $F(s)$ and the corresponding dynamical system can be represented by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ \tilde{y}(t) = CAx(t) + CBu(t). \end{cases} \tag{14}$$

In view of Lemma 2, the transfer function matrix $R(s)$ is FFNI with limit frequency $\bar{\omega}$ if and only if the transfer function matrix $F(s)$ is FFPR with limit frequency $\bar{\omega}$. In view of Theorem 4 of [14], $F(s)$ is FFPR with limit frequency $\bar{\omega}$ if and only if the passivity property

$$\int_{-\infty}^{\infty} \tilde{y}^T(t)u(t)dt \geq 0$$

holds for all square integrable inputs u such that the inequality (12) holds.

On the other hand, it follows from the system equations in (13) and (14) that

$$\tilde{y}(t) = C[Ax(t) + Bu(t)] = C\dot{x}(t) = \dot{y}(t) - D\dot{u}(t).$$

Therefore, $F(s)$ is FFPR with limit frequency $\bar{\omega}$ if and only if the inequality in (11) holds for all square integrable and differentiable inputs u such that (12) holds. ■

Remark 4: When the transfer function matrix $R(s)$ in Theorem 2 is strictly proper, (i.e., $D = 0$), the requirement of differentiability of inputs can be removed.

Remark 5: The statement 2) can be directly obtained from FFNI lemma. Pre- and post-multiplying the inequality (1) by $\dot{x}^T(t)A^{-T}$ and its transpose, respectively, we have

$$\dot{x}^T(t)\left(A^{-T}P + PA^{-1} - Q + \bar{\omega}^2 A^{-T}QA^{-1}\right)\dot{x}(t) \leq 0.$$

In view of the equations in (2) and (3), we obtain

$$\begin{aligned} &\dot{x}^T(t)PA^{-1}\dot{x}(t) \\ &\quad = \dot{x}^T(t)PA^{-1}[Ax(t) + Bu(t)] \\ &\quad = \dot{x}^T(t)Px(t) - [\dot{y} - D\dot{u}(t)]^T u(t) \\ &\dot{x}^T(t)A^{-T}QA^{-1}\dot{x}(t) \\ &\quad = [Ax(t) + Bu(t)]^T A^{-T}QA^{-1}[Ax(t) + Bu(t)] \\ &\quad = x^T(t)Qx(t). \end{aligned}$$

Hence, the above inequality can be written as

$$\bar{\omega}^2 x^T(t)Qx(t) - \dot{x}^T(t)Q\dot{x}(t) + \frac{d}{dt}\left(x^T(t)Px(t)\right) \leq 2[\dot{y} - D\dot{u}(t)]^T u(t).$$

Next, following similar lines to the proof of [14, Theorem 4], the statement 2) in Theorem 2 can be obtained.

Remark 6: In view of statement 2) in Theorem 2, FFNI systems may be considered as systems that possess the property (11) with respect to control inputs that do not drive the states too quickly; here the quickness is quantified by $\bar{\omega}$ in the sense of (12), and it follows from (12) that $\|\dot{x}\|_2 \leq \bar{\omega}\|x\|_2$. Similar interpretations have been given for FFPR systems in [17].

Remark 7: A frequency domain interpretation of the above result can be obtained via Parseval's Theorem. Assume that $D = 0$ to simplify the argument. Then the time domain property (11) can be written as a frequency domain property

$$\frac{1}{2\pi}\int_0^{\infty} \omega\hat{u}^*(\omega)\left\{j\left[R(j\omega) - R^*(j\omega)\right]\right\}\hat{u}(\omega)d\omega \geq 0 \tag{15}$$

and the property (12) can be written as

$$\Re\left[\int_0^{\bar{\omega}} (\bar{\omega}^2 - \omega^2)\hat{x}(\omega)\hat{x}^*(\omega)d\omega\right] \geq \Re\left[\int_{\bar{\omega}}^{\infty} (\omega^2 - \bar{\omega}^2)\hat{x}(\omega)\hat{x}^*(\omega)d\omega\right] \tag{16}$$

where $\hat{u}(\omega)$ and $\hat{x}(\omega)$ are the Fourier transforms of the control input $u(t)$ and the system state $x(t)$, respectively; $\Re[\cdot]$ denotes the real part of a complex matrix. Therefore, FFNI systems are systems that possess the frequency domain property (15) with respect to control inputs that mainly excite the system with natural frequency below $\bar{\omega}$ in the sense of (16). Similar interpretations have been given for FFPR systems in [14].

Remark 8: As the limit frequency $\bar{\omega}$ approaches to infinity, the following can be observed:

1) The statement 1) in Theorem 2 will reduce to that $R(s)$ is negative imaginary.
2) The constraint in (12) will always hold for any square integrable input ($D = 0$ is assumed). Hence, the statement 2) in Theorem 2 will reduce to the condition that the inequality (11) holds for all square integrable inputs.

Because the two statements in Theorem 2 are equivalent, they provide us with an approach to characterize the negative imaginary property in the time domain. Theorem 2 also makes it possible to generalize the negative imaginary results for linear systems to the case of nonlinear systems. Although nonlinear systems usually do not have transfer functions in the frequency domain, they do have the input, the state and the output in the time domain. It should be noticed that the time domain interpretation of negative imaginary systems is related to the property of counter-clockwise input-output dynamical systems introduced in [9].

IV. Illustrative Example

To illustrate the FFNI concept and the FFNI lemma, the piezoelectric tube example studied in [5], [18] is considered in this section. The piezoelectric tube is used in the scanning unit of scanning tunneling microscopes and atomic force microscopes. The inputs to the piezoelectric tube are two voltage signals: V_{x+} and V_{y+}, which are applied to the input ends of the electrodes of the piezoelectric tube. The outputs to the piezoelectric tube are classified into two groups. The first group is the voltages V_{x-} and V_{y-}, which are the voltages at the output ends of the electrodes. The second output group is the distances dx (x-axis direction) and dy (y-axis direction) between an aluminum cube and capacitive sensor heads. These distances are measured by capacitive sensors in terms of the change in the capacitance between the aluminum cube and the heads of the capacitive sensors. Accordingly, the transfer function from input $[V_{x+} \quad V_{y+}]^T$ to output $[V_{x-} \quad V_{y-}]^T$ is called the voltage subsystem transfer function of the tube; the transfer function from input $[V_{x+} \quad V_{y+}]^T$ to output $[dx \quad dy]^T$ is called the capacitance subsystem transfer function of the tube.

For the capacitance subsystem of the tube, the experiment in [5] shows that the transfer functions form V_{x+} to dx and from V_{y+} to dy are given by

$$G^{(v)}_{d_x x}(s) = G^{(v)}_{d_y y}(s) = \frac{c_1 s^2 + c_2 s + c_3}{s^2 + 2\sigma_1\omega_1 s + \omega_1^2}.$$

Note that the equality $G^{(v)}_{d_x x}(s) = G^{(v)}_{d_y y}(s)$ is expected because of the symmetric alignment of the capacitive sensors and the aluminum cube faces in the x and y directions [5]. The parameter values of the transfer function are given by $2\sigma_1\omega_1 = 60.2$, $\omega_1^2 = 2.8488 \times 10^7$, $c_1 = 0.0055$, $c_2 = -112.3$ and $c_3 = 1.807 \times 10^6$ (see [5, Table I]).

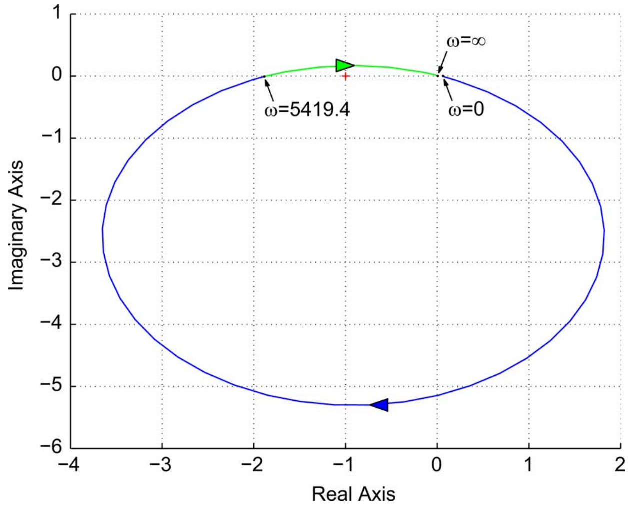

Fig. 1. Nyquist plot of piezo tube frequency response ($\omega \geq 0$).

Now, we show that the above transfer function is actually FFNI, and determine the corresponding limit frequency. Since the transfer function $G_{d_x x}^{(v)}(s)$ has no purely imaginary poles, we only need to consider the imaginary part of $G_{d_x x}^{(v)}(j\omega)$ on $(0, \infty)$. It follows from a direct computation that

$$\Im[G_{d_x x}^{(v)}(j\omega)] = \frac{\omega(-3.3080 \times 10^9 + 112.6311\omega^2)}{(2.8488 \times 10^7 - \omega^2)^2 + 3624\omega^2}$$

where $\Im[\cdot]$ denotes the imaginary part of a complex number. Therefore, $G_{d_x x}^{(v)}(s)$ is FFNI with limit frequency $\bar{\omega} = \sqrt{3.3080 \times 10^9 / 112.6311} = 5419.4$. The limit frequency can also be obtained through a Nyquist plot as shown in Fig. 1. It can be seen from Fig. 1 that the imaginary part of $G_{d_x x}^{(v)}(j\omega)$ is negative for $0 < \omega < 5419.4$.

To verify the FFNI Lemma for this example, we first found a minimal state-space realization of $G_{d_x x}^{(v)}(s)$ with

$$A = \begin{bmatrix} -60.2 & -6955.0781 \\ 4096 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 32 \\ 0 \end{bmatrix}$$
$$C = [-3.5197 \quad 12.5909], \quad D = 0.0055.$$

Now, solving the linear matrix inequality in (1) and the linear matrix equations in (2), (3) with $\bar{\omega} = 5419.4$, we found a feasible solution is given by

$$P = \begin{bmatrix} 1605.013754986344 & -764.993387142813 \\ -764.993387142813 & 2736.584151540312 \end{bmatrix}$$
$$Q = \begin{bmatrix} 0.219981883719332 & 0 \\ 0 & 0 \end{bmatrix}.$$

If we set the limit frequency $\bar{\omega}$ to be a slightly larger number, say 5419.5, then (1)–(3) have no feasible solutions. According to the FFNI lemma, the transfer function $G_{d_x x}^{(v)}(s)$ is FFNI with limit frequency $\bar{\omega} = 5419.4$ but not FFNI with limit frequency $\bar{\omega} = 5419.5$. This confirms the above findings via both direct computations and the Nyquist plot.

V. Conclusions

This technical note has studied the FFNI property of dynamical systems. The concept of FFNI transfer function matrices was first introduced. Then an FFNI lemma was derived for dynamical systems to be FFNI in terms of their minimal state-space realizations. A time-domain interpretation of the FFNI property was also proposed in terms of the system input, output and state. Finally, the FFNI lemma was illustrated by an example found in a piezoelectric tube scanner system. The time domain interpretation opens a door to generalize the negative imaginary theory from linear systems to nonlinear systems. Another area for future research is to develop a stability result for interconnected FFNI systems.

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