Stability Analysis of Impulsive Stochastic Nonlinear Systems

Wei Ren and Junlin Xiong, *Member, IEEE*

***Abstract*—This paper studies stochastic input-to-state stability and stochastic global stability for impulsive stochastic nonlinear systems. Using fixed dwell-time condition and Lyapunov-based approach, sufficient conditions are established for the stability properties. Two cases are studied: 1) the case that the continuous dynamics is stable and 2) the case that the impulsive effects are stable. Furthermore, the relations among different dwell-time conditions are studied. Finally, two examples are used to illustrate the developed theory.**

***Index Terms*—Fixed dwell-time (FDT) condition, nonlinear impulsive systems, stochastic stability, stochastic systems.**

I. Introduction

As a special class of hybrid systems, impulsive systems are dynamical systems involving continuous-time dynamics and instantaneous state jumps; see [1]–[3]. Impulsive systems have been studied extensively in the literature [4]–[8], and have many practical applications such as in networked control systems [9], synchronization of complex network [10], and chaotic secure communication [11].

In practice, numerous physical systems are usually affected by the random noises. For instance, the intervals of information transmission in communication network may be random [9]; financial systems may encounter abrupt changes of volatility rates [3]. These phenomena lead to stochastic modeling and control, which results in impulsive stochastic systems. Many salient results on the impulsive stochastic systems could be found in the literature; see [12]–[16]. For instance, p-moment stability criteria were presented in [13] for impulsive stochastic nonlinear systems with Markovian switching. Stochastic stability and robust control have been addressed in [15] for uncertain impulsive linear systems. Nevertheless, most previous results are for impulsive (stochastic) linear systems [14], [15] or deterministic impulsive nonlinear systems [4], [5], [13]. There are few works on impulsive stochastic nonlinear systems. In this paper, we study the stability properties of impulsive stochastic nonlinear systems.

To study stability of impulsive systems, Lyapunov functions and dwell-time conditions are commonly used in the literature; see [4]–[6] and [16]–[19]. In terms of Lyapunov function, Lyapunov–Krasovskii functions (e.g., [10], [12], and [16]) and Lyapunov-Razumikhin functions (e.g., [16], [19], and [20]) are implemented. However, both of Lyapunov–Krasovskii functions and Lyapunov–Razumikhin functions are generally exponential; see [4], [10]–[13], and [20]. Because exponential Lyapunov functions are not necessarily existent or easy to be constructed [5], the exponential assumption limits the classes of the impulsive systems that could be studied via Lyapunov approaches. On the other hand, different types of dwell time have been considered in the past decades, such as average dwell time (ADT) [4], [10], [12], [18], minimum dwell time (also called fixed dwell time (FDT) in [18]) [5], [14], maximum dwell time [5], [14], and constant dwell time [14], etc. Based on the aforementioned types of dwell time, stability conditions have been obtained in [14] for impulsive stochastic linear systems. However, except for ADT [4], [12], [17], other types of dwell time are seldom applied for impulsive stochastic nonlinear systems.

In this paper, we study a class of impulsive stochastic nonlinear systems using general Lyapunov function and FDT condition. Because the external signals (e.g., external disturbances and controlled input) affect the continuous evolution and/or the state jumps, two situations are considered. The first situation is that the continuous dynamics is stable; the second situation is that the discrete impulses is stable. For these two situations, sufficient conditions are established in Section III for stochastic input-to-state stability (SISS) and stochastic global stability (SGS) of impulsive stochastic nonlinear systems. Compared with the previous works [4], [12], [18], the general Lyapunov function is applied in this paper. That is, our results allow the Lyapunov function to be not exponential, which is more practical. In addition, compared with the ADT condition that could not guarantee the compactness or the sparseness of the discrete jumps, the FDT condition is implemented, which just gives a lower bound on the impulsive intervals. Furthermore, the relation between the FDT condition and the (generalized) ADT condition is discussed in this paper; see Section III-C. As a result, the obtained results allow us to study a richer class of impulsive stochastic nonlinear systems, especially those that cannot be analyzed using exponential Lyapunov functions and ADT condition. Two numerical examples are used in Section IV to illustrate the developed results. Both the two examples could not be analyzed by exponential Lyapunov function and ADT.

Notation: The notation used in this paper is fairly standard. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^+$ (or $\mathbb{N}^+$) stands for the sets of the nonnegative numbers (or integers) and $\mathbb{R}^+_{t_0} = \{t \in \mathbb{R}^+ | t \geq t_0\}$. $|\cdot|$ represents the Euclidean vector norm; $\mathbb{P}\{\cdot\}$ denotes the probability measure; $\mathbb{E}[\cdot]$ denotes the mathematical expectation. Superscript $\top$ denotes the transpose of the vectors or matrices. $\text{tr}(A)$ denotes the trace of a square matrix A. $\mathcal{O}$ represents the set of all functions that are continuous, zero at zero; $\mathcal{P}$ denotes the set of the functions that belong to $\mathcal{O}$ and are positive in $\mathbb{R}^+$. $\mathcal{C}^{2,1}$ stands for the space of the functions that are continuously twice differentiable on the first augment and continuously differentiable on the second augment. Denote by $\alpha^{-1}(t)$ the inverse of the function $\alpha(t) : \mathbb{R}^+ \to \mathbb{R}^+$. A function $\alpha(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{K}$ if it belongs to class $\mathcal{P}$ and strictly increasing; $\alpha(t)$ is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded; $\alpha(t)$ is of class $\mathcal{L}$ if it is continuous and strictly decreasing to zero as $t \to \infty$. A function $\beta(s,t) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{KL}$ if $\beta(s,t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s,t)$ decreases to zero

Manuscript received December 4, 2016; revised December 5, 2016; accepted March 23, 2017. Date of publication March 28, 2017; date of current version August 28, 2017. The work was supported by the National Natural Science Foundation of China under Grant 61374026. Recommended by Associate Editor L. Zhang. *(Corresponding author: Junlin Xiong.)*

The authors are with the Department of Automation, University of Science and Technology of China, Hefei 230026, China (e-mail: gtpp@ustc.mail.edu.cn; junlin.xiong@gmail.com).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TAC.2017.2688350

as $t \to 0$ for each fixed $s \geq 0$. $\mathcal{L}_\infty^n$ denotes the set of all the measurable and locally essentially bounded signal $x \in \mathbb{R}^n$ on $\mathbb{R}_{t_0}^+$ with norm $\|x\| := \sup_{t \geq t_0} \inf_{\{\mathcal{A} \subset \Omega, \mathbb{P}\{\mathcal{A}\}=0\}} \sup\{|x(t,\omega)| | \omega \in \Omega \setminus \mathcal{A}\}$.

II. Problem Formulation

Consider the impulsive stochastic nonlinear system

$$\begin{cases} dx(t) = f(t,x,u)dt + g(t,x,u)dw(t), & t \in \mathbb{R}_{t_0}^+ \backslash \mathcal{T} \\ x(t) = h(x(t^-), u(t^-)), & t \in \mathcal{T} \end{cases} \tag{1}$$

where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^{n_x}$ is the system state, $u(t) \in \mathcal{U} \subseteq \mathcal{L}_\infty^{n_u}$ is the external input, $w(t) \in \mathbb{R}^{n_w}$ is a Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq t_0}, \mathbb{P})$. Impulsive time sequence $\mathcal{T} := \{t_1, t_2, \ldots\}$ is strictly increasing and approaches to infinity. The functions $f : \mathbb{R}_{t_0}^+ \times \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ and $g : \mathbb{R}_{t_0}^+ \times \mathcal{X} \times \mathcal{U} \to \mathbb{R}^{n_x \times n_w}$ are assumed to be continuous with respect to t, x, u and uniformly locally Lipschitz with respect to x, u; $f(\cdot, 0, 0) = 0$, $g(\cdot, 0, 0) = 0$. The function $h : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ is continuous with respect to x and u. Assume that given an initial condition, there is a unique stochastic process satisfying the system (1); see [5] and [13].

Definition 1: Given an impulsive time sequence $\mathcal{T}$, the impulsive stochastic nonlinear system (1) is *SISS*, if for an arbitrary $\varepsilon \in (0,1)$, there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ such that for all $x(t_0) \in \mathcal{X}$, $u \in \mathcal{U}$

$$\mathbb{P}\{|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma(\|u\|)\} \geq 1 - \varepsilon \quad \forall t \in \mathbb{R}_{t_0}^+. \tag{2}$$

Given a set $\mathcal{S}$ of the admissible impulsive time sequences, if the system is SISS for every $\mathcal{T} \in \mathcal{S}$ and β, γ do not depend on the choice of $\mathcal{T}$, then the system (1) is *uniformly SISS* over $\mathcal{S}$.

Definition 2: Given an impulsive time sequence $\mathcal{T}$, the impulsive stochastic nonlinear system (1) is *SGS*, if for an arbitrary $\varepsilon \in (0,1)$, there exist $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that for all $x(t_0) \in \mathcal{X}$, $u \in \mathcal{U}$

$$\mathbb{P}\{|x(t)| \leq \gamma_1(|x(t_0)|) + \gamma_2(\|u\|)\} \geq 1 - \varepsilon \quad \forall t \in \mathbb{R}_{t_0}^+.$$

Given a set $\mathcal{S}$ of the admissible impulsive time sequences, if the system is SGS for every $\mathcal{T} \in \mathcal{S}$ and γ_1, γ_2 do not depend on the choice of $\mathcal{T}$, then the system (1) is *uniformly SISS* over $\mathcal{S}$.

The aforementioned definitions are parallel to those given in [17] for stochastic switched nonlinear systems, to those given in [5] for deterministic impulsive nonlinear systems. To investigate the stochastic stability properties of the system (1), a differential operator of the $\mathcal{C}^{2,1}$ functions and the SISS-Lyapunov function are introduced as follows.

Definition 3 (see[21]): Given any $\mathcal{C}^{2,1}$ function $V : \mathcal{X} \times \mathbb{R}_{t_0}^+ \to \mathbb{R}^+$, the differential operator $\mathscr{L}$ associated with the continuous stochastic differential equation in (1), is defined as

$$\mathscr{L}V(x,t) := \frac{\partial V(x,t)}{\partial t} + \frac{\partial V(x,t)}{\partial x} f(t,x,u) + \frac{1}{2}\left[g^\top(t,x,u) \frac{\partial^2 V(x,t)}{\partial x^2} g(t,x,u)\right].$$

By Itô's formula in [22, ch. IV. 3], it obtains that

$$dV(x,t) = \mathscr{L}V(x,t)dt + \frac{\partial V(x,t)}{\partial x} g(t,x,u)dw(t), \quad t \in \mathbb{R}_{t_0}^+ \backslash \mathcal{T}.$$

Definition 4: A $\mathcal{C}^{2,1}$ function $V : \mathcal{X} \times \mathbb{R}_{t_0}^+ \to \mathbb{R}^+$ is called an *SISS-Lyapunov function*, if there exist $\alpha_1, \alpha_2, \rho \in \mathcal{K}_\infty$, $\psi \in \mathcal{P}$ and $\varphi \in \mathcal{O}$ such that for all $x(t_0) \in \mathcal{X}$, $u \in \mathcal{U}$

$$\alpha_1(|x|) \leq V(x,t) \leq \alpha_2(|x|), \quad t \in \mathbb{R}_{t_0}^+ \tag{3}$$

$$|x| \geq \rho(\|u\|) \Rightarrow \begin{cases} \mathscr{L}V(x,t) \leq -\varphi(V(x,t)), & t \in \mathbb{R}_{t_0}^+ \backslash \mathcal{T} \\ V(h(x,u),t) \leq \psi(V(x,t)), & t \in \mathcal{T}. \end{cases} \tag{4}$$

In addition, if $\varphi(v) = cv$ and $\psi(v) = e^{-d}v$, where $c, d \in \mathbb{R}$, then V is called an *exponential SISS-Lyapunov function*.

The SISS-Lyapunov function in Definition 4 is parallel to the one for the stochastic nonlinear systems in [17] and those for the deterministic impulsive systems in [4] and [5]. If $c, d > 0$, then V is decreasing along the time, which implies that the system (1) is SISS for all the impulsive time sequences. If $c, d < 0$, then V is not decreasing along the time, which means the system (1) is not stable. Therefore, we consider the case of $cd < 0$ in this paper.

Remark 1: For the general stochastic nonlinear systems, SISS-Lyapunov function is equivalent to the SISS property; see [21] and [23]. Though this equivalent relationship does not hold for impulsive/switched stochastic systems, SISS-Lyapunov function still plays an essential role in stability analysis of SISS; see [4], [13], [17], and [24].

III. SISS Analysis

In this section, sufficient conditions are established to guarantee the SISS and SGS properties for the system (1). Both the situation that the continuous dynamics is stable and the situation that the discrete dynamics is stable are considered. In addition, the relationship between the (reverse) FDT condition and the (reverse) ADT condition is discussed.

A. Stable Continuous Dynamics Situation

We start with the first situation. Define $\mathcal{S}_{1\theta} := \{\mathcal{T} = \{t_1, t_2, \ldots\} | \mathcal{T} \subseteq \mathbb{R}_{t_0}^+, t_{k+1} - t_k \geq \theta \; \forall k \in \mathbb{N}^+\}$ for certain $\theta > 0$. In the following, based on the Lyapunov approach and FDT condition (5), SISS of the system (1) is established.

Theorem 1: Consider the impulsive stochastic nonlinear system (1). Suppose that $V : \mathcal{X} \times \mathbb{R}_{t_0}^+ \to \mathbb{R}^+$ is an SISS-Lyapunov function for (1), where $\varphi \in \mathcal{P}$ is convex and $\psi \in \mathcal{P}$ is concave. If there exist certain $\theta, \delta > 0$ such that for all $a > 0$

$$\int_a^{\psi(a)} \frac{ds}{\varphi(s)} \leq \theta - \delta \tag{5}$$

then the system (1) is SISS for all impulsive time sequences $\mathcal{T} \in \mathcal{S}_{1\theta}$.

Proof: Given an impulsive time sequence $\mathcal{T} \in \mathcal{S}_{1\theta}$, SISS of the system (1) will be proven by constructing the functions β and γ such that (2) holds. Therefore, the proof is divided into two cases: $u \equiv 0$ and $u \neq 0$.

Case 1: $u \equiv 0$. In this case, using the FDT condition (5), we first determine a function $F : \mathbb{R}^+ \to \mathbb{R}$ such that F is decreasing along the values of the Lyapunov function at the impulsive times, and then, construct a $\beta \in \mathcal{KL}$ to bound $\mathbb{E}[V(x(t),t)]$ for all $t \in \mathbb{R}_{t_0}^+$. Since $u \equiv 0$, (4) is written as

$$\mathscr{L}V(x,t) \leq -\varphi(V(x,t)), \quad t \in \mathbb{R}_{t_0}^+ \backslash \mathcal{T}$$
$$V(h(x,u),t) \leq \psi(V(x,t)), \quad t \in \mathcal{T}. \tag{6}$$

Because $\varphi \in \mathcal{P}$ is convex and $\psi \in \mathcal{P}$ is concave, it follows from (5) and Jensen's inequality in [22, ch. II. 18.3] that

$$\mathbb{E}[\mathscr{L}V(x,t)] \leq -\varphi(\mathbb{E}[V(x,t)]), \quad t \in \mathbb{R}_{t_0}^+ \backslash \mathcal{T} \tag{7}$$

$$\mathbb{E}[V(h(x,u),t)] \leq \psi(\mathbb{E}[V(x,t)]), \quad t \in \mathcal{T}. \tag{8}$$

If there exists a $\bar{t} \in [t_k, t_{k+1})$ such that $\mathbb{E}[V(x(\bar{t}), \bar{t})] = 0$, then the equilibrium point $x = 0$ implies that $\mathbb{E}[V(x(t),t)] \equiv 0$ for all $t > \bar{t}$. Next, only the case that $\mathbb{E}[V(x(t),t)] > 0$ needs to be studied.

Based on the fact that $d\,\mathbb{E}[V(x(t),t)] = \mathbb{E}[\mathscr{L}V(x(t),t)]dt$ in [17], integrating (7) leads to the fact that for all $t \in [t_k, t_{k+1})$

$$\int_{t_k}^{t} \frac{\mathbb{E}[\mathscr{L}V(x(s),s)]ds}{\varphi(\mathbb{E}[V(x(s),s)])} \leq -(t-t_k). \tag{9}$$

Based on (9), define the following function:

$$F(\varrho) := \int_{\nu}^{\varrho} \frac{ds}{\varphi(s)} \tag{10}$$

where $\nu > 0$ is fixed and $\varrho > 0$. Observe that $F : \mathbb{R}^+ \to \mathbb{R}$ is continuous and strictly increasing, so is its inverse $F^{-1} : \mathbb{R} \to \mathbb{R}^+$.

Using Itô's formula in [22, ch. IV. 3], Fubini's Theorem in [22, ch. II. 12.2] and (10), the inequality (9) can be rewritten as

$$\int_{\mathbb{E}[V(x(t_k),t_k)]}^{\mathbb{E}[V(x(t),t)]} \frac{ds}{\varphi(s)} \leq -(t-t_k), \quad t \in [t_k, t_{k+1})$$

that is, for all $t \in [t_k, t_{k+1})$

$$F(\mathbb{E}[V(x(t),t)]) - F(\mathbb{E}[V(x(t_k),t_k)]) \leq -(t-t_k).$$

Letting $t \to t_{k+1}^-$ yields that

$$F(\mathbb{E}[V(x(t_{k+1}^-),t_{k+1}^-)]) - F(\mathbb{E}[V(x(t_k),t_k)]) \leq -\theta.$$

Because of (8) and the FDT condition (5), it follows that

$$\begin{aligned}
&F(\mathbb{E}[V(x(t_{k+1}),t_{k+1})]) - F(\mathbb{E}[V(x(t_k),t_k)]) \\
&\leq F(\psi(\mathbb{E}[V(x(t_{k+1}^-),t_{k+1}^-)])) - F(\mathbb{E}[V(x(t_{k+1}^-),t_{k+1}^-)]) \\
&\quad + F(\mathbb{E}[V(x(t_{k+1}^-),t_{k+1}^-)]) - F(\mathbb{E}[V(x(t_k),t_k)]) \\
&\leq \theta - \delta - \theta = -\delta.
\end{aligned}$$

That is

$$\mathbb{E}[V(x(t_{k+1}),t_{k+1})] \leq F^{-1}(F(\mathbb{E}[V(x(t_k),t_k)]) - \delta). \tag{11}$$

Iterating (11) from t_1 to t_{k+1}, it obtains that

$$\mathbb{E}[V(x(t_{k+1}),t_{k+1})] \leq F^{-1}(F(\mathbb{E}[V(x(t_1),t_1)]) - k\delta)$$

which is valid for all $k \in \mathfrak{K} := \{k \in \mathbb{N}^+ | F(\mathbb{E}[V(x(t_1),t_1)]) - k\delta \geq \lim_{\varrho \downarrow 0} F(\varrho)\}$. Denote $k_1 := \max_{k\in\mathfrak{K}} k$ (if not exists, $k_1 := \infty$) and $r := V(x(t_0), t_0)$.

In the following, based on (8) and (11), a class $\mathcal{KL}$ function β is constructed as a bound of $\mathbb{E}[V(x(t),t)]$. Define

$$\beta_1(r, t_1 - t_0) := \max\{\mathbb{E}[V(x(t_1),t_1)], \psi(\mathbb{E}[V(x(t_1),t_1)])\}$$
$$\beta_1(r, t_{k+1} - t_0) := F^{-1}(F(\beta_1(r, t_k - t_0)) - k\delta), k \in \{1,\ldots,k_1\}.$$

In the interval $(t_k - t_0, t_{k+1} - t_0)$, where $k \in \{1,\ldots,k_1\}$, $\beta_1(r,s)$ is required to be continuously decreasing and to lie above every solution of (7). In $[0, t_1 - t_0)$, the requirement on $\beta_1(r,s)$ is satisfied by construction. If $k_1 < \infty$, then $\beta_1(r,s)$ in the interval $[t_{k_1} - t_0, \infty)$ is defined to be continuous and decreasing to zero as $s \to \infty$.

From the construction of $\beta_1(r,s)$, we have that for all $t \geq t_0$

$$\mathbb{E}[V(x(t),t)] \leq \beta_1(V(x(t_0),t_0), t - t_0).$$

The function $\beta_1(r,s)$ is continuous and decreasing with respect to s. If $k_1 < \infty$, then $\beta_1(r,s) \to 0$ as $s \to \infty$ by the construction. If $k_1 = \infty$, then $\beta_1(r,s) \to 0$ as $s \to \infty$ needs to be proven.

Claim that if $\beta_1(r, t_k - t_0) \to 0$ as $k \to \infty$, then $\beta_1(r,s) \to 0$ as $s \to \infty$. If the claim is invalid, then $\lim_{k\to\infty} \beta_1(r, t_k - t_0) = \epsilon > 0$, where ϵ is related to the choice of r. Denote $\vartheta := \min_{\epsilon \leq v \leq \beta_1(r,0)} \varphi(v)$. It obtains from the middle-value theorem that

$$\begin{aligned}
\delta &\leq F(\beta_1(r, t_k - t_0)) - F(\beta_1(r, t_{k+1} - t_0)) \\
&\leq \frac{\beta_1(r, t_k - t_0) - \beta_1(r, t_{k+1} - t_0)}{\vartheta}.
\end{aligned}$$

Thus, it follows that

$$\beta_1(r, t_k - t_0) - \beta_1(r, t_{k+1} - t_0) \geq \delta\vartheta > 0$$

which indicates that $\beta_1(r, t_k - t_0)$ is decreasing to zero as $k \to \infty$. This contradicts with the assumption that $\lim_{k\to\infty} \beta_1(r, t_k - t_0) = \epsilon > 0$. Thus, the claim is valid. That is, given $r > 0$, $\beta_1(r,s) \in \mathcal{L}$.

Define $\beta_2(r,t) := \sup_{0 \leq v \leq r} \beta_1(v,t)$. Then, $\beta_2(r,t) \geq \beta_1(r,t)$ for all $r > 0$ and $t > 0$. Furthermore, define $\beta_3(r,t) := \frac{1}{r}\int_r^{2r} \beta_2(s,t)ds + re^{-t}$. Note that $\beta_3(r,t) \in \mathcal{KL}$ and $\beta_3(r,t) \geq \beta_2(r,t)$ for all $r > 0$ and $t > 0$. It obtains that

$$\mathbb{E}[V(x(t),t)] \leq \beta_3(V(x(t_0),t_0), t - t_0) \quad \forall t \geq t_0. \tag{12}$$

Applying Markov's inequality in [22, ch. II, 18.1] to (12) yields that for an arbitrary $\varepsilon_1 \in (0,1)$, there exists a function $\beta_4(v,t) := \beta_3(v,t)/\varepsilon_1 \in \mathcal{KL}$ such that for all $t \geq t_0$

$$\begin{aligned}
&\mathbb{P}\{V(x(t),t) > \beta_4(V(x(t_0),t_0), t - t_0)\} \\
&\leq \frac{\mathbb{E}[V(x(t),t)]}{\beta_4(V(x(t_0),t_0), t - t_0)} \leq \varepsilon_1
\end{aligned}$$

which implies from (3) that for all $t \geq t_0$

$$\mathbb{P}\{|x(t)| > \beta(|x(t_0)|, t - t_0)\} \leq \varepsilon_1 \tag{13}$$

where $\beta(v,t) := \alpha^{-1}(\beta_4(\alpha_2(v),t)) \in \mathcal{KL}$.

Case 2: $u \neq 0$. Define the set $\mathbf{B}_1 := \{x \in \mathcal{X} | |x| \leq \rho(\|u\|)\}$. Both the case $x \in \mathbf{B}_1$ and the case $x \notin \mathbf{B}_1$ are considered. If $x \notin \mathbf{B}_1$ for all $t \geq t_0$, then (6) holds and it follows from Case 1 that

$$\mathbb{P}\{|x(t)| \leq \beta(|x(t_0)|, t - t_0)\} \geq 1 - \varepsilon_1. \tag{14}$$

Define $\bar{t} := \inf\{t \geq t_0 | x(t) \in \mathbf{B}_1\}$, then (14) holds for all $t \leq \bar{t}$.

For $t > \bar{t}$, if the system state escapes from $\mathbf{B}_1$, then it happens at certain impulsive times. Moreover, there exists some $\tilde{t} > \bar{t}$ such that $x(\tilde{t}) \in \partial\mathbf{B}_1$, then $x(\tilde{t}) \in \mathbf{B}_1$ holds due to (4). Define

$$\bar{\psi}_1(v) := \max\left\{\max_{0 \leq v \leq \rho(v)} \alpha_1^{-1}(\psi(\alpha_2(v))), \rho(v)\right\}$$

$$\mathbf{B}_2 := \{x \in \mathcal{X} | |x| \leq \bar{\psi}_1(\|u\|)\} \supseteq \mathbf{B}_1.$$

Next we prove $x(t) \in \mathbf{B}_2$ holds for all $t > t_k$. If there exist an impulsive time $t_k \in \mathcal{T}$, $t_k > \bar{t}$, and $\varepsilon_2 > 0$ such that $x(t_k) \notin \mathbf{B}_1$ and $x(t) \in \mathbf{B}_1$ for $t \in (t_k - \varepsilon_2, t_k)$, then $x(t) \in \mathbf{B}_2$ by the construction of $\mathbf{B}_2$. If $x(t) \notin \mathbf{B}_1$ and $t > t_k$, then it has been proven that $\mathbb{E}[V(x(t),t)] < \mathbb{E}[V(x(t_k),t_k)]$. Therefore, $x(t) \in \mathbf{B}_2$ for all $t > \bar{t}$. That is, for all $t > \bar{t}$, we have

$$\mathbb{E}[V(x(t),t)] \leq \alpha_2(\bar{\psi}_1(\|u\|)).$$

Using Markov's inequality, there exist a $\kappa \in \mathcal{K}_\infty$ and a sufficiently small $\varepsilon_3 = \varepsilon_3(\kappa) > 0$ such that for all $t \in \mathbb{R}_{t_0}^+$

$$\mathbb{P}\{V(x(t),t) > \kappa(\alpha_2(\bar{\psi}_1(\|u\|)))\} \leq \frac{\mathbb{E}[V(x(t),t)]}{\kappa(\alpha_2(\bar{\psi}_1(\|u\|)))} \leq \varepsilon_3. \tag{15}$$

Define $\bar{\gamma}(v) := \alpha_1^{-1}(\kappa(\alpha_2(\bar{\psi}_1(v))))$ and it can be majorized to be a class $\mathcal{K}_\infty$ function γ. Combining (13) and (15) yields that

$$\mathbb{P}\{|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma(\|u\|)\} \geq 1 - \max\{\varepsilon_1, \varepsilon_3\}$$

for all $t \in \mathbb{R}_{t_0}^+$. Thus, the proof is completed. ■

Remark 2: The FDT condition (5) in Theorem 1 first appeared in [7, ch. 3, Th. 48] for asymptotic stability of autonomous impulsive systems. Contrary to the ADT condition bounding the interval of consecutive impulses on average [18], the FDT condition provides the lower bound for the impulsive intervals. Some previous works [4], [5], [20] are recovered as the special cases of Theorem 1. For instance, if the considered system is deterministic, then Theorem 1 is reduced to [5, Th. 1].

Remark 3: For the FDT condition (5), the magnitude of θ affects the SISS gain, which in turn contributes to the decay rate of $V(x(t), t)$. A numerical example is given in Example 1 of Section IV to show the relation between θ and the SISS gain. In addition, because the construction of the function β depends on the impulsive time sequence, the system (1) is not uniform SISS.

Remark 4: Whether the discrete dynamics is stable or not, the result in Theorem 1 always holds. Especially, if the discrete dynamics stabilizes the system (1), then $\psi \leq \mathrm{Id}$, which implies that the left-hand side of (5) is not more than zero. Thus, the FDT condition (5) holds for arbitrary $\theta > 0$. In this case, the system (1) is SISS for all impulsive time sequences. In addition, if the impulsive time sequence is periodic, then the construction of the function β depends only on the period of the impulsive intervals. As a result, the system (1) is uniformly SISS over a set of periodic impulsive time sequences.

Theorem 2: Under the same assumptions as in Theorem 1, if the FDT condition (5) holds for $\delta \equiv 0$, then the system (1) is uniformly SGS over $\mathcal{S}_{1\theta}$.

Proof: Since all the assumptions in Theorem 1 hold for $\delta \equiv 0$, following the proof of Theorem 1, (11) holds for $\delta \equiv 0$. Thus, define $\chi_1(s) := \max\{s, \psi(s)\}$. Instead of (12), it holds that

$$\mathbb{E}[V(x(t), t)] \leq \chi_1(V(x(t_0), t_0)) \leq \chi_1(\alpha_2(|x(t_0)|)), \quad t \leq \bar{t}.$$

For all $t \geq t_0$, it obtains that

$$\mathbb{E}[V(x(t), t)] \leq \chi_1(\alpha_2(|x(t_0)|)) + \rho(\|u\|). \tag{16}$$

Exploiting Markov's inequality to (16) gives that for an arbitrary $\varepsilon \in (0, 1)$ and all $t \geq t_0$

$$\mathbb{P}\{V(x(t), t) \leq \chi_1(\alpha_2(|x(t_0)|))/\varepsilon + \rho(\|u\|)/\varepsilon\} \geq 1 - \varepsilon$$

that is, $\mathbb{P}\{|x(t)| \leq \gamma_1(|x(t_0)|) + \gamma_2(\|u\|)\} \geq 1 - \varepsilon$, where $\gamma_1(v) := \alpha_1^{-1}(2\chi_1(\alpha_2(v))/\varepsilon)$ and $\gamma_2(v) := \alpha_1^{-1}(2\rho(v)/\varepsilon)$.

It obtains from the construction of χ_1 and ρ that γ_1 and γ_2 are not related to the impulsive time sequence $\mathcal{T} \in \mathcal{S}_{1\theta}$. Hence, the proof of uniform SGS is completed. ■

B. Stable Discrete Dynamics Situation

In this subsection, the second situation is studied, where the continuous dynamics is unstable and the discrete dynamics is stable. Define $\mathcal{S}_{2\theta} := \{\mathcal{T} = \{t_1, t_2, \ldots\} | \mathcal{T} \subseteq \mathbb{R}^+_{t_0}, t_{k+1} - t_k \leq \theta, \forall k \in \mathbb{N}^+\}$ for some $\theta > 0$. The following theorems provide sufficient conditions to guarantee SISS and SGS of the system (1).

Theorem 3: Consider the system (1). Suppose that $V : \mathcal{X} \times \mathbb{R}^+_{t_0} \to \mathbb{R}^+$ is an SISS-Lyapunov function for (1), where $-\varphi, \psi \in \mathcal{P}$ is concave. If there exist certain $\theta, \delta > 0$ such that for all $a > 0$

$$\int_{\psi(a)}^{a} \frac{ds}{-\varphi(s)} \geq \theta + \delta \tag{17}$$

then the system (1) is SISS for all impulsive time sequences $\mathcal{T} \in \mathcal{S}_{2\theta}$.

Proof: Since $-\varphi, \psi \in \mathcal{P}$ are concave, for the case $u \equiv 0$, it follows from (4) and Jensen's inequality that

$$\begin{aligned} \mathbb{E}[\mathscr{L}V(x, t)] &\leq -\varphi(\mathbb{E}[V(x, t)]), \quad t \in \mathbb{R}^+_{t_0} \backslash \mathcal{T} \\ \mathbb{E}[V(h(x, u), t)] &\leq \psi(\mathbb{E}[V(x, t)]), \quad t \in \mathcal{T}. \end{aligned} \tag{18}$$

Integrating (18) implies that for any $t \in [t_k, t_{k+1})$

$$\int_{t_k}^{t} \frac{\mathbb{E}[\mathscr{L}V(x, s)]ds}{-\varphi(\mathbb{E}[V(x, s)])} \leq t - t_k.$$

Similar to the proof of Theorem 1, define the following function:

$$F(\varrho) := \int_{\nu}^{\varrho} \frac{ds}{-\varphi(s)}$$

where $\nu > 0$ is fixed and $\varrho > 0$. Thus, $F : \mathbb{R}^+ \to \mathbb{R}$ and its inverse $F^{-1} : \mathbb{R} \to \mathbb{R}^+$ are continuous and strictly increasing.

Analogous to the proof of Theorem 1, it follows from the reverse FDT condition (17) that for all $k > 1$

$$F(\mathbb{E}[V(x(t), t)]) \leq F(\mathbb{E}[V(x(t_k), t_k)]) + \theta, \; t \in (t_k, t_{k+1}) \tag{19}$$

$$\mathbb{E}[V(x(t_{k+1}), t_{k+1})] \leq F^{-1}(F(\mathbb{E}[V(x(t_1), t_1)]) - k\delta). \tag{20}$$

Thus, by the similar construction of the function $\beta_1(r, s)$ as in the proof of Theorem 1, it holds that

$$\mathbb{E}[V(x(t), t)] \leq \beta_1(V(x(t_0), t_0), t - t_0). \tag{21}$$

The remaining part is the same as the proof of Theorem 1 for the case $u \equiv 0$. Thus, the inequality (13) is also obtained.

For the case that $u \neq 0$, define $\mathbf{B}_1$ as in the proof of Theorem 1. Therefore, for an arbitrary $\varepsilon_1 \in (0, 1)$, there exist a $\bar{t} \geq t_0$ and a $\beta \in \mathcal{KL}$ such that for all $t \leq \bar{t}$

$$\mathbb{P}\{|x(t)| \leq \beta(|x(t_0)|, t - t_0)\} \geq 1 - \varepsilon_1.$$

Define $\tilde{t} := \inf\{t_k | t_k > \bar{t}\}$. Thus, for $t \in [\bar{t}, \tilde{t})$, combining $|x| \leq \rho(\|u\|)$ and (19) implies that

$$\begin{aligned} \mathbb{E}[V(x(t), t)] &\leq F^{-1}(F(\mathbb{E}[V(x(\bar{t}), \bar{t})]) + \theta) \\ &\leq F^{-1}(F(\alpha_2(\rho(\|u\|))) + \theta). \end{aligned} \tag{22}$$

Because $\mathbb{E}[V(x(\tilde{t}), \tilde{t})] < \mathbb{E}[V(x(\bar{t}), \bar{t})]$ from (11), the inequality (22) also holds for all $t > \bar{t}$. Applying Markov's inequality to (22), there exist a $\kappa \in \mathcal{K}_\infty$ and a sufficiently small $\varepsilon_3 = \varepsilon_3(\kappa) > 0$ such that

$$\mathbb{P}\{V(x(t), t) > \kappa(\bar{\psi}_2(\|u\|))\} \leq \frac{\mathbb{E}[V(x(t), t)]}{\kappa(\bar{\psi}_2(\|u\|))} \leq \varepsilon_3$$

where $\bar{\psi}_2(v) := F^{-1}(F(\alpha_2(\rho(v))) + \theta)$, which means that

$$\mathbb{P}\{|x(t)| \leq \alpha_1^{-1}(\kappa(\bar{\psi}_2(\|u\|)))\} \geq 1 - \varepsilon_3.$$

Based on the aforementioned analysis and similar to the proof of Theorem 1, we conclude that, for a given impulsive time sequence $\mathcal{T} \in \mathcal{S}_{2\theta}$, the system (1) is SISS. Therefore, the proof is completed. ■

The inequality (17) in Theorem 3 is called the reverse fixed dwell-time (RFDT) condition, which corresponds to the reverse average dwell-time (RADT) condition in [4]. The RFDT condition provides the upper bound for the impulsive intervals. Similar to Theorem 2, a counterpart result is obtained as follows.

Theorem 4: Under the same assumptions as in Theorem 3, if the RFDT condition (17) holds for $\delta \equiv 0$, then the system (1) is uniformly SGS over $\mathcal{S}_{2\theta}$.

Proof: Since all the assumptions in Theorem 3 hold for $\delta \equiv 0$, following the proof of Theorem 3, the inequality (20) holds for $\delta \equiv 0$.

Thus, define $\chi_1(v) := \max\{v, \psi(v)\}$. Instead of (21), it holds that

$$\mathbb{E}[V(x(t),t)] \le \chi_1(V(x(t_0),t_0)) \le \chi_1(\alpha_2(|x(t_0)|)), \quad t \le \bar{t}.$$

Similar to the proof of Theorem 2, it obtains that $\mathbb{E}[V(x(t),t)] \le \chi_1(\alpha_2(|x(t_0)|)) + \rho(\|u\|)$ for all $t \ge t_0$. Applying Markov's inequality yields that for an arbitrary $\varepsilon \in (0,1)$ and all $t \ge t_0$

$$\mathbb{P}\{V(x(t),t) \le \chi_1(\alpha_2(|x(t_0)|))/\varepsilon + \rho(\|u\|)/\varepsilon\} \ge 1-\varepsilon$$

that is, $\mathbb{P}\{|x(t)| \le \gamma_1(|x(t_0)|) + \gamma_2(\|u\|)\} \ge 1-\varepsilon$, where $\gamma_1(v) := \alpha_1^{-1}(2\chi_1(\alpha_2(v))/\varepsilon)$ and $\gamma_2(v) := \alpha_1^{-1}(2\rho(v)/\varepsilon)$.

It follows from the construction of χ_1 and ρ that γ_1 and γ_2 are not related to the impulsive time sequence $\mathcal{T} \in \mathcal{S}_{2\theta}$. Hence, the proof of uniform SGS is completed. ∎

C. Relationship Between FDT Condition and ADT Condition

In this subsection, we first establish the uniform SISS of the system (1) through the exponential SISS-Lyapunov function and the generalized ADT condition. Then, based on the generalized ADT condition, the relation between the FDT (or RFDT) condition and the ADT (or RADT) condition is established.

Proposition 1: Consider the system (1). Suppose that $V : \mathcal{X} \times \mathbb{R}^+_{t_0} \to \mathbb{R}^+$ is an exponential SISS-Lyapunov function for (1) with the coefficients $c, d \in \mathbb{R}$. If there exists a class $\mathcal{L}$ function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(v) \le \psi(v)$ for all $v > 0$ and

$$-dN(t,s) - c(t-s) \le \ln\phi(t-s), \quad t \ge s \ge t_0 \tag{23}$$

where $N(t,s)$ denotes the number of state jumps in $(s,t]$, then the system (1) is uniformly SISS over a given set $\mathcal{S}[\phi]$ of impulsive time sequences satisfying the generalized ADT condition (23).

Proof: Based on the magnitudes of $|x|$ and $\rho(\|u\|)$, the interval $[t_0,\infty)$ is divided to be $\cup_{k=0}^{\infty}[\hat{t}_k, \hat{t}_{k+1})$, $k \in \mathbb{N}^+$, such that

$$|x| \ge \rho(\|u\|), \quad t \in [\hat{t}_{2i}, \hat{t}_{2i+1}),\ i \in \mathbb{N}^+$$
$$|x| < \rho(\|u\|), \quad t \in [\hat{t}_{2i+1}, \hat{t}_{2i+2}),\ i \in \mathbb{N}^+.$$

In the interval $\mathcal{I}_{2i} = [\hat{t}_{2i}, \hat{t}_{2i+1})$, $i \in \mathbb{N}^+$, there are $N(\hat{t}_{2i+1}, \hat{t}_{2i})$ state jumps at t^i_j, where $j \in \{1, 2, \ldots, N(\hat{t}_{2i+1}, \hat{t}_{2i})\}$. Thus, $|x| \ge \rho(\|u\|)$ holds in each interval $(t^i_j, t^i_{j+1}]$. Along the same line as the proof of Theorem 1, it obtains that

$$\mathbb{E}[\mathscr{L}V(x(t),t)] \le -cV(x(t),t), \quad t \in (t^i_j, t^i_{j+1})$$
$$\mathbb{E}[V(x(t^i_{j+1}), t^i_{j+1})] \le e^{-d}\,\mathbb{E}[V(x(t^{i-}_{j+1}), t^{i-}_{j+1})]$$

which implies that

$$\mathbb{E}[V(x(t^i_{j+1}), t^i_{j+1})] \le e^{-d-c(t^i_{j+1}-t^i_j)}\,\mathbb{E}[V(x(t^i_j), t^i_j)].$$

Thus, by iterating, we have that for all $t \in \mathcal{I}_{2i}$

$$\mathbb{E}[V(x(t),t)] \le e^{-dN(t,\hat{t}_{2i})-c(t-\hat{t}_{2i})}\,\mathbb{E}[V(x(\hat{t}_{2i}), \hat{t}_{2i})]$$

combining which with (23) gives that

$$\mathbb{E}[V(x(t),t)] \le \phi(t-\hat{t}_{2i})\,\mathbb{E}[V(x(\hat{t}_{2i}), \hat{t}_{2i})].$$

Pick $\hat{t}_{2i} = t_0$ and define $\beta(v,t) := \phi(t-\hat{t}_{2i})\alpha_2(v) \in \mathcal{KL}$ and $\bar{t} := \inf\{t \ge t_0 \,|\, |x(t)| \le \rho(\|u\|)\}$, it holds that for all $t \in [t_0, \bar{t}]$

$$\mathbb{E}[V(x(t),t)] \le \beta(|x(t_0)|, t-t_0).$$

For all $t \in \mathcal{I}_{2i+1} = [\hat{t}_{2i+1}, \hat{t}_{2i+2})$, it obtains that $\mathbb{E}[V(x(t),t)] < \alpha_2(\rho(\|u\|))$. If there is an impulse at $\hat{t}_{2i+2}$, then it follows that

$$\mathbb{E}[V(x(\hat{t}_{2i+2}), \hat{t}_{2i+2})] < \max\{e^{-d}, 1\}\alpha_2(\rho(\|u\|)).$$

Consider the situation at $\hat{t}_{2i+1}$ and the property of the function ϕ, one has that, for all $t \in \mathcal{I}_{2i+1}$ and $t > \bar{t}$

$$\mathbb{E}[V(x(t),t)] < \phi(0)\max\{e^{-d}, 1\}\alpha_2(\rho(\|u\|)).$$

Therefore, for the overall interval $[t_0,\infty) = \cup_{i=0}^{\infty}(\mathcal{I}_{2i} \cup \mathcal{I}_{2i+1})$, the estimate of $\mathbb{E}[V(x(t),t)]$ is given by

$$\mathbb{E}[V(x(t),t)] < \beta(|x(t_0)|, t-t_0) + \gamma(\|u\|) \tag{24}$$

where $\gamma(v) := \max\{e^{-d}, 1\}\phi(0)\alpha_2(\rho(v))$. Using Markov's inequality to (24) obtains that for all $t \in \mathbb{R}^+_{t_0}$ and any $\varepsilon \in (0,1)$

$$\mathbb{P}\{|x(t)| \le \bar{\beta}(|x(t_0)|, t-t_0) + \bar{\gamma}(\|u\|)\} \ge 1-\varepsilon$$

where $\bar{\beta}(v,t) := \alpha_1^{-1}(2\beta(v,t))/\varepsilon$ and $\bar{\gamma}(v) := \alpha_1^{-1}(2\gamma(v))/\varepsilon$. This completes the proof. ∎

The inequality (23) is called the generalized ADT condition. If $\phi(t) = e^{\mu-\lambda t}$ for certain $\mu, \lambda > 0$, then the generalized ADT condition is same as the ADT condition obtained in [4] and [18]. In this case, denote by $\mathcal{S}[\mu,\lambda]$ the class of the impulsive time sequences.

In the following, to connect the ADT condition and the FDT condition, we assume that there exists an exponential SISS-Lyapunov function for the system (1), that is, $\varphi(v) = cv$, $\psi(v) = e^{-d}v$, where $c, d \in \mathbb{R}$. For the stable discrete dynamics case, it follows that $c < 0$ and $d > 0$. As a result, the RFDT condition (17) is equivalent to

$$\int_a^{\psi(a)} \frac{ds}{\varphi(s)} = \frac{d}{-c} \ge \theta + \delta \tag{25}$$

which implies that there exists a $\lambda > 0$ such that $\frac{d}{-c-\lambda} \ge \theta$. Define $\theta_2 := \frac{d}{-c-\lambda}$, thus $\theta \le \theta_2$. That is, given λ, θ_2 is the largest, then the set $\mathcal{S}_{2\theta_2}$ is the largest. The following lemma provides the relationship between the RADT condition and the RFDT condition.

Proposition 2: If there exists an exponential SISS-Lyapunov function for the system (1) with $c < 0$ and $d > 0$, then $\mathcal{S}_{2\theta_2} = \mathcal{S}[d,\lambda]$.

Proof: First, we prove $\mathcal{S}_{2\theta_2} \subseteq \mathcal{S}[d,\lambda]$. Based the definition of the reverse ADT in [4], if $\mathcal{T} \in \mathcal{S}_{2\theta_2}$, then it follows that

$$N(t,s) \ge N_0 + \frac{-c-\lambda}{d}(t-s) \quad \forall t \ge s \ge t_0 \tag{26}$$

where $N_0 \in \mathbb{N}^+$. The inequality (26) is equivalent to

$$-dN(t,s) - (c+\lambda)(t-s) \le -N_0 d \quad \forall t \ge s \ge t_0. \tag{27}$$

Second, we prove $\mathcal{S}[d,\lambda] \subseteq \mathcal{S}_{2\theta_2}$. Assume $\mathcal{T} \in \mathcal{S}[d,\lambda]$, it obtains that (27) holds. If $t - s = \ell\theta_2$ and $\ell \ge 0$, then it follows from (27) that $N(t,s) \ge N_0 + \ell$, which means that $N(t,s) \ge N_0 + \ell - 1$ for all $t - s \in [(\ell-1)\theta_2, \ell\theta_2)$.

In the sequel, $\mathcal{S}_{2\theta_2} = \mathcal{S}[d,\lambda]$ and the proof is completed. ∎

For the stable continuous dynamics case, it obtains that $c > 0$ and $d < 0$. As a result, the FDT condition (5) is equivalent to

$$\int_a^{\psi(a)} \frac{ds}{\varphi(s)} = \frac{-d}{c} \le \theta - \delta. \tag{28}$$

It follows from (28) that there exists a $\lambda > 0$ such that $\frac{-d}{c-\lambda} \le \theta$. Define $\theta_1 := \frac{-d}{c-\lambda}$, thus $\theta \ge \theta_1$. That is, if λ is given, then θ_1 is the smallest, which indicates the set $\mathcal{S}_{1\theta_1}$ is the largest. Similar to Lemma 2, the following lemma gives the relationship between the ADT condition and the FDT condition and the proof is omitted here.

Proposition 3: If there exists an exponential SISS-Lyapunov function for the system (1) with $c > 0$ and $d < 0$, then $\mathcal{S}_{1\theta_1} = \mathcal{S}[-d,\lambda]$.

Remark 5: If $N(t,s)$ denotes the number of state jumps in $[s,t]$, then the results in Lemmas 2 and 3 are still valid according to [16, Lemma 3.12].

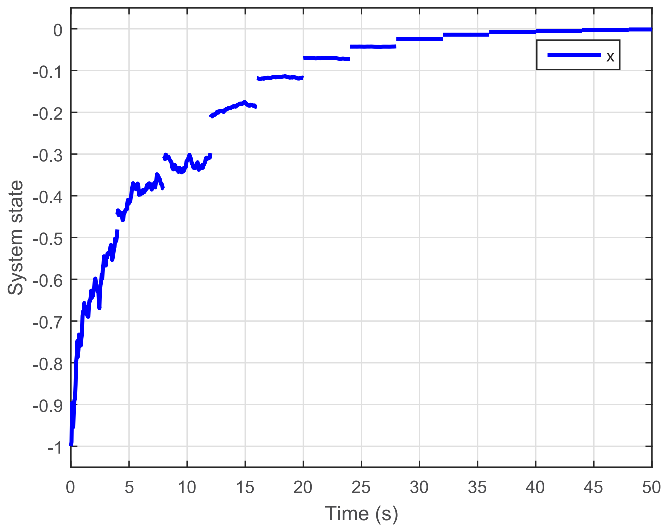

Fig. 1. State response of the system $\mathcal{H}$ with a periodic impulsive time sequence and $\theta = 4$.

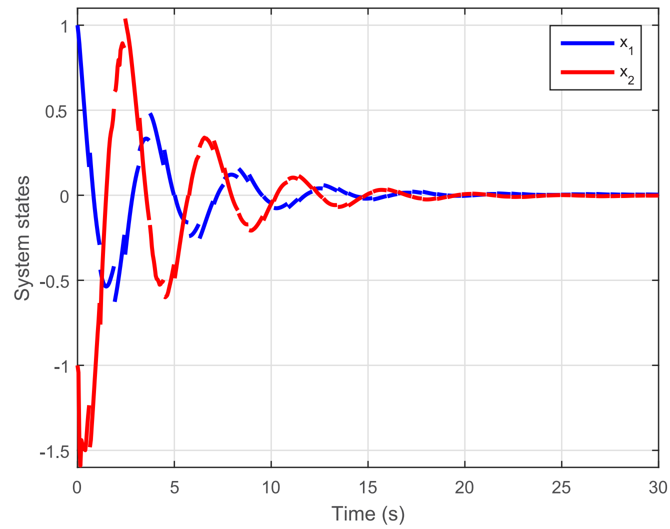

Fig. 2. State response of the system (30)–(31) with an aperiodic impulsive time sequence and $\theta = 0.14$.

IV. Illustrative Examples

In this section, two numerical examples are presented to illustrate the developed results in the previous section.

Example 1: Consider the impulsive stochastic nonlinear system $\mathcal{H}$ with the following form:

$$\begin{cases} dx(t) = [-x^5(t) + x^2(t)u^3(t)/2]dt + (x(t)u(t))^{3/2}dw(t), \ t \notin \mathcal{T} \\ x(t) = x^3(t^-) + x(t^-)/\sqrt{3} + u^3(t^-), \ t \in \mathcal{T} \end{cases}$$

where $\mathcal{T}$ is an impulsive time sequence. Choose Lyapunov function $V(x(t),t) = x^2(t)$ and the function $\rho(v) = b^{-1/3}|v|$, where $b \in [0,1)$. Thus, it follows that $\mathscr{L}V(x(t),t) = -2x^6(t) + 2x^3(t)u^3(t) \leq -2(1-b)(V(x(t)))^3$, for $t \notin \mathcal{T}$ and $V(h(x(t^-),u(t^-)),t) = (x^3(t^-) + x(t^-)/\sqrt{3} + u^3(t^-))^2 \leq 3(1+b^2)(V(x(t^-),t))^3 + V(x(t^-),t)$, for $t \in \mathcal{T}$. Therefore, $V(x(t),t)$ is not exponential.

Computing the FDT condition (5) in Theorem 1 gives that

$$\begin{aligned} \Im(a,b) &= \int_a^{(3+3b^2)a^3+a} \frac{ds}{2(1-b)s^3} \\ &= \frac{3+3b^2}{4(1-b)} \frac{2+(3+3b^2)a^2}{[1+(3+3b^2)a^2]^2} \leq \frac{3+3b^2}{2(1-b)} \leq \theta - \delta. \quad (29) \end{aligned}$$

There exists an appropriate ε such that $\Im(a, b(\varepsilon)) \leq 1 + 2\varepsilon$. Thus, choose $\theta = 1 + \varepsilon$, and the system $\mathcal{H}$ is SISS. It follows from (29) that the smaller θ is, the smaller b is, which implies that the SISS gain is larger. Therefore, the dependence between θ and the SISS gain implies that there is a tradeoff between the density of allowable impulsive times and the magnitude of the SISS. Let $b = 0.5$ and $u(t) = 2^{-1/3}x(t)$, it obtains that $\theta - \delta \geq 3.75$. Therefore, under the periodic impulsive time sequence with the period $\theta = 4$, the initial state $x(0) = -1$ and the Gaussian white noise w with zero-mean and variance of 10, the state response of $\mathcal{H}$ is given in Fig. 1 .

Example 2: Consider a mass–spring system in [25, Sec. 1.2.3] with a hardening spring and linear viscous damping, which is modeled by the Duffing's equation $m\ddot{y}(t) + c\dot{y}(t) + ky(t) + ka^2y^3(t) = A(t)$, where $c\dot{y}(t)$ is the resistive force, $ky(t) + ka^2y^3(t)$ models hardening spring and $A(t)$ is the external force. Define $x^\top(t) := [x_1(t), x_2(t)] = [y(t), \dot{y}(t)]$ and $A(t) = B(t) + C(t)\dot{w}(t)$, where $\dot{w}(t)$ is a 2-D white noise, the Duffing's equation is rewritten as

$$\begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -[cx_2(t) + kx_1(t) + ka^2x_1^3(t) - B(t)]/m \end{bmatrix} dt + \begin{bmatrix} 0 \\ C(t)/m \end{bmatrix} dw(t). \quad (30)$$

For experiments in a viscous medium such as air or lubricant, there exists a external force to keep the mass–spring moving on at the discrete-time sequence $\mathcal{T} := \{t_k : k \in \mathbb{N}^+\}$. The transfer of kinetic energy between the external force and the mass–spring is modeled as the following state impulsive:

$$\begin{bmatrix} x_1(t_k) \\ x_2(t_k) \end{bmatrix} = \begin{bmatrix} b_1x_1(t_k^-) \\ b_2x_2(t_k^-) + z(t_k^-) \end{bmatrix}, \quad t_k \in \mathcal{T} \quad (31)$$

where $1 \leq b_1, b_2 \leq 2$, and $z(t_k^-)$ represents the change in velocity because of the additional force at t_k. As a result, combining (30) and (31) yields the impulsive stochastic system model of the form (1).

Define $B(t) := 2ka^2x_1^3(t)/3 + 0.01$ m, $C(t) := \sqrt{2}amx_1^3(t)x_2(t)$, and $z(t) = 0.2x_2(t)$; let $k = 3m$, $c = 1.5m$ and $a = 1$. Choose the Lyapunov function $V(x(t),t) = x^\top(t)Px(t) + 0.5x_1^4(t)$, where $P = \left[\begin{smallmatrix} 1.5 & 0.5 \\ 0.5 & 1 \end{smallmatrix}\right]$, then it obtains that for $t \notin \mathcal{T}$, $\mathscr{L}V(x(t),t) = -x_1^4(t) - 3x_1^2(t) - 2x_2^2(t) - 4.5x_1(t)x_2(t) + 0.01[x_1(t) + 2x_2(t)] \leq -2V(x(t)) + 1.5\sqrt{2V(x(t))}$ and $V(x(t_k),t_k) \leq bV(x(t_k^-),t_k))$, where $b = \max\{b_1^4, (b_2+0.2)^4\}$. Thus, $V(x(t),t)$ is not exponential. Because $\psi(v) = 2v - 1.5\sqrt{2v}$ is convex, we have

$$\Im(a) = \int_a^{ba} \frac{ds}{2s - 1.5\sqrt{2s}} \leq \theta - \delta. \quad (32)$$

If the aforementioned equation holds, then the system is SISS. Let $b_1 = 1.5$, $b_2 = 1$, and choose $\theta = 0.14$ by computation. Under the initial state $x(0) = [1, -1]^\top$, the Gaussian white noise w with zero-mean and variance of 50 and the aperiodic impulsive time sequence, the state response of the system (30)–(31) is presented in Fig. 2 .

V. Conclusion

In this paper, stochastic stability properties were studied for impulsive stochastic nonlinear systems. Both the situation that the continuous dynamics is stable and the situation that the discrete dynamics is stable were considered. Based on the FDT condition and general Lyapunov function, sufficient conditions were achieved. Moreover, the relation between FDT condition and average dwell-time condition was discussed. Finally, two examples are given to illustrate the developed theory. Future research could be directed to controller/observer design for impulsive stochastic nonlinear systems and stability analysis of impulsive stochastic nonlinear systems with delays or Markovian switching.

References

[1] W. M. Haddad, V. Chellaboina, and S. G. Nersesov, *Impulsive and Hybrid Dynamical Systems: Stability, Dissipativity, and Control*. Princeton, NJ, USA: Princeton Univ. Press, 2014.

[2] T. Yang, *Impulsive Control Theory*. Berlin, Germany: Springer, 2001.

[3] A. R. Teel, A. Subbaraman, and A. Sferlazza, "Stability analysis for stochastic hybrid systems: A survey," *Automatica*, vol. 50, no. 10, pp. 2435–2456, 2014.

[4] J. P. Hespanha, D. Liberzon, and A. R. Teel, "Lyapunov conditions for input-to-state stability of impulsive systems," *Automatica*, vol. 44, no. 11, pp. 2735–2744, 2008.

[5] S. Dashkovskiy and A. Mironchenko, "Input-to-state stability of nonlinear impulsive systems," *SIAM J. Control Optim.*, vol. 51, no. 3, pp. 1962–1987, 2013.

[6] J. Liu, X. Liu, and W.-C. Xie, "Input-to-state stability of impulsive and switching hybrid systems with time-delay," *Automatica*, vol. 47, no. 5, pp. 899–908, 2011.

[7] A. M. Samoilenko, N. Perestyuk, and Y. Chapovsky, *Impulsive Differential Equations*. Singapore: World Scientific, 1995.

[8] E. A. Medina and D. A. Lawrence, "Reachability and observability of linear impulsive systems," *Automatica*, vol. 44, no. 5, pp. 1304–1309, 2008.

[9] D. Antunes, J. Hespanha, and C. Silvestre, "Stability of networked control systems with asynchronous renewal links: An impulsive systems approach," *Automatica*, vol. 49, no. 2, pp. 402–413, 2013.

[10] J. Lu, D. W. Ho, and J. Cao, "A unified synchronization criterion for impulsive dynamical networks," *Automatica*, vol. 46, no. 7, pp. 1215–1221, 2010.

[11] T. Yang and L. O. Chua, "Impulsive stabilization for control and synchronization of chaotic systems: Theory and application to secure communication," *IEEE Trans. Circuits Syst. I, Fundam. Theory Appl.*, vol. 44, no. 10, pp. 976–988, 1997.

[12] X. Wu, Y. Tang, and W. Zhang, "Input-to-state stability of impulsive stochastic delayed systems under linear assumptions," *Automatica*, vol. 66, pp. 195–204, 2016.

[13] H. Wu and J. Sun, "p-moment stability of stochastic differential equations with impulsive jump and Markovian switching," *Automatica*, vol. 42, no. 10, pp. 1753–1759, 2006.

[14] C. Briat, "Stability analysis and stabilization of stochastic linear impulsive, switched and sampled-data systems under dwell-time constraints," *Automatica*, vol. 74, pp. 279–287, 2016.

[15] L. Hu, P. Shi, and B. Huang, "Stochastic stability and robust control for sampled-data systems with Markovian jump parameters," *J. Math. Anal, Appl.*, vol. 313, pp. 504–517, 2006.

[16] S. Dashkovskiy, M. Kosmykov, A. Mironchenko, and L. Naujok, "Stability of interconnected impulsive systems with and without time delays, using Lyapunov methods," *Nonlinear Anal., Hybrid Syst.*, vol. 6, no. 3, pp. 899–915, 2012.

[17] P. Zhao, W. Feng, and Y. Kang, "Stochastic input-to-state stability of switched stochastic nonlinear systems," *Automatica*, vol. 48, no. 10, pp. 2569–2576, 2012.

[18] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proc. IEEE Conf. Decis. Control*, 1999, pp. 2655–2660.

[19] W. Ren and J. Xiong, "Stability and stabilization of switched stochastic systems under asynchronous switching," *Syst. Control Lett.*, vol. 97, pp. 184–192, 2016.

[20] W.-H. Chen and W. X. Zheng, "Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays," *Automatica*, vol. 45, no. 6, pp. 1481–1488, 2009.

[21] X. Mao, *Stochastic Differential Equations and Applications*, 2nd ed. Elsevier, Chichester, U.K.: Horwood, 2007.

[22] L. Rogers and D. Williams, *Diffusions, Markov Processes, and Martingales: Volume 1, Foundations*. Cambridge, U.K.: Cambridge Univ. Press, 2000.

[23] S.-J. Liu, J.-F. Zhang, and Z.-P. Jiang, "A notion of stochastic input-to-state stability and its application to stability of cascaded stochastic nonlinear systems," *Acta Math. Applicatae Sinica, English Series*, vol. 24, no. 1, pp. 141–156, 2008.

[24] D. Chatterjee and D. Liberzon, "Stability analysis of deterministic and stochastic switched systems via a comparison principle and multiple Lyapunov functions," *SIAM J. Control Optim.*, vol. 45, no. 1, pp. 174–206, 2006.

[25] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ, USA: Prentice Hall, 2002.