Technical Notes and Correspondence

Bilinear Transformation for Discrete-Time Positive Real and Negative Imaginary Systems

Mei Liu and Junlin Xiong

Abstract—**This paper studies the connection between discrete-time and continuous-time negative imaginary systems. First, we analyze differences between two statements that are claimed to provide equivalent conditions for systems to be discrete-time positive real. Our conclusion is that one is equivalent to the definition of discrete-time positive real transfer matrices, the other is not. Second, by means of the bilinear transformation, a connection between discrete-time and continuous-time negative imaginary transfer matrices is established. Third, the concept of discrete-time lossless negative imaginary systems is introduced, and a discrete-time lossless negative imaginary lemma is developed to characterize the lossless negative imaginary properties in terms of minimal state-space realization. Some properties of discrete-time lossless negative imaginary transfer matrices are also studied. Several numerical examples illustrate the developed theory.**

Index Terms—**Bilinear transformation, discrete-time (lossless) negative imaginary systems, discrete-time positive real systems.**

I. Introduction

Classical positive real (PR) systems have achieved great success both in continuous-time (CT) and discrete-time (DT) cases. The concept of DT-PR systems was first introduced in [1]. Subsequently, other versions of DT-PR systems were proposed in [2] and [3]. All of them were claimed to be equivalent to the definition of DT-PR systems. However, it was shown in [4] that the two versions of DT-PR systems in [2] and [3] were not consistent with the definition of DT-PR systems by using three single-input single-output examples. Until now, some researchers still adopt the one in [2] and, [3], e.g., see [5, Definition 3], [6, Th. 13.26], and [7, Th. 3]. In this paper, we shall discuss the differences between those versions, and present a detailed proof why the statement in [2] and [3] is not equivalent to the definition of DT-PR systems.

In recent years, negative imaginary (NI) theory, emerged as a complement to PR theory, has attracted much attention of many researchers, e.g., see [8]–[13]. This theory was first introduced in [14] to model linear mechanical systems with force inputs and collocated position outputs. Subsequently, the concept of NI systems was extended to allow poles on the imaginary axis [15]–[17], nonproper case [17], and DT systems [18]–[21], respectively. Also, a mixed passivity, NI, and small-gain approach in DT case have been used to design a resonance compensator in hard disk drive servo system [18].

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In this paper, we are interested in establishing a connection between CT-NI and DT-NI systems in terms of the bilinear transformation. It is well known that a CT-PR transfer matrix transforms into a DT-PR transfer matrix by such a transformation [1], [22], and vice versa. This result motivates the following question: Does this result remain true for NI systems? The answer is "Yes," and a proof is given in this paper. Meanwhile, the discrete-time lossless negative imaginary (DT-LNI) systems, expressed as a special and important case of DT-NI systems, are also studied in this paper. The concept of DT-LNI systems is introduced. Similarly, a connection between CT-LNI and DT-LNI systems is established, and a DT-LNI lemma is derived based on this connection. In addition, a necessary and sufficient condition is developed to characterize the DT-LNI properties.

The rest of this paper is organized as follows. Section II reviews the definition of DT-PR transfer matrices, the DT-PR lemma, and two frequently used "equivalent" conditions. Then, we analyze the differences between those lemmas. Section III studies the connection between CT-NI and DT-NI transfer matrices. Section IV studies the DT-LNI systems. One numerical example is presented in Section V. Section VI concludes the paper.

Notation: $\mathbb{R}^{m\times n}$ and $\mathcal{R}^{m\times n}$ denote the sets of $m\times n$ real matrices and real-rational proper transfer function matrices, respectively. $\mathrm{Re}[.]$ denotes the real part of complex numbers. $\lambda_{\max}$ denotes the maximum eigenvalue of a square complex matrix with only real eigenvalues. A^T and A^* denote the transpose and the complex conjugate transpose of a complex matrix A, respectively. I denotes any identity matrix with compatible dimensions. $A > (\geq) 0$ and $A < (\leq) 0$ denote the symmetric positive (semi)definite matrix and the symmetric negative (semi)definite matrix, respectively.

II. DT-PR Transfer Function Matrices

In this section, our goal is to discuss the differences between the concept and "equivalent" conditions of DT-PR systems. First, we briefly recall the definition of DT-PR systems in z-domain [1], two different versions of DT-PR systems in terms of properties on the unit cycle [1]–[3], and the DT-PR lemma in terms of state-space realization [1].

Definition 1: [1] A square matrix $F(z)$ of real-rational proper functions is called DT-PR if the following statements hold:

1) all elements of $F(z)$ are analytic in $|z| > 1$; and
2) $F^*(z) + F(z) \geq 0$ for all $|z| > 1$.

The following two lemmas are two different restatements of Definition 1 in terms of properties on $|z| = 1$.

Lemma 1: [1, Lemma 2] A square matrix $F(z)$ whose elements are real-rational proper functions analytic in $|z| > 1$ is DT-PR if and only if the following statements hold:

1) poles of elements of $F(z)$ on $|z| = 1$ are simple;
2) $F^*(e^{j\theta}) + F(e^{j\theta}) \geq 0$ for all real θ at which $F(e^{j\theta})$ exists; and

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3) if $z_0 = e^{j\theta_0}$, θ_0 is real, is a pole of an element of $F(z)$, and if K_0 is the residue matrix of $F(z)$ at z_0, then the matrix $e^{-j\theta_0}K_0$ is positive semidefinite Hermitian.

Remark 1: One should note that the following condition is contained in Condition 3 of Lemma 1: If $F(z)$ has a simple pole at -1, then the corresponding residue matrix $\lim_{z\to -1}(z+1)F(z)$ is negative semidefinite Hermitian. This fact follows by a direct calculation that $e^{-j\theta}|_{\theta=0} = 1$ and $e^{-j\theta}|_{\theta=\pi} = -1$.

Lemma 2: [2], [3] A square real-rational proper transfer function matrix $F(z)$ is DT-PR if and only if the following statements hold:

1) $F(z)$ is analytic in $|z| > 1$;
2) $F^*(e^{j\theta}) + F(e^{j\theta}) \geq 0$ for all real θ at which $F(e^{j\theta})$ exists; and
3) the poles of $F(z)$ on $|z| = 1$ are simple and the corresponding residue matrices of $F(z)$ at those poles are positive semidefinite Hermitian.

Remark 2: According to Condition 3 in Lemma 2, we have the following result: If $F(z)$ has a simple pole at -1, then the corresponding residue matrix $\lim_{z\to -1}(z+1)F(z)$ is positive semidefinite Hermitian.

Obviously, it follows from Remarks 1 and 2 that Condition 3 of Lemma 1 is not equivalent to Condition 3 of Lemma 2. When $F(z)$ has poles on $|z| = 1$, Condition 3 of Lemma 1 requires that $e^{-j\theta_0}K_0$ be positive semidefinite Hermitian, whereas Condition 3 of Lemma 2 requires that the associated residue matrix K_0 be positive semidefinite Hermitian.

The following lemma is a classical DT-PR lemma in terms of minimal state-space realization.

Lemma 3: [1] Let $F(z)$ be a square real-rational proper transfer functions of z with no poles in $|z| > 1$ and simple poles only on $|z| = 1$, and let (A, B, C, D) be a minimal realization of $F(z)$. Then, necessary and sufficient conditions for $F(z)$ to be DT-PR are that there exist a real matrix $P = P^T > 0$ and real matrices L and W such that

$$A^T PA - P = -L^T L$$

$$C^T - A^T PB = L^T W$$

$$D^T + D - B^T PB = W^T W.$$

Remark 3: In [4], it has been pointed out that Definition 1 and Lemmas 1 and 3 are agreeable with each other. Lemma 2 is not equivalent to Definition 1 and Lemmas 1 and 3. Xiao and Hill [4] utilized three counterexamples to show this result.

In this section, we provide the detailed reasons why Lemma 2 is not equivalent to Definition 1 and Lemmas 1 and 3. As was mentioned in [4], Lemma 3 plays an important role in the research of DT-PR systems. Hitz and Anderson [1] proved Lemma 3 by applying the bilinear transformation

$$s = \frac{z-1}{z+1}. \quad (1)$$

So, if we use Lemma 3, one should admit that, via the bilinear transformation in (1), the CT-PR transfer matrix $F(s)$ is transformed into a DT-PR transfer matrix $F(z)$, and vice versa. Next, we will show that: under the transformation in (1), Lemma 2 is not equivalent to [23, Th. 2.7.2], which is the CT counterpart of Lemma 2.

According to Lemmas 1 and 2, it can be found that the main distinctions lie in Condition 3 when the transfer matrix $F(z)$ has poles on $|z| = 1$. Hence, without loss of generality, assume that the CT-PR transfer matrix $F(s)$ has some simple poles on the purely imaginary axis. Then, according to the minor decomposition theory in [23, p. 216], $F(s)$ can be written in the following form:

$$F(s) = \Sigma_i \frac{K_i}{s - j\omega_i} + sA + \frac{C}{s} + F_0(s) \quad (2)$$

where $F_0(s)$ is analytic in $\mathrm{Re}[s] > 0$; $K_i = K_i^* \geq 0$, $A = A^* \geq 0$, and $C = C^* \geq 0$ are the associated residue matrix at $j\omega_i$ ($\omega_i > 0$), 0, and ∞, respectively. By means of the transformation in (1), (2) transforms into $F(z) = \Sigma_i \frac{(z+1)K_i}{(1-j\omega_i)z-(1+j\omega_i)} + \frac{z-1}{z+1}A + \frac{z+1}{z-1}C + F_0\left(\frac{z-1}{z+1}\right)$. Because $F(s)$ is CT-PR, it follows from [22, Th. 1] and Lemma 3 that $F(z)$ is DT-PR. However, when $e^{j\theta_1} = \frac{1+j\omega_1}{1-j\omega_1}$ is a simple pole of $F(z)$, the residue matrix of $F(z)$ at $e^{j\theta_1}$ is given by

$$K_0 = \lim_{z\to\frac{1+j\omega_1}{1-j\omega_1}} \left(z - \frac{1+j\omega_1}{1-j\omega_1}\right)F(z) = \frac{2K_1}{1-\omega_1^2-2j\omega_1}$$

which is not positive semidefinite Hermitian, where $K_1 = \lim_{s\to j\omega_1}(s - j\omega_1)F(s)$ is positive semidefinite. This contradicts Condition 3 in Lemma 2. Moreover, the matrix

$$e^{-j\theta_1}K_0 = \left(\frac{1-\omega_1^2-2\omega_1 j}{1+\omega_1^2}\right)\frac{2K_1}{1-\omega_1^2-2\omega_1 j} = \frac{2K_1}{1+\omega_1^2}$$

is positive semidefinite Hermitian. This coincides with Condition 3 in Lemma 1. Although some researchers adopt Lemma 2 in their research and the form of Lemma 2 is similar to the CT case, it follows from above theoretical analysis that Lemma 1 is correct and Lemma 2 cannot be used to test DT-PR properties when the system has poles on $|z| = 1$. In addition, it is noteworthy that some inconsistencies in [3] have been corrected in [24].

Example 1: To illustrate the main results in this section, consider a CT-PR transfer matrix $F(s) = \begin{pmatrix} \frac{s}{s^2+1} & \frac{1}{s^2+1} \\ \frac{-1}{s^2+1} & \frac{s}{s^2+1} \end{pmatrix}$. Using the bilinear transformation in (1) transforms $F(s)$ to $F(z) = \begin{pmatrix} \frac{z^2-1}{2(z^2+1)} & \frac{(z+1)^2}{2(z^2+1)} \\ \frac{-(z+1)^2}{2(z^2+1)} & \frac{z^2-1}{2(z^2+1)} \end{pmatrix}$. A calculation shows that the residue matrix of $F(z)$ at $z = j$ is given by $K = \begin{pmatrix} \frac{j}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{j}{2} \end{pmatrix}$, which is not positive semidefinite Hermitian and contradicts Condition 3 in Lemma 2. However, the matrix $e^{-j\theta}K|_{\theta=\frac{\pi}{2}} = -jK = \begin{pmatrix} \frac{1}{2} & -\frac{j}{2} \\ \frac{j}{2} & \frac{1}{2} \end{pmatrix}$ is positive semidefinite Hermitian and satisfies Condition 3 in Lemma 1.

III. DT-NI Transfer Function Matrices

In this section, our goal is to establish the connection between DT-NI and CT-NI systems. First, the definitions of CT-NI and DT-NI transfer matrices are introduced, respectively.

Definition 2: [17] A square real-rational transfer function matrix $G(s)$ is called CT-NI if the following statements hold:

1) $G(s)$ has no poles in $\mathrm{Re}[s] > 0$;
2) $j[G(j\omega) - G^*(j\omega)] \geq 0$ for all $\omega > 0$ except values of ω where $j\omega$ is a pole of $G(s)$;
3) if $s = 0$ is a pole of $G(s)$, then $\lim_{s\to 0} s^2 G(s)$ is positive semidefinite Hermitian, and $\lim_{s\to 0} s^m G(s) = 0$ for all $m \geq 3$;
4) if $s = j\omega_0$ with $\omega_0 > 0$ is a pole of $G(s)$, ω_0 is finite, it is at most a simple pole and the residue matrix $K = \lim_{s\to j\omega_0}(s - j\omega_0)jG(s)$ is positive semidefinite Hermitian; and
5) if $s = j\infty$ is a pole of $G(s)$, then $\lim_{\omega\to\infty} \frac{G(j\omega)}{(j\omega)^2}$ is negative semidefinite Hermitian, and $\lim_{\omega\to\infty} \frac{G(j\omega)}{(j\omega)^m} = 0$ for all $m \geq 3$.

Definition 3: [20] A square real-rational proper transfer function matrix $G(z)$ is called DT-NI if the following statements hold:

1) $G(z)$ has no poles in $|z| > 1$;
2) $j[G(e^{j\theta}) - G^*(e^{j\theta})] \geq 0$ for all $\theta \in (0, \pi)$ except values of θ where $e^{j\theta}$ is a pole of $G(z)$;
3) if $z_0 = e^{j\theta_0}$, $\theta_0 \in (0, \pi)$, is a pole of $G(z)$, then it is at most a simple pole and the residue matrix $\tilde{K} = \lim_{z\to z_0}(z - z_0)jG(z)$ satisfies that $e^{-j\theta_0}\tilde{K}$ is positive semidefinite Hermitian;

4) if $z = 1$ is a pole of $G(z)$, then it is at most a double pole, $\lim_{z\to 1}(z-1)^2 G(z)$ is positive semidefinite Hermitian, and $\lim_{z\to 1}(z-1)^m G(z) = 0$ for all $m \geq 3$; and
5) if $z = -1$ is a pole of $G(z)$, then it is at most a double pole, $\lim_{z\to -1}(z+1)^2 G(z)$ is negative semidefinite Hermitian, and $\lim_{z\to -1}(z+1)^m G(z) = 0$ for all $m \geq 3$.

The following lemma characterizes a connection between CT-NI and DT-NI transfer matrices.

Lemma 4: A CT-NI transfer matrix $G(s)$ transforms into a DT-NI transfer matrix $G(z)$ by the bilinear transformation $s = \frac{z-1}{z+1}$. Conversely, a DT-NI transfer matrix $G(z)$ transforms into a CT-NI transfer matrix $G(s)$ by the bilinear transformation $z = \frac{1+s}{1-s}$.

Proof: Assume that $G(s)$ is CT-NI. Then, under the bilinear transformation $s = \frac{z-1}{z+1}$, we will show that the five conditions in Definition 3 are satisfied. Conditions 1 and 2 of Definition 3 are immediate.

If $s = 0$ is a pole of $G(s)$, then $z = 1$ is also a pole of $G(z)$. According to the minor decomposition theory in [17], $G(s)$ is of the form $G(s) = \frac{A_2}{s^2} + \frac{A_1}{s} + G_0(s)$, where $A_2 = A_2^* \geq 0$, $A_1 + A_1^T \geq 0$, and $G_0(s)$ has no poles in $\mathrm{Re}[s] > 0$ and at $s = 0$. By means of the bilinear transformation $s = \frac{z-1}{z+1}$, $G(s)$ transforms into $G(z) = \left(\frac{z+1}{z-1}\right)^2 A_2 + \frac{z+1}{z-1} A_1 + G_0\left(\frac{z-1}{z+1}\right)$. Then, $\lim_{z\to 1}(z-1)^2 G(z) = 4A_2 \geq 0$, and $\lim_{z\to 1}(z-1)^m G(z) = 0$ for all $m \geq 3$.

If $s = j\omega_0, \omega_0 > 0$, is a pole of $G(s)$, then $z = \frac{1+j\omega_0}{1-j\omega_0}$ is also a pole of $G(z)$. Decompose $G(s)$ to the form

$$G(s) = \frac{-jK}{s - j\omega_0} + \frac{jK^*}{s + j\omega_0} + G_0(s) \tag{3}$$

where K is the residue matrix of $jG(s)$ at $j\omega_0$, $K = K^* \geq 0$, and $G_0(s)$ has no poles in $\mathrm{Re}[s] > 0$ and at $\pm j\omega_0$. By means of the transformation $s = \frac{z-1}{z+1}$, $G(s)$ in (3) transforms into $G(z) = \frac{-jK(z+1)}{(1-j\omega_0)z-(1+j\omega_0)} + \frac{jK^*(z+1)}{(1+j\omega_0)z-(1-j\omega_0)} + G_0\left(\frac{z-1}{z+1}\right)$. The residue matrix of $jG(z)$ at $e^{j\theta_0} = \frac{1+j\omega_0}{1-j\omega_0}$ is given by

$$\tilde{K} = \lim_{z\to\frac{1+j\omega_0}{1-j\omega_0}} \left(z - \frac{1+j\omega_0}{1-j\omega_0}\right) jG(z) = \frac{2K}{1-\omega_0^2 - 2\omega_0 j}$$

which is not positive semidefinite Hermitian unless $\omega_0 = 0$. Then, we have $e^{-j\theta_0} = \frac{1-\omega_0^2-2\omega_0 j}{1+\omega_0^2}$, and the matrix $e^{-j\theta_0}\tilde{K} = \frac{2K}{1+\omega_0^2}$ is positive semidefinite Hermitian. Similarly, the residue matrix at $\frac{1-j\omega_0}{1+j\omega_0}$ has the same property.

If $s = j\infty$ is a pole of $G(s)$, then $z = -1$ is also a pole of $G(z)$. Decompose $G(s)$ to the form: $G(s) = s^2 C_2 + sC_1 + G_0(s)$, where $C_2 = C_2^* \leq 0$, $C_1 + C_1^T \leq 0$, and $G_0(s)$ has no poles in $\mathrm{Re}[s] > 0$ and at infinity. Under the transformation $s = \frac{z-1}{z+1}$, $G(s)$ transforms into $G(z) = \left(\frac{z-1}{z+1}\right)^2 C_2 + \frac{z-1}{z+1} C_1 + G_0\left(\frac{z-1}{z+1}\right)$. Then, $\lim_{z\to -1}(z+1)^2 G(z) = 4C_2 \leq 0$, and $\lim_{z\to -1}(z+1)^m G(z) = 0$ for all $m \geq 3$.

Conversely, assume that $G(z)$ is DT-NI. We will show that the five conditions in Definition 2 are satisfied by means of the bilinear transformation $z = \frac{1+s}{1-s}$. Also, Conditions 1 and 2 of Definition 2 are immediate.

If $z = e^{j\theta_0}$, $\theta_0 \in (0, \pi)$, is a pole of $G(z)$, then $s = j\frac{\sin\theta_0}{1+\cos\theta_0} > 0$ is also a pole of $G(s)$. Similar to the minor decomposition of CT case in [17], we decompose $G(z)$ as $G(z) = \frac{-j\tilde{K}_0}{z - e^{j\theta_0}} + \frac{j\tilde{K}_0^*}{z - e^{-j\theta_0}} + G_0(z)$, where $\tilde{K}_0$ is the residue matrix of $jG(z)$ at $e^{j\theta_0}$, $e^{-j\theta_0}\tilde{K}_0$ is positive semidefinite Hermitian, and $G_0(z)$ has no poles in $|z| > 1$ and at $e^{\pm j\theta_0}$. Consider the transformation $z = \frac{1+s}{1-s}$. $G(z)$ transforms into $G(s) = \frac{-j\tilde{K}_0(1-s)}{(1+e^{j\theta_0})s+(1-e^{j\theta_0})} + \frac{j\tilde{K}_0^*(1-s)}{(1+e^{-j\theta_0})s+(1-e^{-j\theta_0})} + G_0\left(\frac{1+s}{1-s}\right)$. Then, the residue matrix of $jG(s)$ at $s = \frac{e^{j\theta_0}-1}{e^{j\theta_0}+1}$ is given by

$$K = \lim_{s\to\frac{e^{j\theta_0}-1}{e^{j\theta_0}+1}} \left(s - \frac{e^{j\theta_0}-1}{e^{j\theta_0}+1}\right) jG(s) = \frac{2\tilde{K}_0}{(e^{j\theta_0}+1)^2}$$

$$= \frac{e^{-j\theta_0}\, 2\tilde{K}_0}{e^{-j\theta_0}(1 + 2e^{j\theta_0} + e^{2j\theta_0})} = \frac{e^{-j\theta_0}\tilde{K}_0}{1+\cos\theta_0}$$

which is positive semidefinite Hermitian.

If $z = 1$ is a pole of $G(z)$, then $s = 0$ is a pole of $G(s)$. We decompose $G(z)$ as $G(z) = \frac{A_2}{(z-1)^2} + \frac{A_1}{z-1} + G_0(z)$, where $A_2 = A_2^* \geq 0$ and $G_0(z)$ has no poles in $|z| > 1$ and at $z = 1$. By the same transformation, $G(z)$ transforms into $G(s) = \frac{A_2}{4s^2} + \frac{A_1 - A_2}{2s} + G_1\left(\frac{1+s}{1-s}\right) - \frac{A_1}{2} + \frac{A_2}{4}$. Then, $\lim_{s\to 0} s^2 G(s) = \frac{A_2}{4} \geq 0$, and $\lim_{s\to 0} s^m G(s) = 0$ for all $m \geq 3$.

If $z = -1$ is a pole of $G(z)$, then $s = j\infty$ is a pole of $G(s)$. We write $G(z)$ in the form $G(z) = \frac{C_2}{(z+1)^2} + \frac{C_1}{z+1} + G_0(z)$, where $C_2 = C_2^* \leq 0$. Using the same transformation, $G(z)$ transforms into $G(s) = \frac{C_2}{4}s^2 + \frac{-C_1-C_2}{2}s + G_0\left(\frac{1+s}{1-s}\right) + \frac{C_1}{2} + \frac{C_2}{4}$. Then, $\lim_{\omega\to\infty}\frac{G(j\omega)}{(j\omega)^2} = \frac{C_2}{4} \leq 0$, and $\lim_{\omega\to\infty}\frac{G(j\omega)}{(j\omega)^m} = 0$ for all $m \geq 3$. ■

Example 2: To illustrate the usefulness of Lemma 4, consider a nonsymmetric CT-NI transfer function matrix $G(s) = \begin{pmatrix} -s^2 - s & -s^2 - s \\ -s^2 + s & -2s^2 - s \end{pmatrix}$. By the bilinear transformation in (1), $G(s)$ transforms into $G(z) = \begin{pmatrix} \frac{-2(z^2-z)}{(z+1)^2} & \frac{-2(z^2-z)}{(z+1)^2} \\ \frac{2z-2}{(z+1)^2} & \frac{-3z^2+4z-1}{(z+1)^2} \end{pmatrix}$. Conditions 1 and 2 of Definition 3 are immediate after a direct calculation. $G(z)$ has a double pole at -1, and $\lim_{z\to -1}(z+1)^2 G(z) = \begin{pmatrix} -4 & -4 \\ -4 & -8 \end{pmatrix}$ is negative definite Hermitian, $\lim_{z\to -1}(z+1)^m G(z) = 0$ for all $m \geq 3$. Hence, $G(z)$ is DT-NI in view of Definition 3.

In the following lemma, we present an alternative method to prove the DT-NI lemma in [20].

Lemma 5: [20] Let (A, B, C, D) be a minimal state-space realization of a real-rational proper DT transfer matrix $G(z) \in \mathcal{R}^{m\times m}$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, and $m \leq n$. Suppose $\det(I + A) \neq 0$ and $\det(I - A) \neq 0$. Then, $G(z)$ is DT-NI if and only if the following statements hold:

1) $C(I + A)^{-1}B - D = B^T(I + A^T)^{-1}C^T - D^T$; and
2) there exists a matrix $Y = Y^T > 0, Y \in \mathbb{R}^{n\times n}$, such that

$$Y - AYA^T \geq 0 \text{ and } B = (I - A)Y(I + A^T)^{-1}C^T.$$

Proof: By means of the bilinear transformations $s = \frac{z-1}{z+1}$ and $z = \frac{1+s}{1-s}$, the DT transfer matrix $G(z) = C(zI - A)^{-1}B + D$ is transformed into a CT transfer matrix $G(s) = H(sI - F)^{-1}G + J$, and vice versa, where

$$\begin{cases} F = (A + I)^{-1}(A - I),\ G = \sqrt{2}(I + A)^{-1}B \\ H = \sqrt{2}C(A + I)^{-1}, \quad J = D - C(I + A)^{-1}B. \end{cases} \tag{4}$$

According to [22, Th. 1], (F, G, H, J) is a minimal realization of $G(s)$. Then, one has the following equivalent statements:

$G(z) \sim (A, B, C, D)$ is DT-NI.

$\Leftrightarrow G(s) \sim (F, G, H, J)$ is CT-NI. This equivalence is via Lemma 4.

$\Leftrightarrow J = J^T$ and there exists a real matrix $Y = Y^T > 0, Y \in \mathbb{R}^{n\times n}$, such that $FY + YF^T \leq 0$ and $G + FYH^T = 0$. This equivalence is via the CT-NI lemma in [15, Lemma 7].

$\Leftrightarrow D - C(I + A)^{-1}B = D^T - B^T(I + A^T)^{-1}C^T$ and there exists a matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, such that $Y - AYA^T \geq 0$ and $B = (I - A)Y(I + A^T)^{-1}C^T$. ■

Remark 4: It is worthwhile noting that the proof of Lemma 5 is based on the connection between CT-NI and DT-NI transfer matrices developed in Lemma 4. However, the proof of DT-NI lemma in [20, Lemma 11] and [19, Th. 3.2] is based on the relation between DT-PR and DT-NI systems. Lemma 5 is equivalent to [20, Lemma 11] according to [19, Corollary 3.1]. Moreover, Lemma 4 can be used to develop other properties of DT-NI systems, such as the internal stability results in [19] and [20], and the residue matrix properties with simple pole at $z = \pm 1$ [20, Lemma 4]. The more detailed proofs of Lemmas 4 and 5 can be found in [21].

IV. DT-LNI Transfer Function Matrices

In this section, our goal is to introduce the concept of DT-LNI systems, study some properties of such systems, and present the DT-LNI lemma in terms of minimal state-space realization. First, we recall the definition of CT-LNI systems.

Definition 4: [17] A square real-rational transfer function matrix $G(s)$ is called CT-LNI if the following statements hold:

1) $G(s)$ is CT-NI; and
2) $j[G(j\omega) - G^*(j\omega)] = 0$ for all $\omega > 0$ except values of ω where $j\omega$ is a pole of $G(z)$.

Definition 5: A square real-rational proper transfer function matrix $G(z)$ is called DT-LNI if the following statements hold:

1) $G(z)$ is DT-NI; and
2) $j[G(e^{j\theta}) - G^*(e^{j\theta})] = 0$ for all $\theta \in (0, \pi)$ except values of θ where $e^{j\theta}$ is a pole of $G(z)$.

The following lemma characterizes a connection between CT-LNI and DT-LNI transfer matrices.

Lemma 6: A CT-LNI transfer matrix $G(s)$ transforms into a DT-LNI transfer matrix $G(z)$ by the bilinear transformation $s = \frac{z-1}{z+1}$. Conversely, a DT-LNI transfer matrix $G(z)$ transforms into a CT-LNI transfer matrix $G(s)$ by the bilinear transformation $z = \frac{1+s}{1-s}$.

Proof: Suppose $G(s)$ is CT-LNI. Condition 1 of Definition 4 implies that $G(s)$ is CT-NI. According to Lemma 4, $G(z)$ is DT-NI by the bilinear transformation $s = \frac{z-1}{z+1}$. Furthermore, if $s = j\omega$, $\omega > 0$, is not a pole of $G(s)$, then $z = \frac{1+s}{1-s} = \frac{1+j\omega}{1-j\omega} = \frac{1-\omega^2+2\omega j}{1+\omega^2}$ is not a pole of $G(z)$. Then, for all $\omega > 0$ with $j\omega$ not a pole of $G(s)$, $j[G(j\omega) - G^*(j\omega)] = 0$ implies that $j[G(\frac{1-\omega^2+2\omega j}{1+\omega^2}) - G^*(\frac{1-\omega^2+2\omega j}{1+\omega^2})] = 0$ for all $\theta \in (0, \pi)$ with $e^{j\theta} = \frac{1-\omega^2+2\omega j}{1+\omega^2}$ not a pole of $G(z)$. Therefore, it follows from Definition 5 that $G(z)$ is DT-LNI.

Conversely, suppose $G(z)$ is DT-LNI. Condition 1 of Definition 5 implies that $G(z)$ is DT-NI. It follows from Lemma 4 that $G(s)$ is CT-NI under the transformation $z = \frac{1+s}{1-s}$. Also, for all $\theta \in (0, \pi)$ with $e^{j\theta}$ not a pole of $G(z)$, $j[G(e^{j\theta}) - G^*(e^{j\theta})] = 0$ implies that $j[G(j\frac{\sin\theta}{1+\cos\theta}) - G^*(j\frac{\sin\theta}{1+\cos\theta})] = 0$ for all real $\omega > 0$ such that $j\omega = \frac{e^{j\theta}-1}{e^{j\theta}+1} = j\frac{\sin\theta}{1+\cos\theta}$ is not a pole of $G(s)$. Hence, $G(s)$ is CT-LNI by Definition 4. ■

Example 3: To illustrate the usefulness of Lemma 6, consider a nonsymmetric CT-LNI transfer matrix $G(s) = \begin{pmatrix} \frac{2}{s^2+1} & \frac{-s}{s^2+1} \\ \frac{s}{s^2+1} & \frac{2}{s^2+1} \end{pmatrix}$. By the bilinear transformation $s = \frac{z-1}{z+1}$, $G(s)$ transforms into $G(z) = \begin{pmatrix} \frac{(z+1)^2}{(z^2+1)} & \frac{1-z^2}{2(z^2+1)} \\ \frac{z^2-1}{2(z^2+1)} & \frac{(z+1)^2}{(z^2+1)} \end{pmatrix}$. After a direct calculation, it follows that $j[G(e^{j\theta}) - G^*(e^{j\theta})] = 0$ for all $\theta \in (0, \pi)$ with $e^{j\theta}$ not a pole of $G(z)$. The residue matrix of $jG(z)$ at $e^{j\theta} = j$ ($\theta = \frac{\pi}{2}$) is given by $\tilde{K} = \begin{pmatrix} j & \frac{1}{2} \\ -\frac{1}{2} & j \end{pmatrix}$, which is not positive semidefinite Hermitian. Then, the matrix $e^{-j\theta}\tilde{K} = -j\tilde{K} = \begin{pmatrix} 1 & -\frac{j}{2} \\ \frac{j}{2} & 1 \end{pmatrix}$ is positive semidefinite Hermitian, which satisfies Condition 3 in Definition 3. It follows from Definitions 3 and 5 that $G(z)$ is DT-LNI.

The following lemma provides a necessary and sufficient condition for a system to be DT-LNI.

Lemma 7: A square real-rational proper transfer function matrix $G(z)$ is DT-LNI if and only if the following statements hold:

1) all poles of elements of $G(z)$ lie on $|z| = 1$ and the poles at $e^{j\theta}$, $\theta \in (0, \pi)$, are simple, and the residue matrix $\tilde{K} = \lim_{z\to z_0}(z - z_0)jG(z)$ at any pole $z_0 = e^{j\theta_0}$, $\theta_0 \in (0, \pi)$, satisfies that $e^{-j\theta_0}\tilde{K}$ is positive semidefinite Hermitian;
2) if $z = 1$ is a pole of $G(z)$, then $\lim_{z\to 1}(z-1)^2 G(z)$ is positive semidefinite Hermitian, and $\lim_{z\to 1}(z-1)^m G(z) = 0$ for all $m \geq 3$;
3) if $z = -1$ is a pole of $G(z)$, then $\lim_{z\to -1}(z+1)^2 G(z)$ is negative semidefinite Hermitian, and $\lim_{z\to -1}(z+1)^m G(z) = 0$ for all $m \geq 3$; and
4) $G(z) = G^T(z^{-1})$ for all z such that z is not a pole of $G(z)$.

Proof: (Necessity) Suppose $G(z)$ is DT-LNI. Condition 2 of Definition 5 implies that $j[G(e^{j\theta}) - G^*(e^{j\theta})] = 0$ for all $\theta \in (0, \pi)$ with $e^{j\theta}$ not a pole of $G(z)$. By taking the complex conjugate, we have $\overline{j[G(e^{j\theta}) - G^*(e^{j\theta})]} = 0$. That is, $j[G(e^{j\theta}) - G^*(e^{j\theta})] = 0$ for all $\theta \in (0, -\pi)$. When $G(z)$ has no poles at $\theta = 0, \pi$, one has that $j[G(e^{j\theta}) - G^*(e^{j\theta})] = 0$ for $\theta = 0, \pi$ due to the continuity of $G(z)$. Hence, we have

$$j[G(e^{j\theta}) - G^*(e^{j\theta})] = 0 \tag{5}$$

for all $\theta \in [0, 2\pi]$ with $e^{j\theta}$ not a pole of $G(z)$. In view of the fact that $G(z)$ is real-rational function, one has that $G^*(e^{j\theta}) = G^T(e^{-j\theta})$, and so (5) implies that

$$j[G(z) - G^T(z^{-1})] = 0$$

for all $|z| = 1$, where z is not a pole of any element of $G(z)$. Because $j[G(z) - G^T(z^{-1})]$ is an analytic function of z, it follows from maximum modulus theorem ([25, Th. A4-3]) that $j[G(z) - G^T(z^{-1})] = 0$ for all z with z not a pole of $G(z)$. Hence, $G(z) = G^T(z^{-1})$ for all z such that z is not a pole of $G(z)$. Condition 4 holds.

Suppose z_0 is a pole of $G(z)$. Then, it follows from $G(z) = G^T(z^{-1})$ that z_0^{-1} is also a pole of $G(z)$. However, we know that $G(z)$ has no poles in $|z| > 1$. If $|z_0| < 1$, then $|z_0^{-1}| > 1$. So, the only case is that all poles of elements of $G(z)$ lie on $|z| = 1$. Moreover, Condition 3 of Definition 3 implies that the poles at $e^{j\theta}$, $\theta \in (0, \pi)$, are simple, and the matrix $e^{-j\theta_0}\tilde{K}$ at any pole $e^{j\theta_0}$, $\theta_0 \in (0, \pi)$, is positive semidefinite Hermitian. Thus, Condition 1 holds. Also, Conditions 4 and 5 of Definition 3 imply that Conditions 2 and 3 hold.

(Sufficiency) Suppose Conditions 1–4 hold. First, Conditions 1–3 imply that Condition 1 and Conditions 3–5 of Definition 3 hold. Second, Condition 4 implies that $G(z) = G^T(z^{-1})$. It follows that $G(e^{j\theta}) = G^*(e^{j\theta})$, and so implies that $j[G(e^{j\theta}) - G^*(e^{j\theta})] = 0$ for all $\theta \in (0, \pi)$ with $e^{j\theta}$ not a pole of $G(z)$. Thus, $G(z)$ is DT-LNI according to Definitions 3 and 5. ■

The following lemma characterizes the properties of a sum of DT-LNI transfer matrices.

Lemma 8: Given two DT-LNI transfer matrices $G_1(z)$, $G_2(z)$, and a DT-NI transfer matrix $G(z)$. Then

1) $G_1(z) + G_2(z)$ is DT-LNI; and
2) $G_1(z) + G(z)$ is DT-NI.

Proof: The proof is trivial according to the definition of DT-LNI and DT-NI transfer matrices. ■

The DT-LNI lemma proposed in the following provides a necessary and sufficient condition for a system to be DT-LNI in terms of minimal state-space realization.

Lemma 9: Let (A, B, C, D) be a minimal state-space realization of a real-rational proper DT transfer function matrix $G(z) \in \mathcal{R}^{m\times m}$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{m\times m}$, and $m \leq n$.

Suppose $\det(I + A) \neq 0$ and $\det(I - A) \neq 0$. Then, $G(z)$ is DT-LNI if and only if the following statements hold:

1) $C(I + A)^{-1}B - D = B^T(I + A^T)^{-1}C^T - D^T$; and
2) there exists a matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, such that

$$Y - AYA^T = 0 \text{ and } B = (I - A)Y(I + A^T)^{-1}C^T.$$

Proof: Similar to the proof of Lemma 5, the proof follows from the following sequence of equivalent reformulations:

$G(z) \sim (A, B, C, D)$ is DT-LNI.

$\Leftrightarrow G(s) \sim (F, G, H, J)$ is CT-LNI, where F, G, H, and J are defined in (4). This equivalence is according to Lemma 6.

$\Leftrightarrow J = J^T$ and there exists a real matrix $Y = Y^T > 0, Y \in \mathbb{R}^{n\times n}$, such that $FY + YF^T = 0$ and $G + FYH^T = 0$. This equivalence is via the CT-LNI lemma in [8, Th. 1].

$\Leftrightarrow D - C(I + A)^{-1}B = D^T - B^T(I + A^T)^{-1}C^T$ and there exists a matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n\times n}$, such that $Y - AYA^T = 0$ and $B = (I - A)Y(I + A^T)^{-1}C^T$. ■

In the following result, we consider the internal stability of a positive feedback interconnection of two DT-NI systems in terms of loop gain at $z = 1$. The positive feedback interconnection is denoted by $[G(z), G_s(z)]$, where $G(z)$ is DT-LNI.

Corollary 1: Given a DT-LNI transfer matrix $G(z)$, and a DT strictly NI transfer matrix $G_s(z)$. Suppose $G(z)$ and $G_s(z)$ have no poles at -1 and 1, and that also satisfy $G(-1)G_s(-1) = 0$ and $G_s(-1) \geq 0$. Then, the positive feedback interconnection $[G(z), G_s(z)]$ is internally stable if and only if $\lambda_{\max}(G(1)G_s(1)) < 1$.

Remark 5: The DT-LNI lemma in Lemma 9 can be considered as a modification of the DT-NI lemma in [20] by replacing the inequality with equality. The DT-LNI systems can be considered as a special case of the DT-NI systems with all the systems poles on $|z| = 1$. As a result, all results developed in [20] are valid for DT-LNI systems. The results in Corollary 1 are actually a special case of [19, Th. 8] or [20, Th. 1] with one system being DT-LNI, and hence proof is omitted here. Similar to [8, Corollaries 1 and 2], Corollary 1 can be written in the same form as the small-gain theorem, where one system is DT-LNI; details are omitted here.

V. Numerical Examples

In this section, one numerical example is given to illustrate the DT-(L)NI lemma of the paper.

Example 4: To illustrate Lemmas 5 and 9, consider the DT-(L)NI transfer matrix $G(z)$ in Example 3 (DT-LNI system is also DT-NI system). A minimal state-space realization of $G(z)$ in Example 3 is as follows:

$$A = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 \end{pmatrix},\ D = \begin{pmatrix} 1 & -\frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}.$$

A calculation shows that $C(I + A)^{-1}B - D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus, Condition 1 in Lemmas 5 and 9 holds, respectively. Because $G(s)$ in Example 3 is strictly proper, it leads to $J = 0$, and hence $C(I + A)^{-1}B - D = 0$ always holds. If $G(s)$ is proper, but not strictly proper, then $J \neq 0$, and also $C(I + A)^{-1}B - D \neq 0$. Then, YALMIP and SeDuMi were used to solve the Condition 2 in Lemma 9, and we obtained the following solution:

$$Y = \begin{pmatrix} \frac{2}{3} & 0 & 0 & -\frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ -\frac{1}{3} & 0 & 0 & \frac{2}{3} \end{pmatrix} > 0.$$

Condition 2 in Lemmas 5 and 9 holds, respectively.

Consider the transfer matrix $G(z)$ in Example 2. Because $G(s)$ in Example 2 is nonproper and has a double pole at infinity, $G(z)$ in Example 2 has a double pole at -1. The minimal state-space realization of such $G(z)$ always has poles at -1, and hence the condition $\det(I + A) \neq 0$ does not hold. In this case, we cannot use Lemma 5 to judge whether $G(z)$ is DT-NI. Furthermore, consider the robotic arm example in [16]. The finite dimensional model $G_f(s)$ in [16, eq. (23)] is CT-NI. A calculation shows that $G_f(z)$ is also DT-NI by the bilinear transformation in (1).

VI. Conclusion

This paper has studied three related problems. First, it was shown by theoretical analysis that only the original necessary and sufficient conditions were equivalent to the definition of DT-PR transfer matrices and DT-PR lemma. This result is in line with conclusions in [4]. Second, motivated by the DT-PR case, it was found that DT-NI and CT-NI transfer matrices were equivalent by bilinear transformations. Third, the DT-LNI systems were studied. Finally, the developed theory in this paper was illustrated by examples.

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