Bilinear Transformation for Discrete-Time Positive Real and Negative Imaginary Systems

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Abstract—This paper studies the connection between discrete-time and continuous-time negative imaginary (NI) systems. First, we analyse differences between two statements that are claimed to provide equivalent conditions for systems to be discrete-time positive real (PR). Our conclusion is that one is equivalent to the definition of discrete-time PR transfer matrices, the other is not. Secondly, by means of the bilinear transformation, a connection between discrete-time and continuous-time NI transfer matrices is established. Thirdly, the concept of discrete-time lossless NI systems is introduced, and a discrete-time lossless NI lemma is developed to characterise the lossless NI properties in terms of minimal state-space realization. Some properties of discrete-time lossless NI transfer matrices are also studied. Several numerical examples illustrate the developed theory.

Index Terms—Discrete-time positive real systems, discrete-time (lossless) negative imaginary systems, bilinear transformation.

I. INTRODUCTION

Classical positive real (PR) systems have achieved great success both in continuous-time (CT) and discrete-time (DT) cases. The concept of discrete-time positive real (DT-PR) systems was firstly introduced in [1]. Subsequently, other versions of DT-PR systems were proposed in [2], [3]. All of them were claimed to be equivalent to the definition of DT-PR systems. However, it was shown in [4] that two versions of DT-PR systems in [2], [3] were not consistent with the definition of DT-PR systems by using three SISO examples. Until now, some researchers still adopt the one in [2], [3]; e.g., see [5, Definition 3], [6, Theorem 13.26], [7, Theorem 3]. In this paper, we shall discuss the differences between those versions, and present a detailed proof why the statement in [2], [3] is not equivalent to the definition of DT-PR systems.

In recent years, negative imaginary (NI) theory, emerged as a complement to PR theory, has attracted much attention of many researchers; e.g., see [8]–[13]. This theory was firstly introduced in [14] to model linear mechanical systems with force inputs and collocated position outputs. Subsequently, the concept of NI systems was extended to allow poles on the imaginary axis [15]–[17], non-proper case [17] and DT systems [18]–[21], respectively. Also, a mixed passivity, NI and small-gain approach in DT case has been used to design a resonance compensator in hard disk drive servo system [18].

In this paper, we are interested in establishing a connection between CT-NI and DT-NI systems in terms of the bilinear transformation. It is well-known that a CT-PR transfer matrix transforms into a DT-PR transfer matrix by such a transformation [1], [22], and vice versa. This result motivates the following question: does this result remain true for NI systems? The answer is “Yes”, and a proof is given in this paper. Meanwhile, the discrete-time lossless negative imaginary (DT-LNI) systems, expressed as a special and important case of DT-NI systems, are also studied in this paper. The concept of DT-LNI systems is introduced. Similarly, a connection between CT-LNI and DT-LNI systems is established, and a DT-LNI lemma is derived based on this connection. In addition, a necessary and sufficient condition is developed to characterize the DT-LNI properties.

The rest of the paper is organized as follows: Section II reviews the definition of DT-PR transfer matrices, the DT-PR lemma and two frequently-used “equivalent” conditions. Then, we analyze the differences between those lemmas. Section III studies the connection between CT-NI and DT-NI transfer matrices. Section IV studies the DT-LNI systems. One numerical example is presented in Section V. Section VI concludes the paper.

Notation: $\mathbb{R}^{m \times n}$ and $\mathbb{R}^{m \times n}$ denote the sets of $m \times n$ real matrices and real-rational proper transfer function matrices, respectively. $\Re[.]$ denotes the real part of complex numbers. $\lambda_{\text{max}}$ denotes the maximum eigenvalue of a square complex matrix with only real eigenvalues. $A^T$ and $A^*$ denote the transpose and the complex conjugate transpose of a complex matrix $A$, respectively. $I$ denotes any identity matrix with compatible dimensions. $A > (\geq) 0$ and $A < (\leq) 0$ denote the symmetric positive (semi-)definite matrix and the symmetric negative (semi-)definite matrix, respectively.

II. DISCRETE-TIME POSITIVE REAL TRANSFER FUNCTION MATRICES

In this section, our goal is to discuss the differences between the concept and “equivalent” conditions of DT-PR systems. Firstly, we briefly recall the definition of DT-PR systems in $z$-domain [1], two different versions of DT-PR systems in terms of properties on the unit cycle [1]–[3], and the DT-PR lemma in terms of state-space realization [1].

Definition 1. [1] A square matrix $F(z)$ of real-rational proper functions is called DT-PR if

1) all elements of $F(z)$ are analytic in $|z| > 1$;
2) $F^*(z) + F(z) \geq 0$ for all $|z| > 1$.

The following two lemmas are two different restatements of Definition 1 in terms of properties on $|z| = 1$. This work was supported by National Natural Science Foundation of China under Grant 61374026. The material in this paper was partially presented at the 55th IEEE Conference on Decision and Control, December, pp. 4931–4936, 2016, Las Vegas, USA. Mei Liu and Junlin Xiong are with the Department of Automation, University of Science and Technology of China, Hefei 230027, China (e-mail: Lmaymay@mail.ustc.edu.cn, Xiong77@ustc.edu.cn)
Lemma 1. [1, Lemma 2] A square matrix $F(z)$ whose elements are real-rational proper functions analytic in $|z| > 1$ is DT-PR if and only if

1) poles of elements of $F(z)$ on $|z| = 1$ are simple;
2) $F^*(e^{j\theta}) + F(e^{j\theta}) \geq 0$ for all real $\theta$ at which $F(e^{j\theta})$ exists;
3) if $z_0 = e^{j\theta_0}$, $\theta_0$ is real, is a pole of an element of $F(z)$, and if $K_0$ is the residue matrix of $F(z)$ at $z_0$, then the matrix $e^{-j\theta_0}K_0$ is positive semidefinite Hermitian.

Remark 1. One should note that the following condition is contained in Condition 3 of Lemma 1: if $F(z)$ has a simple pole at $-1$, then the corresponding residue matrix $\lim_{z \to -1}(z+1)F(z)$ is negative semidefinite Hermitian. This fact follows by a direct calculation that $e^{-j\theta}|_{\theta=\pi} = -jK = \left(\begin{array}{cc} 1 & -j \\ -j & 1 \end{array}\right)$ is positive semidefinite Hermitian and satisfies Condition 3 in Lemma 1.

Lemma 2. [2], [3] A square real-rational proper transfer function matrix $F(z)$ is DT-PR if and only if

1) $F(z)$ is analytic in $|z| > 1$;
2) $F^*(e^{j\theta}) + F(e^{j\theta}) \geq 0$ for all real $\theta$ at which $F(e^{j\theta})$ exists;
3) the poles of $F(z)$ on $|z| = 1$ are simple and the corresponding residue matrices of $F(z)$ at those poles are positive semidefinite Hermitian.

Remark 2. According to Condition 3 in Lemma 2, we have the following result: if $F(z)$ has a simple pole at $-1$, then the corresponding residue matrix $\lim_{z \to -1}(z+1)F(z)$ is positive semidefinite Hermitian.

Obviously, it follows from Remarks 1 and 2 that Condition 3 of Lemma 1 is not equivalent to Condition 3 of Lemma 2. When $F(z)$ has poles on $|z| = 1$, Condition 3 of Lemma 1 requires that $e^{-j\theta_0}K_0$ be positive semidefinite Hermitian, while Condition 3 of Lemma 2 requires that the associated residue matrix $K_0$ be positive semidefinite Hermitian.

The following lemma is a classical DT-PR lemma in terms of minimal state-space realisation.

Lemma 3. [1] Let $F(z)$ be a square real-rational proper transfer functions of $z$ with no poles in $|z| > 1$ and simple poles only on $|z| = 1$, and let $(A, B, C, D)$ be a minimal realisation of $F(z)$. Then necessary and sufficient conditions for $F(z)$ to be DT-PR are that there exist a real matrix $P = P^T > 0$ and real matrices $L$ and $W$ such that $A^TPA - P = -L^TL$, $C^TA^TPB = L^TW$, $D^T + D = B^TPB = W^TW$.

Remark 3. In [4], it has been pointed out that Definition 1, Lemmas 1 and 3 are agreeable with each other. Lemma 2 is not equivalent to Definition 1, Lemmas 1 and 3. The authors of [4] utilized three counter-examples to show this result.

In this section, we provide the detailed reasons why Lemma 2 is not equivalent to Definition 1, Lemmas 1 and 3. As was mentioned in [4], Lemma 3 plays an important role in the research of DT-PR systems. The authors of [1] proved Lemma 3 by applying the bilinear transformation

$$s = \frac{z - 1}{z + 1}.$$  \hspace{1cm} (1)

So, if we use Lemma 3, one should admit that, via the bilinear transformation in (1), the CT-PR transfer matrix $F(s)$ is transformed into a DT-PR transfer matrix $F(z)$, and vice versa. Next, we will show that: under the transformation in (1), Lemma 2 is not equivalent to [23, Theorem 2.7.2], which is the CT counterpart of Lemma 2.

According to Lemmas 1 and 2, it can be found that the main distinctions lie in Condition 3 when the transfer matrix $F(z)$ has poles on $|z| = 1$. Hence, without loss of generality, assume that the CT-PR transfer matrix $F(s)$ has some simple poles on the purely imaginary axis. Then, according to the minor decomposition theory in [23, pp. 216], $F(s)$ can be written in the following form

$$F(s) = \Sigma_i \frac{K_i}{s - j\omega_i} + sA + \frac{C}{s} + F_0(s),$$  \hspace{1cm} (2)

where $F_0(s)$ is analytic in $\text{Re}[s] > 0$, $K_i = K_i^* \geq 0$, $A = A^* \geq 0$ and $C = C^* \geq 0$ are the associated residue matrix at $j\omega_i$ ($\omega_i > 0$), $0$ and $\infty$, respectively. By means of the transformation in (1), equation (2) transforms into $F(z) = \Sigma_i \frac{z+1}{(z+1)^2}K_i + \Sigma_i A + \Sigma_i C + F_0(z)$, because $F(s)$ is CT-PR, it follows from [22, Theorem 1] and Lemma 3 that $F(z)$ is DT-PR. However, when $e^{j\theta_1} = \frac{1+j\omega_1}{2\omega_1}$ is a simple pole of $F(s)$, the residue matrix of $F(z)$ at $e^{j\theta_1}$ is given by

$$K_0 = \lim_{z \to \frac{1+j\omega_1}{2\omega_1}} (z - \frac{1 + j\omega_1}{1 - j\omega_1}) F(z) = \frac{2K_i}{1 - \omega_1^2 - 2j\omega_1},$$

which is not positive semidefinite Hermitian, where $K_1 = \lim_{z \to j\omega_1} (s - j\omega_1)F(s)$ is positive semidefinite. This contradicts Condition 3 in Lemma 2. Moreover, the matrix

$$e^{-j\theta_1}K_0 = \left(\begin{array}{cc} 1 - \frac{\omega^2}{\omega^2} - 2\omega j & 2K_i \omega_1 \omega_1 - 2\omega j \\ 2K_i \omega_1 \omega_1 + 1 - \frac{\omega^2}{\omega^2} + 2\omega j & 1 + \omega^2 \end{array}\right)$$

is positive semidefinite Hermitian. This coincides with Condition 3 in Lemma 1. Although some researchers adopt Lemma 2 in their research and the form of Lemma 2 is similar to the CT case, it follows from above theoretical analysis that Lemma 1 is correct and Lemma 2 can not be used to test DT-PR properties when the system has poles on $|z| = 1$. In addition, it is noteworthy that some inconsistencies in [3] have been corrected in [24].

Example 1. To illustrate the main results in this section, consider a CT-PR transfer matrix $F(s) = \left(\begin{array}{cc} \frac{s}{s+1} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{s}{s+1} \end{array}\right)$. Using the bilinear transformation in (1) transforms $F(s)$ to

$$F(z) = \left(\begin{array}{cc} \frac{z^2}{(z+1)^2} & \frac{(z+1)^2}{2(z^2+1)} \\ \frac{(z+1)^2}{2(z^2+1)} & \frac{z^2}{(z+1)^2} \end{array}\right).$$

A calculation shows that the residue matrix of $F(z)$ at $z = j$ is given by $K = \left(\begin{array}{cc} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{array}\right)$, which is not positive semidefinite Hermitian and contradicts Condition 3 in Lemma 2. However, the matrix $e^{-j\theta}K|_{\theta=\pi} = -jK = \left(\begin{array}{cc} -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{array}\right)$ is positive semidefinite Hermitian and satisfies Condition 3 in Lemma 1.
III. DISCRETE-TIME NEGATIVE IMAGINARY TRANSFER FUNCTION MATRICES

In this section, our goal is to establish the connection between DT-NI and CT-NI systems. Firstly, the definition of CT-NI and DT-NI transfer matrices are introduced, respectively.

Definition 2. [17] A square real-rational transfer function matrix \( G(s) \) is called CT-NI if

1) \( G(s) \) has no poles in \( \Re(s) > 0 \);
2) \( jG(j\omega) - G^*(j\omega) \geq 0 \) for all \( \omega > 0 \) except values of \( \omega \) where \( j\omega \) is a pole of \( G(s) \);
3) if \( s = 0 \) is a pole of \( G(s) \), then \( \lim_{s \to 0} s^m G(s) \) is positive semidefinite Hermitian, and \( \lim_{s \to 0} s^m G(s) = 0 \) for all \( m \geq 3 \);
4) if \( s = j\omega_0 \) with \( \omega_0 > 0 \) is a pole of \( G(s) \), \( \omega_0 \) is finite, it is at most a simple pole and the residue matrix \( K = \lim_{s \to j\omega_0} (s - j\omega_0)G(s) \) is positive semidefinite Hermitian;
5) if \( s = j\infty \) is a pole of \( G(s) \), then \( \lim_{s \to j\infty} \) is a positive semidefinite Hermitian, and \( \lim_{s \to j\infty} \) is given by

\[
\begin{align*}
&K = \lim_{z \to 1 - j\omega_0} \\
&\left( 1 - \frac{1 + j\omega_0}{1 - j\omega_0} \right)G(j\omega_0) = \frac{2K_0}{1 - \omega_0^2 - 2\omega_0j},
\end{align*}
\]

which is not positive semidefinite Hermitian unless \( \omega_0 = 0 \). Then, we have

\[ e^{-j\theta_0} = \frac{1 - \omega_0^2 - 2\omega_0j}{1 + \omega_0^2} \]

and the matrix \( e^{-j\theta_0}K_0 = \frac{2K_0}{1 + \omega_0^2} \) is positive semidefinite Hermitian. Similarly, the residue matrix \( \frac{1 - j\omega_0}{1 + j\omega_0} \) has the same property.

If \( s = j\infty \) is a pole of \( G(s) \), then \( s = -1 \) is also a pole of \( G(z) \). Decompose \( G(s) \) to the form: \( G(s) = \frac{s^2C_2 + sc_1 + G_0(s)}{1} \), where \( C_2 = C_2^* \leq 0 \). Using the same transformation, \( G(z) \) is written in the form

\[
G(z) = \frac{-jK + jK^* + G_0(s)}{s - j\omega_0 + s + j\omega_0}.
\]

Conversely, assume that \( G(s) \) is CT-NI. We will show that the five conditions in Definition 2 are satisfied by means of the bilinear transformation \( z = \frac{1 + s}{1 - s} \). Also, Conditions 1 and 2 of Definition 2 are immediate.

If \( z = e^{j\theta_0} \), \( \theta_0 \in (0, \pi) \), is a pole of \( G(z) \), then, \( s = j\theta_0 \cos \theta_0 > 0 \), is also a pole of \( G(s) \). Similar to the minor decomposition of CT case in [17], we decompose \( G(z) \) as

\[
G(z) = \frac{-jK_0 + jK_0^* + G_0(z)}{z - j\omega_0 + z + j\omega_0},
\]

where \( K_0 \) is the residue matrix of \( G(z) \) at \( e^{j\theta_0} \), \( e^{-j\theta_0}K_0 = \text{positive semidefinite Hermitian} \), and \( G_0(z) \) has no poles in \( |z| > 1 \) and at \( e^{j\theta_0} \).

Consider the transformation \( z = \frac{1 + s}{1 - s} \), \( G(z) \) transforms into

\[
G(s) = \frac{-jK_0 + jK_0^* + G_0(z)}{s - j\omega_0 + s + j\omega_0}.
\]

Then, the residue matrix of \( G(s) \) at \( s = e^{j\theta_0} \) is given by

\[
K = \lim_{s \to e^{j\theta_0}} \left( s - e^{j\theta_0} \right) G(s) = \frac{2K_0}{e^{j\theta_0} + 1}
\]

which is positive semidefinite Hermitian.

If \( s = 1 \) is a pole of \( G(s) \), then \( s = 0 \) is a pole of \( G(s) \). We decompose \( G(s) \) as

\[
G(s) = \frac{A_2 + A_1}{s^2 + \frac{A_1}{2} + G_0(s)},
\]

where \( A_2 = A_2^* \geq 0 \) and \( G_0(s) \) has no poles in \( \Re(s) > 0 \) and at \( s = 0 \). By means of the bilinear transformation \( s = \frac{z - 1}{z + 1} \), \( G(s) \) transforms into

\[
G(z) = \left( \frac{z + 1}{z - 1} \right)^2 A_2 + \frac{A_1}{2} + G_0(z),
\]

where \( A_2 = A_2^* \geq 0 \). Then, \( \lim_{s \to 0} s^m G(s) = 0 \) for all \( m \geq 3 \).

If \( s = j\omega_0 \), \( \omega_0 > 0 \) is a pole of \( G(s) \), then \( z = \frac{1 + j\omega_0}{1 - j\omega_0} \) is also a pole of \( G(z) \). Decompose \( G(s) \) to the form:

\[
G(s) = \frac{-jK + jK^* + G_0(s)}{s - j\omega_0 + s + j\omega_0},
\]

where \( K \) is the residue matrix of \( jG(s) \) at \( j\omega_0 \), \( K^* \geq 0 \), and \( G_0(s) \) has no poles in \( \Re(s) > 0 \) and at \( \pm j\omega_0 \). By means of the transformation \( s = \frac{z + 1}{z - 1} \), \( G(s) \) transforms into

\[
G(z) = \frac{-jK + jK^* + G_0(z)}{z - j\omega_0 - j\omega_0} + \frac{jK^* + G_0(z)}{1 + j\omega_0 + 1 - j\omega_0}.
\]

The residue matrix of \( jG(z) \) at \( e^{j\theta_0} \) is given by

\[
K = \lim_{z \to 1 + j\omega_0} \left( z - 1 - j\omega_0 \right) jG(z) = \frac{2K_0}{1 - \omega_0^2 - 2\omega_0j},
\]

where \( K \) is positive semidefinite Hermitian. Similarly, the residue matrix \( \frac{1 - j\omega_0}{1 + j\omega_0} \) has the same property.
transforms into \( G(s) = \frac{C_2}{4} s^2 + \frac{-C_1 - C_2}{2} s + G_0 \left( \frac{1}{s+1} \right) + \frac{C_2}{4} \). Then, \( \lim_{\omega \to \infty} G_j(\omega) = \frac{C_2}{4} \leq 0 \), and \( \lim_{\omega \to \infty} G_j(\omega)^m = 0 \) for all \( m \geq 3 \).

**Example 2.** To illustrate the usefulness of Lemma 4, consider a non-symmetric CT-NI transfer function matrix \( G(s) = \begin{pmatrix} -s^2 - s & -s^2 - s \\ -s^2 + s & -2s - s \end{pmatrix} \). By the bilinear transformation in (1), \( G(s) \) transforms into \( G(z) = \frac{-2z^2 - 2z + 1}{(z+1)^2} \). Conditions 1 and 2 of Definition 3 are immediate after a direct calculation. \( G(z) \) has a double pole at \(-1\), and \( \lim_{z \to -1} (z + 1)^2 G(z) = \begin{pmatrix} -4 & -4 \\ -4 & -8 \end{pmatrix} \) is negative definite Hermitian, \( \lim_{z \to -1} (z + 1)^m G(z) = 0 \) for all \( m \geq 3 \). Hence, \( G(z) \) is DT-NI in view of Definition 3.

In the following lemma, we present an alternative method to prove the DT-NI lemma in [20].

**Lemma 5.** [20] Let \((A, B, C, D)\) be a minimal state-space realization of a real-rational proper DT transfer matrix \( G(z) \in \mathbb{R}^{m \times m} \), where \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{m \times n} \), \( D \in \mathbb{R}^{m \times m} \), and \( m \leq n \). Suppose \( \det(I + A) \neq 0 \) and \( \det(I - A) \neq 0 \). Then, \( G(z) \) is DT-NI if and only if

1. \( C(I + A)^{-1} B - D = B^T (I + A^T)^{-1} C^T - D^T; \)
2. there exists a matrix \( Y = YT > 0 \), \( Y \in \mathbb{R}^{n \times n} \), such that \( Y - AYA^T \geq 0 \) and \( B = (I - A)Y(A + A^T)^{-1} C^T \).

**Proof.** By means of the bilinear transformations \( s = \frac{z - \frac{1}{z+1}}{1} \) and \( z = \frac{1}{1+\xi} \), the DT transfer matrix \( G(z) = C(zI - A)^{-1} B + D \) is transformed into a CT transfer matrix \( G(s) = H(sI - F)^{-1} G + J \), and vice versa, where

\[
\begin{align*}
F &= (A + I)^{-1}(A - I), \quad G = \sqrt{2}(I + A)^{-1} B, \\
H &= \sqrt{2}C(A + I)^{-1}, \quad J = D - C(I + A)^{-1} B.
\end{align*}
\]

According to Theorem 1 in [22], \((F, G, H, J)\) is a minimal realization of \( G(s) \). Then, one has the following equivalent statements:

\( G(z) \sim (A, B, C, D) \) is DT-NI.*

\( \Rightarrow G(s) \sim (F, G, H, J) \) is CT-NI. This equivalence is via Lemma 4.

\( \Leftarrow J = J^T \) and there exists a real matrix \( Y = YT \geq 0 \), \( Y \in \mathbb{R}^{n \times n} \), such that \( FY + YF^T \leq 0 \) and \( G + FYH^T = 0 \). This equivalence is via the CT-NI lemma in [15, Lemma 7].

\( \Rightarrow D - C(I + A)^{-1} B = B^T (I + A^T)^{-1} C^T \) and there exists a matrix \( Y = YT \geq 0 \), \( Y \in \mathbb{R}^{n \times n} \), such that \( Y - AYA^T \geq 0 \) and \( B = (I - A)Y(A + A^T)^{-1} C^T \).

**Remark 4.** It is worthwhile noting that the proof of Lemma 5 is based on the connection between CT-NI and DT-NI transfer matrices developed in Lemma 4. However, the proof of DT-NI lemma in [20, Lemma 11] and [19, Theorem 3.2] is based on the relation between DT-PR and DT-NI systems. Lemma 5 is equivalent to Lemma 11 in [20] according to Corollary 3.1 in [19]. Moreover, Lemma 4 can be used to develop other properties of DT-NI systems, such as the internal stability results in [19, 20], the residue matrix properties with simple pole at \( z = \pm 1 \) [20, Lemma 4]. The more detailed proofs of Lemmas 4 and 5 can be found in [21].

**IV. DISCRETE-TIME LOSSLESS NEGATIVE IMAGINARY TRANSFER FUNCTION MATRICES**

In this section, our goal is to introduce the concept of DT-LNI systems, study some properties of such systems, and present the DT-LNI lemma in terms of minimal state-space realization. Firstly, we recall the definition of CT-LNI systems.

**Definition 4.** [17] A square real-rational transfer function matrix \( G(s) \) is called CT-LNI if

1. \( G(s) \) is CT-NI;
2. \( jG(j\omega) - G^*(j\omega) = 0 \) for all \( \omega > 0 \) except values of \( \omega \) where \( j\omega \) is a pole of \( G(s) \).

**Definition 5.** A square real-rational proper transfer function matrix \( G(z) \) is called DT-LNI if

1. \( G(z) \) is DT-NI;
2. \( j[G(e^{j\theta}) - G^*(e^{j\theta})] = 0 \) for all \( \theta \in (0, \pi) \) except values of \( \theta \) where \( e^{j\theta} \) is a pole of \( G(z) \).

The following lemma characterizes a connection between CT-LNI and DT-LNI transfer matrices.

**Lemma 6.** A CT-LNI transfer matrix \( G(s) \) transforms into a DT-LNI transfer matrix \( G(z) \) by the bilinear transformation \( s = \frac{z - 1}{z + 1} \). Conversely, a DT-LNI transfer matrix \( G(z) \) transforms into a CT-LNI transfer matrix \( G(s) \) by the bilinear transformation \( z = \frac{1 + \omega}{1 - \omega} \).

**Proof.** Suppose \( G(s) \) is CT-LNI. Condition 1 of Definition 4 implies that \( G(s) \) is CT-NI. According to Lemma 4, \( G(z) \) is DT-NI by the bilinear transformation \( s = \frac{z - 1}{z + 1} \). Furthermore, if \( s = j\omega \), \( \omega > 0 \) is not a pole of \( G(s) \), then \( z = \frac{1 + \omega}{1 - \omega} = 1 + \omega \frac{1}{1 + \omega} \) is not a pole of \( G(z) \). Then, for all \( \omega > 0 \) with \( j\omega \) not a pole of \( G(s) \), \( jG(j\omega) = G^*(j\omega) = 0 \) implies that \( jG(e^{j\theta}) = G^*(e^{j\theta}) = 0 \) for all \( \theta \in (0, \pi) \) with \( e^{j\theta} \) not a pole of \( G(z) \). Therefore, it follows from Definition 5 that \( G(z) \) is DT-LNI.

Conversely, suppose \( G(z) \) is DT-LNI. Condition 1 of Definition 5 implies that \( G(z) \) is DT-NI. It follows from Lemma 4 that \( G(s) \) is CT-NI under the transformation \( z = \frac{1 + \omega}{1 - \omega} \). Also, for all \( \theta \in (0, \pi) \) with \( e^{j\theta} \) not a pole of \( G(z) \), \( jG(e^{j\theta}) - G^*(e^{j\theta}) = 0 \) implies that \( jG(e^{j\sin \theta}) - G^*(e^{j\sin \theta}) = 0 \) for all real \( \omega > 0 \) such that \( j\omega = e^{j\sin \theta} \) is not a pole of \( G(z) \). Hence, \( G(z) \) is CT-LNI by Definition 4.

**Example 3.** To illustrate the usefulness of Lemma 6, consider a non-symmetric CT-LNI transfer matrix \( G(s) = \begin{pmatrix} \frac{1}{s+1} & -\frac{s}{s+1} \\ \frac{s}{s+1} & \frac{-s}{s+1} \end{pmatrix} \). By the bilinear transformation \( s = \frac{z - 1}{z + 1} \), \( G(s) \) transforms into \( G(z) = \begin{pmatrix} \frac{1 + z}{(z+1)^2} & \frac{1}{z+1} \\ \frac{z}{1+z} & \frac{-1}{z+1} \end{pmatrix} \). After a direct calculation, it follows that \( jG(e^{j\theta}) - G^*(e^{j\theta}) = 0 \) for all \( \theta \in (0, \pi) \) with \( e^{j\theta} \) not a pole of \( G(z) \). The residue matrix of \( jG(z) \) at \( e^{j\theta} = j \) \( (\theta = \frac{\pi}{2}) \) is given by \( K = \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & j \end{pmatrix} \), which is not positive semidefinite Hermitian. Then, the matrix
Lemma 7. A square real-rational proper transfer function matrix $G(z)$ is DT-LNI if and only if

1) all poles of elements of $G(z)$ lie on $|z| = 1$ and are simple, and the residue matrix $\tilde{K} = \lim_{z \to z_0}(z - z_0)jG(z)$ at any pole $z_0 = e^{j\theta_0}$, $\theta_0 \in (0, \pi)$, satisfies that $e^{-j\theta_0}\tilde{K}$ is positive semidefinite Hermitian; 
2) if $z = 1$ is a pole of $G(z)$, then $\lim_{z \to 1}(z - 1)^2G(z)$ is positive semidefinite Hermitian, and $\lim_{z \to 1}(z - 1)^mG(z) = 0$ for all $m \geq 3$;
3) if $z = -1$ is a pole of $G(z)$, then $\lim_{z \to -1}(z + 1)^2G(z)$ is negative semidefinite Hermitian, and $\lim_{z \to -1}(z + 1)^mG(z) = 0$ for all $m \geq 3$;
4) $G(z) = G^T(z^{-1})$ for all $z$ such that $z$ is not a pole of $G(z)$.

Proof. (Necessity) Suppose $G(z)$ is DT-LNI. Condition 2 of Definition 3 implies that $j[G(e^{j\theta}) - G^*(e^{j\theta})] = 0$ for all $\theta \in (0, \pi)$ with $e^{j\theta}$ not a pole of $G(z)$. By taking the complex conjugate, we have

$$j[G(e^{j\theta}) - G^*(e^{j\theta})] = 0$$

for all $\theta \in [0,2\pi]$ with $e^{j\theta}$ not a pole of $G(z)$. In view of the fact that $G(z)$ is real-rational function, one has that $G^*(e^{j\theta}) = G^T(e^{-j\theta})$, and so equation (5) implies that

$$j[G(z) - G^T(z^{-1})] = 0$$

for all $|z| = 1$, where $z$ is not a pole of any element of $G(z)$. Because $j[G(z) - G^T(z^{-1})]$ is an analytic function of $z$, it follows from maximum modulus theorem (25, Theorem A4-3) that $j[G(z) - G^T(z^{-1})] = 0$ for all $z$ with $z$ not a pole of $G(z)$. Hence, $G(z) = G^T(z^{-1})$ for all $z$ such that $z$ is not a pole of $G(z)$. Condition 4 holds.

Suppose $z_0$ is a pole of $G(z)$. Then, it follows from $G(z) = G^T(z^{-1})$ that $z_0^{-1}$ is also a pole of $G(z)$. However, we know that $G(z)$ has no poles in $|z| > 1$. If $|z_0| < 1$, then $|z_0^{-1}| > 1$. So, the only case is that all poles of elements of $G(z)$ lie on $|z| = 1$. Moreover, Conditions 3 of Definition 3 implies that the poles are simple, and the matrix $e^{-j\theta_0}\tilde{K}$ at any pole $e^{j\theta_0}$, $\theta_0 \in (0, \pi)$, is positive semidefinite Hermitian. Thus, Condition 1 holds. Also, Conditions 4 and 5 of Definition 3 imply Conditions 2 and 3 hold.

(Sufficiency) Suppose Conditions 1-4 hold. Firstly, Conditions 1-3 imply that Condition 1 and Conditions 3-5 of Definition 3 hold. Secondly, Condition 4 implies that $G(z) = G^T(z^{-1})$. It follows that $G(e^{j\theta}) = G^*(e^{j\theta})$, and so implies that $j[G(e^{j\theta}) - G^*(e^{j\theta})] = 0$ for all $\theta \in (0, \pi)$ with $e^{j\theta}$ not a pole of $G(z)$. Thus, $G(z)$ is DT-LNI according to Definitions 3 and 5.

The following lemma characterizes the properties of a sum of DT-LNI transfer matrices.

Lemma 8. Given two DT-LNI transfer matrices $G_1(z)$, $G_2(z)$, and a DT-NI transfer matrix $G(z)$. Then

1) $G_1(z) + G_2(z)$ is DT-LNI;
2) $G_1(z) + G(z)$ is DT-NI.

Proof. The proof is trivial according to the definition of DT-LNI and DT-NI transfer matrices.

The DT-LNI lemma proposed in the following provides a necessary and sufficient condition for a system to be DT-LNI in terms of minimal state-space realization.

Lemma 9. Let $(A, B, C, D)$ be a minimal state-space realization of a real-rational proper DT transfer function matrix $G(z) \in \mathbb{R}^{m \times m}$, where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{m \times m}$, and $m \leq n$. Suppose $\det(I + A) \neq 0 \neq \det(I + B)$. Then, $G(z)$ is DT-LNI if and only if

1) $C(I + A)^{-1}B - D = B^T(I + A^T)^{-1}C^T - DT$;
2) there exists a matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n \times n}$, such that

$$Y - AY^T A^T - B(I - A)Y(I + A^T)^{-1}C^T = 0$$

Proof. Similar to the proof of Lemma 5, the proof follows from the following sequence of equivalent reformulations:

$$G(z) \sim (A, B, C, D)$$ is DT-LNI.

$\Rightarrow G(x) \sim (F, G, H, J)$ is CT-LNI, where $F, G, H, J$ are defined in (4). This equivalence is according to Lemma B. $\Rightarrow J = JT$ and there exists a real matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n \times n}$, such that $FY + Y^TF = 0$ and $G + FY^TH = 0$.

This equivalence is via the CT-LNI lemma in [8, Theorem 1.1].

$\Rightarrow D - C(I + A)^{-1}B = B^T(I + A^T)^{-1}C^T$ and there exists a matrix $Y = Y^T > 0$, $Y \in \mathbb{R}^{n \times n}$, such that $Y - AY^T A^T - B(I - A)Y(I + A^T)^{-1}C^T = 0$.

In the following result, we consider the internal stability of a positive feedback interconnection of two DT-NI systems in terms of loop gain at $z = 1$. The positive feedback interconnection is denoted by $[G(z), G_s(z)]$, where $G(z)$ is DT-LNI.

Corollary 1. Given a DT-LNI transfer matrix $G(z)$, and a DT strictly NI transfer matrix $G_s(z)$. Suppose $G(z)$ and $G_s(z)$ have no poles at $-1$ and $1$, and that also satisfy $G(-1)G_s(-1) = 0$ and $G_s(-1) \geq 0$. Then, the positive feedback interconnection $[G(z), G_s(z)]$ is internally stable if and only if $\lambda_{\text{max}}(G(1)G_s(1)) < 1$.

Remark 5. The DT-LNI lemma in Lemma 9 can be considered as a modification of the DT-NI lemma in [20] by replacing the inequality with equality. The DT-LNI systems can be considered as a special case of the DT-NI systems with all the systems poles on $|z| = 1$. As a result, all results developed in [20] are valid for DT-LNI systems. The results in Corollary 1 are actually a special case of Theorem 8 in [19] or Theorem 1 in [20] with one system being DT-LNI, and hence proof is omitted here. Similar to Corollaries 1 and 2 in [8], Corollary 1 can be written in the same form as the small-gain theorem, where one system is DT-LNI; details are omitted here.
V. NUMERICAL EXAMPLES

In this section, one numerical example is given to illustrate the DT-(L)NI lemma of the paper.

Example 4. To illustrate Lemmas 5 and 9, consider the DT-(L)NI transfer matrix $G(z)$ in Example 3 (DT-LNI system is also DT-NI system). A minimal state-space realization of $G(z)$ in Example 3 is as follows:

$A = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$,

$C = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & -1 & 0 \end{pmatrix}$, $D = \begin{pmatrix} 1 & -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$.

A calculation shows that $C(I + A)^{-1}B - D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Thus, Condition 1 in Lemmas 5 and 9 hold, respectively. Because $G(s)$ in Example 3 is strictly proper, it leads to $J = 0$, and hence $C(I + A)^{-1}B - D = 0$ always holds. If $G(s)$ is proper, but not strictly proper, then $J \neq 0$, and also $C(I + A)^{-1}B - D \neq 0$. Then, YALMIP and SeDuMi were used to solve the Condition 2 in Lemma 9, and we obtained the following solution

$Y = \begin{pmatrix} \frac{2}{3} & 0 & 0 & -\frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & 0 & 0 & \frac{2}{3} \end{pmatrix} > 0.$

Condition 2 in Lemmas 5 and 9 hold, respectively.

Consider the transfer matrix $G(z)$ in Example 2. Because $G(s)$ in Example 2 is non-proper and has a double pole at infinity, $G(z)$ in Example 2 has a double pole at $-1$. The minimal state-space realization of such $G(z)$ always has poles at $-1$, and hence the condition det $(I + A) \neq 0$ does not hold. In this case, we can not use Lemma 5 to judge whether $G(z)$ is DT-NI. Furthermore, consider the robotic arm example in [16]. The finite dimensional model $G_f(s)$ in equation (23) of [16] is CT-NI. A calculation shows that $G_f(z)$ is also DT-NI by the bilinear transformation in (1).

VI. CONCLUSIONS

This paper has studied three related problems. First, it was shown by theoretical analysis that only the original necessary and sufficient conditions were equivalent to the definition of DT-PR transfer matrices and DT-PR lemma. This result is in line with conclusions in [4]. Secondly, motivated by the DT-PR case, it was found that DT-NI and CT-NI transfer matrices were equivalent by bilinear transformations. Thirdly, the DT-LNI systems were studied. Finally, the developed theory in this paper was illustrated by examples.

REFERENCES


