Quantized Feedback Stabilization of Nonlinear Systems With External Disturbance

Wei Ren and Junlin Xiong, *Member, IEEE*

***Abstract*—This paper studies the quantized feedback stabilization problem for nonlinear systems with external disturbance. A dynamic quantizer with a quantization parameter is implemented in this paper. To extract more information, the quantization parameter is updated according to an updating protocol at the discrete-time instants. Three quantization cases are studied in this paper, i.e., state quantization case, input quantization case, and output quantization case. Based on the updating protocol and Lyapunov approach, sufficient conditions are developed for quantized feedback stabilization of the closed-loop system.**

***Index Terms*—Feedback stabilization, input-to-state stability, nonlinear systems, quantized measurement.**

I. Introduction

Quantized feedback control, which refers to feedback control of systems with discrete-valued signal measurements [1]–[3], arises in system and control field due to limited capacity or security constraints on communication between the plant and the controller. Quantized feedback control has gradually become an active research area over the past few decades and could be found in numerous applications, such as mechanical systems [4], networked control systems [5], and switched systems [6], [7]. One fundamental topic of quantized feedback control is how to choose the quantizer and design the quantized control strategy. The quantizers in the control systems are usually classified into two types: static quantizers and dynamic quantizers. Results on feedback control with static quantizers can be found in the literature like [8]–[10]. However, due to time-invariant quantization regions, static quantizers only provide simple structures for coding/decoding schemes. Therefore, dynamic quantizers with quantization parameters are introduced, which adjust the quantization levels dynamically; see [11]–[13]. In addition, for dynamic quantizers, the quantization mechanism with zooming-out stage and zooming-in stage is implemented; see [6], [12], and [14]. Following the similar line as in [12] and [14], the dynamic quantizer is applied in this paper.

In a control system, quantization may cause many phenomena [2], such as limit circle, saturation, chaos, and dead zone, which have great impacts on system stability and performances. To study the effects of the quantization on system stability, there are two approaches to system modeling and stability analysis in the literature. The first approach is based on discrete-time parameter-varying systems; see [15]–[20]. In such an approach, besides quantization (or encoding/decoding [18]–[20]), sampling is also considered, thereby leading to quantized sampled-data control. In the sequel, system models are discrete-time and parameter varying, and the stability analysis is trajectory-based. It is not easy to apply such approach to nonlinear systems due to an intrinsic difficulty: exact sampled-data models of nonlinear systems cannot be found. For nonlinear systems, some additional conditions are required; see [16] and [17] for more details. The second approach is based on jump-flow systems; see [7], [12]–[14], and [21]. In this approach, sampling is not studied and discrete-time events are treated as the jumps in control systems. Hence, system models are hybrid and system stability can be proceeded via the Lyapunov approach. Furthermore, according to the second approach, quantization effects can be studied together with other network-induced phenomena; see [5] and [8]. In the literature, salient results based on the first approach can be found for both linear systems [7], [19], [20] and nonlinear systems [16]–[18]. However, little attention has been given to quantized feedback control of nonlinear systems using the second approach.

This paper studies the quantized feedback stabilization problem for nonlinear systems with external disturbance using the second approach. Based on the dynamic quantizer, the feedback control law is designed. Three quantization cases are studied, that is, state quantization, input quantization, and output quantization. For different quantization cases, stability conditions are established based on Lyapunov approach. The contributions of this paper are threefold. First, contrary to [16] and [18] following the first approach, and [6] and [14] on linear systems, the second approach is applied for nonlinear systems in this paper. Moreover, we extend the quantization feedback stabilization results from the linear system case [14] to the nonlinear system case. Second, the external disturbance is studied in this paper. The external disturbance is inevitable in control systems and may lead the systems state to escape from the quantization regions, which further deteriorates system stability [13], [14]. Therefore, it is necessary to study the effects of the disturbance on system stability. However, the disturbance is not considered in [6] and [12], thereby simplifying stability analysis. Third, besides state quantization [13], [14], input quantization and output quantization are studied in this paper. In addition, the quantization mechanism with zooming-out stage and zooming-in stage is applied in this paper, whereas the encoding and the decoding processes are used in [13].

This paper is organized as follows. In Section II, the problem is formulated and the updating protocol of the quantization parameter is given. The main results, presented in Sections III, give sufficient conditions for quantized feedback stabilization of nonlinear systems with external disturbance. The aforementioned three cases are studied sequentially. Conclusions and future works are stated in Section IV.

Notation: $\mathbb{R} := (-\infty, +\infty)$; $\mathbb{R}_{\geq 0} := [0, +\infty)$; $\mathbb{R}_{>0} := (0, +\infty)$; $\mathbb{N} := \{0, 1, 2, \ldots\}$; $\mathbb{N}_{>0} := \{1, 2, \ldots\}$. Symbols $\wedge$ and $\vee$ denote "AND" and "OR" in logic, respectively. Given a piecewise continuous function $f : \mathbb{R} \to \mathbb{R}$ and $t \in \mathbb{R}$, $f(t^-) := \lim_{s \nearrow t} f(s)$. f^- denotes $f(t^-)$ simply if the time argument is ignored. $|\cdot|$ stands for Euclidean norm; $\|f\|_{[a,b]} := \text{ess. sup}_{t\in[a,b]} |f(t)|$ and $\|f\|_{[a,b]}$ is denoted by $\|f\|$ if $a = t_0$ is given and $b = \infty$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$

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The authors are with the Department of Automation, University of Science and Technology of China, Hefei 230026, China (e-mail: gtpp@mail.ustc.edu.cn; junlin.xiong@gmail.com).

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if it is continuous, zero at zero, and strictly increasing; it is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(s,t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s,t)$ decreases to zero as $t \to 0$ for each fixed $s \geq 0$. For arbitrary functions α_1, α_2 and $v \in \mathbb{R}_{\geq 0}$, $\alpha_1 \circ \alpha_2(v) := \alpha_1(\alpha_2(v))$.

II. Problem Formulation

Consider the nonlinear system of the form

$$\dot{x}(t) = f(t, x, u, w) \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the system state, $u \in \mathbb{R}^{n_u}$ is the control input, and $w \in \mathbb{R}^{n_w}$ is an unknown disturbance. Suppose that both u and w are Lebesgue measurable and locally bounded, and that f is locally Lipschitz in x and w.

In this paper, the uncontrolled system $\dot{x}(t) = f(t, x, 0, w)$ is assumed to be forward complete, i.e., the solution $x(t, x_0, w)$ from initial state $x_0 = x(t_0) \in \mathbb{R}^{n_x}$ and locally bounded $w \in \mathbb{R}^{n_w}$ exists for all $t \geq t_0$; see [22]. Also, assume that there exists a feedback control law $u = \kappa(x)$ such that the closed-loop system is input-to-state stable (ISS). That is, there exist a Lyapunov function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2, \alpha_3, \sigma_1, \sigma_2 \in \mathcal{K}_\infty$ such that for all $x, e \in \mathbb{R}^{n_x}$, $w \in \mathbb{R}^{n_w}$, (see also [12], and [23, Th. 4.19])

$$\alpha_1(|x|) \leq V(t,x) \leq \alpha_2(|x|) \tag{2}$$

$$|x| \geq \sigma_1(|e|) + \sigma_2(\|w\|) \Rightarrow$$
$$V_t(t,x) + V_x(t,x) f(t, x, \kappa(x+e), w) \leq -\alpha_3(|x|) \tag{3}$$

where $V_t(t,x) := \partial V(t,x)/\partial t$ and $V_x(t,x) := \partial V(t,x)/\partial x$.

In control systems, especially digital control systems, because of the limited transmission captivity of the communication channel, the signal has to be quantized before being transmitted. A quantizer is a piecewise constant function $q : \mathbb{R}^{n_x} \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}^{n_x}$ [12]. That is, the quantizer divides $\mathbb{R}^{n_x}$ into a finite number of quantization regions of the form $\{z \in \mathbb{R}^{n_x} : q(z) = \jmath \in \mathcal{Q}\}$. Assume that the quantizer satisfies the following conditions [12], [14]:

$$|z| \leq M \Rightarrow |q(z) - z| \leq \Delta \tag{4}$$

$$|z| > M \Rightarrow |q(z)| > M - \Delta \tag{5}$$

$$|z| \leq \Delta_0 \Rightarrow q(z) \equiv 0 \tag{6}$$

where $M > \Delta > 0$; M is called the range of the quantizer, and Δ is called the bound of the quantization error $q(z) - z$. The small number $\Delta_0 > 0$ in (6) is called the dead-zone of q, in which the signal is so small that it is reasonable to quantize the signal as zero directly. Condition (4) implies that if the signal does not saturate, then the quantization error is bounded by Δ. Condition (5) provides an approach to detecting whether the signal saturates or not.

The dynamic quantizer used in this paper is given by

$$q_\mu(z) := \mu q\left(\frac{z}{\mu}\right), \quad \mu > 0 \tag{7}$$

where μ is called the quantization parameter. For the quantizer (7), the range is $M\mu$ and the bound of the quantization error is $\Delta\mu$. To extract more information, μ is updated at the discrete-time instants and evolved according to the following updating protocol.

In the continuous-time domain, the system state is evolved according to (1) and μ is kept constant. In the discrete-time instants, the updating protocol for μ is presented as follows; see [14]. Some auxiliary variables are introduced first. Let μ be initialized as $\mu_0 > 0$, and $\Omega_{\text{in}} \in (0,1)$, $\Omega_{\text{out}} > 1$, $T_{\text{out}} < \log \Omega_{\text{out}}/L_x$, where L_x is a given constant. Let $l_{\text{out}} = M - \Delta$ and $l_{\text{in}} = \Omega_{\text{in}}(M - 2\Delta) - \Delta$. The time-dependent logic variable $\mathtt{c}$ is used to distinguish zooming-out stage and zooming-in stage. $\mathtt{c} \in \{\mathtt{yes}, \mathtt{no}\}$ and is initialized at $\mathtt{no}$. Based on the above auxiliary variables, μ gets update according to the following three discrete-time events:

Zoom-out event:

If $[(\tau_{\text{out}}^- = T_{\text{out}}) \wedge (\mathtt{c}^- = \mathtt{no})] \vee [(|q_{\mu^-}(z)| \geq l_{\text{out}}\mu^-) \wedge (\mathtt{c}^- = \mathtt{yes})]$

then set $\mu = \Omega_{\text{out}}\mu^-$ and $\tau_{\text{out}} = 0$.

Capture event:

If $(|q_{\mu^-}(z)| \leq l_{\text{out}}\mu^-) \wedge (\tau_{\text{out}}^- \in (0, T_{\text{out}})) \wedge (\mathtt{c}^- = \mathtt{no})$

then set $\mu = \Omega_{\text{out}}\mu^-$ and $\mathtt{c} = \mathtt{yes}$.

Zoom-in event:

If $(|q_{\mu^-}(z)| \leq l_{\text{in}}\mu^-) \wedge (\min\{\tau_{\text{out}}^-, \tau_{\text{in}}^-\} \geq T_{\text{in}}) \wedge (\mathtt{c}^- = \mathtt{yes})$

then set $\mu = \Omega_{\text{in}}\mu^-$ and $\tau_{\text{in}} = 0$.

In the above updating protocol, τ_{out} and τ_{in} are two auxiliary variables to distinguish whether it is time to reset. Both τ_{out} and τ_{in} are initialized at 0; $\tau_{\text{out}} \in [0, T_{\text{out}}]$ and $\tau_{\text{in}} \in [0, T_{\text{in}}]$, where $T_{\text{out}} \geq T_{\text{in}}$. The variables τ_{out} and τ_{in} evolve according to

$$\dot{\tau}_{\text{out}} = 1, \quad \text{if } \tau_{\text{out}} < T_{\text{out}}; \quad \dot{\tau}_{\text{in}} = 1, \quad \text{if } \tau_{\text{in}} < T_{\text{in}}.$$

If τ_{out} or τ_{in} reaches its upper bound, then it is reset as zero and restarts to record the time, and μ is updated.

If a zoom-out or capture event occurs, then the quantizer enters into the zooming-out stage. In the zooming-out stage, μ increases and the quantization regions are expanding to recover the system state. As a result, the quantization error is enlarged accordingly. If a capture event occurs, it means that the quantizer will enter into the zooming-in stage in the finite time. That is, the capture event is the switch from the zooming-out stage to the zooming-in stage. If a zoom-in event occurs, then μ decreases and the quantization regions are contracted to drive the system state to converge to the neighbor of the origin sequentially. Based on the variable $\mathtt{c}$, the feedback control law applied in this paper is of the following form:

$$u(t) = \begin{cases} 0, & \mathtt{c}(t) = \mathtt{no} \\ \kappa\left(q_{\mu(t)}(x(t))\right), & \mathtt{c}(t) = \mathtt{yes}. \end{cases} \tag{8}$$

That is, the system is open-loop in the zooming-out stage and closed-loop in the zooming-in stage.

In this paper, our objective is to find sufficient conditions such that under the dynamic quantizer (7) with the preceding updating protocol and the control law (8), the quantized closed-loop system is ISS with respect to $w \in \mathbb{R}^{n_w}$. In view of [24] and [25], system (1) is ISS if there exist $\beta \in \mathcal{KL}$, $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that for all $x_0 \in \mathbb{R}^{n_x}$ and all bounded $w \in \mathbb{R}^{n_w}$

$$|x(t)| \leq \beta(|x_0|, t - t_0) + \gamma_1(\|w\|) \quad \forall t \geq t_0 \tag{9}$$

$$\limsup_{t\to\infty} |x(t)| \leq \gamma_2\left(\limsup_{t\to\infty} |w(t)|\right). \tag{10}$$

Because of μ, the closed-loop system is not continuous all the time. As an additional discrete state, μ needs to be considered. With a slight abuse of terminology, properties (9) and (10) are thought of as ISS of the considered quantized closed-loop system.

III. Stabilization of Nonlinear Systems With Quantization

In this section, sufficient conditions are derived to guarantee ISS of nonlinear quantized systems. Three quantization cases are studied sequentially, i.e., state quantization case, input quantization case, and output quantization case. In what follows, the state quantization case is studied.

A. State Quantization Case

For the state quantization case, the following theorem implies that, under the appropriate conditions, input-to-state stability of the quantized closed-loop system is guaranteed by the control law (8).

Theorem 1: Consider system (1) satisfying conditions (2) and (3). Suppose that the quantizer (7) satisfies conditions (4)–(6) and that the control law (8) is applied. If the following conditions hold:

$$\alpha_2^{-1} \circ \alpha_1((M - 2\Delta)\mu) > 2\Delta\mu + \sigma_1(\Delta\mu) \tag{11}$$

$$\Omega_{\text{out}}\alpha_1((M - 2\Delta)\mu) > \alpha_2(M\mu) \tag{12}$$

$$\Omega_{\text{in}}\alpha_1((M - 2\Delta)\mu) > \alpha_2 \circ \sigma_1(\Delta\mu) + \alpha_1(2\Delta\mu) \tag{13}$$

where $\alpha_1, \alpha_2, \sigma_1$, and σ_2 are from (2) and (3), then for the nonlinear quantized closed-loop system, the continuous dynamics is ISS, and the quantization parameter μ is bounded for all $t \geq t_0$.

Remark 1: In Theorem 1, inequality (11) guarantees the existence of all the variables and quantities defined in Section II. It also implies that for the quantizer (7), compared with the bound $\Delta\mu$, the range $M\mu$ should be large enough. If the system state is always in the quantization regions, then (11) can be relaxed to only hold for some bounded μ like $\mu \in (\varepsilon, \mu_0]$ with some small $\varepsilon > 0$; see [26].

To prove Theorem 1, some preliminary lemmas are necessary. In the following, all the assumptions in Theorem 1 are assumed to be satisfied. The next lemma implies that there exists a time instant such that the system state enters into the quantization regions and $\mathtt{c}$ switches to $\mathtt{yes}$.

Lemma 1: There exists $t_1 \geq t_0$ such that for some $\rho_x, \rho_\mu \in \mathcal{KL}$, $\gamma_x, \gamma_\mu \in \mathcal{K}$

$$\|x\|_{[t_0,t_1]} \leq \rho_x(|x_0|, t_1 - t_0) + \gamma_x(\|w\|) \tag{14}$$

$$\mu(t_1) \leq \rho_\mu(|x_0|, t_1 - t_0) + \gamma_\mu(\|w\|). \tag{15}$$

In addition, $\mathtt{c}(t) = \mathtt{yes}$ and $|x(t)| \leq M\mu(t)$ for all $t \geq t_1$.

Proof: At the initial time t_0, $\mathtt{c}(t_0) = \mathtt{no}$. There is no control input and μ starts to increase. Thus, the nonlinear system is $\dot{x}(t) = f(t, x, 0, w)$. Let us introduce the autonomous systems as follows:

$$\dot{\hat{x}}(t) = f(t, \hat{x}, 0, 0) \tag{16}$$

and the initial observer state is $\hat{x}(t_0) = x_0$.

Based on the forward completeness of the function f, the system state $x(t)$ and the observer state $\hat{x}(t)$ exist on $[t_0, \infty)$. Furthermore, it follows from the local Lipschitz property of f that, for all locally bounded $w \in \mathbb{R}^{n_w}$, there exist $\check{t}_1 > t_0$ and $\delta_1 > 0$ such that, for all $t \in [t_0, \check{t}_1]$, $x(t), \hat{x}(t) \in \mathcal{B}(\delta_1) := \{z \in \mathbb{R}^{n_x} : |z - x_0| \leq \delta_1\}$

$$\begin{aligned}|\dot{x}(t) - \dot{\hat{x}}(t)| &= |f(t, x, 0, w) - f(t, \hat{x}, 0, 0)| \\ &\leq L_{x1}|x(t) - \hat{x}(t)| + L_{w1}\|w\|_{[t_0,\check{t}_1]}\end{aligned} \tag{17}$$

where $L_{x1}, L_{w1} \geq 0$ are local Lipschitz constants in $[t_0, \check{t}_1)$. According to [23, Th 3.1] and the standard comparison lemma, it follows from (17) that for all $t \in [t_0, \check{t}_1]$

$$|x(t)| \leq |\hat{x}(t)| + e^{L_{x1}(t-t_0)} L_{w1} L_{x1}^{-1} \|w\|_{[t_0,\check{t}_1]}.$$

In addition, $\hat{x}(t)$ is bounded in $[t_0, \check{t}_1)$ based on [22, Section 2.1]. We can write the bound of $\hat{x}(t)$ as a function of $t - t_0$ and $|x_0|$.

With the increase of the time, the system state and the observer state will escape the region $\mathcal{B}(\delta_1)$. Then, for all locally bounded $w \in \mathbb{R}^{n_w}$, there exist $\delta_2 > \delta_1$ and $\check{t}_2 > \check{t}_1$ such that, for all $t \in [\check{t}_1, \check{t}_2)$, $x(t), \hat{x}(t) \in \mathcal{B}(\delta_2) := \{z \in \mathbb{R}^{n_x} : |z - x_0| \leq \delta_2\}$

$$|\dot{x}(t) - \dot{\hat{x}}(t)| \leq L_{x2}|x(t) - \hat{x}(t)| + L_{w2}\|w\|_{[\check{t}_1,\check{t}_2]}$$

where $L_{x2}, L_{w2} \geq 0$ are local Lipschitz constants in $[\check{t}_1, \check{t}_2)$. Similarly, it obtains that for all $t \in [\check{t}_1, \check{t}_2)$, $\hat{x}(t)$ is bounded and

$$\begin{aligned}|x(t)| &\leq |\hat{x}(t)| + e^{L_{x2}(t-\check{t}_1)}[|x(\check{t}_1) - \hat{x}(\check{t}_1)| \\ &\quad + L_{w2}L_{x2}^{-1}\|w\|_{[\check{t}_1,\check{t}_2]}] \\ &\leq |\hat{x}(t)| + e^{\max\{L_{x1},L_{x2}\}(t-t_0)} \\ &\quad \times (L_{w1}L_{x1}^{-1} + L_{w2}L_{x2}^{-1})\|w\|_{[t_0,\check{t}_2]}.\end{aligned}$$

Repeating the above mechanism, it follows that for all $t \geq t_0$

$$|x(t)| \leq |\hat{x}(t)| + e^{L_x(t-t_1)} L_w \|w\| \tag{18}$$

where $L_x := \sup_{i \in \mathbb{N}_{>0}}\{L_{xi}\}$ and $L_w := \sum_{i \in \mathbb{N}_{>0}} L_{wi}L_{xi}^{-1}$. Since the bound of $\hat{x}(t)$ can be written as a function of $t - t_0$ and $|x_0|$, there exist $\rho_x, \rho_\mu \in \mathcal{KK}$, and $\gamma_x, \gamma_\mu \in \mathcal{K}_\infty$ such that (14) and (15) hold.

Next, suppose $\mathtt{c} = \mathtt{no}$ for all $t \geq t_0$. Because zoom-out events occur successively at least every T_{out} units of time, it obtains that $\mu(t_0 + kT_{\text{out}}) = \Omega_{\text{out}}^k\mu_0$, where $k \in \mathbb{N}$. Combining the boundedness of the disturbance, $T_{\text{out}} < \log\Omega_{\text{out}}/L_x$ and (18), it follows that $\mu(t_0 + kT_{\text{out}})$ grows faster than $\|x\|_{[t_0+kT_{\text{out}}, t_0+(k+1)T_{\text{out}})}$. As a result, if μ stays in zooming-out stage all the time, there exists an infinite time sequence $\{kT_{\text{out}} + T_c : T_c \in (0, T_{\text{out}}), k > k_1 \in \mathbb{N}_{>0}\}$ such that at these time instants, $|q_\mu(x)| > (M - \Delta)\mu$, which implies that $|x| > (M - 2\Delta)\mu$. This is a contradiction to (18). Therefore, $\mathtt{c} = \mathtt{no}$ does not hold for all $t \geq t_0$, and there exists $t_1 \geq t_0$ such that $\mathtt{c}(t_1) = \mathtt{yes}$.

Since $|q_{\mu(t_1^-)}(x(t_1^-))| \leq (M - \Delta)\mu(t_1^-)$ at $t_1 \geq t_0$, it follows that $|x(t_1)| \leq M\mu(t_1^-) < M\mu(t_1)$. Claim that $|x(t)| \leq M\mu(t)$ for all $t \geq t_1$. The claim does not fail at the zoom-out event, because $\mu(t)$ increases at the zoom-out event. Due to the definition of l_{in} and conditions (4) and (5), the claim also does not fail at the zoom-in event. Even between two successive discrete-time events, if $|x(t^-)| \geq M\mu(t^-)$ holds for some $t > t_1$, then the zoom-out events occur, which in turn implies that $|x(t)| \leq M\mu(t)$. Therefore, the system state is in the quantization regions and $\mathtt{c} = \mathtt{yes}$ for all $t \geq t_1$.

Finally, based on the state trajectory in $[t_0, t_1]$, ρ_x and ρ_μ can be majorized to be of class $\mathcal{KL}$. ■

Lemma 1 provides bounds for the overshoots of the system state and the quantization parameter. Thanks to these bounds, the capture event does not last infinitely. As long as the capture event occurs, we do not need to repeat it more than twice. For the linear case, Lemma 1 is more direct because conditions (2) and (3) hold simply for each stabilizing feedback; see [14, Lemma IV.1].

After $t_1 \geq t_0$, $\mathtt{c} = \mathtt{yes}$ and the state feedback law (8) is implemented, then the closed-loop system becomes

$$\dot{x}(t) = f(t, x, \kappa(q_\mu(x)), w) = f(t, x, \kappa(x + e), w)$$

where $e = q_\mu(x) - x$ is the quantization error. If the system state satisfies $|x| \geq \sigma_1(|e|) + \sigma_2(\|w\|)$ for $t \geq t_1$, then it follows from (3) that $\dot{V}(t, x) < -\alpha_3(|x|)$. In the sequel, define a ball $\mathcal{B} := \{x \in \mathbb{R}^{n_x} : |x| = \sigma_1(|e|) + \sigma_2(\|w\|)\}$. Based on the definition of the invariant set and the properties of the Lyapunov function in [23, Ch. 4.2], if the sublevel set $\{x \in \mathbb{R}^{n_x} : V(t, x) \leq \alpha(|x|), \alpha \in \mathcal{K}\}$ of V contains $\mathcal{B}$ and is contained in the quantization regions, then the sublevel set of V is invariant for the system. Based on the sublevel sets of V, the occurrence of the zoom-in event is studied in the following lemma.

Lemma 2: Suppose there exists $t \geq t_1$ such that $x(t) \in \mathcal{R}_1(\mu) := \{x \in \mathbb{R}^{n_x} : V(t, x) < \alpha_1((M - 2\Delta)\mu)\}$. If $\mu(t)$ satisfies

$$\alpha_1((M - 2\Delta)\mu(t)) > \alpha_2 \circ [\sigma_1(\Delta\mu(t)) + \sigma_2(\|w\|)] \tag{19}$$

then the next can only be a zoom-in event. Moreover, if $\mu(t)$ satisfies

$$\Omega_{\rm in}\alpha_1((M-2\Delta)\mu(t)) - \alpha_1(2\Delta\mu(t)) > \alpha_2 \circ [\sigma_1(\Delta\mu(t)) + \sigma_2(\|w\|)] \tag{20}$$

then the next zoom-in event will occur in the finite time.

Proof: It follows from Lemma 1 that $\mathtt{c} = \mathtt{yes}$ and the system state is in the quantization regions for all $t \geq t_1$. Thus, $|e(t)| \leq \Delta\mu(t)$ for all $t \geq t_1$.

Define two balls and an ellipsoid as follows:

$$\mathcal{B}_1(\mu) := \{x \in \mathbb{R}^{n_x} : |x| \leq (M-2\Delta)\mu\}$$

$$\mathcal{B}_2(\mu) := \{x \in \mathbb{R}^{n_x} : |x| \leq \sigma_1(\Delta\mu) + \sigma_2(\|w\|)\}$$

$$\mathcal{R}_2(\mu) := \{x \in \mathbb{R}^{n_x} : V(t,x) \leq \Omega_{\rm in}\alpha_1((M-2\Delta)\mu) - \alpha_1(2\Delta\mu)\}.$$

It can be observed from (19) that $\mathcal{B}_2(\mu) \subset \mathcal{R}_1(\mu) \subset \mathcal{B}_1(\mu)$. Because $\mathcal{B}$ is contained in $\mathcal{B}_2(\mu)$, it is observed that as long as μ remains constant, $\mathcal{R}_1(\mu)$ is the invariant set for the continuous dynamics. Hence, the zoom-out event cannot occur in the next update, which implies that the next discrete-time event is a zoom-in event.

After finite zoom-in events, there exists $\bar{t} > t \geq t_1$ such that $x(\bar{t}) \in \mathcal{R}_2(\mu(\bar{t}))$. Define the following ball:

$$\mathcal{B}_3(\mu) := \{x \in \mathbb{R}^{n_x} : |x| \leq \Omega_{\rm in}(M-2\Delta)\mu - 2\Delta\mu\}.$$

It can be observed from (20) that $\mathcal{B}_2(\mu) \subset \mathcal{R}_2(\mu) \subset \mathcal{B}_3(\mu) \subset \mathcal{R}_1(\mu)$. Observe that $\mathcal{R}_2(\mu)$ is an invariant set for the continuous dynamics and contained in $\mathcal{B}_3(\mu)$. Since the next event is zoom-in event, the trajectory of the system state from $\mathcal{R}_1(\mu)$ will converge to $\mathcal{R}_2(\mu)$. Therefore, the arrival time from $\mathcal{R}_1(\mu)$ to $\mathcal{R}_2(\mu)$ is earlier than $\bar{t} + T_{\rm in}$. That is, the next zoom-in event happens in finite time. ∎

Because the zoom-in events occur successively, the sublevel sets of V become smaller gradually and converge to the ball $\mathcal{B}$. In addition, the ball $\mathcal{B}$ becomes smaller with the decrease of the quantization error e. Therefore, the next lemma shows that if the external disturbance is sufficiently small, then the origin is stable.

Lemma 3: For every $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x_0| \leq \delta$ and $\|w\| \leq \delta$, then there exists certain $t_2 \geq t_1$ such that the following properties hold.

1) $\mathcal{R}_1(\mu(t_2)) \subset \{x : |x| \leq \varepsilon\}$.
2) If $t = t_2$, then inequality (19) holds.
3) $x(t) \in \mathcal{R}_1(\mu(t_2))$ for all $t \in [t_0, t_2]$.

Proof: First, for a fixed $\varepsilon > 0$, since the zoom-in event occurs and $\Omega_{\rm in} \in (0,1)$, there exists $k \in \mathbb{N}_{>0}$ such that $\tilde{\mu} = \mu(t_0 + (k+1)T_{\rm in}) = \Omega_{\rm in}^k \Omega_{\rm out}\mu_0$ and $(M-2\Delta)\tilde{\mu} < \varepsilon$. Thus, $\mathcal{R}_1(\tilde{\mu}) \subset \{x : |x| < \varepsilon\}$, which implies that the first property holds.

Second, It follows from (11) that $\alpha_2^{-1} \circ \alpha_1((M-2\Delta)\mu) > 2\Delta\mu + \sigma_1(\Delta\mu)$. Thus, there exists $\delta_w > 0$ such that $\sigma_2(\delta_w) \leq 2\Delta\tilde{\mu}$ and $\alpha_2^{-1} \circ \alpha_1((M-2\Delta)\tilde{\mu}) > \sigma_1(\Delta\tilde{\mu}) + \sigma_2(\delta_w)$. That is, the second property is valid.

Third, pick a $\delta_x > 0$ such that $\tilde{x} = x(t_0 + (k+1)T_{\rm in} + T_c) = e^{L_x(T_c + kT_{\rm in})}(\delta_x + L_w L_x^{-1}\delta_w)$. Moreover, $\tilde{x}$ satisfies $|\tilde{x}| \leq \Omega_{\rm in}^{k-1}\mu_0\Delta_0$ and $\alpha_2(|\tilde{x}|) < \alpha_1((M-2\Delta)\tilde{\mu})$.

Define $\delta := \max\{\delta_x, \delta_w\}$ and $t_2 := t_0 + T_c + kT_{\rm in}$. It follows that the first and second properties are established at $t = t_2$. At the initial time t_0, a zoom-out event happens. After T_c units of time, a capture event follows. Sequentially, there are k zoom-in events occurring at $t_0 + T_c + iT_{\rm in}$ successively, where $i \in \{1, 2, \ldots, k\}$. If the system state and the disturbance are sufficiently small, then because of (6) and (8), the closed-loop system has no control law. Therefore, the system state remains in $\mathcal{R}_1(\tilde{\mu}) = \mathcal{R}_1(\mu(t_2))$ for all $t \in [t_0, t_2]$. It implies that the third property is guaranteed. ∎

Based on Lemmas 1–3, the proof of Theorem 1 is given as follows.

Proof of Theorem 1 Let inequality (19) be the following equation:

$$\alpha_1((M-2\Delta)\mu(t)) = \alpha_2 \circ [\sigma_1(\Delta\mu(t)) + \sigma_2(\|w\|)]. \tag{21}$$

The solution to (21) is denoted by $\bar{\mu}(t)$. Thus, it is easy to verify that (19) holds as long as $\mu(t) > \bar{\mu}(t)$.

First, claim that $\mu(t) \leq \Omega_{\rm out}\max\{\mu(t_1), \bar{\mu}(t)\}$ for all $t \geq t_1$. If not, then there exists $t' > t_1$ such that $\mu(t') > \max\{\mu(t_1), \bar{\mu}(t')\}$. The existence of t' implies that there is at least one zoom-out event occurs after t_1 with $\mu(t^-) > \max\{\mu(t_1), \bar{\mu}(t)\}$ for certain $t > t_1$. Before this zoom-out event is either a zoom-out event or a zoom-in event, which results in the fact that $\mu(t) \geq \max\{\mu(t_1), \bar{\mu}(t)\}$ for $t > t_1$ by the virtue of right continuity of μ. However, it follows from Lemma 1 that $|x(t)| \leq M\mu(t)$ for all $t \geq t_1$. Under the condition that $|x(t)| \leq M\mu(t)$, it is a zoom-in event that after a zoom-out or zoom-in event by Lemma 2. This contradicts with the occurrence of the zoom-out event after either a zoom-out or a zoom-in event, which implies that the claim is true.

According to Lemma 1, the boundedness of the system state and the quantization parameter is established in $[t_0, t_1]$. Because of the above claim, the boundedness of the system state and the quantization parameter is valid in $[t_1, \infty)$. Therefore, it follows that (9) holds with continuous and increasing functions β and γ_1.

Furthermore, consider the neighborhood of the origin. For every $\varepsilon > 0$, there exists $\delta > 0$ such that the three properties in Lemma 3 hold. From Lemma 2, the next event after $t_2 \geq t_1$ is a zoom-in event. Thus, $\mu(t) \leq \mu(t_2)$ for all $t \geq t_2$. Otherwise, it follows from Lemma 2 that the whole system can still be kept in zooming-in stage from certain time instant, which guarantees the boundedness of μ. Furthermore, it implies from Lemmas 1 and 3 that if $|x_0|$ and $\|w\|$ are sufficient small, then an arbitrarily small bound on $|x(t)|$ can be obtained for all time. Therefore, we can choose appropriate functions β and γ_1 such that they are of class $\mathcal{KL}$ and class $\mathcal{K}_\infty$, respectively. That is, inequality (9) is proven.

In the following, inequality (20) is written into the equation as follows:

$$\Omega_{\rm in}\alpha_1((M-2\Delta)\mu(t)) - \alpha_1(2\Delta\mu(t)) = \alpha_2 \circ \left[\sigma_1(\Delta\mu(t)) + \sigma_2\left(\limsup_{t\to\infty}|w(t)| + \epsilon\right)\right] \tag{22}$$

where $\epsilon > 0$ is arbitrarily small. The solution to (22) is denoted by $\hat{\mu}(t)$. By the boundedness of $w(t)$, there exists $\hat{t}_1 \geq t_1$ ($\hat{t}_1$ may depend on ϵ) such that $\|w\| \leq \limsup_{t\to\infty}|w(t)| + \epsilon$ for all $t \geq \hat{t}_1$. Hence, as long as $t \geq \hat{t}_1$ and $\mu(t) > \hat{\mu}(t)$, (20) holds by the fact that $\mathrm{ess.\ sup} \leq \limsup$.

Assert that $\mu(t) \leq \Omega_{\rm out}\hat{\mu}(t)$ for all $t \geq \hat{t}_1$. If not, there exists $t \geq \hat{t}_1$ such that $\mu(t) > \hat{\mu}(t)$. If $x(t) \in \mathcal{R}_1(\mu)$, then the zoom-in events occur repeatedly by Lemma 2 and $\mu(t) \leq \Omega_{\rm out}\hat{\mu}(t)$ holds from some $\hat{t} \geq \hat{t}_1$. Otherwise, the following two cases may occur. The first case is that the system state $x(t)$ enters into $\mathcal{R}_1(\mu)$ directly before the next event. The second case is that a zoom-out event happens. For both of the cases, $x(t)$ will enter into $\mathcal{R}_1(\mu)$ with a new value $\mu(t) = \Omega_{\rm out}\mu(t^-)$ for certain $t \geq \hat{t}_1$. Once $x(t) \in \mathcal{R}_1(\mu)$, it obtains from Lemma 2 that the zoom-in events occur repeatedly and $\mu(t) \leq \Omega_{\rm out}\hat{\mu}(t)$. Thus, along the same argument, the zoom-out event does not occur. Therefore, the assertion is true.

Above assertion provides the bound for μ from $\hat{t} \geq \hat{t}_1 \geq t_1$. Since the system state remains in the quantization regions for all $t \geq t_1$, it follows that $|x(t)| \leq (M-2\Delta)\mu(t)$. Taking the upper limit from both sides yields inequality (10). The gain function γ_2 is obtained by $\limsup_{t\to\infty}|x(t)| \leq \limsup_{t\to\infty}(M-2\Delta)\Omega_{\rm out}\hat{\mu}(t)$, where the solution $\hat{\mu}(t)$ to (22) is related to $\limsup_{t\to\infty}|w(t)|$.

In the above analysis, μ is also proven to be bounded for all $t \geq t_0$. As a result, the proof is completed. ■

Remark 2: Theorem 1 recovers the previous works [12], [14] as the special cases. For instance, quantized control for linear systems with external disturbance was considered in [14]; quantized control for nonlinear systems without external noise was studied in [12].

Example 1: Consider the nonlinear system of the following form:

$$\dot{x} = x^3 + xu + w$$

where $x \in \mathbb{R}$ is the state, $u \in \mathbb{R}$ is the control input, and $w \in \mathbb{R}$ is the external disturbance. Based on the example in [27, Section V], the controller $u = -x^2 - 1$ is designed to stabilize the system without disturbance. Moreover, set $V(t,x) = x^2/2, \alpha_1(v) = \alpha_2(v) = v^2/2, \alpha_3(v) = av^2, \sigma_1(v) = bv, \sigma_2(v) = cv$, where $b > 1, c > 1, 0 < a < 1 - 1/c$. The differential of V satisfies

$$\dot{V}(t,x) \leq -x^2 + |x|\|w\| \leq -(1-1/c)x^2$$

which implies that conditions (2) and (3) hold. Moreover, it obtains from Theorem 1 that $M > (c+4)\Delta$, $\Omega_{\text{out}} > M^2/(M-2\Delta)^2$, and $\Omega_{\text{in}} > (c^2+4)\Delta^2/(M-2\Delta)^2$.

B. Input Quantization Case

We study the input quantization case in this section. In the input quantization case, the available information for the plant is the quantized state feedback law $q_\mu(u)$. Thus, different from (8), the control law in this case is of the following form:

$$u(t) = \begin{cases} 0, & \mathtt{c}(t) = \mathtt{no} \\ q_{\mu(t)}(\kappa(x(t))), & \mathtt{c}(t) = \mathtt{yes}. \end{cases} \tag{23}$$

Assume that there exists a continuous differential Lyapunov function $V : \mathbb{R}_{\geq 0} \times \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ and $\alpha_1, \alpha_2, \alpha_3, \sigma_1, \sigma_2 \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^{n_x}$, $e \in \mathbb{R}^{n_u}$, and $w \in \mathbb{R}^{n_w}$, (2) holds and

$$\begin{aligned} &|x| \geq \sigma_1(|e|) + \sigma_2(\|w\|) \Rightarrow \\ &\quad V_t(t,x) + V_x(t,x) f(t,x,\kappa(x)+e,w) \leq -\alpha_3(|x|). \end{aligned} \tag{24}$$

Hence, the closed-loop system $\dot{x}(t) = f(t,x,\kappa(x)+e,w)$ is ISS with respect to $e \in \mathbb{R}^{n_u}$ and $w \in \mathbb{R}^{n_w}$.

In the following, pick a function $\varphi \in \mathcal{K}_\infty$ such that $\varphi(r) \geq \max_{0 \leq |x| \leq r} |\kappa(x)|$ for all $r \geq 0$. Choosing $r = |x|$, it follows that $|\kappa(x)| \leq \varphi(|x|)$ for all $x \in \mathbb{R}^{n_x}$. With the quantized feedback control law (23), the quantized closed-loop system is given by

$$\dot{x}(t) = f(t,x,q_\mu(\kappa(x)),w) = f(t,x,\kappa(x)+e,w) \tag{25}$$

where $e := q_{\mu(t)}(\kappa(x)) - \kappa(x)$.

In the stability analysis, the following lemma plays the similar role as Lemma 2. Its proof is proceeded along the similar proof strategy of Lemma 2, and hence omitted here.

Lemma 4: Suppose there exists $t \geq t_1$ such that $x(t) \in \mathcal{R}_1(\mu) := \{x \in \mathbb{R}^{n_x} : V(t,x) < \alpha_1 \circ \varphi^{-1}((M-2\Delta)\mu)\}$. If $\mu(t)$ satisfies

$$\alpha_1 \circ \varphi^{-1}((M-2\Delta)\mu(t)) > \alpha_2 \circ [\sigma_1(\Delta\mu(t)) + \sigma_2(\|w\|)] \tag{26}$$

then the next can only be a zoom-in event. Moreover, if $\mu(t)$ satisfies

$$\begin{aligned} &\Omega_{\text{in}}\alpha_1 \circ \varphi^{-1}((M-2\Delta)\mu(t)) - \alpha_1 \circ \varphi^{-1}(2\Delta\mu(t)) \\ &\quad > \alpha_2 \circ [\sigma_1(\Delta\mu(t)) + \sigma_2(\|w\|)] \end{aligned} \tag{27}$$

then the next zoom-in event will occur in finite time.

According on Lemmas 1, 3, and 4, the following theorem is established for the input quantization case.

Theorem 2: Consider system (1) satisfying (2) and (24). Suppose that the quantizer (7) satisfies (4)–(6) and the control law (23) is applied. If the following hold:

$$\begin{aligned} &\alpha_2^{-1} \circ \alpha_1 \circ \varphi^{-1}((M-2\Delta)\mu) > 2\Delta\mu + \sigma_1(\Delta\mu) \\ &\Omega_{\text{out}}\alpha_1 \circ \psi^{-1}((M-2\Delta)\mu) > \alpha_2(M\mu) \\ &\Omega_{\text{in}}\alpha_1 \circ \varphi^{-1}((M-2\Delta)\mu) > \alpha_2 \circ \sigma_1(\Delta\mu) + \alpha_1 \circ \varphi^{-1}(2\Delta\mu) \end{aligned}$$

where $\alpha_1, \alpha_2, \sigma_1$, and σ_2 are given in (2) and (24), then the continuous dynamics of system (25) is ISS and μ is bounded for all $t \geq t_0$.

Proof: First, (26) is transformed to be the equation as follows:

$$\alpha_1 \circ \varphi^{-1}((M-2\Delta)\mu(t)) = \alpha_2 \circ [\sigma_1(\Delta\mu(t)) + \sigma_2(\|w\|)]. \tag{28}$$

Denote by $\bar{\mu}_1(t)$ the solution to (28). Similar to the proof of Theorem 1, we can prove that $\mu(t) \leq \Omega_{\text{out}} \max\{\mu(t_1), \bar{\mu}_1(t)\}$ for all $t \geq t_1$.

Second, (27) is written as the following equation:

$$\begin{aligned} &\Omega_{\text{in}}\alpha_1 \circ \varphi^{-1}((M-2\Delta)\mu(t)) - \alpha_1 \circ \varphi^{-1}(2\Delta\mu(t)) \\ &\quad = \alpha_2 \circ \left[\sigma_1(\Delta\mu(t)) + \sigma_2\left(\limsup_{t\to\infty} |w(t)| + \epsilon_1\right)\right] \end{aligned} \tag{29}$$

where $\epsilon_1 > 0$ is arbitrarily small. The solution to (29) is denoted by $\hat{\mu}_1(t)$. There exists $\hat{t}_2 > t_1$ such that if $t \geq \hat{t}_2$ and $\mu(t) > \hat{\mu}_1(t)$, then (27) holds. As a result, $\mu(t) \leq \Omega_{\text{out}}\hat{\mu}_1(t)$ for all $t \geq \hat{t}_2$.

The rest of the proof is proceeded along the similar line as the proof of Theorem 1, and hence omitted here. ■

Remark 3: For the input quantization case, the initial state is not necessarily known exactly. Thus, the control input is unknown and has to be quantized before transmitted. However, if the system state is known exactly, then whether the control input is in the quantization regions can be identified directly. In this case, the updating protocol is simplified greatly. For instance, the variable c is not needed, and the zooming-out stage is only introduced when the effects of the disturbance on the convergence of the system state cannot be ignored.

If both the state and the control input are quantized, then a direct approach is to combine the results of Theorems 1 and 2. In the sequel, the feedback control law is of the following form:

$$u(t) = \begin{cases} 0, & \mathtt{c}(t) = \mathtt{no} \\ q^u_{\mu(t)}(\kappa(q^x_{\mu(t)}(x(t)))), & \mathtt{c}(t) = \mathtt{yes} \end{cases}$$

where q^x_μ and q^u_μ are a state quantizer and an input quantizer, respectively. For the state quantizer and the input quantizer, the ranges are $M_x\mu$ and $M_u\mu$; the bounds of the quantization errors are $\Delta_x\mu$ and $\Delta_u\mu$. Assume that there exists a controller $u = \kappa(x)$ such that the closed-loop system $\dot{x}(t) = f(t,x,\kappa(x+e_1)+e_2,w)$ is ISS with respect to e_1, e_2, w, where $e_1 := q^x_\mu(x) - x$ and $e_2 := q^u_\mu(\kappa(q^x_\mu(x))) - \kappa(q^x_\mu(x))$. Combining the state quantizer and the input quantizer yields that if $|x| \leq \min\{M_x\mu, M_u\mu, \varphi^{-1}(M_u\mu) - \Delta_u\mu\}$, then $|e_1| \leq \Delta_x\mu$ and $|e_2| \leq \Delta_u\mu$. The proceeding analysis is a combination of the analysis strategies for Theorems 1 and 2.

C. Output Quantization Case

The quantized output feedback case is studied in this section. That is, the system output is quantized and sent to the controller to generate the control input. Assume that the output of nonlinear system (1) is of the form $y(t) = h(x(t))$, where $y \in \mathbb{R}^{n_y}$ and h is locally Lipschitz. The system output is initialized as zero.

To study ISS of system (1) in the output quantization case, the observer-based quantized output feedback controller is given by [26]

$$\dot{p}(t) = g(t,p,u,q_\mu(y)), \quad u(t) = \kappa(p(t)) \tag{30}$$

where $p \in \mathbb{R}^{n_c}$ is the controller state. Combining (1) and (30) yields the augmented quantized closed-loop system

$$\begin{cases} \dot{x}(t) = f(t, x, \kappa(p), w) \\ \dot{p}(t) = g(t, p, \kappa(p), h(x) + e) \end{cases} \tag{31}$$

where $e := q_\mu(h(x)) - h(x)$. Moreover, assume that the p-subsystem is a full-order state observer for the x-subsystem.

Assume that conditions (2) and (3) are satisfied for the augmented state $\mathfrak{X} := (x^\top, p^\top)^\top$, i.e.,

$$\alpha_1(|\mathfrak{X}|) \le V(t, \mathfrak{X}) \le \alpha_2(|\mathfrak{X}|) \tag{32}$$

$$\begin{aligned} |\mathfrak{X}| \ge \sigma_1(|e|) + \sigma_2(\|w\|) \Rightarrow & V_t(t, \mathfrak{X}) + V_x(t, \mathfrak{X}) f(t, x, \kappa(p), w) \\ & + V_p(t, \mathfrak{X}) g(t, p, \kappa(p), h(x) + e) \le -\alpha_3(|\mathfrak{X}|) \end{aligned} \tag{33}$$

which means that system (31) is ISS with respect to e and w. Pick $\psi \in \mathcal{K}_\infty$ such that $\psi(r) \ge \max_{|x| \le r} |h(x)|$ for all $r \ge 0$. As a result, $|h(x)| \le \psi(|x|)$ holds for all $x \in \mathbb{R}^{n_x}$.

Just like Lemma 4 and Theorem 2, the following lemma and theorem are established, and the proofs are omitted here.

Lemma 5: Suppose there exists $t \ge t_1$ such that $\mathfrak{X}(t) \in \mathcal{R}_1(\mu) := \{\mathfrak{X} : V(t, \mathfrak{X}) < \alpha_1 \circ \psi^{-1}((M - 2\Delta)\mu)\}$. If $\mu(t)$ satisfies $\alpha_1 \circ \psi^{-1}((M - 2\Delta)\mu(t)) > \alpha_2 \circ (\sigma_1(\Delta\mu(t)) + \sigma_2(\|w\|))$, then the next is a zoom-in event. If $\mu(t)$ satisfies $\Omega_{\text{in}} \alpha_1 \circ \psi^{-1}((M - 2\Delta)\mu(t)) - \alpha_1 \circ \psi^{-1}(2\Delta\mu(t)) > \alpha_2 \circ (\sigma_1(\Delta\mu(t)) + \sigma_2(\|w\|))$, then the next zoom-in event will occur in the finite time.

Theorem 3: Consider systems (1) and (30). Suppose that conditions (32) and (33) hold for (x, p), and that the quantizer $q_\mu(y)$ satisfies conditions (4)–(6). If the following conditions hold:

$$\alpha_2^{-1} \circ \alpha_1 \circ \psi^{-1}((M - 2\Delta)\mu(t)) > 2\Delta\mu(t) + \sigma_1(\Delta\mu(t))$$

$$\Omega_{\text{out}} > \frac{\alpha_2(M\mu(t))}{\alpha_1 \circ \psi^{-1}((M - 2\Delta)\mu(t))}$$

$$\begin{aligned} &\Omega_{\text{in}} \alpha_1 \circ \psi^{-1}((M - 2\Delta)\mu(t)) - \alpha_1 \circ \psi^{-1}(2\Delta\mu(t)) \\ &\quad > \alpha_2 \circ \sigma_1(\Delta\mu(t)) \end{aligned}$$

where $\alpha_1, \alpha_2, \sigma_1$, and σ_2 are the same as those in (32) and (33); then, the continuous dynamics of system (31) is ISS, and μ is bounded for all $t \ge t_0$.

Remark 4: For the output quantization case, the ISS property is achieved by two steps. First, there exists a state feedback law such that the x-subsystem is ISS with respect to the measurement disturbance and the external disturbance. Second, a full-order state observer is constructed for the x-subsystem. The full-order state observer ensures that the difference between the system state and the observer state converges asymptotically to the origin or the neighborhood of the origin. This is why the p-subsystem is assumed to be a full-order state observer. Thus, the essence is to design the state feedback controller and the full-order state observer for system (1). However, not all the nonlinear systems have such a controller and an observer satisfying (32) and (33). See [26] and references therein for more details.

IV. Conclusion

This paper addressed quantized feedback stabilization of nonlinear systems with external disturbance. Three quantization cases were studied and stability conditions were derived for quantized closed-loop nonlinear systems with external disturbance. Future works focus on the controller and observer design for quantized nonlinear systems and extension of the obtained theory to nonlinear switched systems.

References

[1] R. A. Gupta and M.-Y. Chow, “Networked control system: Overview and research trends,” *IEEE Trans. Ind. Electronics*, vol. 57, no. 7, pp. 2527–2535, Jul. 2010.

[2] L. Zhang, H. Gao, and O. Kaynak, “Network-induced constraints in networked control systems: A survey,” *IEEE Trans. Ind. Informat.*, vol. 9, no. 1, pp. 403–416, Feb. 2013.

[3] J. Lunze, *Control Theory of Digitally Networked Dynamic Systems*. New York, NY, USA: Springer, 2014.

[4] G. Giachetta, L. Mangiarotti, and G. Sardanashvily, “Geometric quantization of mechanical systems with time-dependent parameters,” *J. Math. Physics*, vol. 43, no. 6, pp. 2882–2894, Feb. 2002.

[5] D. Nešić and D. Liberzon, “A unified framework for design and analysis of networked and quantized control systems,” *IEEE Trans. Autom. Control*, vol. 54, no. 4, pp. 732–747, Apr. 2009.

[6] M. Wakaiki and Y. Yamamoto, “Stabilization of switched linear systems with quantized output and switching delays,” *IEEE Trans. Autom. Control*, vol. 62, no. 6, pp. 2958–2964, Jun. 2017.

[7] D. Liberzon, “Finite data-rate feedback stabilization of switched and hybrid linear systems,” *Automatica*, vol. 50, no. 2, pp. 409–420, 2014.

[8] S. van Loon, M. Donkers, N. van de Wouw, and W. Heemels, “Stability analysis of networked and quantized linear control systems,” *Nonlinear Anal.: Hybrid Syst.*, vol. 10, pp. 111–125, Nov. 2013.

[9] D. F. Coutinho, M. Fu, and C. E. de Souza, “Input and output quantized feedback linear systems,” *IEEE Trans. Autom. Control*, vol. 55, no. 3, pp. 761–766, Mar. 2010.

[10] H. Gao and T. Chen, “A new approach to quantized feedback control systems,” *Automatica*, vol. 44, no. 2, pp. 534–542, 2008.

[11] S.-I. Azuma and T. Sugie, “Dynamic quantization of nonlinear control systems,” *IEEE Trans. Autom. Control*, vol. 57, no. 4, pp. 875–888, Apr. 2012.

[12] D. Liberzon, “Hybrid feedback stabilization of systems with quantized signals,” *Automatica*, vol. 39, no. 9, pp. 1543–1554, 2003.

[13] A. Franci and A. Chaillet, “Quantised control of nonlinear systems: Analysis of robustness to parameter uncertainty, measurement errors, and exogenous disturbances,” *Int. J. Control*, vol. 83, no. 12, pp. 2453–2462, 2010.

[14] D. Liberzon and D. Nešic, “Input-to-state stabilization of linear systems with quantized state measurements,” *IEEE Trans. Autom. Control*, vol. 52, no. 5, pp. 767–781, May 2007.

[15] H. Ishii and B. A. Francis, “Quadratic stabilization of sampled-data systems with quantization,” *Automatica*, vol. 39, no. 10, pp. 1793–1800, Oct. 2003.

[16] T. Kameneva and D. Nešic, “Input-to-state stabilization of nonlinear systems with quantized feedback,” in *Proc. IFAC World Congr.*, 2008, pp. 12 480–12 485.

[17] T. Kameneva and D. Nešic, “On l_2 stabilization of linear systems with quantized control,” *IEEE Trans. Autom. Control*, vol. 53, no. 1, pp. 399–405, Feb. 2008.

[18] D. Liberzon and J. P. Hespanha, “Stabilization of nonlinear systems with limited information feedback,” *IEEE Trans. Autom. Control*, vol. 50, no. 6, pp. 910–915, Jun. 2005.

[19] D. Liberzon, “On stabilization of linear systems with limited information,” *IEEE Trans. Autom. Control*, vol. 48, no. 2, pp. 304–307, Feb. 2003.

[20] T. Kameneva and D. Nešić, “Robustness of quantized control systems with mismatch between coder/decoder initializations,” *Automatica*, vol. 45, no. 3, pp. 817–822, 2009.

[21] R. W. Brockett and D. Liberzon, “Quantized feedback stabilization of linear systems,” *IEEE Trans. Autom. Control*, vol. 45, no. 7, pp. 1279–1289, Jul. 2000.

[22] D. Angeli and E. D. Sontag, “Forward completeness, unboundedness observability, and their Lyapunov characterizations,” *Syst. Control Lett.*, vol. 38, no. 4, pp. 209–217, 1999.

[23] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, NJ, USA: Prentice-Hall, 2002.

[24] E. D. Sontag and Y. Wang, “New characterizations of input-to-state stability,” *IEEE Trans. Autom. Control*, vol. 41, no. 9, pp. 1283–1294, Sep. 1996.

[25] E. D. Sontag, “Further facts about input to state stabilization,” *IEEE Trans. Autom. Control*, vol. 35, no. 4, pp. 473–476, Apr. 1990.

[26] D. Liberzon, “Observer-based quantized output feedback control of nonlinear systems,” in *Proc. Mediterranean Conf. Control Autom.*, 2007, pp. 1–5.

[27] Z.-P. Jiang, I. Mareels, and D. Hill, “Robust control of uncertain nonlinear systems via measurement feedback,” *IEEE Trans. Autom. Control*, vol. 44, no. 4, pp. 807–812, Apr. 1999.