Stability Analysis of Impulsive Switched Time-Delay Systems With State-Dependent Impulses

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***Abstract*—This paper studies the stability for impulsive switched time-delay systems with state-dependent impulses. Since the impulses and the switches are not necessarily synchronous, we start from a stability analysis of impulsive switched time-delay systems with time-dependent impulses. Sufficient conditions are derived to guarantee the stability property, which extends the previous results for the synchronous switch and impulse case. For the state-dependent impulse case, using the B-equivalent method, impulsive switched time-delay systems with state-dependent impulses are transformed into impulsive switched time-delay systems with time-dependent impulses. The equivalence between the original system and the transformed system is established, and stability conditions are obtained for impulsive switched time-delay systems with state-dependent impulses. Finally, a numerical example is given to demonstrate the obtained results.**

***Index Terms*—B-equivalence, Lyapunov function, state-dependent impulses, switched systems, time delays.**

I. Introduction

Hybrid systems are dynamic systems that combine both continuous evolution and instant state jumps, see [1]. Two important classes of hybrid systems are impulsive systems [2] and switched systems [3]. Impulsive systems are composed of continuous-time dynamics with instantaneous state jumps. Switched systems consist of a family of subsystems and a switching signal that orchestrates the switching among them. Many physical or man-made systems can be modeled as impulsive or switched systems, such as networked control systems [4], mechanical systems [5], multiagent systems [6], and neural networks [7]. In the literature, there are numerous works on impulsive or switched systems, and many salient results can be found on the stability analysis of impulsive or switched systems, see, e.g., [3], [8], [9], and references therein.

If both impulses and switches exist synchronously in dynamic systems, then such dynamic systems are called impulsive switched systems, see [10] and [11]. For instance, impulses and switches coexist in many physical and man-made systems, such as chaotic systems and networked control systems [4]. Due to numerous applications in diverse fields of sciences and engineering, impulsive switched systems have attracted increasing attention, see [10] and [11]. On the other hand, time delays are frequently encountered in numerous engineering systems, and affect stability and performances of control systems. For instance, time delays are indispensable in networks due to data sampling, data coding, and long distances among different subsystems [4]; finite switching speed of amplifiers or information processing leads to time delays in hardware implementations [12]. Recently, great efforts have been devoted to impulsive or/and switched time-delay systems, and many salient results can be found, see [6], [8], and [11].

In the literature on impulsive switched time-delay systems, there are generally two types of switches and impulses. The first type is that impulses and switches are time-dependent. That is, the occurrence of impulses and switches depends on the time. Many salient results can be found on such a type. See [3], [9], and [13] for switched systems, [8], [14]–[16] for impulsive systems, and [7], [11], and [12] for impulsive switched systems. The second type is that impulses or switches are state-dependent. In this type, the occurrence of impulses or switches is related to the system state [2]. For instance, the velocity of the bouncing ball jumps when the ball hits the ground [17]; the occurrence of the quantization in a switched system depends on whether the system state is large enough [18]. Comparing with the first type, the second type is more practical because switches and impulses of physical systems (e.g., biological and physiological systems) do not occur at fixed times [19]. Since the state-dependent switches result in design of switching rule and the state-dependent impulses result in a beating phenomenon, the second type is of much more theoretical and technical challenges than the first type. Up to now, only a few works are on the second type, see [19]–[22].

In this paper, we study the stability of impulsive switched time-delay systems with state-dependent impulses. To this end, we start with the time-dependent impulse case, which is our first contribution. Both the stable continuous dynamics case and the stable discrete dynamics case are studied. Sufficient conditions are established to guarantee global asymptotical stability (GAS). Hence, we extend the previous results on impulsive or/and switched (time-delay) systems in [3], [8], and [11] to impulsive switched time-delay systems. Next, we turn to the state-dependent impulse case, which is the second contribution. Since impulses are state-dependent, the first difficulty is the beating phenomena of solutions of the system at certain impulsive surface [19], [21]. The second difficulty is that the solution does not depend on the initial condition continuously in such a way that this continuity can be uniform on a finite interval [23]. Therefore, we first rule out the beating phenomenon and guarantee the continuity of the solution on a finite interval. Using the B-equivalent method in [24] and [25], we develop an equivalent impulsive switched time-delay system with time-dependent impulses. Based the obtained results on the time-dependent impulse case, stability criteria are obtained for the state-dependent impulse case. In the sequel, we extend the existing results in [19]–[22] in terms of both the system model and stability analysis.

Notation: $\mathbb{R} := (-\infty, +\infty)$; $\mathbb{R}^+_{t_0} := [t_0, +\infty)$; $\mathbb{N} := \{0, 1, \ldots\}$; $\mathbb{N}^+ := \{1, 2, \ldots\}$. Given a vector or matrix P, $P^\top$ denotes its transpose. For a matrix $P \in \mathbb{R}^{n\times n}$, $\mathrm{tr}[P]$ denotes the trace of P. $|\cdot|$

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represents the Euclidean norm. Let $\mathbb{PC}([a,b];\mathbb{R}^n)$ denote the class of piecewise continuous functions mapping $[a,b]$ to $\mathbb{R}^n$ and having finite right-hand continuous jumps on $[a,b]$. Given a function $f:\mathbb{R}\to\mathbb{R}$, denote $f(t^+):=\limsup_{s\to 0^+} f(t+s)$. Given a function $f:\mathbb{R}^+_{t_0}\to\mathbb{R}^n$ with $t_0\geq\tau>0$, $\|f\|_\tau:=\sup_{t\in[t_0-\tau,t_0]}|f(t)|$; $\|f\|_{[t_0,t)}:=\sup_{t\in[t_0,t]}|f(t)|$; $\|f\|$ denotes the supremum norm on $[t_0,\infty)$. A function $\alpha:\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing; $\alpha(t)$ is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A function $\beta:\mathbb{R}_{\geq 0}\times\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(s,t)$ is of class $\mathcal{K}$ for each fixed $t\geq 0$ and decreases to zero as $t\to\infty$ for each fixed $s\geq 0$.

II. Preliminaries

Consider the following impulsive switched time-delay system:

$$\begin{cases} \dot{x}(t)=f_{\sigma(t)}(t,x,x_t), & t\notin\mathcal{I} \quad (1a)\\ \Delta x(t)=h_{\delta(t)}(x(t)), & t\in\mathcal{I} \quad (1b)\\ x(t)=\eta(t), & t\in[t_0-\tau,t_0] \quad (1c)\end{cases}$$

where $x(t)\in\mathbb{R}^{n_x}$ is the system state. Denoted $x_t:=x(t-\tau(t))$, where the time delay $\tau(t):\mathbb{R}^+_{t_0}\to[0,\tau]$ is piecewise continuous and bounded with a constant $\tau>0$. $\Delta x(t):=x(t^+)-x(t)$ with $x(t^+):=\lim_{s\to t^+}x(s)$. $\mathcal{I}:=\{\xi_1,\xi_2,\ldots\}$ and $\mathcal{S}:=\{s_1,s_2,\ldots\}$ are given impulsive time sequence and switching time sequence, respectively. The function $\sigma:\mathbb{R}^+_{t_0}\to\mathfrak{L}=:\{1,\ldots,L\}$ is the switching signal, which is piecewise and left-continuous. The function $\delta:\mathbb{R}^+_{t_0}\to\mathcal{Q}=:\{1,\ldots,Q\}$ is piecewise and left-continuous, and used to decide which impulsive function to be applied at the impulsive times. The initial function is $\eta\in\mathbb{PC}([-\tau,0],\mathbb{R}^{n_x})$ with finite $\|\eta\|^2_\tau$. For all $l\in\mathfrak{L}$ and $q\in\mathcal{Q}$, $f_l:\mathbb{R}^+_{t_0}\times\mathbb{R}^{n_x}\times\mathbb{R}^{n_x}\to\mathbb{R}^{n_x}$, and $h_q:\mathbb{R}^{n_x}\to\mathbb{R}^{n_x}$ are assumed to be locally Lipschitz. Assume that $f_l(t,0,0)\equiv 0$ and $h_q(0)\equiv 0$ for all $t\in\mathbb{R}^+_{t_0}$. That is, $x(t)\equiv 0$ is a trivial solution of the system (1). Also, assume that the system (1) has a unique solution for all the time, see [3] and [9].

Definition 1 (see [26]): Given an impulsive time sequence $\mathcal{I}$ and a switching time sequence $\mathcal{S}$, the system (1) is *GAS*, if there exists $\beta\in\mathcal{KL}$ such that

$$|x(t)|\leq\beta(\|\eta\|_\tau,t-t_0)\quad\forall t\in\mathbb{R}^+_{t_0}. \tag{2}$$

In this paper, our goal is to study GAS of the system (1) with time-dependent or state-dependent impulses. To this end, multiple Lyapunov functions and dwell-time condition are applied. In the following, the infinitesimal operator of Lyapunov functions is defined, and then a new version of average dwell-time (ADT) is proposed for the discontinuities of the system (1).

Definition 2 (see [3]): Given any continuous function $V_l:\mathbb{R}^+_{t_0}\times\mathbb{R}^{n_x}\to\mathbb{R}_{\geq 0}$, $l\in\mathfrak{L}$, the differential operator $\mathscr{L}$ associated with the continuous dynamics in (1), is defined as

$$\mathcal{L}V_l(t,x_t):=\frac{\partial V_l(t,x)}{\partial t}+\frac{\partial V_l(t,x)}{\partial x}f_l(t,x,x_t).$$

Since the discontinuities (e.g., switches and impulses) may result in instability of the system (1), we need to introduce the ADT constrain the frequencies of the discontinuities. The following definition extends the one in [27] for switching times and the one in [8] for impulsive times.

Definition 3: For a piecewise signal $\chi(t)$ and any $t_2>t_1>t_0$, let $N_\chi(t_2,t_1)$ be the number of discontinuities of $\chi(t)$ over the interval $[t_1,t_2)$. If there exist $N_0\geq 1,\tau_a>0$ such that

$$\frac{t_2-t_1}{\tau_a}-N_0\leq N_\chi(t_2,t_1)\leq\frac{t_2-t_1}{\tau_a}+N_0 \tag{3}$$

then N_0 and τ_a are called the *chatter bound* and the ADT, respectively.

As observed from the system (1), there are two types of discrete-time signals: impulses and switches, which affect system stability. For such two types of discrete-time signals, we introduce their chatter bounds and ADTs. For the switching time sequence $\mathcal{S}$, denoted by $N_1(T_2,T_1)$ the number of switches in $[T_1,T_2)$, N_{01} the chatter bound, and τ_{a1} the ADT. For the impulsive time sequence $\mathcal{I}$, denoted by $N_2(T_2,T_1)$ the number of impulses in $[T_1,T_2)$, N_{02} the chatter bound, and τ_{a2} the ADT. The whole discrete-time sequence combining both $\mathcal{I}$ and $\mathcal{S}$ is denoted as $\mathcal{T}:=\{t_1,t_2,\ldots\}$ and the discrete-time number in $[T_1,T_2)$ is denoted by $N(T_2,T_1)$. The overlapping discrete-time sequence between $\mathcal{I}$ and $\mathcal{S}$ is denoted as $\mathcal{O}:=\{o_1,o_2,\ldots\}$. For the sequence $\mathcal{O}$, τ_{ao} denotes the ADT, and $\bar{N}(T_2,T_1)$ denotes the number of the discrete times at which both impulses and switches occur synchronously in $[T_1,T_2)$. Note that the set $\mathcal{O}=\emptyset$ if $\mathcal{I}\cap\mathcal{S}=\emptyset$. In this case, $\bar{N}(T_2,T_1)\equiv 0$ and $\tau_{ao}=+\infty$.

III. Time-Dependent Impulse Case

In this section, Lyapunov-based stability conditions are established for the system (1) with time-dependent impulses. Both the stable continuous dynamics case and the stable discrete dynamics case are addressed.

A. Stable Continuous Dynamics Case

Theorem 1: Consider the system (1) with time-dependent impulses. If there exist continuous Lyapunov functions $V_l:\mathbb{R}^+_0\times\mathbb{R}^{n_x}\to\mathbb{R}^+_0$, $l\in\mathfrak{L}$, $\alpha_1,\alpha_2\in\mathcal{K}_\infty$, and constants $\lambda_1>\lambda_2>0$ and $\mu_1,\mu_2\geq 1$, such that for all $l\in\mathfrak{L}$

A.1) for all $t\in\mathbb{R}^+_{t_0}$, $\alpha_1(|x(t)|)\leq V_l(t,x(t))\leq\alpha_2(|x(t)|)$;

A.2) for all $t\in\mathbb{R}^+_{t_0}\backslash\mathcal{T}$, $\mathcal{L}V_l(t,x_t)\leq-\lambda_1V_l(t,x(t))+\lambda_2\sup_{s\in[-\tau,0]}V_l(t+s,x(t+s))$;

A.3) for all $t\in\mathcal{S}$, $V_{\sigma(t^+)}(t^+,x(t^+))\leq\mu_1V_{\sigma(t)}(t,x(t))$;

A.4) for all $t\in\mathcal{I}\backslash\mathcal{S}$, $V_{\sigma(t^+)}(t^+,x(t)+h_{\delta(t)}(x(t)))\leq\mu_2V_{\sigma(t)}(t,x(t))$;

A.5) the ADTs τ_{a1} and τ_{a2} satisfy $\tau^{-1}_{a1}\ln\mu_1+\tau^{-1}_{a2}\ln\mu_2<\lambda_0$, where $\lambda_0\in(0,\bar{\lambda})$ and $\bar{\lambda}>0$ is the solution to the equation $\lambda-\lambda_1+\lambda_2e^{\lambda\tau}=0$,

then the system (1) with time-dependent impulses is GAS.

Proof: The proof is divided as following three steps.

Step 1: Define $\Gamma(\lambda):=\lambda-\lambda_1+\lambda_2e^{\lambda\tau}$. Observe that $\Gamma(0)=-\lambda_1+\lambda_2<0$ and that $\Gamma(\lambda)\to\infty$ as $\lambda\to\infty$. In addition, $\Gamma'(\lambda):=1+\lambda_2\tau e^{\lambda\tau}\geq 0$. Thus, there exists a unique $\bar{\lambda}>0$ such that $\Gamma(\bar{\lambda})=0$, and $\Gamma(\lambda_0)<0$ for all $\lambda_0\in(0,\bar{\lambda})$.

Step 2: In this step, define $W_{\sigma(t)}(t):=e^{\lambda_0(t-t_0)}V_{\sigma(t)}(t,x(t))$ for $t\in\mathbb{R}^+_{t_0}$, where λ_0 is from (A.5). It is obvious that $W_{\sigma(t)}(t)\leq V_{\sigma(t)}(t,x(t))\leq\alpha_2(\|\eta\|_\tau)$ for $t=t_0$. Next, we prove that

$$W_{\sigma(t)}(t)\leq\mu_1^{N_1(t,t_0)}\mu_2^{N_2(t,t_0)}\alpha_2(\|\eta\|_\tau)\quad\forall t\in\mathbb{R}^+_{t_0}. \tag{4}$$

If (4) does not hold for all $t\in\mathbb{R}^+_{t_0}$, then there are two scenarios such that (4) fails. The first scenario is that (4) fails due to discrete-time events, and the second scenario is that (4) fails in the continuous-time interval. For the first scenario, there are two cases: (4) fails at the switching time and (4) fails at the impulsive time. At the switching time

$s_i \in \mathcal{S}, i \in \mathbb{N}^+$, we have from (A.3) that

$$W_{\sigma(s_i^+)}(s_i^+) \le \mu_1 W_{\sigma(s_i)}(s_i, x(s_i)) \le \mu_1^{N_1(s_i^+, t_0)} \mu_2^{N_2(s_i, t_0)} \alpha_2(\|\eta\|_\tau). \quad (5)$$

At the impulsive time $\xi_i \in \mathcal{I}\backslash\mathcal{S}$, we get from (A.4) and (4) that

$$W_{\sigma(\xi_i^+)}(\xi_i^+) \le \mu_2 W_{\sigma(\xi_i)}(\xi_i, x(\xi_i)) \le \mu_1^{N_1(\xi_i, t_0)} \mu_2^{N_2(\xi_i^+, t_0)} \alpha_2(\|\eta\|_\tau). \quad (6)$$

Observe that $N_\jmath(t^-, t_0) \le N_\jmath(t, t_0) \le N_\jmath(t^+, t_0)$ for all $t > t_0$ and $\jmath = 1, 2$. In sequel, for all $t \in \mathcal{I} \cup \mathcal{S}$, we have that

$$W_{\sigma(t^+)}(t^+) \le \mu_1^{N_1(t^+, t_0)} \mu_2^{N_2(t^+, t_0)} \alpha_2(\|\eta\|_\tau). \quad (7)$$

That is, the inequality (4) holds for all $t \in \mathcal{S} \cup \mathcal{I}$.

On the other hand, consider the second scenario that (4) fails in certain continuous-time interval. Define $t^* := \inf\{t \in \mathbb{R}_{t_0}^+ \backslash \mathcal{T} : W_{\sigma(t)}(t) = \mu_1^{N_1(t,t_0)} \mu_2^{N_2(t,t_0)} \alpha_2(\|\eta\|_\tau)\}$. Therefore, it follows from the definition of t^* that

$$W_{\sigma(t)}(t) \le \mu_1^{N_1(t,t_0)} \mu_2^{N_2(t,t_0)} \alpha_2(\|\eta\|_\tau) \quad \forall t \le t^* \quad (8)$$

$$W_{\sigma(t^*)}(t^*) = \mu_1^{N_1(t^*,t_0)} \mu_2^{N_2(t^*,t_0)} \alpha_2(\|\eta\|_\tau) \quad (9)$$

$$W_{\sigma(t)}(t) > \mu_1^{N_1(t,t_0)} \mu_2^{N_2(t,t_0)} \alpha_2(\|\eta\|_\tau) \quad (10)$$

where $t \in (t^*, t^* + \Delta t)$ in (10) and $\Delta t > 0$ is arbitrarily small. It follows from the upper Dini derivative that

$$\begin{aligned} D^+ W_{\sigma(t^*)}(t^*) &= \limsup_{\Delta t \to 0^+} \frac{W_{\sigma(t^*)}(t^* + \Delta t) - W_{\sigma(t^*)}(t^*)}{\Delta t} \\ &> \limsup_{\Delta t \to 0^+} \\ &\quad \times \frac{\mu_1^{N_1(t^*+\Delta t, t_0)} \mu_2^{N_2(t^*+\Delta t, t_0)} - \mu_1^{N_1(t^*,t_0)} \mu_2^{N_2(t^*,t_0)}}{\Delta t} \\ &\quad \times \alpha_2(\|\eta\|_\tau) = 0 \end{aligned} \quad (11)$$

where Δt is so small that no impulse or switch exists in $(t^*, t^* + \Delta t)$.

Since (4) holds for all $t < t^*$, we have that for all $l \in \mathfrak{L}$

$$\begin{aligned} V_l(t^* - \tau(t^*), x_{t^*}) &= e^{-\lambda_0(t^* - \tau(t^*) - t_0)} W_l(t^* - \tau(t^*)) \\ &\le e^{-\lambda_0(t^* - \tau(t^*) - t_0)} W_l(t^*) \\ &\le e^{\lambda_0 \tau} V_l(t^*, x(t^*)). \end{aligned} \quad (12)$$

It follows from (A.2) and (12) that

$$\begin{aligned} D^+ W_{\sigma(t^*)}(t^*) &\le e^{\lambda_0(t^* - t_0)} \Big(\lambda_0 V_{\sigma(t^*)}(t^*, x(t^*)) \\ &\quad - \lambda_1 V_{\sigma(t^*)}(t^*, x(t^*)) \\ &\quad + \lambda_2 \sup_{s \in [-\tau, 0]} V_{\sigma(t^*)}(t^* + s, x(t^* + s)) \Big) \\ &\le e^{\lambda_0(t^* - t_0)} (\lambda_0 V_{\sigma(t^*)}(t^*, x(t^*)) - \lambda_1 V_{\sigma(t^*)}(t^*, x(t^*)) \\ &\quad + \lambda_2 e^{\lambda_0 \tau} V_{\sigma(t^*)}(t^*, x(t^*))) < 0 \end{aligned} \quad (13)$$

which contradicts with (11). Hence, (4) holds for all $t \in \mathbb{R}_{t_0}^+ \backslash (\mathcal{S} \cup \mathcal{I})$.

Step 3: According to the above-mentioned analysis, (4) holds for all $t \in \mathbb{R}_{t_0}^+$. We yield from (4) that for all $t \in \mathbb{R}_{t_0}^+$

$$V_{\sigma(t)}(t, x(t)) \le \mu_1^{N_1(t,t_0)} \mu_2^{N_2(t,t_0)} e^{-\lambda_0(t - t_0)} \alpha_2(\|\eta\|_\tau). \quad (14)$$

Furthermore, from Definition 3, we obtain that

$$\begin{aligned} &\mu_1^{N_1(t,t_0)} \mu_2^{N_2(t,t_0)} e^{-\lambda_0(t-t_0)} \\ &\le e^{N_{10} \ln \mu_1 + N_{20} \ln \mu_2} e^{(\tau_{a1}^{-1} \ln \mu_1 + \tau_{a2}^{-1} \ln \mu_2 - \lambda_0)(t - t_0)}. \end{aligned} \quad (15)$$

It follows from (A.5) that $\tau_{a1}^{-1} \ln \mu_1 + \tau_{a2}^{-1} \ln \mu_2 - \lambda_0 < 0$. In addition, $N_{10} \ln \mu_1 + N_{20} \ln \mu_2$ is a positive constant. Define $\omega := \tau_{a1}^{-1} \ln \mu_1 + \tau_{a2}^{-1} \ln \mu_2 - \lambda_0 < 0$. Combining (A.1), (14),and (15) yields that

$$\begin{aligned} |x(t)| &\le \alpha_1^{-1}(\mu_1^{N_{10}} \mu_2^{N_{20}} e^{\omega(t - t_0)} \alpha_2(\|\eta\|_\tau) \\ &=: \beta(\|\eta\|_\tau, t - t_0) \quad \forall t \in \mathbb{R}_{t_0}^+ \end{aligned}$$

where $\beta(v, s) := \alpha_1^{-1}(2\mu_1^{N_{10}} \mu_2^{N_{20}} e^{\omega s} \alpha_2(v))$ and α_1^{-1} is the reverse of α_1. As a result, the system (1) is GAS. ■

Remark 1: Since there are two types of discrete-time events in the system (1), ADTs for impulses and switches are coupled in (A.5). In the sequel, there is a tradeoff between τ_{a1} and τ_{a2}. (A.5) also provides a design method to balance impulses and switches to guarantee system stability. In addition, (A.5) is different from these in the existing works. For instance, only impulses or switches were considered in [8], [9], [16], and [28], and both impulses and switches were assumed to be simultaneous in [10] and [11]. These previous works are included as special cases of this paper. ■

B. Stable Discrete Dynamics Case

In this section, the stable impulsive dynamics case is studied and stability conditions are established.

Theorem 2: Consider the system (1) with time-dependent impulses. If there exist continuous Lyapunov functions $V_l : \mathbb{R}_{t_0}^+ \times \mathbb{R}^{n_x} \to \mathbb{R}_0^+, l \in \mathfrak{L}, \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\lambda_1, \lambda_2 > 0, \mu_1 \ge 1, \mu_2 \in (0, 1)$, such that for all $l \in \mathfrak{L}$, (A.1) holds, and

B.1) for all $t \in \mathbb{R}_{t_0}^+ \backslash \mathcal{T}$, $\mathcal{L}V_l(t, x_t) \le \lambda_1 V_l(t, x) + \lambda_2 \sup_{s \in [-\tau, 0]} V_l(t + s, x(t + s))$;

B.2) for all $t \in \mathcal{S} \backslash \mathcal{I}$, $V_{\sigma(t^+)}(t^+, x(t^+)) \le \mu_1 V_{\sigma(t)}(t, x(t))$;

B.3) for all $t \in \mathcal{I}$, $V_{\sigma(t^+)}(t^+, x(t) + h_{\delta(t)}(x(t))) \le \mu_2 V_{\sigma(t)}(t, x(t))$;

B.4) the ADTs τ_{a1} and τ_{a2} satisfy $-\tau_{a1}^{-1} \ln \mu_1 - \tau_{a2}^{-1} \ln \mu_2 > \lambda_1 + \lambda_2 \mu_1^{N_{10}} \mu_2^{-N_{20}}$,

then the system (1) with time-dependent impulses is GAS.

Proof: The proof is based on the comparison principle and the techniques applied in [8]. First, using the comparison principle and along the similar techniques used in [8], the bounds of the Lyapunov functions are established. Second, based on the bounds of Lyapunov functions and (B.4), we ensure the convergence of the system state.

Step 1: Define $\mathfrak{U}_{\sigma(t)}(t) := V_{\sigma(t)}(t, x(t))$. Consider the continuous-time interval first. It follows from (B.1) that for all $t \in (t_k, t_{k+1}]$

$$\begin{aligned} \mathfrak{U}_{\sigma(t)}(t) &\le e^{\lambda_1(t - t_k)} \mathfrak{U}_{\sigma(t_k^+)}(t_k^+) \\ &\quad + \int_{t_k}^t e^{\lambda_1(s - t_k)} \lambda_2 \sup_{s \in [-\tau, 0]} \mathfrak{U}_{\sigma(t)}(v - s) dv. \end{aligned} \quad (16)$$

If t_k is an impulsive time instant, then we obtain from (B.3) that for $t \in (t_k, t_{k+1}]$

$$\mathfrak{U}_{\sigma(t)}(t) \le \mu_1 e^{\lambda_1(t-t_k)} \mathfrak{U}_{\sigma(t_k)}(t_k) + \int_{t_k}^{t} e^{\lambda_1(s-t_k)} \lambda_2 \sup_{s\in[-\tau,0]} \mathfrak{U}_{\sigma(t)}(v-s)dv \tag{17}$$

if t_k is a switching time instant, then it follows from (B.2) that for all $t \in (t_k, t_{k+1}]$

$$\mathfrak{U}_{\sigma(t)}(t) \le \mu_2 e^{\lambda_1(t-t_k)} \mathfrak{U}_{\sigma(t_k)}(t_k) + \int_{t_k}^{t} e^{\lambda_1(s-t_k)} \lambda_2 \sup_{s\in[-\tau,0]} \mathfrak{U}_{\sigma(t)}(v-s)dv. \tag{18}$$

Combining (17) and (18) and using mathematical induction yield that for all $t \in \mathbb{R}^+_{t_0}$

$$\mathfrak{U}_{\sigma(t)}(t) \le \mu_1^{N_1(t,t_0)} \mu_2^{N_2(t,t_0)} e^{\lambda_1(t-t_0)} \mathfrak{U}_{\sigma(t_0)}(t_0) + \int_{t_k}^{t} \mu_1^{N_1(t,s)} \times \mu_2^{N_2(t,s)} e^{\lambda_1(t-s)} \lambda_2 \sup_{s\in[-\tau,0]} \mathfrak{U}_{\sigma(t)}(v-s)dv. \tag{19}$$

Step 2: According to Definition 3, it follows from $\mu_1 \ge 1$ and $\mu_2 \in (0,1)$ that

$$\begin{aligned}\mu_1^{N_1(t,t_0)} \mu_2^{N_2(t,t_0)} e^{\lambda_1(t-t_0)} &\le \mu_1^{(t-t_0)/\tau_{a1}+N_{01}} \mu_2^{(t-t_0)/\tau_{a2}-N_{02}} e^{\lambda_1(t-t_0)} \\ &\le \zeta e^{-\lambda_3(t-t_0)}\end{aligned} \tag{20}$$

where $\zeta := \mu_1^{N_{01}} \mu_2^{-N_{02}}$ and $\lambda_3 := -\lambda_1 - \tau_{a1}^{-1} \ln \mu_1 - \tau_{a2}^{-1} \ln \mu_2$. Observe from (B.4) that $\zeta > 1$ and $\lambda_3 > 0$. As a result, we obtain that for all $t \in \mathbb{R}^+_{t_0}$

$$\mathfrak{U}_{\sigma(t)}(t) \le e^{-\lambda_3(t-t_0)} \zeta \alpha_2(\|\eta\|_\tau) + \int_{t_k}^{t} \zeta e^{-\lambda_3(t-s)} \lambda_2 \sup_{s\in[-\tau,0]} \mathfrak{U}_{\sigma(t)}(v-s)dv. \tag{21}$$

Define $\bar{\Gamma}(\lambda) := \lambda - \lambda_3 + \zeta\lambda_2 e^{\lambda\tau}$. Observe that $\bar{\Gamma}(0) = -\lambda_3 + \lambda_2 < 0$ and that $\bar{\Gamma}(\lambda) \to \infty$ as $\lambda \to \infty$. In addition, $\bar{\Gamma}'(\lambda) := 1 + \zeta\lambda_2\tau e^{\lambda\tau} \ge 0$. Thus, there exists a unique $\bar{\lambda} > 0$ such that $\bar{\Gamma}(\bar{\lambda}) = 0$, and $\bar{\Gamma}(\lambda_0) < 0$ for all $\lambda_0 \in (0, \bar{\lambda})$.

Based on (21) and the similar strategy of the proof in [8, Th. 2, pp. 201–202], it obtains that for all $t \in \mathbb{R}^+_{t_0}$

$$V_{\sigma(t)}(t, x(t)) \le \mathfrak{U}_{\sigma(t)}(t) \le \zeta e^{-\bar{\lambda}(t-t_0)} \alpha_2(\|\xi\|_\tau). \tag{22}$$

The remaining is along the same fashion as in the proof of Theorem 1, and thus, the system (1) is GAS. ■

IV. State-Dependent Impulse Case

In this section, we study another class of impulsive switched time-delay systems, that is, the system (1) with state-dependent impulses. Since the impulses are state-dependent, the impulsive event may occur successively at certain time instant, which results in the beating phenomenon. In addition, the solution to the system (1) may do not depend on the initial state uniformly continuously, and different initial states lead to different system state and impulsive times. Hence, we need to determine the existence of the solution to the system (1) with state-dependent impulses, and then study the system stability.

To this end, assume that the impulsive times are $s_i + \tau_i(x)$, $i \in \mathbb{N}^+$, where $\tau_i(x) : \mathbb{R}^{n_x} \to \mathbb{R}^+_0$ is a nonnegative function of the system state. In the sequel, define the impulsive surface $\Pi_i := \{(t, x(t)) \in \mathbb{R}^+_{t_0} \times \mathbb{R}^{n_x} : t = s_i + \tau_i(x(t)), t \in \mathbb{R}^+_{t_0}, i \in \mathbb{N}^+\}$. That is, if $t = s_i + \tau_i(x(t))$, then an impulse occurs. If the system state meets an impulsive surface many times, then such a phenomenon is called beating, see [24], [25], and [29]. The beating phenomenon implies that two different states are merged into one (see [23, Sec. 1.3]), which results in the difficulties in the stability analysis. To avoid the beating phenomenon and to guarantee the existence of the solution to the system (1) with state-dependent impulses, the following assumption is required in this section.

Assumption 1: For the system (1) with state-dependent impulses, the following conditions are assumed to be satisfied.

1) For all $x \in \mathbb{R}^{n_x}$ and all $i \in \mathbb{N}$, $\tau_i(x)$ is continuous and bounded in $[0, \rho]$.
2) For all $i \in \mathbb{N}$, $\tau_i(x)$ satisfies the Lipschitz condition, that is, there exists $L_i > 0$ such that $|\tau_i(x) - \tau_i(\bar{x})| \le L_i|x - \bar{x}|$ for all $x, \bar{x} \in \mathbb{R}^{n_x}$.
3) For any $i \in \mathbb{N}^+$, one of the following two conditions holds:

$$\text{(a)} \begin{cases} \dot{\tau}_i(x) > 1, & x \in \mathbb{R}^{n_x} \\ \tau_i(x(\xi_i^+)) \ge \tau_i(x(\xi_i)) \end{cases} \quad \text{(b)} \begin{cases} \dot{\tau}_i(x) < 1, & x \in \mathbb{R}^{n_x} \\ \tau_i(x(\xi_i^+)) \le \tau_i(x(\xi_i)) \end{cases}$$

where $x(t)$ is the solution to the system (1) in $(s_i, s_i + \rho]$ and $\xi_i := s_i + \tau_i(x(\xi_i))$ is an impulsive time.
4) The switching intervals satisfy $s_{i+1} - s_i > \rho$ for all $i \in \mathbb{N}$.

Let us state Assumption 1 clearly. The items 1)–3) imply the smoothness, monotonicity, and boundedness of $\tau_i(x)$, see [23], [25]. Combining the item 2) and the Lipschitz property of f_l, h_q, $l \in \mathfrak{L}, q \in \mathcal{Q}$, and using [24, Th. 5.3.1], we yield that the system (1) has a unique solution from any initial condition as in (1c). The item 4) is for switching intervals, and indicates that switching intervals are larger than ρ. The items 1) and 4) show that the impulses only occur in the switching intervals. According to Assumption 1, we first show that the beating phenomenon is ruled out for the system (1) with state-dependent impulses. Next, we develop a B-equivalent system, which can be treated as a comparison-like system for the system (1) and, then, establish stability conditions for the system (1) via the developed B-equivalent system.

A. B-Equivalent Transformation

Using the B-equivalent method in [24], we transform the system (1) with state-dependent impulses into a system with time-dependent impulses. Under Assumption 1, we first study the intersection between the solution with the impulsive surfaces to avoid the beating phenomenon. The following result shows that the solution of the system (1) intersects with each impulsive surface at most once.

Theorem 3: Under Assumption 1, the solution $x(t)$ intersects with every impulsive surface Π_i exactly once, where $i \in \mathbb{N}^+$.

Proof: Assume the solution $x(t)$ does not intersect with every impulsive surface Π_i, $i \in \mathbb{N}^+$. In the sequel, there exists $j \in \mathbb{N}^+$ such that $x(t)$ does not intersect with the surface Π_j. Define $d(t) := t - s_j - \tau_j(x(t))$. Since $s_j + \tau_j(x(t))$ is in the switching interval (s_j, s_{j+1}), there exist $a < s_j < s_j + \rho < b$ such that $d(a) < 0$ and $d(b) > 0$. Due to the continuity of $d(t)$, there exists $t \in (a, b)$ such that $d(t) = 0$, that is, $t = s_j + \tau_j(x(t))$, which is a contradiction. Therefore, $x(t)$ intersects with every impulsive surface Π_i, $i \in \mathbb{N}^+$.

Suppose that the solution $x(t)$ intersects with the surface Π_j at least twice. Thus, we can find at least two time instants $I_1 := s_j + \tau_j(x(I_1))$

and $I_2 := s_j + \tau_j(x(I_2))$ with $I_1 \neq I_2$ such that the solution $x(t)$ intersects with the impulsive surface Π_j. For the time instants I_1, I_2, it follows from (a) of the item 3) in Assumption 1 that

$$\begin{aligned} I_2 - I_1 &= \tau_j(x(I_2)) - \tau_j(x(I_1)) \geq \tau_j(x(I_2)) - \tau_j(x(I_1^+)) \\ &= \dot{\tau}_j(x(t))(I_2 - I_1) > I_2 - I_1 \end{aligned} \tag{23}$$

where $t \in (I_1, I_2]$. Therefore, (23) is a contradiction, which implies that the solution $x(t)$ intersects with the impulsive surface Π_j less than twice. Similarly, if (b) of the item 3) in Assumption 1 holds, then we can, along the same line, obtain that $I_2 - I_1 < I_2 - I_1$, which is also a contradiction. In the sequel, the solution $x(t)$ intersects with the impulsive surface Π_j less than twice. ∎

In the following, the system (1) with state-dependent impulses is transformed as an equivalent impulsive switched time-delay system with time-dependent impulses. The applied transformation technique is the B-equivalent method, see [24, Ch. 5]. To this end, some notations are introduced. Let $x^0(t) := x(t; s_i, x^0(s_i))$ denote a solution to the system (1) in $[s_i, s_{i+1}]$. The intersection time between the solution $x^0(t)$ and the impulsive surface Π_i is denoted by $\xi_i := s_i + \tau_i(x)$. In addition, $x^1(t)$ denotes the solution to the system (1) without impulses in $[s_i, s_{i+1}]$, and $x^1(t)$ satisfies $x^1(\xi_i) = x^0(\xi_i^+) = x^0(\xi_i) + h_{\delta(\xi_i)}(x^0(\xi_i))$. That is, $x^1(t)$ is the solution to the switched system (1a) in $[s_i, s_{i+1}]$, such that the initial (1c) holds and $x^1(\xi_i) = x^0(\xi_i^+) = x^0(\xi_i) + h_{\delta(\xi_i)}(x^0(\xi_i))$. In the sequel, define the mapping $U_{\sigma(s_i)} : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ as

$$\begin{aligned} &U_{\sigma(s_i)}(x^0(s_i)) = x^1(s_i) - x^0(s_i) \\ &= x^0(\xi_i) + h_{\delta(\xi_i)}(x^0(\xi_i)) + \int_{\xi_i}^{s_i} f_{\sigma(t)}(t, x^1, x_t^1)dt - x^0(s_i) \\ &= x^0(s_i) + \int_{s_i}^{\xi_i} f_{\sigma(t)}(t, x^0, x_t^0)dt + h_{\delta(\xi_i)}(x^0(\xi_i)) \\ &\quad + \int_{\xi_i}^{s_i} f_{\sigma(t)}(t, x^1, x_t^1)dt - x^0(s_i) \\ &= \int_{s_i}^{\xi_i} \left[f_{\sigma(t)}(t, x^0, x_t^0) - f_{\sigma(t)}(t, x^1, x_t^1)\right] dt \\ &\quad + h_{\delta(\xi_i)}(x^0(\xi_i)). \end{aligned} \tag{24}$$

Therefore, the mapping $U_{\sigma(s_i)}(x^0(s_i))$ is the difference between the solutions $x^0(t)$ and $x^1(t)$ at the switching times.

Using the mapping $U_{\sigma(s_i)}(x^0(s_i))$ in (24), we construct the following transformed impulsive switched time-delay system with time-dependent impulses:

$$\begin{cases} \dot{y}(t) = f_{\sigma(t)}(t, y, y_t), & t \notin \mathcal{S} \\ \Delta y(t) = U_{\sigma(t)}(y(t)), & t \in \mathcal{S} \\ y(t) = \eta(t), & t \in [t_0 - \tau, t_0] \end{cases} \tag{25}$$

where $y(t) \in \mathbb{R}^{n_x}$ is the system state. Observe the developed system (25), we find that the impulsive times and the switching times are synchronous. In addition, $x^0(t) := x(t; s_i, x^0(s_i))$ can be extended as the solution of the system (1) in $\mathbb{R}_{t_0}^+$, whereas $x^1(t) := x(t; \xi_i, x^0(\xi_i^+))$ can be extended as the solution of the system (25) in $\mathbb{R}_{t_0}^+$. In the following, the B-equivalence between the systems (1) and (25) is studied.

Definition 4 (see [24]): The systems (1) and (25) are *B-equivalent* in the set $G \subset \mathbb{R}^{n_x}$ if there exists a set $G_1 \subset G$ such that for each solution $x(t)$ of the system (1) defined on an interval $\mathfrak{I}$, with discontinuities ξ_i and $x(t) \in G_1, t \in \mathfrak{I}$, then there exists a solution $y : \mathfrak{I} \to G$ of the system (25), satisfying the following conditions:

$$x(t) = y(t), \quad t \notin \widehat{(\xi_i, s_i]}, \quad \xi_i \neq s_i \tag{26a}$$

$$x(s_i) = y(\xi_i), \quad \xi_i = s_i \tag{26b}$$

$$x(s_i) = y(s_i),\ x(\xi_i^+) = y(\xi_i), \quad \xi_i > s_i \tag{26c}$$

$$x(s_i) = y(s_i^+),\ x(\xi_i) = y(\xi_i), \quad \xi_i < s_i \tag{26d}$$

where the oriented interval is defined as $\widehat{(a, b]} := (a, b]$ if $a \leq b$; otherwise, $\widehat{(a, b]} := (b, a]$. Conversely, if there exists a set $G_1 \subset G$ such that for each solution $y(t)$ of the system (25) defined on an interval $\mathfrak{I}$, then there exist a solution $x : \mathfrak{I} \to G$ of the system (1), satisfying (26a)–(26d).

According to Definition 4, the following result shows that the B-equivalence between the systems (1) and (25) under Assumption 1.

Theorem 4: Under Assumption 1, the original system (1) and the transformed system (28) are B-equivalent in $\mathbb{R}^{n_x}$.

Proof: Since $\xi_i \geq s_i$, it follows that $\widehat{(\xi_i, s_i]} = (s_i, \xi_i]$. Thus, we only need to verify (26a)–(26c).

First, from the construction of the system (25), we have that $x^0(t) = x^1(t)$ for all $t \in (\xi_i, s_{i+1}]$. That is, $x^0(t) = x^1(t)$ for all $t \notin (t_i, \xi_i] = \widehat{(\xi_i, s_i]}$. We also obtain that $x^0(t) \equiv x^1(t)$ for all $t = s_i$. Therefore, (26a) holds. Second, for the case that $\xi_i = s_i$, we have $\tau_i(x) = 0$. Thus, the switch and impulse are synchronous, and $x(t_i) = y(\xi_i)$, which implies that (26b) holds. Third, since $\xi_i \geq s_i$, it follows from the relation between $x^0(t)$ and $x^1(t)$ that $x^1(\xi_i) = x^0(\xi_i^+) = x^0(\xi_i) + h_{\sigma(\xi_i)}(x^0(\xi_i))$. That is, $y(\xi_i) = x(\xi_i^+)$, which means that (26c) holds.

In contrast, given the solution to the system (25), we can, along the similar line, find a solution to the system (1) such that the solution satisfies (26a)–(26c). Therefore, the system (1) and the transformed system (25) are B-equivalent. ∎

Remark 2: Using the B-equivalent method, state-dependent impulses are transformed as time-dependent impulses. However, if there are more than two state-dependent impulses in a switching interval, then the B-equivalent method may not be applied. Even if the B-equivalent method is applied, some information on impulses may be lost in the construction of the B-equivalent system. Therefore, other approaches are needed, which deserves further study. ∎

B. Stability Analysis

In this section, we investigate the stability of the system (1) with state-dependent impulses. We first establish the relationships between the system (1) and the transformed system (25) in view of the stability properties. According to the results in Section III, stability criteria are derived for the system (25), which in turn implies the stability of the system (1) with state-dependent impulses.

To begin with, we study the difference of the system (1) and the transformed system (25) in the interval $(s_i, \xi_i]$, $i \in \mathbb{N}^+$. For all $t \in (s_i, \xi_i]$, we have that

$$\begin{aligned} x^1(t) - x^0(t) &= x^0(s_i) + U_{\sigma(t_i)}(x^0(s_i)) \\ &\quad + \int_{s_i}^t f_{\sigma(v)}(v, x^1, x_v^1)dv - x^0(s_i) - \int_{s_i}^t f_{\sigma(v)}(v, x^0, x_v^0)dv \\ &= U_{\sigma(s_i)}(x^0(s_i)) + \int_{s_i}^t [f_{\sigma(v)}(v, x^1, x_v^1) - f_{\sigma(v)}(v, x^0, x_v^0)]dv. \end{aligned}$$

Following the Lipschitz property of the function f_l, $l \in \mathfrak{L}$, we have that for all $t \in (s_i, \xi_i]$, $i \in \mathbb{N}^+$

$$\begin{aligned}&|x^1(t) - x^0(t)|\\&\le |U_{\sigma(s_i)}(x^0(s_i))| + \int_{s_i}^t |f_{\sigma(v)}(v, x^1, x^1_s) - f_{\sigma(v)}(v, x^0, x^0_s)|dv\\&\le |U_{\sigma(s_i)}(x^0(s_i))| + L_{\sigma(s_{i+1})}\int_{s_i}^t |x^1_v - x^0_v|dv\end{aligned} \tag{27}$$

where L_l is the Lipschitz constant for the function f_l, $l \in \mathfrak{L}$. Using the Gronwall–Bellman inequality in [30, Corollary 2.5], it obtains that for all $t \in (s_i, \xi_i]$, $i \in \mathbb{N}^+$

$$|x^1(t) - x^0(t)| \le \max\left\{|U_{\sigma(s_i)}(x^0(s_i))|\right.\\ \left.\sup_{t\in[s_i-\tau, s_i]} |x^1(t) - x^0(t)|\right\} e^{\rho L_{\sigma(s_{i+1})}}. \tag{28}$$

Observe from (28) that the bound of $|x^1(t) - x^0(t)|$ is related to the delayed system state in $[s_i - \tau, s_i]$. In the following, sufficient conditions are established for the boundedness of $U_{\sigma(s_i)}(x^0(s_i))$ and $x^1(t) - x^0(t)$ for $t \in (s_i, \xi_i]$, $i \in \mathbb{N}^+$.

Theorem 5: Consider the system (1) with state-dependent impulses. Assume that there exist continuous Lyapunov functions $V_l : \mathbb{R}^+_0 \times \mathbb{R}^{n_x} \to \mathbb{R}^+_0$, $l \in \mathfrak{L}$, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and constants $\lambda, \mu \in \mathbb{R}$ such that for all $l \in \mathfrak{L}$, (A.1) holds, and

C.1) for all $t \in \mathbb{R}^+_{t_0}\backslash\mathcal{I}$, $\mathcal{L}V_l(t, x_t) \le \lambda V_l(t, x)$;

C.2) for all $t \in \mathcal{I}$, $V_{\sigma(t^+)}(t^+, x(t^+)) \le \mu V_{\sigma(t)}(t, x(t))$;

C.3) the function $v - \rho L_l \alpha_1^{-1}(e^{\lambda\rho}\alpha_2(v))$ is of class $\mathcal{K}_\infty$,

then there exist $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that for all $t \in (s_i, \xi_i]$

$$\begin{aligned}|x^0(s_i) + U_{\sigma(s_i)}(x^0(s_i))| &\le \gamma_1(|x^0(s_i)|)\\ |x^1(t) - x^0(t)| &\le \zeta_1(|x^0(s_i)|).\end{aligned}$$

Proof: According to (C.1), we have that

$$|V_l(t)| \le e^{\lambda(t-s_i)}|V_l(s_i^+)| \quad \forall t \in (s_i, \xi_i] \quad \forall l \in \mathfrak{L}$$

combining that with (A.1) yields that

$$|x^0(t)| \le \alpha_1^{-1}(e^{\lambda(t-s_i)}\alpha_2(|x^0(s_i^+)|)) \tag{29}$$

$$|x^1(t)| \le \alpha_1^{-1}(e^{\lambda(t-s_i)}\alpha_2(|x^1(s_i^+)|)). \tag{30}$$

Since $x^1(s_i) = x^0(s_i) = x^0(s_i^+)$ and $x^1(s_i^+) = x^0(s_i) + U_{\sigma(s_i)}(x^0(s_i))$, we have from the norm triangle inequality, (C.2), and the definition of x^1 at ξ_i that

$$\begin{aligned}&|x^0(s_i) + U_{\sigma(s_i)}(x^0(s_i))|\\&\le |x^0(s_i) + h_{\delta(s_{i+1})}(x^0(s_i))|\\&\quad + L_{\sigma(\xi_i)}\rho\alpha_1^{-1}(e^{\lambda\rho}\alpha_2(|x^0(t_i) + U_i(x^0(s_i))|))\\&\le \alpha_1^{-1}(\mu\alpha_2(|x^0(s_i)|))\\&\quad + \rho L_{\sigma(\xi_i)}\alpha_1^{-1}(e^{\lambda\rho}\alpha_2(|x^0(s_i) + U_i(x^0(s_i))|)).\end{aligned} \tag{31}$$

It follows from (C.3) that there is $\gamma_a \in \mathcal{K}_\infty$ such that $0 < \gamma_a(v) \le v - \rho L_l \alpha_1^{-1}(e^{\lambda\rho}\alpha_2(v))$. In the sequel, we have that for all $i \in \mathbb{N}^+$

$$|x^0(s_i) + U_{\sigma(s_i)}(x^0(s_i))| \le \gamma_1(|x^0(s_i)|) \tag{32}$$

where $\gamma_1(v) := \gamma_a^{-1}(\alpha_1^{-1}(\mu\alpha_2(v)))$. Obviously, both γ_1 and γ_2 are of class $\mathcal{K}_\infty$. It implies from (32) that for all $i \in \mathbb{N}^+$

$$|U_{\sigma(s_i)}(x^0(s_i))| \le |x^0(s_i)| + \gamma_1(|x^0(s_i)|). \tag{33}$$

In the following, consider the bound of $|x^1(t) - x^0(t)|$ in $(s_i, \xi_i]$. If $[s_i - \tau, s_i] \subset (\xi_{i-1}, s_i]$, then $x^1(t) = x^0(t)$ for all $t \in [s_i - \tau, s_i]$, and $\sup_{t\in[s_i-\tau,s_i]} |x^1(t) - x^0(t)| = 0$. In this case, we obtain from (28) and (33) that for all $(t_i, \xi_i]$

$$|x^1(t) - x^0(t)| \le \left[|x^0(s_i)| + \gamma_1(|x^0(s_i)|)\right] e^{\rho L_{\sigma(s_{i+1})}}. \tag{34}$$

If $[s_i - \tau, s_i] \cap (\xi_{i-1}, s_i] \ne \emptyset$ and $|U_{\sigma(s_i)}(x^0(s_i))| \ge \sup_{t\in[s_i-\tau,\xi_{i-1})} |x^1(t) - x^0(t)|$ for all $i \in \mathbb{N}^+$, then (34) still holds.

Next, we only need to study the case that $[s_i - \tau, s_i] \cap (\xi_{i-1}, s_i] \ne \emptyset$ and $|U_{\sigma(s_i)}(x^0(s_i))| < \sup_{t\in[s_i-\tau,\xi_{i-1})} |x^1(t) - x^0(t)|$ for all $i \in \mathbb{N}^+$. In this case, we have from (28) that for all $(s_i, \xi_i]$, $i \in \mathbb{N}^+$

$$|x^1(t) - x^0(t)| \le \sup_{t\in[s_i-\tau,\xi_{i-1})} |x^1(t) - x^0(t)|e^{\rho L_{\sigma(s_{i+1})}}.$$

For all $t \in [t_0 - \tau, s_1]$, $x^1(t) = x^0(t)$, which implies that for all $t \in (t_1, \xi_1]$, $|x^1(t) - x^0(t)| = 0$. Consequently, for all $t \in (t_2, \xi_2]$

$$|x^1(t) - x^0(t)| \le \sup_{t\in[s_2-\tau,\xi_1)} |x^1(t) - x^0(t)|e^{\rho L_{\sigma(\xi_2)}} = 0.$$

We get from iteration that $|x^1(t) - x^0(t)| \equiv 0$ for $t \in (t_i, \xi_i]$, $i \in \mathbb{N}^+$.

Combining the previous analysis yields that

$$|x^1(t) - x^0(t)| \le \zeta_1(|x^0(s_i)|) \quad \forall t \in (t_i, \xi_i] \tag{35}$$

where $\zeta_1(v) := (v + \gamma_1(v))e^{L\rho}$ and $L := \max_{l\in\mathfrak{L}}\{L_l\}$. ■

Remark 3: In Theorem 5, (C.2) is to bound the jumps due to impulses. As a result, (C.2) can be rewritten as: There exists $K_q > 0$ such that $|x + h_q(x)| \le K_q|x|$ holds for all $x \in \mathbb{R}^{n_x}$ and all $q \in \mathcal{Q}$, where the existence of K_q follows the Lipschitz property of h_q. ■

According to Theorem 5, the next theorem establishes the relation between the original system (1) and the transformed system (25) in terms of the system stability.

Theorem 6: Consider the system (1) with state-dependent impulses and the transformed system (25), and let Assumption 1 hold. If the conditions in Theorem 5 are satisfied, then GAS of the system (25) implies GAS of the system (1) with state-dependent impulses.

Proof: Since the transformed system (25) is GAS, then there exists $\beta \in \mathcal{KL}$ such that

$$|x^1(t)| \le \beta(\|\eta\|_\tau, t - t_0) \quad \forall t \in \mathbb{R}^+_{t_0}.$$

In the sequel, we have from Theorem 5 that for all $t \in \mathbb{R}^+_{t_0}$

$$\begin{aligned}|x^0(t)| &\le |x^1(t) - x^0(t)| + |x^1(t)|\\&\le \zeta_1(\beta(\|\eta\|_\tau, s_i - t_0)) + \beta(\|\eta\|_\tau, t - t_0)\\&=: \bar\beta(\|\eta\|_\tau, t - t_0).\end{aligned}$$

Hence, the system (1) is GAS. ■

In the following, according to the previous results in Section III, stability conditions are derived for the system (25).

Theorem 7: Consider the system (25). If there exist continuous Lyapunov functions $V_l : \mathbb{R}^+_0 \times \mathbb{R}^{n_x} \to \mathbb{R}^+_0, l \in \mathfrak{L}, \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\lambda_1 > \lambda_2 \ge 0, \mu \ge 1$, such that (A.1) holds, and

D.1) for all $t \in \mathbb{R}^+_{t_0}\backslash\mathcal{S}$ and all $l \in \mathfrak{L}$, $\mathcal{L}V_l(t, y_t) \le -\lambda_1 V_l(t, y(t)) + \lambda_2 \sup_{s\in[-\tau,0]} V_l(t + s, y(t + s))$;

D.2) for all $t \in \mathcal{S}$, $V_{\sigma(t^+)}(t^+, y(t^+)) \le \mu V_{\sigma(t)}(t, y(t))$;

D.3) the ADT satisfies $\tau_a > \lambda_0^{-1}\ln\mu$, where $\lambda_0 \in (0, \bar\lambda)$ and $\bar\lambda$ is the solution to the equation $\lambda - \lambda_1 + \lambda_2 e^{\lambda\tau} = 0$,

then the system (25) is GAS.

Theorem 8: Consider the system (25). If there exist continuous Lyapunov functions $V_l : \mathbb{R}_0^+ \times \mathbb{R}^{n_x} \to \mathbb{R}_0^+, l \in \mathfrak{L}, \alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\lambda_1, \lambda_2 > 0, \mu \in (0,1)$, such that (A.1) holds, and

E.1) for all $t \in \mathbb{R}_{t_0}^+ \backslash \mathcal{S}$ and all $l \in \mathfrak{L}$, $\mathcal{L}V_l(t, y_t) \leq \lambda_1 V_l(t,y) + \lambda_2 \sup_{s\in[-\tau,0]} V_l(t+s, y(t+s))$;

E.2) for all $t \in \mathcal{S}$, $V_{\sigma(t^+)}(t^+, y(t^+)) \leq \mu V_{\sigma(t)}(t, y(t))$;

E.3) the ADT satisfies $\tau_a < -\frac{\ln \mu}{\lambda_1 + \lambda_2 \mu^{-N_0}}$,

then the system (25) is GAS.

The proofs of Theorems 7 and 8 are similar to these of Theorems 1 and 2, and hence omitted here. Since impulses and switches in (25) are synchronous, (D.2) and (E.2) are for both impulses and switches. Thus, the tradeoff is not needed for the ADTs for impulses and switches in (25). Comparing the conditions in Theorems 6 and 7, we observe that (D.1) is similar to (C.1). In addition, (D.2) bounds the jumps at the impulsive switching times, thereby bounding the jumps at the impulsive times. Therefore, if all the conditions in Theorem 7 and (C.3) hold, then GAS of the system (1) with state-dependent impulses is guaranteed. Similar case can be obtained from the conditions in Theorem 8 and (C.3).

V. Illustrative Example

In this section, a numerical example is provided to illustrate the developed results in the previous sections. Consider impulsive switched time-delay neural system of the form

$$\begin{cases} \dot{x}(t) = -C_{\sigma(t)}x(t) + A_{\sigma(t)}f_{\sigma(t)}(x(t)) \\ \qquad +B_{\sigma(t)}g_{\sigma(t)}(x_t), \quad t \notin \mathcal{I} \\ \Delta x(t) = E_{\delta(t)}x(t), \quad t \in \mathcal{I} \end{cases} \tag{36}$$

where $x(t) \in \mathbb{R}^n$ is the neural system state, $f_{\sigma(t)}(x(t)) \in \mathbb{R}^n$ is the neuron activation function, and $g_{\sigma(t)}(x_t)$ is the function to describe the delay kernel. The switching function $\sigma(t) : \mathbb{R}_0^+ \to \mathfrak{L} = \{1, \ldots, L\}$ is piecewise continuous and the switching time sequence is denoted by $\mathcal{S}$. The impulsive time sequence is $\mathcal{I} := \{\xi_i : \xi_i = s_i + \tau_i(x(t)), i \in \mathbb{N}^+\}$. The matrices in (36) are of appropriate dimensions.

Assume there are two subsystems and $x(t) \in \mathbb{R}^2$. Set $\tau_i(x) = 0.2\text{arccot}\,(x_1^2)$ and

$$C_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.1 & -0.1 \\ -0.15 & 0.2 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.15 & -0.2 \\ 0.1 & 0.25 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -1.5 & -0.3 \\ 0.8 & 0.7 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0.1 & 1.15 \\ -0.4 & 1.5 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.7 \end{bmatrix}$$

$$f_1(x(t)) = f_2(x(t)) = \begin{bmatrix} \sin(x_1(t)) \\ \sin(x_2(t)) \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.45 \end{bmatrix}$$

$$g_1(x_t) = g_2(x_t) = \begin{bmatrix} \tanh(x_1(t-\tau)) \\ \tanh(x_2(t-\tau)) \end{bmatrix}.$$

For the function $\tau_i(x)$, we have that $\tau_i(x)$ is continuous and has the upper bound $\rho = 0.1\pi$. The Lipschitz constant for $\tau_i(x)$ is 0.2. In addition, for all $l \in \{1, 2\}$

$$\begin{aligned} \frac{d\tau_i(x(t))}{dt} &= \frac{-0.4x_1(t)}{1 + x_1^4(t)}[-x_1(t) + 0.1\sin(x_1(t)) - 0.1\sin(x_2(t)) \\ &\quad - 1.5\tanh(x_1(t-\tau)) - 0.3\tanh(x_2(t-\tau))] \\ &\leq \frac{x_1^4(t) + 0.5204}{1 + x_1^4(t)} < 1 \end{aligned}$$

and at the impulsive times, $\tau_i(x + E_q(x)) = 0.2\text{arccot}\,(2.25x_1^2) \leq \tau_i(x_1) = \tau_i(x)$. Therefore, the items 1)–3) in Assumption 1 hold and the beating phenomenon is ruled out.

Choosing Lyapunov functions as follows:

$$V_l(t, x(t)) := x^\top(t)P_l x(t) + \int_{t-\tau}^{t} x^\top(s)Q_l x(s)ds + \int_{-\tau}^{0}\int_{t+\theta}^{t} x^\top(s)R_l x(s)dsd\theta$$

with $P_l = \left[\begin{smallmatrix} 0.7233 & 0.0743 \\ 0.0743 & 0.5955 \end{smallmatrix}\right]$, $Q_l = \left[\begin{smallmatrix} 1.2352 & 0.4077 \\ 0.4077 & 1.2074 \end{smallmatrix}\right]$, and $R_l = \left[\begin{smallmatrix} 0.7736 & 0.2604 \\ 0.2604 & 0.7981 \end{smallmatrix}\right]$. As a result, (A.1) holds with $\alpha_1(v) := 0.5614v^2$ and $\alpha_2(v) := 1.1251v^2$. For all $t \notin \mathcal{I}$, $\mathcal{L}V_l(t, x_t) \leq -0.5V_l(t, x(t)) + 0.1\sup_{s\in[-\tau,0]} V_l(s, x(s))$. For all $t \notin \mathcal{I}$, $V_{\sigma(t^+)}(t^+, x(t^+)) \leq 1.8225V_{\sigma(t)}(t, x(t))$. Therefore, from Theorem 7, if $\tau_a > 0.7578$, then the system (36) is GAS. Under the initial state $z(t) = [-3, 4]^\top$ for $t \in [-0.2, 0]$, and the periodic switching time sequence with $\tau_a = 0.8$, the state response of the system (36) is given in Fig. 1.

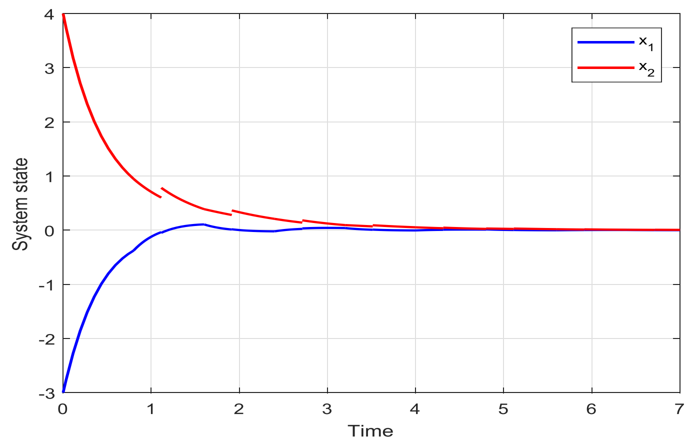

Fig. 1. State responses of the system (36) with initial condition $x(t) = (-3, 4)^\top$ for $t \in [-0.2, 0]$, periodic switching intervals, and $\tau_a = 0.8$.

VI. Conclusion

In this paper, we studied the stability of impulsive switched time-delay systems with state-dependent impulses. For impulsive switched time-delay systems with time-dependent impulses, sufficient conditions were derived to guarantee system stability. For impulsive switched time-delay systems with state-dependent impulses, we applied the B-equivalent method to construct the transformed system, and derived stability conditions based on the transformed system. Future research can be directed to impulsive switched time-delay systems where multiple state-dependent impulses exist in a switching interval.

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