Globally Optimal State-Feedback LQG Control for Large-Scale Systems With Communication Delays and Correlated Subsystem Process Noises

Yan Wang, Junlin Xiong, *Member, IEEE*, and Daniel W. C. Ho, *Fellow, IEEE*

***Abstract*—This paper studies the optimal decentralized state-feedback control of large-scale systems. The large-scale system is composed of subsystems and defined over a connected digraph. One step time is required for information to travel across an edge in the graph. Under the above-mentioned setup, when subsystem process noises are uncorrelated, the explicit optimal state-feedback controller can be designed by independence decomposition based on information hierarchy graph. However, this decomposition method fails when the subsystem process noises are correlated. In this paper, we propose a new decomposition method for system state and control input, and split the optimal state-feedback control problem with correlated process noises into two subproblems that can be solved separately. The solution to the first subproblem can be obtained by solving a linear matrix equation. The second subproblem is solved by algebraic Ricatti equation.**

***Index Terms*—Communication delay, decentralized control, large-scale systems, optimal control.**

I. Introduction

Large-Scale systems comprise of numerous subsystems. The subsystems have a common performance criterion, and do not share the same system information. To achieve the best performance, the optimal decentralized control is an effective technique for large-scale systems. However, the optimal decentralized controller is difficult to design in general [1]. The reason is that the optimal decentralized control policies may be nonlinear, and that the optimal decentralized control problem may be nonconvex [2]. Fortunately, two related conditions, partial nestedness [3] and quadratic invariance [4], [5], have been found to ensure that the optimal linear quadratic Gaussian (LQG) control policies are linear and the optimal decentralized control problem can be cast as a convex optimization problem.

The optimal decentralized LQG control of large-scale systems with delayed sharing pattern has been studied in [6]–[11]. In more practical case, the delay pattern in large-scale systems should follow a geometry topology. Consequently, the study of the large-scale systems with communication delays arising in a connected digraph has attracted lots of research attention [12]–[15]. In particular, the optimal state-feedback LQG controller was constructed in [12] by decomposition of the noise history based on information hierarchy graph. The varying communication delay case was studied in [13]. The results of [13] are only for the two-player systems. The results of [12] were extended to delay and sparsity case in [14]. The results in [12]–[14] were developed under the assumption that different subsystem process noises were uncorrelated. Feyzmahdavian *et al.* [15] removed this assumption, and studied the optimal state-feedback LQG control problem. However, the results of [15] are only suitable for three-player systems with chain structure. The optimal output-feedback LQG controller was designed using a system augmentation approach in [16]. The design method in [16] is suitable for correlated noises case, but is proposed under the assumption that each subsystem can obtain all subsystem control input in current time. This assumption seems unrealistic for large-scale systems with communication delay. Also, the decentralized state-feedback LQG problem with correlated noises was studied in [17] and [18]. However, Lessard [17], [18] did not consider the case that the communication delays between the subsystems arise from a digraph. In conclusion, the optimal LQG problem for large-scale systems with communication delay and correlated subsystem process noises has not been completely solved.

In this paper, we are interested in the optimal state-feedback LQG controller design problem for a large-scale system. The system is defined over a connected digraph. One step time is required for the information to travel across an edge in the graph. The subsystem control input is constructed using the available subsystem state, and is not transmitted to other subsystems. It is shown that both the system state and the control input are functions of the noise history and the initial state. A new decomposition of the system state and control input is proposed. In particular, we decompose the system state and the control input into several components based on the noise history. Based on the decomposition, the optimal state-feedback control problem is spilt into two subproblems that can be solved separately. The first subproblem is a standard LQR problem whose solution can be obtained by solving algebraic Riccati equations directly. The second subproblem can be cast as a convex optimization problem, and the solution is obtained by solving a linear matrix equation.

The main contribution of this paper is to unify and generalize the results in [12] and [15] to a more general case. The proposed methods are suitable for the problems considered in [12] and [15], but not vice versa. Compared with Lamperski and Doyle [12] and Lamperski and Lessard [14], the assumption of different subsystem noises being uncorrelated is not required. The results in [15] are only for three-player system, whereas our work is suitable for general large-scale systems. Compared with Wang [16], this paper removes the unrealistic assumption of each subsystem obtaining all subsystem control input in current time.

Notation: For a connected digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, $\mathcal{V} = \{1, \dots, N\}$ is the node set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set. The sequence $\{x_0, \dots, x_t\}$ is denoted by $x_{0:t}$ for short. Let $\mathrm{tr}(X)$ denote the trace of the square matrix X. $\mathbb{E}(x)$ is the expectation of the random variable x. Let A^T

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Y. Wang and J. Xiong are with the Department of Automation, University of Science and Technology of China, Hefei 230026, China (e-mail: wangyany@mail.ustc.edu.cn; junlin.xiong@gmail.com).

D. W. C. Ho is with the Department of Mathematics, City University of Hong Kong, Hong Kong (e-mail: madaniel@cityu.edu.hk).

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denote the transpose of the matrix A. The notations $X \succ 0$ and $X \succeq 0$ mean that X is a positive definite matrix and a positive semidefinite matrix, respectively. $X^{(n)}$ is the nth power of X. $0_{m\times n}$ is the $m \times n$ zero matrix, and 0_n is the $n \times n$ zero matrix. $(X)_{ij}$ is the ith row, jth column element of X.

II. Problem Statement

Consider a large-scale system composed of N subsystems defined over a strongly connected digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The nodes represent the subsystems and the edges represent the network connection. The dynamic of the ith subsystem is given by

$$x_{t+1}^i = A^{ii} x_t^i + \sum_{j\in\mathcal{N}_i} A^{ij} x_t^j + B^i u_t^i + \omega_t^i \tag{1}$$

where $x_t^i \in \mathbb{R}^{n_i}$ is the state of the ith subsystem; $u_t^i \in \mathbb{R}^{l_i}$ is the control input; $\omega_t^i \in \mathbb{R}^{n_i}$ is the process noise, and ω_t^i is related with ω_t^j for $i \neq j$; $\mathcal{N}_i = \{j : (j, i) \in \mathcal{E}\}$; and (j, i) is an edge in the graph (an arrow from node j to node i).

Remark 1: The system parameters A^{ij} and B^i are time invariant. However, if the graph $\mathcal{G}$ is constant, the results to be established in this paper can be extended to those cases suitable for the time-varying system parameters straightforwardly.

Define the following vectors:

$$x_t = \begin{bmatrix} x_t^1 \\ \vdots \\ x_t^N \end{bmatrix}, \quad u_t = \begin{bmatrix} u_t^1 \\ \vdots \\ u_t^N \end{bmatrix}, \quad \omega_t = \begin{bmatrix} \omega_t^1 \\ \vdots \\ \omega_t^N \end{bmatrix}$$

and the matrices

$$A = \begin{bmatrix} A^{11} & \cdots & A^{1N} \\ \vdots & \ddots & \vdots \\ A^{N1} & \cdots & A^{NN} \end{bmatrix}, \ A^{ij} = 0, \ \text{if } (j,i) \notin \mathcal{E}$$

$$B = \mathrm{diag}(B^1, \cdots, B^N).$$

The large-scale system can be rewritten as follows:

$$x_{t+1} = Ax_t + Bu_t + \omega_t \tag{2}$$

where the initial state x_0 is a Gaussian variable with $x_0 \sim N(0, \Sigma_0)$, $\Sigma_0 \succ 0$. The noise ω_t is an independent Gaussian process with $\omega_t \sim \mathcal{N}(0, W_t)$, $W_t \succ 0$.

Remark 2: In this paper, ω_t^i is correlated with ω_t^j for $i \neq j$, which implies that the covariance matrix W_t is not a block diagonal matrix. In this case, the independence decomposition method based on the information hierarchy graph used in [12] and [14] is not valid.

In large-scale system (2), the subsystem state is transmitted to other subsystems through the network connection (edge in the graph). Assume that one step time is required for the information to travel across an edge in the graph. Then, the subsystem state available to the ith subsystem is

$$\mathcal{I}_t^i = \{x_{0:t-\tau_{ij}}^j : j \in \mathcal{V}\} \tag{3}$$

where τ_{ij} is the length of the shortest path from the jth subsystem to the ith subsystem.

Remark 3: For example, if the system is defined over the graph in Fig. 1 , then

$$\mathcal{I}_t^1 = \{x_{0:t}^1, x_{0:t-2}^2, x_{0:t-1}^3, x_{0:t-3}^4, x_{0:t-5}^5, x_{0:t-4}^6\}$$

$$\vdots$$

$$\mathcal{I}_t^6 = \{x_{0:t-4}^1, x_{0:t-3}^2, x_{0:t-5}^3, x_{0:t-2}^4, x_{0:t-1}^5, x_{0:t}^6\}.$$

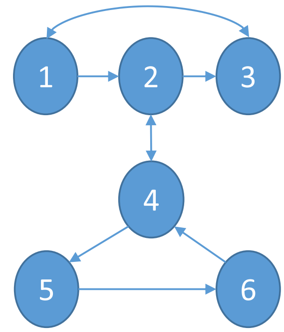

Fig. 1. Strongly connected directed graph with six nodes.

The control input of the ith subsystem is restricted to the following form:

$$u_t^i = \gamma_t^i(\mathcal{I}_t^i) \tag{4}$$

where $\gamma_t^i(\cdot)$ is a Borel-measurable function to be designed. In this paper, we assume that the subsystem control input u_t^i is not transmitted to other subsystems.

Remark 4: A function $f : X \to Y$ between two topological spaces is called Borel measurable if $f^{-1}(\Omega)$ is a Borel set for any open space Ω. The definition of Borel set is given in [19, Sec. 2.3, Ch. 2].

Define the following cost function:

$$J = \mathbb{E}\left\{\sum_{t=0}^{T-1}\left(x_t^\top Q_t x_t + u_t^\top R_t u_t\right) + x_T^\top Q_T x_T\right\} \tag{5}$$

where $Q_t \succeq 0$ for $t \in \{0, \ldots, T\}$ and $R_t \succ 0$ for $t \in \{0, \ldots, T-1\}$ are the weight matrices. The objective of this paper is to find a global optimal control law $\gamma_t^{i*}(\cdot)$, $i \in \mathcal{V}$, to minimize the cost function (5).

Problem 1:

$$\min_{\gamma_t^i} \quad J$$

$$\text{subject to} \quad (1) \text{ and } (4).$$

III. Main Result

In this section, the main results are presented. In Section III-A, the decomposition of the system state and control input is presented. The global optimal controller is derived in Section III-B.

A. State and Input Decomposition

To decompose the system state and control input, we first state two lemmas (Lemmas 1 and 2). The decomposition results are presented by Lemmas 3 and 4.

For ease of notation, we define $\omega_{-1}^j = x_0^j$, and $\omega_{-1} = x_0$.

Lemma 1: Consider the information set $\mathcal{I}_t^i$ defined in (3). The following results hold for any $i, j \in \mathcal{V}$:

i) $\{x_{0:t-\tau_{ij}}^j\} \subseteq \mathcal{I}_t^i$.

ii) $x_{t-\tau_{ij}-1}^k \in \mathcal{I}_t^i$, for $k \in \mathcal{N}_j$.

iii) $\mathcal{I}_{t-\tau_{ij}}^j \subseteq \mathcal{I}_t^i$.

Proof: The proof is obvious, thus is omitted. ∎

According to (2) and (4), it follows that x_t is a combination of the elements of $\{\omega_{-1:t-1}\}$ for any $t \geq 0$, where $\omega_{-1} = x_0$. Thus, we can use a subsect of the set $\{\omega_{-1:t-1}\}$ to design the control input u_t^i satisfying (4), as shown in the following Lemma.

Lemma 2: Consider the system (1), the control input (4), and the cost function (5). The optimal control input has the following form:

$$u_t^i = \hat{\gamma}_t^i(\hat{\mathcal{I}}_t^i) \tag{6}$$

where the optimal $\hat{\gamma}_t^i(\cdot)$ is a linear function; $\hat{\mathcal{I}}_t^i = \{\omega_{-1:t-1-\tau_{ij}}^j : j \in \mathcal{V}\}$.

Proof: See the Appendix. ■

According to Lemma 2, a decomposition of control input is presented, as the following lemma shows.

Lemma 3: The control input (6) can be decomposed as follows:

$$\begin{aligned} u_t &= F_t^{t-1}\omega_{t-1} + F_t^{t-2}\omega_{t-2} + \cdots + F_t^{t-\tau_t}\omega_{t-\tau_t} \\ &\quad + \eta_t(\omega_{-1:t-1-\tau_t}) \end{aligned} \tag{7}$$

where $\tau_t = \min\{t, \max_{i,j}\tau_{ij}\}$; $\eta_t(\cdot)$ is a linear function; $F_t^{t-\mu} \in \mathbb{R}^{(\sum_{i=1}^N l_i)\times(\sum_{i=1}^N n_i)}$, $\mu \in \{1,\ldots,\tau_t\}$, is a gain matrix satisfying

$$F_t^{t-\mu} = \begin{bmatrix} (F_t^{t-\mu})^{11} & \cdots & (F_t^{t-\mu})^{1N} \\ \vdots & \ddots & \vdots \\ (F_t^{t-\mu})^{N1} & \cdots & (F_t^{t-\mu})^{NN} \end{bmatrix} \tag{8}$$

where $(F_t^{t-\mu})^{ij} \in \mathbb{R}^{l_i \times n_j}$, and $(F_t^{t-\mu})^{ij} = 0$ if $\omega_{t-\mu}^j \notin \hat{\mathcal{I}}_t^i$.

Proof: This lemma follows from Lemma 2 directly. ■

The state can be decomposed into several components, as the following lemma shows.

Lemma 4: The system state x_t can be decomposed as follows:

$$\begin{aligned} x_t &= \sum_{r=0}^{\tau_t-1}\left(A^{(r)} + \sum_{q=0}^{r-1} A^{(q)}BF_{t-1-q}^{t-r-1}\right)\omega_{t-r-1} \\ &\quad + f_t(\omega_{-1:t-1-\tau_t}) \end{aligned} \tag{9}$$

where $f_t(\cdot)$ is a linear function.

Proof: From (2), we have

$$\begin{aligned} x_t &= A^{(\tau_t)}x_{t-\tau_t} + \sum_{r=0}^{\tau_t-1} A^{(r)}Bu_{t-r-1} \\ &\quad + \sum_{r=0}^{\tau_t-1} A^{(r)}\omega_{t-r-1}. \end{aligned} \tag{10}$$

From (7), we have

$$\begin{aligned} u_{t-r-1} &= \sum_{z=1}^{\tau_t-r-1} F_{t-r-1}^{t-r-1-z}\omega_{t-r-1-z} \\ &\quad + \bar{\eta}_{t-r-1}(\omega_{-1:t-\tau_t-1}) \end{aligned} \tag{11}$$

where $\bar{\eta}_{t-r-1}(\cdot)$ is a linear function. It follows from (10) and (11) that

$$\begin{aligned} x_t &= A^{(\tau_t)}x_{t-\tau_t} + \sum_{r=0}^{\tau_t-1} A^{(r)}\omega_{t-r-1} \\ &\quad + \sum_{r=0}^{\tau_t-1} A^{(r)}B\left(\sum_{z=1}^{\tau_t-r-1} F_{t-r-1}^{t-r-1-z}\omega_{t-r-1-z}\right. \\ &\quad \left. + \bar{\eta}_{t-r-1}(\omega_{-1:t-\tau_t-1})\right) \end{aligned} \tag{12}$$

where $\omega_{-1} = x_0$. Combining like terms, we have

$$\begin{aligned} &\sum_{r=0}^{\tau_t-1} A^{(r)}B\left(\sum_{z=1}^{\tau_t-r-1} F_{t-r-1}^{t-r-1-z}\omega_{t-r-1-z}\right) \\ &= \sum_{r=0}^{\tau_t-1}\sum_{q=0}^{r-1} A^{(q)}BF_{t-1-q}^{t-r-1}\omega_{t-r-1}. \end{aligned} \tag{13}$$

Note that $x_{t-\tau_t}$ is a linear combination of $\omega_{-1:t-1-\tau_t}$. Thus, we can denote

$$\begin{aligned} &A^{(\tau_t)}x_{t-\tau_t} + \sum_{r=0}^{\tau_t-1} A^{(r)}B\bar{\eta}_{t-r-1}(\omega_{-1:t-\tau_t-1}) \\ &= f_t(\omega_{-1:t-1-\tau_t}). \end{aligned} \tag{14}$$

Substitute (13) and (14) into (12), one has (9). The proof is completed. ■

Remark 5: A similar decomposition for system state and control input has been proposed for three-player systems in [15]. In [15], both the system state and the control input are decomposed into two terms based on the proposition that $\mathbb{E}(x|y)$ and $x - \mathbb{E}(x|y)$ are independent, where x and y are zero-mean random vectors with a jointly Gaussian distribution. It is difficult to extend such a decomposition to large-scale systems composed of N subsystems. The reason is that the term "$x - \mathbb{E}(x|y)$" is hard to derive when N is large. In this paper, we decompose the system state and control input into several components from the viewpoint of noise history. In particular, the decomposition leads to that different component depends on the noise of different time.

B. Optimal Controller Design

Based on the decomposition of the system state and control input established in Section III-A, we derive the global optimal controller in this section.

To derive the optimal controller, the cost function (5) is first decomposed into two terms.

Lemma 5 (see [20]): Consider the system (2) and the cost function (5). The cost function (5) can be written as follows:

$$\begin{aligned} J &= \underbrace{\sum_{t=0}^{T-1}\mathbb{E}\left\{(u_t - L_tx_t)^{\mathsf{T}}H_t(u_t - L_tx_t)\right\}}_{J_u} \\ &\quad \underbrace{+ \mathbb{E}(x_0^{\mathsf{T}}X_0x_0) + \sum_{t=0}^{T-1}\mathrm{tr}(X_{t+1}W_t)}_{J_\omega} \end{aligned} \tag{15}$$

where J_ω is independent of u_t; and

$$\begin{aligned} X &= A^{\mathsf{T}}X^+A + Q - (A^{\mathsf{T}}X^+B) \\ &\quad \times (B^{\mathsf{T}}X^+B + R)^{-1}(B^{\mathsf{T}}X^+A),\ X_T = Q_T \\ H &= B^{\mathsf{T}}X^+B + R \\ L &= -(B^{\mathsf{T}}X^+B + R)^{-1}B^{\mathsf{T}}X^+A \end{aligned}$$

where the time index t is omitted here; and the superscript $+$ means that the time index is $t+1$. The matrix L is the optimal feedback gain matrix of the standard LQR problem; X is the solution of corresponding discrete time algebra Riccati equation; and H is a matrix defined for simplicity of presentation.

Note that J_w does not depend on u_t. Thus, we only need to deal with J_u. For ease of notation, denote

$$\bar{u}_t = \sum_{z=1}^{\tau_t} F_t^{t-z}\omega_{t-z},\ \tilde{u}_t = \eta_t(\omega_{-1:t-1-\tau_t})$$

$$\bar{x}_t = \sum_{r=0}^{\tau_t-1}\left(A^r + \sum_{q=0}^{r-1} A^q B F_{t-1-q}^{t-r-1}\right)\omega_{t-r-1}$$

$$\tilde{x}_t = f_t(\omega_{-1:t-1-\tau_t}).$$

It is known that $\omega_{-1:t-1-\tau_t}$ is independent of $\omega_{t-\tau_t:t}$. Thus, $\mathbb{E}(\bar{u}_t\tilde{u}_t^\mathsf{T}) = 0, \mathbb{E}(\bar{u}_t\tilde{x}_t^\mathsf{T}) = 0, \mathbb{E}(\bar{x}_t\tilde{x}_t^\mathsf{T}) = 0, \mathbb{E}(\bar{x}_t\tilde{u}_t^\mathsf{T}) = 0$. Substituting $u_t = \bar{u}_t + \tilde{u}_t$ and $x_t = \bar{x}_t + \tilde{x}_t$ into J_u, one has that

$$J_u = \underbrace{\sum_{t=0}^{T-1}\mathbb{E}\left\{(\tilde{u}_t - L_t\tilde{x}_t)^\mathsf{T}H_t(\tilde{u}_t - L_t\tilde{x}_t)\right\}}_{\tilde{J}_u} + \underbrace{\sum_{t=0}^{T-1}\mathbb{E}\left\{(\bar{u}_t - L_t\bar{x}_t)^\mathsf{T}H_t(\bar{u}_t - L_t\bar{x}_t)\right\}}_{\bar{J}_u}. \tag{16}$$

Define the following two subproblems.

Problem 2:

$$\min_{\tilde{u}_t}\ \tilde{J}_u$$
$$\text{subject to}\ \ \tilde{u}_t = \eta_t(\omega_{-1:t-1-\tau_t})$$
$$\tilde{x}_t = f_t(\omega_{-1:t-1-\tau_t}).$$

Problem 3:

$$\min_{\bar{u}_t}\ \bar{J}_u$$

$$\text{subject to}\ \ \bar{u}_t = \sum_{z=1}^{\tau_t} F_t^{t-z}\omega_{t-z}$$

$$\bar{x}_t = \sum_{r=0}^{\tau_t-1}\left(A^{(r)} + \sum_{q=0}^{r-1} A^{(q)} B F_{t-1-q}^{t-r-1}\right)\omega_{t-r-1}.$$

The optimal solution to Problem 1 can be found by solving the above-mentioned two subproblems separately, as the following theorem shows.

Theorem 1: The global optimal solution to Problem 1 is $u_t = \tilde{u}_t^* + \bar{u}_t^*$, where $\tilde{u}_t^*$ and $\bar{u}_t^*$ are the global optimal solutions to Problems 2 and 3, respectively. Moreover, $\tilde{u}_t^* = L_t\tilde{x}_t$.

Proof: See the Appendix. ■

Remark 6: Note that $\{\omega_{-1:t-1-\tau_t}^j : j \in \mathcal{V}\}$ is available to all subsystems, and called common information. Problem 2 defined by $\{\omega_{-1:t-1-\tau_t}^j : j \in \mathcal{V}\}$ is a standard LQR problem, thus, can be solved directly. Problem 3 is with respect to $\{\omega_{t-\tau_t:t}^j : j \in \mathcal{V}\}$ named noncommon information. For Problem 3, the gain matrix F_t^{t-z} must be designed to satisfy certain sparse structure constraint defined in Lemma 3, because for any $\vartheta \in \{\omega_{t-\tau_t:t}^j : j \in \mathcal{V}\}$, there exists a subsystem that cannot obtain ϑ. As depicted in Fig. 2, based on Theorem 1, Problem 1 is decomposed into two subproblems from the viewpoint of common information and noncommon information.

The optimal solution to Problem 2 has been found. Now, we focus on solving Problem 3. That is to find the optimal gains $F_{t-\tau_t+1}^{t-\tau_t}, \ldots, F_t^{t-\tau_t}$ for Problem 3.

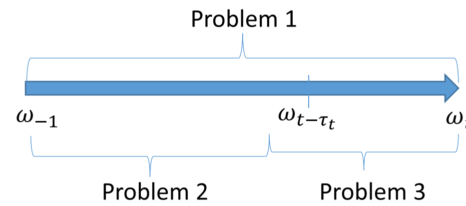

Fig. 2. Diagram of problem decomposition.

Theorem 2: Consider Problem 3. The global optimal gain matrices $F_{t-\tau_t+1}^{t-\tau_t}, \ldots, F_t^{t-\tau_t}$ can be obtained by solving the following linear matrix equation:

$$\sum_{r=1}^{\tau_t}(\Lambda_t^r)^\mathsf{T}H_{t-\tau_t+r} \times \left(\Lambda_t^r\Pi_t - L_{t-\tau_t+r}A^{r-1}\right)W_{t-\tau_t+r} + \Psi = 0 \tag{17}$$

where

$$\Pi_t = \begin{bmatrix} F_t^{t-\tau_t} \\ \vdots \\ F_{t-\tau_t+1}^{t-\tau_t} \end{bmatrix},\ \Lambda_t^1 = \begin{bmatrix} 0_{l\times(\tau_t-1)l} & I_l \end{bmatrix}$$

$$\Lambda_t^2 = \begin{bmatrix} 0_{l\times(\tau_t-2)l} & I_l & -L_{t-\tau_t+2}B \end{bmatrix}$$

$$\Lambda_t^3 = \begin{bmatrix} 0_{l\times(\tau_t-3)l} & I_l & -L_{t-\tau_t+3}B & -L_{t-\tau_t+3}AB \end{bmatrix}$$

$$\vdots$$

$$\Lambda_t^{\tau_t} = \begin{bmatrix} I_l & -L_tB & -L_tAB & \cdots & -L_tA^{(\tau_t-2)}B \end{bmatrix}$$

and Ψ has the same dimension with Π_t and satisfies $(\Psi)_{ij} = 0$ if $(\Pi_t)_{ij} \neq 0$.

Proof: Define $\bar{J}_u^t = \mathbb{E}\left\{(\bar{u}_t - L_t\bar{x}_t)^\mathsf{T}H_t(\bar{u}_t - L_t\bar{x}_t)\right\}$, then we have $\bar{J}_u = \sum_{t=0}^{T-1}\bar{J}_u^t$. According to the definition of $\bar{x}_t$ and $\bar{u}_t$, $\bar{J}_u^t$ can be expanded as follows:

$$\mathbb{E}\Bigg\{\Bigg(\left(\sum_{z=1}^{\tau_t}F_t^{t-z}\omega_{t-z}\right) - L_t\left(\sum_{r=0}^{\tau_t-1}(A^{(r)} + \sum_{q=0}^{r-1}A^{(q)}BF_{t-1-q}^{t-r-1})\omega_{t-r-1}\right)\Bigg)^\mathsf{T} \times H_t\Bigg(\left(\sum_{z=1}^{\tau_t}F_t^{t-z}\omega_{t-z}\right) - L_t\left(\sum_{r=0}^{\tau_t-1}(A^{(r)} + \sum_{q=0}^{r-1}A^{(q)}BF_{t-1-q}^{t-r-1})\omega_{t-r-1}\right)\Bigg)\Bigg\}$$

$$= \mathbb{E}\left\{\left(\sum_{r=1}^{\tau_t}\Big(F_t^{t-r} - L_t(A^{(r-1)} + \sum_{q=0}^{r-2}A^{(q)}BF_{t-1-q}^{t-r})\Big)\omega_{t-r}\right)^\mathsf{T} \times H_t\left(\sum_{r=1}^{\tau_t}\Big(F_t^{t-r} - L_t(A^{(r-1)} + \sum_{q=0}^{r-2}A^{(q)}BF_{t-1-q}^{t-r})\Big)\omega_{t-r}\right)\right\}$$

$$= \sum_{r=1}^{\tau_t} g_t^{t-r},\ \text{where}$$

$$g_t^{t-r} = \mathrm{tr}\left\{\left(F_t^{t-r} - L_t\left(A^{(r-1)} + \sum_{q=0}^{r-2} A^{(q)} B F_{t-1-q}^{t-r}\right)\right)^{\mathsf{T}} \times H_t\left(F_t^{t-r} - L_t(A^{(r-1)} + \sum_{q=0}^{r-2} A^{(q)} B F_{t-1-q}^{t-r})\right) W_{t-r}\right\}. \tag{18}$$

As a result, $\bar{J}_u = \sum_{t=0}^{T-1}\sum_{r=1}^{\tau_t} g_t^{t-r}$. From the expression of g_t^{t-r}, we have that the optimal $F_t^{t-r}, F_{t-1}^{t-r}, \ldots, F_{t-r+1}^{t-r}$ are coupled. Thus, the optimal $F_t^{t-\tau_t}, F_{t-1}^{t-\tau_t}, \ldots, F_{t-\tau_t+1}^{t-\tau_t}$ can only be obtained by solving the following optimization problem.

Problem 4:

$$\min_{F_{t-\tau_t+1}^{t-\tau_t},\ldots,F_t^{t-\tau_t}} \sum_{r=1}^{\tau_t} g_{t-\tau_t+r}^{t-\tau_t}$$

subject to $F_{t-\tau_t+1}^{t-\tau_t}, \ldots, F_t^{t-\tau_t}$ satisfy (8).

According to (18) and the expressions of Π_t and Λ_t^r, one has that $g_{t-\tau_t+r}^{t-\tau_t}$ has the following form:

$$g_{t-\tau_t+r}^{t-\tau_t} = \mathrm{tr}\left\{(\Lambda_t^r \Pi_t - L_{t-\tau_t+r} A^{(r-1)})^{\mathsf{T}} H_{t-\tau_t+r} \times (\Lambda_t^r \Pi_t - L_{t-\tau_t+r} A^{(r-1)}) W_{t-\tau_t+r}\right\}.$$

Then, the Hamiltonian function for Problem 4 can be written by

$$\mathcal{H} = \sum_{r=1}^{\tau_t} \mathrm{tr}\left\{(\Lambda_t^r \Pi_t - L_{t-\tau_t+r} A^{(r-1)})^{\mathsf{T}} H_{t-\tau_t+r} \times (\Lambda_t^r \Pi_t - L_{t-\tau_t+r} A^{(r-1)}) W_{t-\tau_t+r}\right\} + \mathrm{tr}(2\Pi_t \Psi^{\mathsf{T}}).$$

Note that $H_{t-\tau_t+r} \succeq 0$ and $W_{t-\tau_t+r} \succeq 0$. According to the formula $\mathrm{tr}(AXBX^{\mathsf{T}}) = \mathrm{vec}^{\mathsf{T}}(X)(B^{\mathsf{T}} \otimes A)\mathrm{vec}(X)$, where $\mathrm{vec}(\cdot)$ is the vectorization operator, $\otimes$ is the Kronecker product, and the fact that if $A \succeq 0$, $B \succeq 0$, then $B^{\mathsf{T}} \otimes A \succeq 0$, one has that $\mathcal{H}$ with respect to Π_t is convex. As a result, the global optimal Π_t exits, and is computed by solving $\frac{\partial \mathcal{H}}{\partial \Pi_t} = 0$. Note that $\frac{\partial \mathcal{H}}{\partial \Pi_t} = 0$ is equivalent to (17). The proof is completed.

Remark 7: The construction of the Hamiltonian function $\mathcal{H}$ is motivated by [6, Eq. (61)]. In Hamiltonian function $\mathcal{H}$, Ψ that has been defined in Theorem 2 is the Lagrange multipliers matrix. Note that under the constraints on Π_t and Ψ, $\mathrm{tr}(2\Pi_t\Psi^{\mathsf{T}}) = 0$, thus, $\mathcal{H} = \sum_{r=1}^{\tau_t} g_{t-\tau_t+r}^{t-\tau_t}$. As a result, $\mathcal{H}$ can serve as the Hamiltonian function for Problem 4.

Remark 8: For Problem 4, all terms in $\bar{J}_u$ depending on $F_t^{t-\tau_t}$, $F_{t-1}^{t-\tau_t}, \ldots, F_{t-\tau_t+1}^{t-\tau_t}$ are retained in $\sum_{r=1}^{\tau_t} g_{t-\tau_t+r}^{t-\tau_t}$. Thus, $F_t^{t-\tau_t}$, $F_{t-1}^{t-\tau_t}, \ldots, F_{t-\tau_t+1}^{t-\tau_t}$ minimizes $\bar{J}_u$ if and only if they minimizes $\sum_{r=1}^{\tau_t} g_{t-\tau_t+r}^{t-\tau_t}$. This implies that the optimal solution to Problem 3 can be obtained by solving Problem 4. In addition, Problem 2 can be solved directly. Then, according to Theorem 1, the optimal solution to Problem 1 is found.

When the global optimal gain matrices $F_t^{t-\mu}, \mu \in \{1, 2, \ldots, \tau_t\}$ are constructed offline based on Theorem 2, the global optimal subsystem control input u_t^i can be realized by the following algorithm online.

Remark 9:

i) Equation (19) is obtained by combining (7), (10), $u_t = \bar{u}_t + \tilde{u}_t$ and $x_t = \bar{x}_t + \tilde{x}_t$.

ii) In Algorithm 1, the history state $\{x_{-1:t-\tau_t-1}\}$ is discarded. The size of the set $\{x_{t-\tau_t:t-\tau_{ij}}^j : j \in \mathcal{V}\}$ does not grow with the time horizon when $t > \tau_t$.

iii) When $\tau_t = 2$ for any $t > 1$ (the three-player system case), Algorithm 1 is reduced to [15, Th. 1].

Algorithm 1.

1) Given $\{x_{t-\tau_t:t-\tau_{ij}}^j : j \in \mathcal{V}\}$, obtain $x_{t-\tau_t}$ and $\{\omega_{t-\tau_t:t-1-\tau_{ij}}^j : j \in \mathcal{V}\}$.
2) Compute

$$\tilde{x}_t = A^{\tau_t} x_{t-\tau_t} + \sum_{r=0}^{\tau_t-1} A^{(r)} B L_{t-r-1} \tilde{x}_{t-r-1} + \sum_{r=0}^{\tau_t-1} A^{(r)} B\Big(\sum_{z=0}^{r-\tau_t+\tau_{t-r-1}} F_{t-r-1}^{t-r-1-\tau_{t-r-1}+z} \times \omega_{t-r-1-\tau_{t-r-1}+z}\Big),\ \tilde{x}_0 = x_0. \tag{19}$$

3) The optimal subsystem control input is computed by

$$u_t^i = \sum_{r=1}^{\tau_t} (\Omega^i F_t^{t-r*})\omega_{t-r} + (\Omega^i L_t)\tilde{x}_t \tag{20}$$

where $\Omega^i = \begin{bmatrix} 0_{l_i \times \varrho} & I_{l_i} & 0_{l_i \times \bar{\varrho}} \end{bmatrix}$, $\varrho = l_1 + \cdots + l_{i-1}$, $\bar{\varrho} = l_{i+1} + \cdots + l_N$.

The global optimal value of the cost function is given by the following theorem.

Theorem 3: Consider Problem 1. The value of the cost function (5) achieved by the global optimal controller (20) is presented by

$$J^* = \mathrm{tr}(X_0\Sigma_0) + \sum_{t=0}^{T-1} \mathrm{tr}(X_{t+1}W_t) + \sum_{t=0}^{T-1}\sum_{r=1}^{\tau_t} g_t^{t-r*}$$

where

$$g_t^{t-r*} = \mathrm{tr}\left\{(\Lambda_{t+\tau_t-r}^r \Pi_{t+\tau_t-r}^* - L_t A^{(r-1)})^{\mathsf{T}} \times H_t(\Lambda_{t+\tau_t-r}^r \Pi_{t+\tau_t-r}^* - L_t A^{(r-1)}) W_t\right\}.$$

IV. Conclusion

In this paper, the global optimal state feedback controller was constructed by a new decomposition of system state and control input. Both the system state and the control input were decomposed into several components depending on noises of different times. Based on the decomposition, the optimal state feedback control problem was split into two subproblems, which can be solved separately. The results in this paper are the extension of the results in [12] and [15]. In the future, we will extend this work to random communication delay case and more general delay models.

Appendix

A. Proof of Lemma 2

The proof includes two steps. First, we prove that the optimal control input has the form (6). Second, we show that the optimal $\hat{\gamma}_t^i$ is linear.

Step 1: We need to prove that $\hat{\mathcal{I}}_t^i$ can be restored from $\mathcal{I}_t^i$ perfectly by the ith subsystem, and vice versa. Note that $\mathcal{I}_0^i = \hat{\mathcal{I}}_0^i = \{x_0^j : j \in \mathcal{V}\}$ for any $i \in \mathcal{V}$. In addition, the initial state $\{x_0^j : j \in \mathcal{V}\}$ is known to all subsystems. Using the mathematical induction, we only need to prove that $\hat{\mathcal{I}}_t^i$ can be restored from $\mathcal{I}_t^i$ and $\hat{\mathcal{I}}_{t-1}^i$ perfectly, and $\mathcal{I}_t^i$ can be restored from $\hat{\mathcal{I}}_t^i$ and $\mathcal{I}_{t-1}^i$ perfectly. Given $\mathcal{I}_t^i$ and $\hat{\mathcal{I}}_{t-1}^i$. From Lemma 1, we have $x_{t-\tau_{ij}}^j \in \mathcal{I}_t^i$; $x_{t-\tau_{ij}-1}^{\bar{j}} \in \mathcal{I}_t^i$ for any $\bar{j} \in \mathcal{N}_j$. According to the proof of (iii) in Lemma 1, we have $\hat{\mathcal{I}}_{t-\tau_{ij}-1}^j \subseteq \hat{\mathcal{I}}_{t-1}^i$. Hence, the ith subsystem can compute $u_{t-\tau_{ij}-1}^j$ at time t, because $u_{t-\tau_{ij}-1}^j$ is a function of the elements in $\hat{\mathcal{I}}_{t-\tau_{ij}-1}^j$. It has been shown that $x_{t-\tau_{ij}}^j, x_{t-\tau_{ij}-1}^{\bar{j}}$ ($\bar{j} \in \mathcal{N}_j$) and $u_{t-\tau_{ij}-1}^j$ can be known if $\mathcal{I}_t^i$ and $\hat{\mathcal{I}}_{t-1}^i$ are given. This implies that the ith subsystem can obtain $\omega_{t-\tau_{ij}-1}^j$ perfectly by (1), that is

$$\omega_{t-\tau_{ij}-1}^j = x_{t-\tau_{ij}}^j - A^{jj} x_{t-\tau_{ij}-1}^j - \sum_{\bar{j} \in \mathcal{N}_j} A^{j\bar{j}} x_{t-\tau_{ij}-1}^{\bar{j}} - B^j u_{t-\tau_{ij}-1}^j. \tag{21}$$

As a result, given $\mathcal{I}_t^i$ and $\hat{\mathcal{I}}_{t-1}^i$, the ith subsystem can obtain $\hat{\mathcal{I}}_t^i$ perfectly. Similarly, we can prove that given $\hat{\mathcal{I}}_t^i$ and $\mathcal{I}_{t-1}^i$, the ith subsystem can obtain $\mathcal{I}_t^i$ perfectly.

Step 2: From (1), we know that x_t^i is affected by $x_{t-\tau_{ij}}^j$, for $j \in \mathcal{V}$. It follows from Lemma 1 that $\mathcal{I}_t^i \supseteq \mathcal{I}_{t-\tau_{ij}}^j$ holds for any $j \in \mathcal{V}$. As a result, the information pattern of Problem 1 is partially nested. According to [3, Th. 3], the optimal $\gamma_t^i(\cdot)$ is linear. Note that $\hat{\mathcal{I}}_t^i$ is a linear transformation of $\mathcal{I}_t$. Thus, the optimal $\hat{\gamma}_t^i(\cdot)$ is linear. The proof is completed.

B. Proof of Theorem 1

To prove Theorem 1, we need the following lemma.

Lemma 6: If there exists a u_t^* satisfying (4), and leading to $J_u = \bar{J}_u^*$, then u_t^* is the optimal solution to Problem 1, where $\bar{J}_u^*$ is the optimal value of the objective function of Problem 3.

Proof: From (15), we have that $J = J_u + J_\omega$, and J_ω is independent of u_t. Thus, u_t is the optimal solution to Problem 1 (u_t minimizes J and satisfies (4)) if and only if u_t minimizes J_u and satisfies (4). As a result, we only need to show that $J_u \geq \bar{J}_u^*$ always holds. Note that $\tilde{J}_u \geq 0$ always holds, because $H_t \succ 0$. According to (16), we have $J_u = \tilde{J}_u + \bar{J}_u \geq \bar{J}_u \geq \bar{J}_u^*$. The proof is completed. ∎

Now, we start to prove Theorem 1. Consider Problem 2. Both $\tilde{u}_t$ and $\tilde{x}_t$ are linear combinations of $\omega_{-1:t-1-\tau_t}$. As a result, we can choose $\tilde{u}_t = L_t \tilde{x}_t$ such that $\tilde{J}_u = 0$. Because $H_t \succ 0$, one has that $\tilde{J}_u \geq 0$ always holds. This implies that $\tilde{u}_t = L_t \tilde{x}_t$ is the global optimal solution to Problem 2. Substitute $u_t = L_t \tilde{x}_t + \bar{u}_t^*$ into J_u gives that $J_u = \bar{J}_u^*$. In addition, it is easy to see that $u_t = L_t \tilde{x}_t + \bar{u}_t^*$ has the form (7), thus satisfies (4). Hence, according to Lemma 6, one has that $u_t = L_t \tilde{x}_t + \bar{u}_t^*$ is the global optimal solution to Problem 1. The proof is completed.

REFERENCES

[1] J. Tsitsiklis and M. Athans, "On the complexity of decentralized decision making and detection problems," *IEEE Trans. Autom. Control*, vol. 30, no. 5, pp. 440–446, May 1985.

[2] H. S. Witsenhausen, "A counterexample in stochastic optimum control," *SIAM J. Control*, vol. 6, no. 1, pp. 131–147, 1968.

[3] Y. C. Ho and K.-H. Chu, "Team decision theory and information structures in optimal control problems—Part I," *IEEE Trans. Autom. Control*, vol. 17, no. 1, pp. 15–22, Feb. 1972.

[4] M. Rotkowitz and S. Lall, "A characterization of convex problems in decentralized control," *IEEE Trans. Autom. Control*, vol. 51, no. 2, pp. 274–286, Feb. 2006.

[5] L. Lessard and S. Lall, "An algebraic approach to the control of decentralized systems," *IEEE Trans. Control Netw. Syst.*, vol. 1, no. 4, pp. 308–317, Dec. 2014.

[6] B.-Z. Kurtaran and R. Sivan, "Linear-quadratic-Gaussian control with one-step-delay sharing pattern," *IEEE Trans. Autom. Control*, vol. 19, no. 5, pp. 571–574, Oct. 1974.

[7] T. Yoshikawa and H. Kobayashi, "Separation of estimation and control for decentralized stochastic control systems," *Automatica*, vol. 14, no. 6, pp. 623–628, 1978.

[8] A. Nayyar, A. Mahajan, and D. Teneketzis, "Optimal control strategies in delayed sharing information structures," *IEEE Trans. Autom. Control*, vol. 56, no. 7, pp. 1606–1620, Jul. 2011.

[9] H. R. Feyzmahdavian, A. Gattami, and M. Johansson, "Distributed output-feedback LQG control with delayed information sharing," in *Proc. IFAC Workshop Distrib. Estimation Control Networked Syst.*, 2012, pp. 192–197.

[10] A. Nayyar, A. Mahajan, and D. Teneketzis, "Decentralized stochastic control with partial history sharing: A common information approach," *IEEE Trans. Autom. Control*, vol. 58, no. 7, pp. 1644–1658, Jul. 2013.

[11] N. Nayyar, D. Kalathil, and R. Jain, "Optimal decentralized control with asymmetric one-step delayed information sharing," *IEEE Trans. Control Netw. Syst.*, vol. 5, no. 1, pp. 653–663, Mar. 2018.

[12] A. Lamperski and J. C. Doyle, "Dynamic programming solutions for decentralized state-feedback LQG problems with communication delays," in *Proc. Amer. Control Conf.*, 2012, pp. 6322–6327.

[13] N. Matni and J. C. Doyle, "Optimal distributed LQG state feedback with varying communication delay," in *Proc. IEEE Conf. Decis. Control*, 2013, pp. 5890–5896.

[14] A. Lamperski and L. Lessard, "Optimal decentralized state-feedback control with sparsity and delays," *Automatica*, vol. 58, pp. 143–151, 2015.

[15] H. R. Feyzmahdavian, A. Alam, and A. Gattami, "Optimal distributed controller design with communication delays: Application to vehicle formations," in *Proc. IEEE Conf. Decis. Control*, 2012, pp. 2232–2237.

[16] Y. Wang, Y. Zhang, and J. Xiong, "Optimal decentralized output-feedback control with communication delay: A system augmentation approach," in *Proc. IEEE Conf. Decis. Control*, 2017, pp. 3537–3342.

[17] L. Lessard, "Optimal control of a fully decentralized quadratic regulator," in *Proc. Allerton Conf. Commun., Control, Comput.*, 2012, pp. 48–54.

[18] L. Lessard, "A separation principle for decentralized state-feedback optimal control," in *Proc. Allerton Conf. Commun., Control, Comput.*, 2013, pp. 528–534.

[19] H. L. Royden and P. Fitzpatrick, *Real Analysis*. New York, NY, USA: Macmillan, 1988.

[20] K. J. Åström, *Introduction to Stochastic Control Theory*. Chelmsford, MA, USA: Courier Corporation, 2012.