Optimal Decentralized Output-Feedback LQG Control With Random Communication Delay

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***Abstract*—This paper is concerned with the optimal decentralized output-feedback control of the large-scale systems. A random information pattern is considered, where the information is transmitted among the subsystems with random communication delays. For the random information pattern, the optimal LQG problems for both global estimation case and local estimation case are studied. It is difficult to derive the optimal controller under random framework, because the gains of the controller must be designed to satisfy the random sparse structure constraints. In this paper, we design the optimal controller by Hadamard product method. For global estimation case, the gains of the controller are obtained by solving linear matrix equation. For local estimation case, an iterative algorithm is exploited to compute the gains. In addition, the value of the cost function achieved by the designed controller is found and shown to monotonically increase with the increase of the delay probability for both global and local estimation cases. Finally, the theoretical results are illustrated by two numerical examples.**

***Index Terms*—Communication delay, decentralized control, optimal, output-feedback, random.**

I. Introduction

LARGE-SCALE systems have been found in many applications, such as autonomous vehicles [1], [2]; electric power systems [3]; satellite formations [4], [5]; and robotics [6], [7]. In large-scale systems, subsystems usually exchange information through a communication network [8]. The communication network used in the large-scale systems offers numerous advantages like simple installation, easy maintenance, and low cost [9]. However, the information transmitted through the network may suffer from communication delay [10], [11]. As a result, the information available to each subsystem is incomplete at each time step. To achieve the best system performance with the incomplete information, decentralized control has been proved to be a useful control technique. However, the design of the optimal decentralized control strategy is a challenging task, because it is computationally intractable in general [12]. For example, the optimal decentralized control policies may be nonlinear even for the linear system [13].

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A. Related Work and Motivation

Decentralized control with communication delay has attracted a lot of research attentions since 1970s. The optimal decentralized control with one step delay sharing pattern was studied by the matrix minimum principle in [14] and by the second-guessing technique in [15]. For the multiple step delays sharing pattern, two structural results to the optimal decentralized control design were established in [16]. The decentralized stochastic control with symmetric delay and asymmetric delay has been studied by a common information approach in [17]. A common feature of the results in [14]–[17] is that each subsystem estimates not only its own subsystem state but also the others (global estimation). This implies that the subsystem state is estimated more than one time. For the case that each subsystem only estimates its own subsystem state (local estimation), Wang *et al.* [18] studied the decentralized output feedback control for a two-player system with one step communication delay.

For delay patterns arising from a communication graph, the optimal decentralized LQG control problems were investigated by the information hierarchy graph in [19] and [20] and by the independence decomposition technique in [21]. The results in [19]–[21] are for the state-feedback controller design. The output-feedback case was considered in [22]. However, the result in [22] is only suitable for a three-player system with a chain structure. The sufficient statistics of linear control strategies was studied in [23] for large-scale systems with delay pattern defined over a communication graph. In [14]–[23], the communication delay is assumed to be determinate. In practice, the network environment is affected by random factors, and the communication delay is naturally random. The LQG problem with varying communication delay has been studied in [24] and [25]. The results proposed in [24] and [25] are sound, but are only for two-player system with state feedback case. The framework of the randomized information pattern studied in [26] can be used to model state-feedback LQG problem with random communication delay, but the realization of the optimal controller design is not derived. Thus, the optimal output feedback LQG control with random communication delay is not fully studied.

B. Our Work

This paper focuses on the optimal decentralized output-feedback control of a large-scale system. The information is transmitted from one subsystem to other subsystems through a network with random delays. The random delay satisfies Bernoulli distributions. Under this setup, we derive a linear matrix equation used to design the optimal controller for global estimation case. Also, the optimal value of the cost function is obtained, and is shown to strictly increase as the delay probability increases. The optimal LQG problem under local estimation case is also studied. An iterative algorithm is exploited to design the gains of the optimal controller for local estimation case. It is shown that the algorithm converges to person-by-person optimum. The optimal value of the cost function has the same monotonicity as the one of global estimation case. Finally, two numerical examples are given to illustrate the effectiveness of the theoretical results.

C. Main Contribution

The contribution of this work is summarized as follows.

1) Compared to [14]–[17], the local estimation case is also studied in this paper, while Kurtaran and Sivan [14], Toda and Aoki [15], Nayyar *et al.* [16], and Mahajan and Nayyar [17] only studied the global estimation case. The local estimation is important for large-scale systems. One advantage of local estimation is that it consumes less computational resources. Compared to the global estimation case, the main challenge of the local estimation case is that the available estimated state is incomplete, thus, is not a sufficient statistic for optimal decision. Then, the corresponding optimization problem is nonconvex. Our contribution is to establish the framework of the optimal decentralized controller under local estimation, and exploit an iterative algorithm to compute the gains of the controller in the sense of person-by-person optimum.
2) Compared to [14]–[23], we study the random delay case instead of the determinate delay case considered in [14]–[23]. The main challenge of the random delay case is to deal with the random sparse structure constraints induced by the random communication delay. Our contribution is to propose the method of Hadamard product derivative to design the optimal decentralized controller under the random sparse structure constraints.
3) We investigate the property of the cost function with respect to the delay probability. Our contribution is to prove that the optimal value of the cost function is monotonically increasing with the increase of the delay probability. Based on this result, we can find the critical delay probability effectively by the binary search method for a given value of the cost function, such that the optimal value of the cost function is smaller than the given one when the delay probability is below the critical delay probability. To the best of our knowledge, the similar results do not appear in the related literature.
4) Compared to [24] and [25], our contribution is to design the optimal decentralized output-feedback controller for more general large-scale systems. However, the results established in [24] and [25] are only for two-player systems under state feedback and uncorrelated subsystem noises. In other words, the methods proposed in [24] and [25] are unsuitable for our problem.

Notation: For a time-varying matrix $X(t)$ or a time-varying vector $x(t)$, to save space, we sometimes omit the time index and write X or x, respectively. Furthermore, $X(t+1)$ and $x(t+1)$ are denoted by X^+ and x^+, respectively. $\mathbb{E}(x)$ denotes the expectation of the random variable x. Let $\mathbb{E}(x \mid y)$ denote the conditional expectation of x given y. The transpose of the matrix A is denoted by A^T. $X \succ 0$ and $X \succeq 0$ mean that X is a symmetric positive definite matrix and a symmetric positive semi-definite matrix, respectively. Let $y(0:t)$ denote the sequence $\{y(0), y(1), \dots, y(t)\}$. The trace of a square matrix X is denoted by $\mathrm{tr}(X)$. $1_{i\times j}$ denote the matrix of which all the elements are 1, and $1_{i\times j} \in \mathbb{R}^{i\times j}$. The $n \times n$ identity matrix is denoted by I_n. Define $A\backslash B \triangleq \{x : x \in A \text{ and } x \notin B\}$. $\Pr(\cdot)$ is the probability measure. For a denumerable set S, the numbers of the elements in S is denoted by $|S|$. For a matrix X, $(X)_{ij}$ denotes the i-row, j-column element of matrix X. For matrices $A, B \in \mathbb{R}^{m\times n}$, the Hadamard product is defined as $(A \circ B)_{ij} = (A)_{ij}(B)_{ij}$.

II. Problem Description

Consider a large-scale system composed of N subsystems, where the ith subsystem is of the form

$$x_i(t+1) = A_{ii}(t)x_i(t) + \sum_{j\in\Omega\backslash\{i\}} A_{ij}(t)x_j(t) + B_i(t)u_i(t) + \omega_i(t),\ \ i \in \Omega \tag{1}$$

$$y_i(t) = C_i(t)x_i(t) + \upsilon_i(t),\ \ i \in \Omega \tag{2}$$

where $\Omega = \{1, 2, \dots, N\}$. For the ith subsystem, $x_i \in \mathbb{R}^{n_i}$ is the state; $u_i \in \mathbb{R}^{l_i}$ and $y_i \in \mathbb{R}^{m_i}$ are the control input and the measurement output, respectively; $\omega_i \in \mathbb{R}^{n_i}$ is the process noise; $\upsilon_i \in \mathbb{R}^{m_i}$ is the measurement noise. In (1) and (2), A_{ij}, B_i, and C_i are the matrices of proper dimensions for $i, j \in \Omega$.

Define the following augmented vectors and matrices:

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix},\quad u = \begin{bmatrix} u_1 \\ \vdots \\ u_N \end{bmatrix},\quad \omega = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_N \end{bmatrix},\quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix}$$

$$\upsilon = \begin{bmatrix} \upsilon_1 \\ \vdots \\ \upsilon_N \end{bmatrix},\quad A = \begin{bmatrix} A_{11} & \cdots & A_{1N} \\ \vdots & \ddots & \vdots \\ A_{N1} & \cdots & A_{NN} \end{bmatrix}$$

$$B = \mathrm{diag}\{B_1, \dots, B_N\},\quad C = \mathrm{diag}\{C_1, \dots, C_N\}.$$

The large-scale system (1), (2) can be rewritten as

$$x(t+1) = A(t)x(t) + B(t)u(t) + \omega(t) \tag{3}$$

$$y(t) = C(t)x(t) + \upsilon(t) \tag{4}$$

where the initial state $x(0)$ is a Gaussian variable with $x(0) \sim \mathcal{N}(0, \Sigma(0))$, $\Sigma(0) \succ 0$. The noises $\omega(t)$ and $\upsilon(t)$ are the independent Gaussian processes with $\omega(t) \sim \mathcal{N}(0, \mathbb{W}(t))$, $\mathbb{W}(t) \succ 0$ and $\upsilon(t) \sim \mathcal{N}(0, \mathbb{V}(t))$, $\mathbb{V}(t) \succ 0$, respectively. In addition, $x(0)$, $\omega(t_1)$, and $\upsilon(t_2)$ are mutually independent for all t_1, t_2.

It is assumed that the system parameter matrices $A(t)$, $B(t)$, and $C(t)$ and the statistical properties of $\omega(t)$, $\upsilon(t)$, and $x(0)$ are known to all subsystems.

We now describe the information pattern. For system (1) and (2), the information transmitted from the jth subsystem to the ith subsystem is delayed by $\tau_{ij}(t)$ time-step, where $\tau_{ij}(t)$ obeys a Bernoulli distribution with the probability

$$\Pr(\tau_{ij}(t)=1)=\begin{cases}\lambda, & i\neq j\\ 0, & i=j\end{cases} \tag{5}$$

$$\Pr(\tau_{ij}(t)=0)=\begin{cases}1-\lambda, & i\neq j\\ 1, & i=j\end{cases} \tag{6}$$

where $0\leq\lambda\leq 1$. We assume that the delay probability λ is known. This is a standard assumption, and is commonly used in [24], [25], and [27]–[30]. Additionally, we assume that $\tau_{i_1j_1}(t_1)$ is independent of $\tau_{i_2j_2}(t_2)$ for any $i_1\neq i_2$ or $j_1\neq j_2$ or $t_1\neq t_2$.

Remark 1: One step communication delay is commonly assumed in [20] and [27]–[31]. In this paper, the assumption of one step communication delay is to ensure that the information is transmitted no slower than the dynamics propagate through the plant. As a result, the communication network design conforms to the standard practice [32].

Due to the one step random communication delay, the measurement output available to the ith subsystem is

$$\begin{aligned}\mathcal{Y}_i(t)&=\{y_i(0:t)\}\bigcup_{j\in\Omega\backslash\{i\}}\left\{y_j\big(0:t-\tau_{ij}(t)\big)\right\}\\ &=\Xi_i(t)\cup\Delta(t)\end{aligned}$$

where $\Xi_i(t)=\{y_j(t):\tau_{ij}(t)=0,\ j\in\Omega\}$, and $\Delta(t)=\{y_j(0:t-1),\ j\in\Omega\}$. Note that $\Delta(t)$ does not depend on the random variable $\tau_{ij}(t)$ and is the common information of all the subsystems.

Define the cost function

$$J\triangleq\mathbb{E}\left\{\sum_{t=0}^{T-1}x(t)^{\mathsf{T}}Q(t)x(t)+u(t)^{\mathsf{T}}R(t)u(t)+x(T)^{\mathsf{T}}Q(T)x(T)\right\} \tag{7}$$

where $Q(t)\succeq 0$ $(t=0,\ldots,T)$ and $R(t)\succ 0$ $(t=0,\ldots,T-1)$ are known to all subsystems. The expectation operation $\mathbb{E}$ in (7) is taken over both $x(0)$, $\omega(t)$, $\upsilon(t)$, and $\tau_{ij}(t)$, $i,j\in\Omega$.

III. Global Estimation Case

In this section, our objective is to design the controller of the form

$$u_i(t)=\mu_i(t,\mathcal{Y}_i(t)) \tag{8}$$

to minimize the cost function (7). That is, we need to solve the following problem.

Problem 1:

$$\begin{aligned}&\min_{\mu_i(t,\cdot)} J\\ &\text{s.t. } (1),\ (2),\ (8).\end{aligned}$$

Remark 2:

1) Due to the random communication delay, the information available to the ith subsystem is $\mathcal{Y}_i(t)$. Thus, the subsystem controller input $u_i(t)$ is restricted to the form (8).
2) The information pattern induced by $\mathcal{Y}_i(t)$ is partially nested ([33, Definition 3]), because $\mathcal{Y}_j(t-1)\subsetneqq\mathcal{Y}_i(t)$ holds for any $i,j\in\Omega$. Thus, the optimal control law $\mu_i(t,\cdot)$ is linear ([33, Th. 2]).

A. Offline Design

In this section, we solve Problem 1 in an offline fashion. That is, $\tau_{ij}(0:T-1)$ is unknown when we design $\mu_i(t,\cdot)$ offline for any $t\geq 0$.

For ease of notations, define

$$\Upsilon\triangleq\begin{bmatrix}\gamma_{11}&\cdots&\gamma_{1N}\\ \vdots&\ddots&\vdots\\ \gamma_{N1}&\cdots&\gamma_{NN}\end{bmatrix},\quad \gamma_{ij}=1-\tau_{ij} \tag{9}$$

$$\Gamma_\Upsilon\triangleq\begin{bmatrix}\gamma_{11}1_{l_1\times m_1}&\cdots&\gamma_{1N}1_{l_1\times m_N}\\ \vdots&\ddots&\vdots\\ \gamma_{N1}1_{l_N\times m_1}&\cdots&\gamma_{NN}1_{l_N\times m_N}\end{bmatrix} \tag{10}$$

$$F\triangleq\begin{bmatrix}F_{11}&\cdots&F_{1N}\\ \vdots&\ddots&\vdots\\ F_{N1}&\cdots&F_{NN}\end{bmatrix},\quad F_{ij}\in\mathbb{R}^{l_i\times m_j} \tag{11}$$

$$\Psi=\left\{\Upsilon:\gamma_{ij}\in\{0,1\},i,j\in\Omega\right\}. \tag{12}$$

The optimal solution to Problem 1 is given by the following theorem.

Theorem 1: Consider Problem 1. The optimal controller (8) under offline design is given by

$$\begin{aligned}u(t)=&\ (F(t)\circ\Gamma_\Upsilon(t))y(t)\\ &+(F(t)\circ\Gamma_\Upsilon(t))\hat{x}(t)+L(t)\hat{x}(t)\end{aligned} \tag{13}$$

where $\hat{x}_t$ is computed by

$$\hat{x}^+=A\hat{x}+Bu+K(y-C\hat{x}),\quad \hat{x}(0)=0 \tag{14}$$

$$\begin{aligned}P^+=&\ APA^{\mathsf{T}}-APC^{\mathsf{T}}\left(CPC^{\mathsf{T}}+\mathbb{V}\right)^{-1}\\ &\times CPA^{\mathsf{T}}+\mathbb{W},\quad P(0)=\Sigma(0)\end{aligned} \tag{15}$$

$$K=APC^{\mathsf{T}}(CPC^{\mathsf{T}}+\mathbb{V})^{-1} \tag{16}$$

the optimal gains $F(t)$ and $L(t)$ are computed by

$$\begin{aligned}\sum_{\Upsilon\in\Psi}\Pr(\Upsilon)\Big\{&(Y(F\circ\Gamma_\Upsilon)^{\mathsf{T}}R)\circ\Gamma_\Upsilon^{\mathsf{T}}\\ &+\Big(Y(B(F\circ\Gamma_\Upsilon)+K)^{\mathsf{T}}X^+B\Big)\circ\Gamma_\Upsilon^{\mathsf{T}}\Big\}=0\end{aligned} \tag{17}$$

$$Y=CPC^{\mathsf{T}}+\mathbb{V} \tag{18}$$

$$\begin{aligned}X=&\ A^{\mathsf{T}}X^+A+Q-(A^{\mathsf{T}}X^+B)(B^{\mathsf{T}}X^+B+R)^{-1}\\ &(B^{\mathsf{T}}X^+A),\ X(N)=Q(N)\end{aligned} \tag{19}$$

$$L=-(B^{\mathsf{T}}X^+B+R)^{-1}B^{\mathsf{T}}X^+A. \tag{20}$$

Proof: See Appendix A. ■

Remark 3:

1) The optimal controller to Problem 1 is of the form (13). This implies that using $\{\hat{x}_j(0:t):j\in\Omega\}$ instead of $\Delta(t)$ is no loss of optimality.
2) Note that the matrix P is the estimation error covariance: $P=\mathbb{E}[(x-\hat{x})(x-\hat{x})^{\mathsf{T}}]$; K is the Kalman filter gain; Y

is defined as $Y = \mathbb{E}[(y - C\hat{x})(Y - C\hat{x})^{\mathsf{T}}]$. The matrices P, K, Y, X, and L can be computed offline, because the system parameter matrices and the noise statistics are known to all subsystems.

Remark 4: One challenge for extending the one-step delay to the delay defined in a communication graph is that the separation principle fails [34]. For the delay defined over a graph, the delay corresponding to subsystems is equal to the shortest length of the path between the subsystems. For this delay model, the ith subsystem may not be able to compute the estimated state $\hat{x}(t)$ by (14). The reason is that $u_j(t-1)$ may not be available to the ith subsystem if $l_{ij} > 1$, where l_{ij} is the length of the shortest path from the jth subsystem to the ith subsystem, and $u_j(t-1)$ is required for computing $\hat{x}(t)$. Then, the large-scale system (3), (4) cannot ran Kalman filter (14). It follows that the separation principle fails.

Remark 5: We design the optimal controller (13) by the methods of independence decomposition and Hadamard product, while, in [14], the methods of matrix minimum principle and Lagrange multiplier are used to derive the optimal controller. The Lagrange multiplier method is not suitable for Problem 1 under offline design, because the Lagrange multiplier method can only deal with one certain sparse structure constraint of gain matrices.

Theorem 2: Consider Problem 1. The optimal value of the cost function J achieved by the optimal controller (13) is

$$J^* = \sum_{t=0}^{T-1} \sum_{\Upsilon \in \Psi} \Pr(\Upsilon) \Big\{ \mathrm{tr}((F^* \circ \Gamma_\Upsilon)^{\mathsf{T}} R (F^* \circ \Gamma_\Upsilon) Y) + \mathrm{tr}\big(X^+ \big(B(F^* \circ \Gamma_\Upsilon) + K\big) \times Y \big(B(F^* \circ \Gamma_\Upsilon) + K\big)^{\mathsf{T}}\big) \Big\} + \sum_{t=0}^{T} \mathrm{tr}(QP) \tag{21}$$

where F^* is the optimal gain obtained by solving (17).

Proof: From the proof of Theorem 1, (21) is obtained directly. ■

Remark 6: When $\lambda = 1$, i.e., $\Pr(\Upsilon = I_N) = 1$, the results presented by Theorems 1 and 2 are reduced to the deterministic one-step delay case which has been derived in [14].

Remark 7: Consider (21), we take λ and F as variables. The optimal F depends on λ. The optimal value of the cost function is denoted by $J^*(\lambda, F_\lambda^*)$. Note that F_1^* is a block diagonal matrix. According to the fact that $\Pr(\gamma_{ii}) = 1$ holds for any $i \in \Omega$, one has that $F_1^* \circ \Gamma_\Upsilon = F_1^*$ holds for any Υ satisfying $\Pr(\Upsilon) \neq 0$. Recall (21), it follows that $J^*(1, F_1^*) = J(\lambda, F_1^*) \geq J^*(\lambda, F_\lambda^*)$, where the inequality follows that F_λ^* is the optimal gain corresponding to λ. This shows that under offline design, the optimal value of the cost function with random delay is less than the one with deterministic delay.

Remark 8: When $\lambda = 0$, i.e., $\Pr(\Upsilon = 1_{N\times N}) = 1$, the optimal controller (13) is reduced to the optimal centralized output feedback controller in [34].

B. Online Design

For online design, the realization of $\Upsilon(t)$ denoted by $\bar{\Upsilon}(t)$ is known when we design $\mu_i(t, \cdot)$. Thus, the results established for offline design in Section III-A can be reduced to the ones for online design by letting $\Pr(\Upsilon(t) = \bar{\Upsilon}(t)) = 1$ and $\Pr(\Upsilon(t) \neq \bar{\Upsilon}(t)) = 0$.

Theorem 3: Consider Problem 1. For online design, the optimal controller (8) to Problem 1 is given by (13), where $\hat{x}(t)$ is computed by (14); $L(t)$ is given by (20); $F(t)$ is obtained by solving (17) by letting $\Pr(\Upsilon(t) = \bar{\Upsilon}(t)) = 1$ and $\Pr(\Upsilon(t) \neq \bar{\Upsilon}(t)) = 0$. Moreover, the optimal value of the cost function is given of the form (21).

Remark 9: Note that the optimal $F^*(t)$ obtained offline in (21) are identical for different realizations of $\Upsilon(t)$. However, for online design, $F^*(t)$ in (21) corresponding to different realizations of $\Upsilon(t)$ are different. The reason is that for online design, $F^*(t)$ is designed based on the realizations of $\Upsilon(t)$, while, for offline design, $F^*(t)$ is designed based on the statistical property of $\Upsilon(t)$.

For online design, if we take the delay probability λ as a variable, then we can show that the optimal value of the cost function J increases as λ increases.

Theorem 4: Consider (21), and take the delay probability λ as a variable. Under online design fashion, the optimal value of the cost function $J^*(\lambda)$ is monotonically increasing with the increase of λ.

Proof: See Appendix B. ■

Remark 10: When extending the one step delay to the delay defined over a graph, it is impossible to analyze the property of the optimal value of the cost function J^* with respect to the delay probability λ in theory. For the delay defined over a graph, τ_{ij} is a random variable taking value in $\{0, 1, \ldots, l_{ij}\}$, where τ_{ij} is the time steps for the information to be transmitted from the jth subsystem to the ith subsystem; and l_{ij} is the length of the shortest path from the jth subsystem to the ith subsystem. Note that $\tau_{i_1j_1}$ is dependent of $\tau_{i_2j_2}$, if the information transmitted from the j_1th subsystem to the i_1th subsystem and the one transmitted from the j_2th subsystem to the i_2th subsystem travel across a common edge. This implies that the explicit expression of $\Pr(\Upsilon)$ cannot be derived for a general large-scale system. Hence, it is a challenging task to analyze the property of J^* with respect to λ.

Remark 11: The optimal controller (13) is designed based on the global estimation. That is, all subsystem need to compute the global estimated state $\hat{x}(t)$ by (14), because the subsystem control input $u_i(t)$ in (13) depends on $\hat{x}(t)$ for any $i \in \Omega$. In this case, $\hat{x}(t)$ need to be computed repeatedly over the whole system.

IV. Local Estimation Case

For global estimation case, the estimated state $\hat{x}(t)$ is computed repeatedly. To save computational resources, in this section, we propose a controller design scheme under local estimation as shown in Fig. 1. It is assumed that the ith subsystem only computes the estimated subsystem state $\hat{x}_i(t)$, and transmits $\hat{x}_i(t)$ to other subsystems through communication network with random delay.

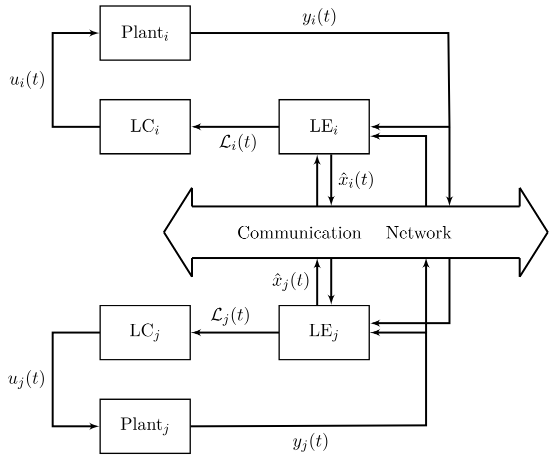

Fig. 1. Configuration of large-scale systems. Each system comprises of a plant, a local estimator, and a local controller (LC). The information is transmitted from the jth subsystem to the ith subsystem with τ_{ij} time-step delay. The information used to construct the ith LC is denoted by $\mathcal{L}_i$.

Due to the random communication delay, the estimated subsystem state available to the ith subsystem is

$$\mathcal{X}_i(t) = \{\hat{x}_i(0:t)\} \bigcup_{j\in\Omega\setminus\{i\}} \{\hat{x}_j(0:t-\tau_{ij}(t))\}.$$

Then, the controller is designed based on the information set: $\mathcal{L}_i(t) = \Xi_i(t) \cup \mathcal{X}_i(t)$, that is,

$$u_i(t) = \tilde{\mu}_i(t, \mathcal{L}_i(t)) \tag{22}$$

where we restrict that $\tilde{\mu}_i(t,\cdot)$ is chosen to be a linear function.

Remark 12: The information set $\Xi_i(t)$ is not used for estimation, but is used to construct the controller directly. Thus, $\Xi_i(t)$ is not discarded.

In this section, our aim is to design the optimal controller of the form (22) to minimize the cost function (7). That is, to solve the following problem.

Problem 2:

$$\min_{\tilde{\mu}_i(t,\cdot)} J$$
$$\text{s.t.}\quad (1),\ (2),\ (22).$$

Remark 13: The information structure induced by $\mathcal{L}_i(t)$ may be nonpartially nested, because $\mathcal{L}_j(t-1) \nsubseteq \mathcal{L}_i(t)$. Thus, the optimal control law $\tilde{\mu}_i(t,\cdot)$ may be nonlinear ([33, Th. 2]). Nevertheless, for implementation simplicity in engineering practice, we focus on designing the linear $\tilde{\mu}_i(t,\cdot)$ in this paper.

A. Offline Design

1) Form of the Optimal Controller: In this section, we solve Problem 2 in an offline fashion ($\Upsilon(t)$ is unknown). Because $\tilde{\mu}_i(t,\cdot)$ is chosen linear, the controller input (22) can be rewritten as

$$u_i(t) = [F(t)\circ\Gamma_\Upsilon(t)]_i y(t) + \eta_i(t, \mathcal{X}_i(t)),\quad i\in\Omega \tag{23}$$

where $[F(t)\circ\Gamma_\Upsilon(t)]_i$ is the ith row of $[F(t)\circ\Gamma_\Upsilon(t)]$.

To find the optimal controller to Problem 2, the form of the optimal controller is found as the following theorem shows.

Theorem 5: If $\tilde{u}^*(t)$ is the optimal solution to the following optimization problem:

$$\min_{\tilde{u}(t)} \quad \tilde{J} = \mathbb{E}\left\{\sum_{t=0}^{T}\hat{x}(t)^{\mathsf{T}}Q(t)\hat{x}(t) + \sum_{t=0}^{T-1}\tilde{u}(t)^{\mathsf{T}}R(t)\tilde{u}(t)\right\} \tag{24}$$
$$\text{subject to}\quad \hat{x}(t+1) = A(t)\hat{x}(t) + B(t)\tilde{u}(t) + \hat{\omega}_\Upsilon(t),$$
$$\tilde{u}(t) = \begin{bmatrix}\tilde{\eta}_1(t,\mathcal{X}_1(t))\\ \vdots \\ \tilde{\eta}_N(t,\mathcal{X}_N(t))\end{bmatrix}$$

where $\hat{\omega}_\Upsilon(t) = (B(t)(F(t)\circ\Gamma_\Upsilon(t)) + K(t))\phi(t)$, $\phi(t) = y(t) - C(t)\hat{x}(t)$, $\tilde{\eta}_i(t,\cdot)$ is a linear function for $i\in\Omega$, then the optimal control input for Problem 2 is of the form

$$u(t) = (F(t)\circ\Gamma_\Upsilon(t))\phi(t) + \tilde{u}^*(t). \tag{25}$$

Proof: This proof is similar to the first part of the proof of Theorem 1, so is omitted. ■

Remark 14: Compared to problem (39) in the proof of Theorem 1, the modified local control input $\tilde{u}(t)$ in problem (24) is of different form $\tilde{u}_i(t) = \tilde{\eta}_i(t,\mathcal{X}_i(t))$, where $\mathcal{X}_i(t)$ is incomplete information set due to the communication delay.

Define

$$\tilde{\Gamma}_\Upsilon \triangleq \begin{bmatrix}\gamma_{11}1_{l_1\times n_1} & \cdots & \gamma_{1N}1_{l_1\times n_N}\\ \vdots & \ddots & \vdots\\ \gamma_{N1}1_{l_N\times n_1} & \cdots & \gamma_{NN}1_{l_N\times n_N}\end{bmatrix} \tag{26}$$

$$H \triangleq \begin{bmatrix}H_{11} & \cdots & H_{1N}\\ \vdots & \ddots & \vdots\\ H_{N1} & \cdots & H_{NN}\end{bmatrix},\quad H_{ij}\in\mathbb{R}^{l_i\times n_j}. \tag{27}$$

From the expression of $\mathcal{X}_i(t)$, we have that the optimal $\tilde{u}(t)$ is of the form

$$\tilde{u}(t) = \left(H(t)\circ\tilde{\Gamma}_\Upsilon(t)\right)\hat{x}(t) + \chi\left(t, \hat{x}(0:t-1)\right) \tag{28}$$

where $\chi(t,\cdot)$ is a linear function.

To solve problem (24), we present a decomposition result via two lemmas first. In particular, both the estimated state $\hat{x}(t)$ and the control input $\tilde{u}(t)$ can be decomposed into two independent parts.

Lemma 1: Consider problem (24). The estimated state $\hat{x}(t)$ is decomposed into two parts

$$\hat{x}(t) = \underbrace{A(t-1)\hat{x}(t-1) + B(t-1)\tilde{u}(t-1)}_{\hat{x}^2(t)} + \underbrace{\hat{\omega}_\Upsilon(t-1)}_{\hat{x}^1(t)}$$

and $\hat{x}^1(t)$ is independent of $\hat{x}^2(t)$.

Proof: See Appendix C. ■

Lemma 2: Consider problem (24). The control input $\tilde{u}(t)$ in (28) is decomposed as

$$\tilde{u}(t) = \underbrace{(H(t)\circ\tilde{\Gamma}_\Upsilon(t))\hat{\omega}_\Upsilon(t-1)}_{\tilde{u}^1(t)} + \tilde{u}^2(t) \tag{29}$$

where $\tilde{u}^2(t) = \mathbb{E}[\tilde{u}(t) \mid \hat{x}(0:t-1), \Upsilon(0:t)]$ is a linear function of $\hat{x}(0:t-1)$. In addition, $\tilde{u}^1(t)$ is independent of $\tilde{u}^2(t)$.

Proof: See Appendix D. ■

According to the decomposition results established in Lemmas 1 and 2, finding the optimal control input (23) is reduced to construct two gain matrices. This is shown by the following theorem.

Theorem 6: Consider Problem 2. The optimal control input (23) to Problem 2 is of the form

$$u(t) = (F(t) \circ \Gamma_\Upsilon(t))\phi(t) + \tilde{u}^*(t) \tag{30}$$

$$\tilde{u}^*(t) = \Big(H(t) \circ \tilde{\Gamma}_\Upsilon(t)\Big)\hat{\omega}_\Upsilon(t-1) + \tilde{u}^{2*}(t) \tag{31}$$

$$\tilde{u}^{2*}(t) = L(t)\hat{x}^2(t)$$

for $t = 0, \ldots, T-1$, where $\tilde{u}^*(0) = 0$; the optimal gain matrices $F(t)$ and $H(t)$ are given by solving the following optimization problem:

$$\min_{F,H}\left(\sum_{t=0}^{T-2} h^1(t) + \sum_{t=0}^{T-1} h^2(t)\right) \tag{32}$$

where

$$\begin{aligned} h^1 = &\sum_{\Upsilon\in\Psi}\sum_{\Upsilon^+\in\Psi^+}\Pr(\Upsilon)\Pr\big(\Upsilon^+\big) \\ &\times \operatorname{tr}\Big(G^+\Big(\Big(H^+ \circ \tilde{\Gamma}^+_\Upsilon\Big) - L^+\Big)(B(F\circ\Gamma_\Upsilon) + K) \\ &\qquad \times Y(B(F\circ\Gamma_\Upsilon)+K)^\mathsf{T}\Big(\Big(H^+\circ\tilde{\Gamma}^+_\Upsilon\Big) - L^+\Big)^\mathsf{T}\Big) \\ h^2 = &\sum_{\Upsilon\in\Psi}\Pr(\Upsilon)\Big\{\operatorname{tr}\Big((F\circ\Gamma_\Upsilon)^\mathsf{T}R(F\circ\Gamma_\Upsilon)Y\Big) \\ &\qquad + \operatorname{tr}\Big(X^+(B(F\circ\Gamma_\Upsilon)+K)Y(B(F\circ\Gamma_\Upsilon)+K)^\mathsf{T}\Big)\Big\} \end{aligned}$$

where $G = B^\mathsf{T}X^+B + R$.

Proof: See Appendix E. ■

Remark 15: The decomposition results presented by Lemmas 1 and 2 are not the main contributions of this paper. Some similar results have been proposed in [21], [35], and [36].

For ease of notations, denote $h^1(T-1) = 0$, and define

$$h = h^1 + h^2.$$

An optimality condition to problem (32) is derived, as the following theorem presents.

Theorem 7: Consider problem (32). The optimal F and H satisfy

$$\begin{cases} \mathcal{F}_1 = 0 & (33a) \\ \mathcal{F}_2 = 0 & (33b) \end{cases}$$

where

$$\begin{aligned} \mathcal{F}_1 = &\sum_{\Upsilon\in\Psi}\sum_{\Upsilon^+\in\Psi^+}\Pr(\Upsilon)\Pr\big(\Upsilon^+\big) \\ &\times\Big(Y\Delta_1^\mathsf{T}\Big(X^+B + \Delta_2^\mathsf{T}G^+\Delta_2B\Big) + Y(F\circ\Gamma_\Upsilon)^\mathsf{T}R\Big)\circ\Gamma_\Upsilon^\mathsf{T} \\ \mathcal{F}_2 = &\sum_{\Upsilon\in\Psi}\sum_{\Upsilon^+\in\Psi^+}\Pr(\Upsilon)\Pr\big(\Upsilon^+\big) \\ &\times\Big(\Delta_1Y\Delta_1^\mathsf{T}\Delta_2^\mathsf{T}G^+\Big)\circ\big(\bar{\Gamma}^+_\Upsilon\big)^\mathsf{T} \end{aligned}$$

where $\Delta_1 = B(F\circ\Gamma_\Upsilon) + K$, $\Delta_2 = (H^+\circ\tilde{\Gamma}^+_\Upsilon) - L^+$.

Algorithm 1 Design of the Gains F and H

for $t = 0 : T-1$ **do**
  if $t \neq T-1$ **then**
    Initialization: Given the original values $F^{[0]}$, $H^{[0]}$ and choose a small constant $\varepsilon > 0$. Define $h\Big(F^{[-1]}, H^{+[-1]}\Big) = +\infty$. Let $k = 0$.
    while $\Big|h\Big(F^{[k-1]}, H^{+[k-1]}\Big) - h\Big(F^{[k]}, H^{+[k]}\Big)\Big| > \varepsilon$ **do**
      Set $k = k+1$.
      Let F in (33b) be $F^{[k-1]}$, obtain $H^{+[k]}$ by solving (33b).
      Let H^+ in (33a) be $H^{+[k]}$, obtain $F^{[k]}$ by solving (33a).
    end while
    $F^* = F^{[k]}$, $H^{+*} = H^{+[k]}$
  else
    Let $G(T) = 0$ in (33a). Then $F^*(T-1)$ is obtained by solving (33a) directly.
  end if
end for

Proof: The optimal F and H are obtained by solving $\partial h/\partial F = 0$ and $\partial h/\partial H = 0$. Similar to the derivation in (41), we can see that $\partial h/\partial F = 0$ and $\partial h/\partial H = 0$ give (33). This completes the proof. ■

Remark 16: Note that the gain of (29) has the sparse structure constraint $H(t)\circ\tilde{\Gamma}_\Upsilon(t)$ due to the information set $\mathcal{X}_i(t)$. Thus, $\hat{\omega}_\Upsilon$ is not a sufficient statistic for optimal decision. This implies that the optimal $H(t)$ depends on $\mathbb{E}(\hat{\omega}_\Upsilon\hat{\omega}_\Upsilon^\mathsf{T})$. From the definition of $\hat{\omega}_\Upsilon$, one has that the optimal $F(t-1)$ and $H(t)$ are coupled as problem (32) shows. Taking F and H as variables, the highest index of the variables in h is four. Thus, the function h with respect to F and H is nonconvex, and it is difficult to obtain the jointly optimal F, H to problem (32).

2) Computing the Gains F and H: Equation (33) is hard to be solved, because the highest index of the variables (F, H^+) in (33) is three. However, when F is given, the optimal H^+ can be obtained effectively by solving (33), and vice versa. Then, we exploit the following iterative algorithm (Algorithm 1) to compute the gain matrices F and H.

Now, we show that Algorithm 1 converges to a person-by-person optimal solution to the problem (32). The definition of the person-by-person optimal solutions is given as follows.

Definition 1 ([37] Person-by-Person Optimal Solutions): Consider problem (32), a pair of gains (F^*, H^{+*}) is person-by-person optimal if

$$h\Big(F^*, H^{+*}\Big) \le h\Big(F, H^{+*}\Big),\ \text{ for any } F \text{ defined in } (11)$$

$$h\Big(F^*, H^{+*}\Big) \le h\Big(F^*, H^{+}\Big),\ \text{ for any } H^+ \text{ defined in } (27).$$

Convergence Analysis: Given $F = F^{[k-1]}$, the optimal H^+ denoted by $H^{+[k]}$ is obtained by solving (33b), because h with respect to H^+ is convex when F is given. Thus, we have $h(F^{[k-1]}, H^{+[k]}) \le h(F^{[k-1]}, H^{+[k-1]})$. Similarly, we have $h(F^{[k]}, H^{+[k]}) \le h(F^{[k-1]}, H^{+[k]})$. As a result

$$h\Big(F^{[k]}, H^{+[k]}\Big) \le h\Big(F^{[k-1]}, H^{+[k-1]}\Big) \tag{34}$$

holds for all k. According to inequality (34) and Definition 1, it turns out that $\lim_{k\to+\infty}(F^{[k]}, H^{+[k]})$ is a person-by-person optimal solution to the problem (32). This indicates that Algorithm 1 converges to person-by-person optimum.

The optimal value of the cost function for the local estimation case can be concluded as follows.

Theorem 8: Consider Problem 2. The optimal value of the cost function (7) achieved by the controller (30) is

$$\begin{aligned} J^* = &\sum_{t=0}^{T-2} h^1\left(F^*, H^{+*}\right) \\ &+ \sum_{t=0}^{T-1}\left(h^2\left(F^*, H^{+*}\right)\right) + \sum_{t=0}^{T} \mathrm{tr}(QP) \end{aligned} \tag{35}$$

where $h^1(F^*, H^{+*})$ and $h^2(F^*, H^{+*})$ are the values of h^1 and h^2, respectively, by letting $F = F^*$, $H^+ = H^{+*}$.

Remark 17: Similar to the discussion in Remark 6, for the local estimation case, it also holds that the optimal value of the cost function with random delay is less than the corresponding one with deterministic delay under offline design.

B. Online Design

For online design, the realization of $\Upsilon(t)$, i.e., $\bar{\Upsilon}(t)$, is known to the designer. Thus, let $\Pr(\Upsilon(t) = \bar{\Upsilon}(t)) = 1$ and $\Pr(\Upsilon(t) \neq \bar{\Upsilon}(t)) = 0$. Then, we can design the gain matrices F and H by Algorithm 1 with a minor modification. Other parameters design methods are the same to the offline case.

Under local estimation, the optimal value of the cost function J has the same property as the one under global estimation case.

Theorem 9: Consider (35). Taking λ as a variable, the optimal value of the cost function $J^*(\lambda)$ is monotonically increasing as λ increases, when the gain matrices F^*, H^{+*} in h^1 and h^2 are designed online.

Proof: The proof is almost identical to the proof of Theorem 2, thus, is omitted. ∎

Remark 18: To minimize the cost function (7), the global estimation case is better than the local estimation case. However, under local estimation, the subsystem only needs to estimate its own estimated subsystem state. In other words, compared to global estimation, local estimation consumes less computational resources. Local estimation can be viewed as a tradeoff between the system performance and the computational resources.

V. Numerical Examples

In this section, two numerical examples are given to illustrate the effectiveness of the proposed theoretical results. Example 1 is used to show that the system state and output can be maintained in the neighborhood of the origin by the proposed controller (30). Example 2 illustrates that the optimal value of the cost function gets higher with a larger delay probability. In addition, Example 2 is used to compare our controller (13), (30) with the optimal controller designed by the approach in [14] and [34]. For simplicity, in these two examples, the controllers are designed online.

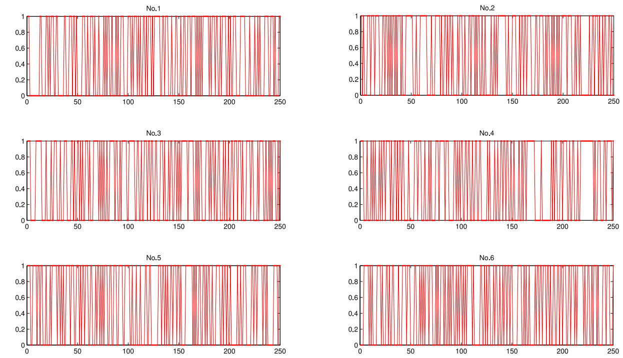

Fig. 2. Values of the random binary numbers τ_{12}, τ_{13}, τ_{21}, τ_{23}, τ_{31}, τ_{32} are illustrated by no.1–no.6, respectively.

Example 1: Consider a linear time-invariant large-scale system composed of three subsystems of the form (1) and (2). The system parameter matrices are given by

$$A = \begin{bmatrix} 0.9 & 0.2 & -0.9 & -0.5 & 0.2 & -0.1 \\ 0.1 & 0.3 & 0.6 & 0.2 & 0 & 0.3 \\ 0 & -0.3 & 0.4 & 0.6 & 0.8 & -0.1 \\ 0 & 0 & 0.2 & 0.3 & 0.4 & 0.1 \\ 0.3 & 0.2 & 0.1 & 0.5 & 0.7 & 0.8 \\ 0.2 & -0.3 & 0 & 0.1 & 0.2 & -0.1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The weight matrices Q and R in (7) are chosen to be

$$Q = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Assume that the noises ω and υ are zero-mean white Gaussian noises with identity covariance matrix. In order to highlight the simulation results, we choose the initial system state to be $x(0) = \begin{bmatrix} 2 & 2 & 2 & 2 & 2 & 2 \end{bmatrix}^{\mathrm{T}} \times 10^3$. Let the time horizon be $T = 250$. The delay indicator $\tau_{ij}(t)$ is generated randomly by $\lambda(t) = 0.5$ (see Fig. 2). We choose $\varepsilon = 0.002$ in Algorithm 1. The controller (30) was designed successfully according to Theorem 6 and Algorithm 1. The value of the control input is shown as Fig. 3.

The system states with controller (30) in contrast with the ones without control are simulated by Figs. 4–6. In addition, the measurement outputs with controller (30) compared to the ones without control are shown by Fig. 7. The simulation results show that the system is unstable without control. With the proposed controller (30), the system state and the control input are maintained in the neighborhood of the origin even though there are system noise and measurement noise. This illustrates the effectiveness of the proposed controller (30).

Example 2: Consider a large-scale system of the form (1) and (2). The parameters of the system and Algorithm 1 are the

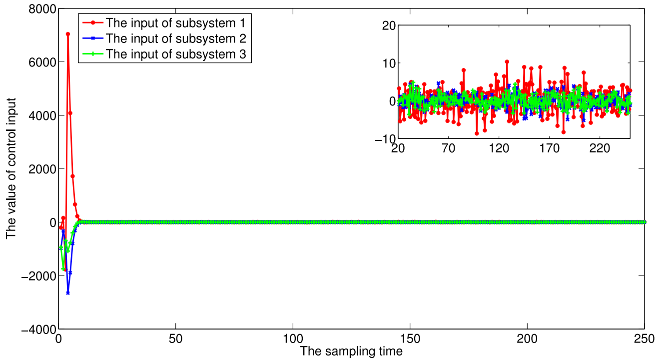

Fig. 3. Control input.

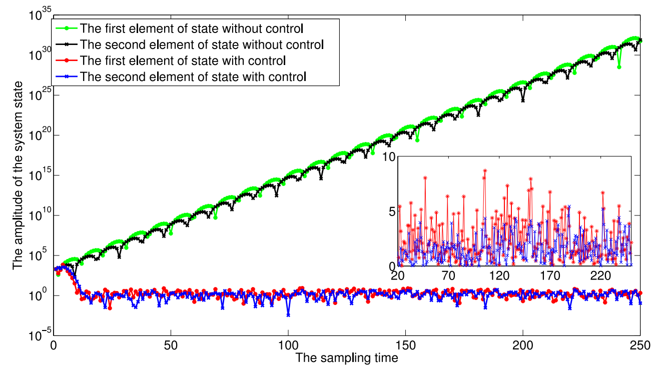

Fig. 4. State of subsystem 1.

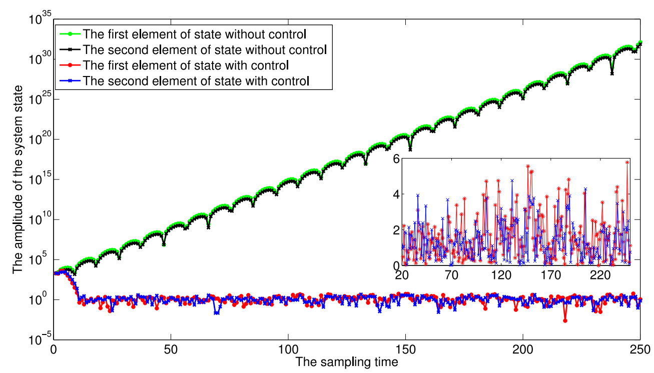

Fig. 5. State of subsystem 2.

same as in Example 1 except that the time horizon T and the delay indicator τ_{ij} are different. The time horizon is chosen to be $T = 50$. Recall that $\gamma_{ij} = 1 - \tau_{ij}$. We denote

$$\Upsilon(t) = \begin{bmatrix} 1 & \gamma_{12}(t) & \gamma_{13}(t) \\ \gamma_{21}(t) & 1 & \gamma_{23}(t) \\ \gamma_{31}(t) & \gamma_{32}(t) & 1 \end{bmatrix}.$$

Define $\daleth = [\gamma_{12}, \gamma_{13}, \gamma_{21}, \gamma_{23}, \gamma_{31}, \gamma_{32}]$, which has totally 64 possible value and is given by

$$\begin{aligned} \daleth_0 &= [0, 0, 0, 0, 0, 0] \\ \daleth_1 &= [0, 0, 0, 0, 0, 1] \\ \daleth_2 &= [0, 0, 0, 0, 1, 0] \\ &\vdots \\ \daleth_{63} &= [1, 1, 1, 1, 1, 1]. \end{aligned}$$

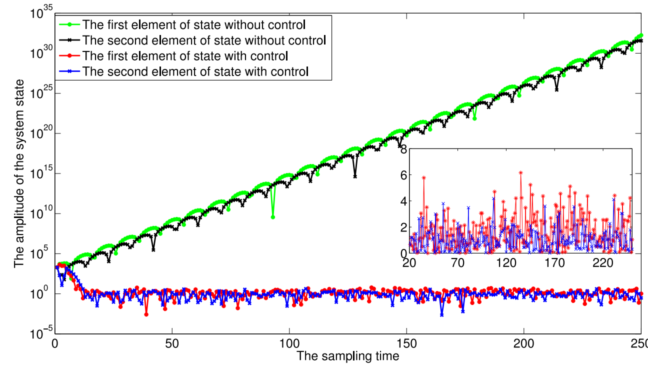

Fig. 6. State of subsystem 3.

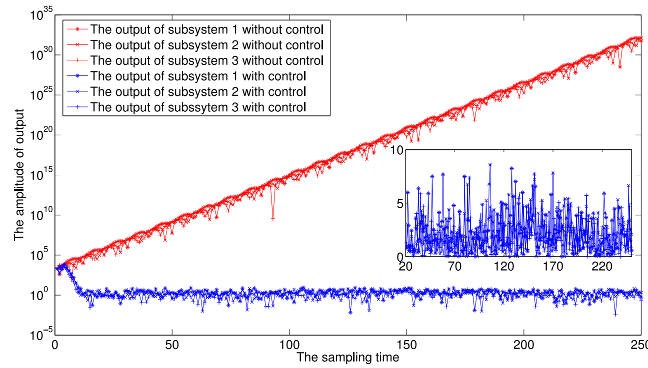

Fig. 7. Measurement output.

For simplicity, we assume that $\Upsilon(t_1) = \Upsilon(t_2)$ for all t_1, t_2. Recall that $\Pr(\gamma_{ij}(t) = 0) = \lambda$ and $\Pr(\gamma_{ij}(t) = 1) = 1 - \lambda$ for $i \neq j$. It follows that:

$$\begin{aligned} \Pr(\daleth_0) &= \lambda^6 \\ \Pr(\daleth_1) &= \lambda^5(1-\lambda) \\ \Pr(\daleth_2) &= \lambda^5(1-\lambda) \\ &\vdots \\ \Pr(\daleth_{63}) &= (1-\lambda)^6. \end{aligned}$$

The optimal value of the cost functions achieved by controller (13) and (30) are presented by the line marked by square and the red solid line in Fig. 8, respectively. The simulation result shows that the optimal value of the cost functions achieved by both (13) and (30) monotonically increase as the delay probability λ increases. The simulation result confirms the results in Theorems 4 and 9. Note that for both global and local estimation cases, J^* has the form $J^* = \sum_{i=0}^{N^2} \alpha_i \lambda^i$, where α_i is a scalar, and in this example, $\alpha_i \gg 1$. Then, J^* varies approximately linearly with respect to λ for $\lambda \in [0, 1]$.

We compare our controller (13), (30) and the optimal controllers in [14] and [34]. In [14], the optimal controller with one step delay sharing pattern was designed. The centralized controller and the centralized controller with one step delay were derived in [34]. The comparison result is given by Fig. 8. The centralized controller in [34] has the best performance as expected, because the full information is used. When $\lambda = 0$ ($\lambda = 1$), the value of the cost function with controller (13) is equal to the one with centralized controller (one

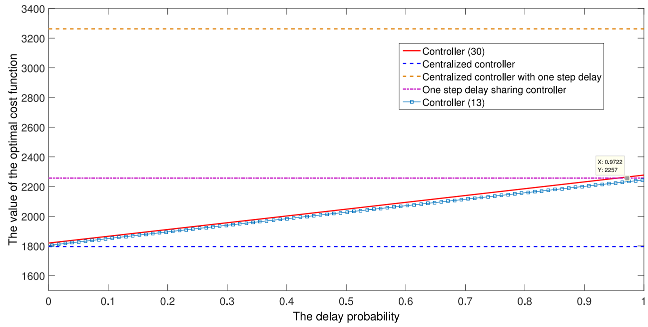

Fig. 8. Optimal value of the cost function with the delay probability.

step delay sharing controller). The reasons are discussed in Remarks 6 and 8. That is, when letting $\lambda = 0$ ($\lambda = 1$), the controller (13) is reduced to the centralized controller (one step delay sharing controller). Fig. 8 shows that the value of cost function with controller (30) is always close to the one with controller (13). When $\lambda = 0$, the value of the cost function with controller (30) is only a little higher than the one with the centralized controller. Compared to the one step delay sharing controller in [14], Fig. 8 shows that controller (30) is better when $\lambda < 0.9722$, and is only a little worse when $\lambda > 0.9722$. Compared to the centralized controller with one step delay in [34], controller (30) is much better even though the delay probability is $\lambda = 1$. The above comparison implies that the designed controllers (13) and (30) in this paper achieves good performances.

VI. Conclusion

This paper studied the optimal decentralized output-feedback control with a random information pattern. Both the global estimation case and the local estimation case were studied. Using Hadamard product method, the optimal controller under global estimation was designed by solving linear matrix equation. The one under the local estimation was designed by an algorithm. It was shown that the algorithm converges to a person-by-person optimum. For both cases, the optimal value of the cost function was derived. Moreover, it was verified that the optimal value of the cost function increases monotonously with respect to the delay probability. Finally, two numerical examples were given to illustrate the theoretical results.

Appendix A
Proof of Theorem 1

To prove Theorem 1, we need the following lemmas.

Lemma 3 [21]: Consider the system (3). The cost function J of the form (7) can be rewritten as

$$J = \underbrace{\sum_{t=0}^{T-1} \mathbb{E}\Big\{(u - Lx)^{\mathsf{T}} G (u - Lx)\Big\}}_{J_u} + \underbrace{x^{\mathsf{T}}(0)X(0)x(0) + \sum_{t=0}^{T-1} \operatorname{tr}\big(X^{+}\mathbb{W}\big)}_{J_\omega}, \tag{36}$$

where $\mathbb{W} = \mathbb{E}(\omega\omega^{\mathsf{T}})$; $G = B^{\mathsf{T}}X^{+}B + R$; L and X are defined in (19) and (20), respectively.

Lemma 4: For matrices $\mathbf{X} \in \mathbb{R}^{n\times m}$, $\mathbf{Y} \in \mathbb{R}^{m\times n}$, and $\mathbf{Z} \in \mathbb{R}^{m\times n}$, the following equality holds:

$$\operatorname{tr}(\mathbf{X}(\mathbf{Y} \circ \mathbf{Z})) = \operatorname{tr}\Big(\Big(\mathbf{X} \circ \mathbf{Y}^{\mathsf{T}}\Big)\mathbf{Z}\Big).$$

Proof: Denote $\mathbf{X} = [\mathbf{X}_{ij}]$, $\mathbf{Y} = [\mathbf{Y}_{ij}]$, and $\mathbf{Z} = [\mathbf{Z}_{ij}]$. We have

$$\begin{aligned}\operatorname{tr}(\mathbf{X}(\mathbf{Y} \circ \mathbf{Z})) &= \operatorname{tr}\big([\mathbf{X}_{ij}][\mathbf{Y}_{ij}\mathbf{Z}_{ij}]\big)\\ &= \sum_{i=1}^{n}(\mathbf{X}_{i1}\mathbf{Y}_{1i}\mathbf{Z}_{1i} + \cdots + \mathbf{X}_{im}\mathbf{Y}_{mi}\mathbf{Z}_{mi}).\\ \operatorname{tr}\Big(\Big(\mathbf{X} \circ \mathbf{Y}^{\mathsf{T}}\Big)\mathbf{Z}\Big) &= \operatorname{tr}\big([\mathbf{X}_{ij}\mathbf{Y}_{ji}][\mathbf{Z}_{ij}]\big)\\ &= \sum_{i=1}^{n}(\mathbf{X}_{i1}\mathbf{Y}_{1i}\mathbf{Z}_{1i} + \cdots + \mathbf{X}_{im}\mathbf{Y}_{mi}\mathbf{Z}_{mi}).\end{aligned}$$

The proof is completed. ∎

Now, we begin to prove Theorem 1.

Proof: According to the assertion and its proof in [14], we have that the optimal controller (8) is of the form

$$u(t) = (F(t) \circ \Gamma_{\Upsilon}(t))y(t) + Z(t)\hat{x}(t) \tag{37}$$

where $\hat{x}_t$ is given by Kalman filter (14), and $Z(t)$ is a gain matrix with proper dimension.

For offline design, $\Upsilon(t)$ is unknown. Define $e(t) = x(t) - \hat{x}(t)$, $\phi(t) = y(t) - C(t)\hat{x}(t)$, and $\tilde{u}(t) = [Z(t) - (F(t) \circ \Gamma_{\Upsilon}(t))]\hat{x}(t)$. Plug $x(t) = \hat{x}(t) + e(t)$, and $u(t) = (F(t) \circ \Gamma_{\Upsilon}(t))\phi(t) + \tilde{u}(t)$ into the cost function J, it follows from [14, Lemmas 1 and 2], we have:

$$\begin{aligned}J = \mathbb{E}\Bigg\{&\sum_{t=0}^{T-1}\Big(\hat{x}(t)^{\mathsf{T}}Q(t)\hat{x}(t) + \tilde{u}(t)^{\mathsf{T}}R(t)\tilde{u}(t)\Big)\\ &+ \hat{x}(T)^{\mathsf{T}}Q(T)\hat{x}(T) + \sum_{t=0}^{T} e(t)^{\mathsf{T}}Q(t)e(t)\Bigg\}\\ &+ \sum_{t=0}^{T-1}\sum_{\Upsilon(t)} \Pr(\Upsilon(t)) \operatorname{tr}\Big((F(t) \circ \Gamma_{\Upsilon}(t))^{\mathsf{T}}R(t)\\ &\times (F(t) \circ \Gamma_{\Upsilon}(t))Y(t)\Big).\end{aligned} \tag{38}$$

Note that, in (38), only $\tilde{J} = \mathbb{E}\{\sum_{t=0}^{T}\hat{x}(t)^{\mathsf{T}}Q(t)\hat{x}(t) + \sum_{t=0}^{T-1}\tilde{u}(t)^{\mathsf{T}}R(t)\tilde{u}(t)\}$ depends on $\tilde{u}(t)$. As a result, the optimal $\tilde{u}(t)$ minimizing (7) is the optimal solution to the following optimization problem:

$$\begin{aligned}&\min_{\tilde{u}(t)}\ \tilde{J}\\ &\text{subject to}\ \ \hat{x}(t+1) = A(t)\hat{x}(t) + B(t)\tilde{u}(t) + \hat{\omega}_{\Upsilon}(t)\\ &\qquad\qquad\quad \tilde{u}(t) = [H(t) - (F(t) \circ \Gamma_{\Upsilon}(t))]\hat{x}(t)\end{aligned} \tag{39}$$

where $\hat{\omega}_{\Upsilon}(t) = \big[B(t)(F(t) \circ \Gamma_{\Upsilon}(t)) + K(t)\big]\phi(t)$. According to Lemma 3 and the formulation of the mathematical expectation, we have

$$\tilde{J} = \underbrace{\sum_{t=0}^{T-1} \mathbb{E}\Big\{(\tilde{u} - L\hat{x})^{\mathsf{T}} G(\tilde{u} - L\hat{x})\Big\}}_{J_{\tilde{u}}} + \underbrace{\hat{x}^{\mathsf{T}}(0)X(0)\hat{x}(0) + \sum_{t=0}^{T-1}\sum_{\Upsilon\in\Psi} \Pr(\Upsilon)\,\mathrm{tr}(X^{+}\hat{\mathrm{W}}_{\Upsilon})}_{J_{\hat{\omega}}} \tag{40}$$

where $\hat{\mathrm{W}}_{\Upsilon} = (B(F\circ\Gamma_{\Upsilon}) + K)Y(B(F\circ\Gamma_{\Upsilon}) + K)^{\mathsf{T}}$. Note that only $J_{\tilde{u}}$ depends on $\tilde{u}(t)$, which implies that the optimal $\tilde{u}(t)$ minimizes J if and only if it minimizes $J_{\tilde{u}}$. Thus, the optimal $\tilde{u}(t)$ is $\tilde{u}(t) = [Z(t) - (F(t)\circ\Gamma_{\Upsilon}(t))]\hat{x}(t) = L(t)\hat{x}(t)$, because $J_{\tilde{u}} \geq 0$ always holds and $J_{\tilde{u}}$ achieves 0 by $\tilde{u}(t) = L(t)\hat{x}(t)$. This shows that $Z(t)\hat{x}(t) = (F(t)\circ\Gamma_{\Upsilon}(t))\hat{x}(t) + L(t)\hat{x}(t)$, thus the optimal $u(t)$ is of the form (13). In the following, we only need to prove that the optimal $F(t)$ is given by solving (17).

According to (38) and (40), all terms of J depending on $F(t)$ is retained in

$$\ell = \sum_{\Upsilon\in\Psi}\Pr(\Upsilon)\Big\{\mathrm{tr}((F\circ\Gamma_{\Upsilon})^{\mathsf{T}}R(F\circ\Gamma_{\Upsilon})Y) + \mathrm{tr}\Big(X^{+}(B(F\circ\Gamma_{\Upsilon}) + K)Y(B(F\circ\Gamma_{\Upsilon}) + K)^{\mathsf{T}}\Big)\Big\}.$$

The optimal $F(t)$ is given by $\partial\ell/\partial F(t) = 0$. That is,

$$\begin{aligned}\partial\ell &= \sum_{\Upsilon\in\Psi}\Pr(\Upsilon)\Big\{2\,\mathrm{tr}\Big(Y(F\circ\Gamma_{\Upsilon})^{\mathsf{T}}R\partial[(F\circ\Gamma_{\Upsilon})]\Big) + 2\,\mathrm{tr}\Big(Y(B(F\circ\Gamma_{\Upsilon}) + K)^{\mathsf{T}}X^{+}B\partial[(F\circ\Gamma_{\Upsilon})]\Big)\Big\}\\
&= \sum_{\Upsilon\in\Psi}\Pr(\Upsilon)\Big\{2\,\mathrm{tr}\Big(Y(F\circ\Gamma_{\Upsilon})^{\mathsf{T}}R(\Gamma_{\Upsilon}\circ\partial(F))\Big) + 2\,\mathrm{tr}\Big(Y(B(F\circ\Gamma_{\Upsilon}) + K)^{\mathsf{T}}X^{+}B(\Gamma_{\Upsilon}\circ\partial(F))\Big)\Big\}\\
&= \sum_{\Upsilon\in\Psi}\Pr(\Upsilon)\Big\{2\,\mathrm{tr}\Big(\Big((Y(F\circ\Gamma_{\Upsilon})^{\mathsf{T}}R)\circ\Gamma_{\Upsilon}^{\mathsf{T}}\Big)\partial(F)\Big) + 2\,\mathrm{tr}\Big(\Big((Y(B(F\circ\Gamma_{\Upsilon}) + K)^{\mathsf{T}}X^{+}B)\circ\Gamma_{\Upsilon}^{\mathsf{T}}\Big)\partial(F)\Big)\Big\}\end{aligned} \tag{41}$$

where the second equality follows from Lemma 4. From (41), one has that $\partial\ell/\partial F(t) = 0$ gives (17). The proof is completed. ■

Appendix B
Proof of Theorem 4

For ease of notations, we define the following sets.

1) $\mathcal{I} = \{(i,j) : i,j\in\Omega\}$.
2) Note that the set of all possible values of Υ is denoted by Ψ, where Ψ is defined in (12). We decompose all possible values of Υ into $\xi+1$ classes: $\Psi = \bigcup_{\alpha=0}^{\xi}\Psi^{(\alpha)}$, where $\xi = N^2$, and
$$\Psi^{(\alpha)} = \Big\{\Upsilon : \sum_{i,j\in\Omega}\gamma_{ij} = \alpha\Big\}.$$
$\Psi^{(\alpha)}$ is the set of Υ satisfying that there is exactly α different γ_{ij} taking value 1, and $\xi-\alpha$ different γ_{ij} taking value 0.
3) Let $\Lambda^{(\alpha)}$ denote the set of all matrices $\bar{F}$, where $\bar{F}$ satisfies $\bar{F} = F\circ\Upsilon$, F is defined in (11) and $\Upsilon\in\Psi^{(\alpha)}$. That is,
$$\Lambda^{(\alpha)} = \left\{\begin{array}{l}\bar{F} : \bar{F} = F\circ\Gamma_{\Upsilon},\\ \Upsilon\in\Psi^{(\alpha)}.\end{array}\right\}.$$
4) Let $\mathcal{I}^{(\alpha)}$ denote the set whose elements are the sets of the form $\{(i_1,j_1),\ldots,(i_\alpha,j_\alpha)\}$, where $i_\varsigma, j_\varsigma\in\Omega$, $\varsigma = 1,2,\ldots,\alpha$. That is,
$$\mathcal{I}^{(\alpha)} = \left\{\begin{array}{l}\underbrace{\{(i_1,j_1),\ldots,(i_\alpha,j_\alpha)\}}_{\alpha\text{ differrent two-tuples}}:\\ i_\varsigma, j_\varsigma\in\Omega,\ \text{for } \varsigma = 1,\ldots,\alpha.\end{array}\right\}.$$
5) Denote $|\mathcal{I}^{(\alpha)}| = \varrho_\alpha$, $\Pi_\alpha = \{1,\ldots,\varrho_\alpha\}$. Note that there are ϱ_α different sets of the form $\{(i_1,j_1),\ldots,(i_\alpha,j_\alpha)\}$ belonging to $\mathcal{I}^{(\alpha)}$. We denote them by $\Xi^{(\alpha)(1)},\ldots,\Xi^{(\alpha)(\varrho_\alpha)}$. For $m\in\Pi_\alpha$, we define
$$\Lambda^{(\alpha)(m)} = \left\{\begin{array}{l}\bar{F} : \bar{F} = F\circ\Gamma_{\Upsilon}\\ \gamma_{ij} = 1 \text{ if and only if } (i,j)\in\Xi^{(\alpha)(m)}\end{array}\right\}.$$
$\Lambda^{(\alpha)(m)}$ is a set whose elements are the matrices of the form $F\circ\Gamma_{\Upsilon}$, where Γ_{Υ} depends on Υ, and Υ satisfies that $\gamma_{ij} = 1$ if and only if $(i,j)\in\Xi^{(\alpha)(m)}$.

Note that,

$$\begin{aligned}\gamma_{ij} &= 1, \text{ if and only if } (i,j)\in\Xi^{(\alpha)(m)}\\ &\Longrightarrow \sum_{i,j\in\Omega}\gamma_{ij} = \alpha.\end{aligned}$$

This implies that $\Lambda^{(\alpha)(m)}\subseteq\Lambda^{(\alpha)}$. From the definition of $\Lambda^{(\alpha)(m)}$, we know that $\Lambda^{(\alpha)(m)}$ corresponds to a certain realization of $\Upsilon\in\Psi^{(\alpha)}$. Thus, we have $\Lambda^{(\alpha)} = \bigcup_{m\in\Pi_\alpha}\Lambda^{(\alpha)(m)}$.

Define $J^* = \sum_{t=0}^{T-1}\sum_{\Upsilon\in\Psi}\Pr(\Upsilon)J_1^* + \sum_{t=0}^{T}\mathrm{tr}(QP)$, where

$$J_1^* = \mathrm{tr}\Big(\big(F^*\circ\Gamma_{\Upsilon}\big)^{\mathsf{T}}R\big(F^*\circ\Gamma_{\Upsilon}\big)Y\Big) + \mathrm{tr}\Big(X^{+}\big(B\big(F^*\circ\Gamma_{\Upsilon}\big) + K\big)Y\big(B\big(F^*\circ\Gamma_{\Upsilon}\big) + K\big)^{\mathsf{T}}\Big).$$

Let $J_1^*(\Lambda^{(\alpha)(m)})$ denote the value of J_1^* in which $F\circ\Gamma_{\Upsilon}$ is chosen in $\Lambda^{(\alpha)(m)}$. Define $\bar{J}_1^*(\Lambda^{(\alpha)}) \triangleq (1/\varrho_\alpha)\sum_{m\in\Pi_\alpha}J_1^*(\Lambda^{(\alpha)(m)})$. It follows that:

$$\begin{aligned}\sum_{\Upsilon\in\Psi}\Pr(\Upsilon)J_1^* &= \sum_{\alpha=0}^{\xi}\sum_{\Upsilon\in\Psi^{(\alpha)}}\Pr(\Upsilon)J_1^*\\
&= \sum_{\alpha=0}^{\xi}\sum_{m\in\Pi_\alpha}\Pr\Big(\Lambda^{(\alpha)(m)}\Big)J_1^*\Big(\Lambda^{(\alpha)(m)}\Big)\\
&= \sum_{\alpha=0}^{\xi}(1-\lambda)^{\alpha}\lambda^{\xi-\alpha}\left(\sum_{m\in\Pi_\alpha}J_1^*\Big(\Lambda^{(\alpha)(m)}\Big)\right)\\
&= \sum_{\alpha=0}^{\xi}\varrho_\alpha(1-\lambda)^{\alpha}\lambda^{\xi-\alpha}\bar{J}_1^*\Big(\Lambda^{(\alpha)}\Big)\\
&= \sum_{\alpha=0}^{\xi}f_\alpha(\pi)\bar{J}_1^*\Big(\Lambda^{(\alpha)}\Big)\end{aligned}$$

where $\pi = 1-\lambda$, and $f_\alpha(\pi) = \varrho_\alpha\pi^{\alpha}(1-\pi)^{\xi-\alpha}$; the second equality follows from that $\Lambda^{(\alpha)(m)}$ corresponds to a certain realization of $\Upsilon \in \Psi^{(\alpha)}$; the third equality follows from that $\Pr(\Lambda^{(\alpha)(m)}) = (1-\lambda)^{\alpha}\lambda^{\xi-\alpha}$ holds for any $m \in \Pi_\alpha$; the fourth equality follows from the definition of $\bar{J}_1^*(\Lambda^{(\alpha)})$.

Now we divide our proof into two steps. First, we prove that $\bar{J}_1^*(\Lambda^{(\alpha+1)}) \leq \bar{J}_1^*(\Lambda^{(\alpha)})$. Second, we show that $\sum_{\alpha=0}^{\xi} f_\alpha(\pi)\bar{J}_1^*(\Lambda^{(\alpha)})$ is monotonically decreasing with the increase of π.

Before step 1, we first state a fact, that is

$$\text{If } \Xi^{(\alpha)(m)} \subseteq \Xi^{(\alpha+1)(k)}, \text{ then } \Lambda^{(\alpha)(m)} \subseteq \Lambda^{(\alpha+1)(k)}. \tag{42}$$

For example, given

$$F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \quad \Upsilon = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}.$$

When $\Xi^{(2)(1)} = \{(1,1),(1,2)\}$, $\Xi^{(3)(1)} = \{(1,1),(1,2),(2,1)\}$, then

$$\Lambda^{(2)(1)} = \left\{\begin{bmatrix} f_{11} & f_{12} \\ 0 & 0 \end{bmatrix} : f_{11}, f_{12} \in \mathbb{R}\right\}$$
$$\Lambda^{(3)(1)} = \left\{\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & 0 \end{bmatrix} : f_{11}, f_{12}, f_{21} \in \mathbb{R}\right\}.$$

It is shown that $\Lambda^{(2)(1)} \subseteq \Lambda^{(3)(1)}$.

Now, we continue the proof.

Step 1: Given any $\Xi^{(\alpha)(m)} \in \mathcal{I}^{(\alpha)}$, we know that

$$\Xi^{(\alpha)(m)} \cup \{(i_{\alpha+1}, j_{\alpha+1})\} \in \mathcal{I}^{(\alpha+1)} \tag{43}$$

where $\Xi^{(\alpha)(m)}$ is given, and $(i_{\alpha+1}, j_{\alpha+1}) \in \mathcal{I} \setminus \Xi^{(\alpha)(m)}$. Note that $(i_{\alpha+1}, j_{\alpha+1})$ taking value in $\mathcal{I} \setminus \Xi^{(\alpha)(m)}$ has totally $|\mathcal{I} \setminus \Xi^{(\alpha)(m)}| = \xi - \alpha$ possible values. Thus, without loss of generality, we denote the set of the form $\Xi^{(\alpha)(m)} \cup \{(i_{\alpha+1}, j_{\alpha+1})\}$ by $\Xi^{(\alpha+1)(m_1)}, \ldots, \Xi^{(\alpha+1)(m_{\xi-\alpha})}$, respectively. According to the fact that $\Xi^{(\alpha)(m)} \subseteq \Xi^{(\alpha+1)(k)}$ if and only if $k \in \{m_1, \ldots, m_{\xi-\alpha}\}$, and the fact (42), one has that

$$J_1^*\left(\Lambda^{(\alpha)(m)}\right) \geq J_1^*\left(\Lambda^{(\alpha+1)(k)}\right), \text{ for } k \in \Delta^m$$

where $\Delta^m = \{m_1, \ldots, m_{\xi-\alpha}\}$. This implies that

$$J_1^*\left(\Lambda^{(\alpha)(m)}\right) \geq \frac{1}{\xi-\alpha}\sum_{k\in\Delta^m} J_1^*\left(\Lambda^{(\alpha+1)(k)}\right).$$

On the other hand, given any $\Xi^{(\alpha+1)(k)} \in \mathcal{I}^{(\alpha+1)}$, we have

$$\Xi^{(\alpha+1)(k)} \setminus \{(i,j)\} \subseteq \Xi^{(\alpha+1)(k)} \tag{44}$$

where $(i,j) \in \Xi^{(\alpha+1)(k)}$, and $|\Xi^{(\alpha+1)(k)}| = \alpha+1$. Note that (i,j) taking value in $\Xi^{(\alpha+1)(k)}$ has totally $\alpha+1$ possible values. Denote the set of the form $\Xi^{(\alpha+1)(k)} \setminus \{(i,j)\}$ by $\Xi^{(\alpha)(k_1)}, \ldots, \Xi^{(\alpha)(k_{\alpha+1})}$, respectively. As a result, for any $k \in \Pi_{\alpha+1}$, $\Xi^{(\alpha+1)(k)} \supseteq \Xi^{(\alpha)(m)}$ holds if and only if $m \in \bar{\Delta}^k$, where $\bar{\Delta}^k = \{k_1, \ldots, k_{\alpha+1}\}$. According to (42), one has that

$$\begin{aligned}\sum_{m\in\Pi_\alpha} J_1^*\left(\Lambda^{(\alpha)(m)}\right) &\geq \frac{1}{\xi-\alpha}\sum_{m\in\Pi_\alpha}\sum_{k\in\Delta^m} J_1^*\left(\Lambda^{(\alpha+1)(k)}\right) \\ &= \frac{\alpha+1}{\xi-\alpha}\sum_{k\in\Pi_{\alpha+1}} J_1^*\left(\Lambda^{(\alpha+1)(k)}\right).\end{aligned} \tag{45}$$

Note that $\varrho_\alpha = |\ \mathcal{I}^\alpha\ | = ([\xi(\xi-1)\times\cdots\times(\xi-\alpha+1)]/[\alpha(\alpha-1)\times\cdots\times 1])$. Thus, $\varrho_{\alpha+1} = \varrho_\alpha \times ([\xi-\alpha]/[\alpha+1])$. The inequality (45) can be written as

$$\frac{1}{\varrho_\alpha}\sum_{m\in\Pi_\alpha} J_1^*\left(\Lambda^{(\alpha)(m)}\right) \geq \frac{1}{\varrho_{\alpha+1}}\sum_{k\in\Pi_{\alpha+1}} J_1^*\left(\Lambda^{(\alpha+1)(k)}\right)$$

which implies that $\bar{J}_1^*(\Lambda^{(\alpha+1)}) \leq \bar{J}_1^*(\Lambda^{(\alpha)})$.

Step 2: The interval [0, 1] can be decomposed into the form $[0, (1/\xi)] \cup [(1/\xi), (2/\xi)] \cup \cdots \cup [(\xi-1)/\xi, 1]$. For any $\pi \in [0,1]$, there exists a $\tilde{\alpha} \in \{0, \ldots, \xi\}$, such that $(\tilde{\alpha}/\xi) \leq \pi < \pi + \epsilon < ([\tilde{\alpha}+1]/\xi)$, where ϵ is a small positive number. In addition, we have

$$\begin{aligned}&f_\alpha(\pi+\epsilon) > f_\alpha(\pi),\ \alpha \in \{\tilde{\alpha}+1, \ldots, \xi\} \\ &f_\alpha(\pi+\epsilon) < f_\alpha(\pi),\ \alpha \in \{0, \ldots, \tilde{\alpha}\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{\alpha=0}^{\xi} f_\alpha(\pi+\epsilon)\bar{J}_1^*\left(\Lambda^{(\alpha)}\right) - \sum_{\alpha=0}^{\xi} f_\alpha(\pi)\bar{J}_1^*\left(\Lambda^{(\alpha)}\right) \\ &= \sum_{\alpha=0}^{\tilde{\alpha}}(f_\alpha(\pi+\epsilon) - f_\alpha(\pi))\bar{J}_1^*\left(\Lambda^{(\alpha)}\right) \\ &\quad + \sum_{\alpha=\tilde{\alpha}}^{\xi}\left(f_\alpha(\pi+\epsilon) - f_\alpha(\pi)\bar{J}_1^*\left(\Lambda^{(\alpha)}\right)\right. \\ &< \left[\sum_{\alpha=0}^{\tilde{\alpha}}(f_\alpha(\pi+\epsilon) - f_\alpha(\pi))\right. \\ &\qquad \left. + \sum_{\alpha=\tilde{\alpha}}^{\xi}(f_\alpha(\pi+\epsilon) - f_\alpha(\pi))\right]\bar{J}_1^*\left(\Lambda^{(\tilde{\alpha})}\right) \\ &= 0\end{aligned}$$

where the last equality holds from $\sum_{\alpha=0}^{\xi} f_\alpha(x) = 1$ for $\forall x \in [0,1]$. Hence, $\sum_{\alpha=0}^{\xi} f_\alpha(\pi)\bar{J}_1^*(\Lambda^{(\alpha)})$ is monotonically decreasing with respect to π, $\pi \in [0,1]$. This means that J^* is monotonically increasing with respect to λ. The proof is completed.

Appendix C
Proof of Lemma 1

First, $\hat{x}^1(t)$ is a linear function of $\phi(t-1)$. Second, $\hat{x}^2(t)$ is a linear function of $\hat{x}(t-1)$, where $\hat{x}(t-1)$ is linear combinations of $y(0:t-2)$. Thus, the independence of $\hat{x}^1(t)$ and $\hat{x}^2(t)$ is true following from Lemma 3 in [14].

Appendix D
Proof of Lemma 2

$\tilde{u}^2(t)$ has the following form:

$$\begin{aligned}\tilde{u}^2(t) &= \mathbb{E}\left[\tilde{u}(t) \mid \hat{x}(0:t-1), \Upsilon(0:t)\right] \\ &= \mathbb{E}\Big[\left(H(t) \circ \tilde{\Gamma}_\Upsilon(t)\right)\hat{x}(t) \\ &\quad + \chi\left(t, \hat{x}(0:t-1)\right) \mid \hat{x}(0:t-1), \Upsilon(0:t)\Big] \\ &= \mathbb{E}\left[\left(H(t) \circ \tilde{\Gamma}_\Upsilon(t)\right)\hat{x}(t) \mid \hat{x}(0:t-1), \Upsilon(0:t)\right] \\ &\quad + \chi\left(t, \hat{x}(0:t-1)\right).\end{aligned}$$

Thus, we have

$$\begin{aligned}
\tilde{u}^1(t) &= u(t) - \tilde{u}^2(t) \\
&= \Big(H(t) \circ \tilde{\Gamma}_\Upsilon(t)\Big)\hat{x}(t) + \hbar\big(t, \hat{x}(0:t-1)\big) \\
&\quad - \mathbb{E}\Big[\Big(H(t) \circ \tilde{\Gamma}_\Upsilon(t)\Big)\hat{x}(t) \mid \hat{x}(0:t-1), \Upsilon(0:t)\Big] \\
&\quad - \chi\big(t, \hat{x}(0:t-1)\big) \\
&= \Big(H(t) \circ \tilde{\Gamma}_\Upsilon(t)\Big)\Big(A(t-1)\hat{x}(t-1) + B(t-1)\tilde{u}(t-1) \\
&\quad + \hat{\omega}_\Upsilon(t-1)\Big) - \Big(H(t) \circ \tilde{\Gamma}_\Upsilon(t)\Big)\Big(A(t-1)\hat{x}(t-1) \\
&\quad + B(t-1)\tilde{u}(t-1)\Big) \\
&= \Big(H(t) \circ \tilde{\Gamma}_\Upsilon(t)\Big)\hat{\omega}_\Upsilon(t-1).
\end{aligned}$$

Note that $\tilde{u}^1(t)$ is a linear function of $\phi(t-1)$. We know that $\tilde{u}^2(t)$ is a linear function of $\hat{x}(0:t-1)$, and that $\hat{x}(0:t-1)$ is linear combinations of $y(0:t-2)$. As a result, $\tilde{u}^2(t)$ is linear combinations of $y(0:t-2)$. It follows from [14, Lemma 3] that $\tilde{u}^1(t)$ is independent of $\tilde{u}^2(t)$. The proof is completed.

Appendix E
Proof of Theorem 6

We only need to prove that the optimal $\tilde{u}(t)$ is of the form (31), and the optimal F and H are given by solving (32).

According to Lemmas 1 and 2, we have that $\hat{x}^1(t)$ is independent of $\hat{x}^2(t)$, and $\tilde{u}^1(t)$ is independent of $\tilde{u}^2(t)$. Furthermore, $\hat{x}^i(t)$ is independent of $\tilde{u}^j(t)$, $i \neq j$, because $\tilde{u}^j(t)$ is a function of $\hat{x}^j(t)$. In addition, following Lemma 3, the cost function $\tilde{J}$ in (24) can be written by (40). Note that only $J_{\tilde{u}}$ depends on $\tilde{u}(t)$, where $J_{\tilde{u}}$ is defined in (40). Thus, to derive the optimal $\tilde{u}(t)$, we only need to deal with $J_{\tilde{u}}$. Plug $\tilde{u}(t) = \tilde{u}(t)^1 + \tilde{u(t)}^2$ and $\hat{x}(t) = \hat{x}(t)^1 + \hat{x}(t)^2$ into $J_{\tilde{u}}$, we obtain

$$\begin{aligned}
J_{\tilde{u}} &= \underbrace{\sum_{t=0}^{T-1} \mathbb{E}\Big\{\Big(\tilde{u}^1 - L\hat{x}^1\Big)^\mathsf{T} G\Big(\tilde{u}^1 - L\hat{x}^1\Big)\Big\}}_{J_{\tilde{u}}^1} \\
&\quad + \underbrace{\sum_{t=0}^{T-1} \mathbb{E}\Big\{\Big(\tilde{u}^2 - L\hat{x}^2\Big)^\mathsf{T} G\Big(\tilde{u}^2 - L\hat{x}^2\Big)\Big\}}_{J_{\tilde{u}}^2}.
\end{aligned}$$

Then, the problem (24) can be decomposed into two independent subproblems, that is,

$$\begin{aligned}
&\min_{\tilde{u}^1} \; J_{\tilde{u}}^1 \\
&\text{subject to } \; \tilde{u}^1(t) = \Big(H(t) \circ \tilde{\Gamma}_\Upsilon(t)\Big)\hat{\omega}_\Upsilon(t-1)
\end{aligned} \tag{46}$$

and

$$\begin{aligned}
&\min_{\tilde{u}^2} \; J_{\tilde{u}}^2 \\
&\text{subject to } \; \tilde{u}^2(t) = \bar{\chi}\big(t, \hat{x}(0:t-1)\big)
\end{aligned} \tag{47}$$

where $\bar{\chi}(t, \cdot)$ is a linear function.

Consider problem (47), both $\tilde{u}^2(t)$ and $\hat{x}^2(t)$ are linear functions of $\hat{x}(0:t-1)$. Thus, problem (47) is a centralized LQG control problem. Using the result in [34], we have

$$\tilde{u}^{2*}(t) = L(t)\hat{x}^2(t), \; \text{and} \; J_{\tilde{u}}^{2*} = 0.$$

It follows from $\hat{x}^1(0) = 0$ and $\hat{x}^2(0) = 0$ that $\tilde{u}^*(0) = 0$. As a result, the optimal $\tilde{u}(t)$ is of the form (31).

Consider problem (46), one has

$$\begin{aligned}
J_{\tilde{u}}^1 &= \sum_{t=1}^{T-1} \mathbb{E}\Big\{\Big(\Big(\Big(H \circ \tilde{\Gamma}_\Upsilon\Big) - L\Big)\hat{\omega}_\Upsilon^-\Big)^\mathsf{T} G\Big(\Big(\Big(H \circ \tilde{\Gamma}_\Upsilon\Big) - L\Big)\hat{\omega}_\Upsilon^-\Big)\Big\} \\
&= \sum_{t=0}^{T-2} \sum_{\Upsilon \in \Psi} \sum_{\Upsilon^+ \in \Psi^+} \Pr(\Upsilon) \Pr\big(\Upsilon^+\big) \\
&\quad \times \operatorname{tr}\Big(G^+\Big(\Big(H^+ \circ \tilde{\Gamma}_\Upsilon^+\Big) - L^+\Big) \\
&\quad \times (B(F \circ \Gamma_\Upsilon) + K)Y(B(F \circ \Gamma_\Upsilon) + K)^\mathsf{T} \\
&\quad \times \Big(\Big(H^+ \circ \tilde{\Gamma}_\Upsilon^+\Big) - L^+\Big)^\mathsf{T}\Big)
\end{aligned} \tag{48}$$

where the superscript "−" means that the time index is $t-1$. From (48), we can see that $H(t)$ and $F(t-1)$ are coupled, because $\hat{\omega}_\Upsilon(t-1)$ depends on $F(t-1)$. Hence, $H(t)$ and $F(t-1)$ cannot be designed separately.

Following (38), (40), (48), and $J_{\tilde{u}} = J_{\tilde{u}}^1$ ($J_{\tilde{u}}^{2*} = 0$), one has that all terms in the cost function J which depend on F and H are retained in $\sum_{t=0}^{T-2} h^1 + \sum_{t=0}^{T-1} h^2$. As a result, the optimal F and H are given by solving problem (32). The proof is completed.

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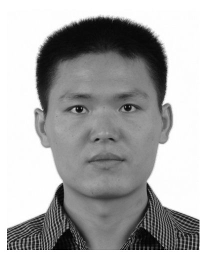

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