Lyapunov Conditions for Stability of Stochastic Impulsive Switched Systems

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***Abstract*— This paper studies stability of stochastic impulsive switched systems. Different from exponential Lyapunov function and average dwell-time in the previous works, general Lyapunov function and fixed dwell-time are implemented in this paper to analyze input-to-state stability and global stability of stochastic impulsive switched systems. Two cases are investigated, that is, the case that the continuous dynamics is stable and the case that the discrete dynamics is stable. To implement multiple Lyapunov functions, there are two subcases considered in this paper: the subcase that the estimates on the derivatives of the multiple Lyapunov functions are the same and the subcase that the growths of the multiple Lyapunov functions by the jumps are the same. For aforementioned different cases, sufficient stability conditions are established for stochastic impulsive switched systems. Finally, the developed theory is illustrated through examples from networked control systems and synchronization problem of 3-D novel chaotic circuit systems.**

***Index Terms*— Fixed dwell-time, impulsive switched systems, stochastic hybrid systems, stochastic stability, Lyapunov function.**

I. Introduction

HYBRID systems are dynamical systems involving both continuous-time and discrete-time behaviors; see survey paper [1]. There are two important types of hybrid systems: impulsive systems and switched systems. Impulsive systems are composed of continuous-time dynamics with instantaneous state jumps [2], [3]. Switched systems consist of a family of subsystems and a switching signal that orchestrates the switching among them [4]. In practice, a rich range of physical and man-made systems could be modeled as impulsive systems or switched systems, such as networked control systems [5], [6], fuzzy systems [7], electronic circuit systems [8]–[10], Lorenz systems [11], aircraft [12] and medical systems [13], [14].

In the real world, both the impulses and the switches may appear synchronously in physical systems [3], [8], [10], [15]. For instance, the phenomena of impulsive switching in non-autonomous piecewise constant circuit have been studied in [10]. Impulsive switching control has been addressed in [8] for synchronization problem of chaotic systems. On the other hand, both continuous dynamics and discrete dynamics may be affected by random noises. For instance, the information transmission in communication network may be subjected to random dropouts or congestions [6]. The switched-capacitor circuits have the intrinsic noises generated in the MOS transistors and the extrinsic noises originating from the on-chip digital circuitry [16]. Therefore, a more general and comprehensive system model is proposed, that is, stochastic impulsive switched systems [17], [18], which include those in [19]–[25] as the special cases. As a fundamental problem, different stability properties have been studied for both impulsive systems and switched systems. See [22]–[24] for impulsive systems, [4], [25]–[28] for (stochastic) switched systems, and [17], [18], [29] for impulsive switched systems. Some salient results have been applied in practical systems like chaotic systems [8], [9], [11], neural network [30]–[32], secure communication [33] and complex network [5], [6], [29], [34]. However, there are few works on stochastic impulsive switched systems, which are the study subject in this paper.

To study stability of impulsive/switched systems, Lyapunov-based approach and dwell-time condition are commonly used in the literature. In terms of Lyapunov function, both Lyapunov-Krasovskii functions (LKFs) [18], [20], [34] and Lyapunov-Razumikhin functions (LRFs) [19], [27], [35] are involved. For instance, input-to-state stability (ISS) for impulsive switched delayed systems has been considered in [18] based on LKF. Using both LKF and LRF, ISS of interconnected impulsive systems has been addressed in [35]. However, the Lyapunov functions used in the previous works [18]–[20], [25]–[27], [34]–[36] are exponential, which limits the applicable range of the obtained results. On the other hand, different types of dwell-time have been proposed in the past decades, such as minimum dwell-time [17], [24] (also called fixed dwell-time (FDT) in [4]), maximum dwell-time [17], [24], average dwell-time (ADT) [4], [20], [23], [34], ranged dwell-time [17] and constant dwell-time [17]. Based on these types of dwell-time, stability conditions have been established for impulsive systems [17], [23], [24] and switched systems [4], [27]. However, as far as we know, except for average dwell-time [20], [23], [36], up to now, there is no work using other types of dwell-time to study stability of stochastic impulsive switched systems.

The objective of this paper is to study stability of stochastic impulsive switched systems using multiple general Lyapunov

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functions and fixed dwell-time (FDT) condition. Sufficient conditions are established for ISS and global stability of stochastic impulsive switched systems. In comparison to the previous works like [19]–[21], [23], [35]–[37], the contributions of this paper are three-fold. (**I**) The combination of the impulses and the switches and the external inputs (e.g., control input and random noises) are studied in this paper. As a result, two cases are considered: the case that the continuous dynamics is stable and the case that the discrete dynamics is stable. For such two cases, stability conditions are established. (**II**) Different from the previous works [19], [20], [23], [35]–[37] using the exponential Lyapunov functions and ADT, the stability analysis in this paper is based on general Lyapunov functions and FDT condition, which are motivated by [21], [24], [38]. Because exponential Lyapunov functions are not necessarily existent or easy to be constructed [24], the exponential assumption limits the classes of the dynamic systems that could be studied via Lyapunov approaches. In addition, on contrary to the ADT which limits the impulses or switches on average, the FDT provides lower bound for impulsive switching intervals. In some dynamic systems like networked control systems [5], [6], the bound of the transmission intervals (i.e, FDT or reverse FDT [24], [27]) is needed, and more useful than the ADT [4], [25]. Furthermore, the FDT in this paper is not given *a priori* but can be calculated and adjusted. Therefore, our results are more general and allowed to study a richer class of stochastic impulsive switched systems, even those that could not be analyzed via exponential Lyapunov functions; see Example 1 in Section IV. A special case of our results was already used in [21] to analyze stability of stochastic impulsive systems. (**III**) Due to the switching and impulsive effects on general Lyapunov functions, we cannot implement multiple general Lyapunov functions directly. To avoid such a difficulty, we consider the following two cases in this paper: the case that the estimates on the derivatives of the multiple Lyapunov functions are the same and the case that the growths of the multiple Lyapunov functions by the jumps are the same. As a result, this paper extends the previous work [22], which is on deterministic impulsive switched systems and only considered the latter case; see also Remark 5.

The obtained results provide an alternative approach for stability analysis and could be applied in various fields, such as stability of neural networks [29], [31], synchronization of chaotic and/or hyperchaotic circuits [8], [33], stabilization of Lorenz systems under impulsive switched control [11], tracking control of networked control systems [39]. Furthermore, the range of the applicability of the obtained results includes not only linear or piecewise constant system models like in [8], [11], [32], and [33], but also nonlinear, parameter uncertain, coupled and even complex system models. In this paper, synchronization problem of switched 3-D novel chaotic circuit systems and the estimation of remote networked control systems are presented to illustrate the derived results.

The rest of this work is organized as follows. In Section II, the considered problem is formulated and some preliminaries are given. Based on general multiple Lyapunov functions and FDT condition, the stability conditions are derived in Section III for stochastic impulsive switched systems in different cases. Two numerical examples are given in Section IV to illustrate the derived results. Finally, conclusions and future works are stated in Section V.

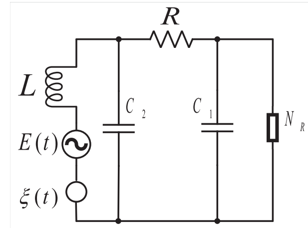

Fig. 1. Chua's circuit driven by noise. N_R is the nonlinear resistor; $E(t)$ is the external deterministic forcing signal; $\xi(t)$ is the noise.

Notation: $\mathbb{R} := (-\infty, +\infty)$; $\mathbb{R}_{>t} := (t, +\infty)$ and $\mathbb{R}_{\geq t} := [t, +\infty)$ for a given $t \geq 0$. $\mathbb{N} := \{0, 1, \ldots\}$; $\mathbb{N}_{>0} := \{1, 2, \ldots\}$. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $|\cdot|$ represents the Euclidean vector norm; $\mathbb{P}\{\cdot\}$ denotes the probability measure; $\mathbb{E}[\cdot]$ denotes the mathematical expectation. For a given vector or matrix A, $A^\top$ denotes its transpose. Given a square matrix A, $\mathrm{tr}[A]$ denotes its trace; $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ are the minimum and maximum eigenvalues of A, respectively. Id denotes the identity function. $\mathcal{P}$ denotes the set of the functions that are continuous, zero at zero, and positive definite in $\mathbb{R}_{\geq 0}$. $\mathcal{C}^{1,2}$ stands for the space of the functions that are continuously differentiable on the first augment and continuously twice differentiable on the second augment. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it belongs to class $\mathcal{P}$ and strictly increasing; $\alpha(t)$ is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{KL}$ if $\beta(s, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s, t)$ decreases to zero as $t \to \infty$ for each fixed $s \geq 0$. $\mathcal{L}_\infty^n$ denotes the set of all the measurable and locally essentially bounded signal $x \in \mathbb{R}^n$ on $\mathbb{R}_{\geq t_0}$ with norm $\|x\| := \sup_{t \geq t_0} \inf_{\{\mathcal{A} \subset \Omega, \mathbb{P}\{\mathcal{A}\}=0\}} \sup\{|x(t, \omega)| | \omega \in \Omega \setminus \mathcal{A}\}$.

II. Problem Formulation

There are different ways to model the circuit systems according to the structure of the circuit systems and the expected performances. For instance, as shown in Fig. 1, the nonautonomous Chua's circuit has the nonlinear resistor N_R, the external input $E(t)$ and the noise $\xi(t)$. If the nonlinear resistor is piecewise continuous and the noise is uncertain but bounded, then the system model is deterministic [8], [33]. If the noise is random, then the system model is stochastic. Furthermore, to generate different attractors by varying the frequency of the external input, the modeled systems are switched systems [40, Ch. 4]. To establish synchronization of chaotic systems through impulsive (switched) control, the constructed system is impulsive (switched) systems; see e.g., [8], [11], [33]. Considering these possible phenomena in physical systems like chaotic systems and neural systems, we study a general system model in this paper, that is, stochastic impulsive switched system with Markovian switching:

$$\begin{cases} dx(t) = f_{\sigma(t)}(t, x, u)dt \\ \qquad\quad + g_{\sigma(t)}(t, x, u)dB(t), \quad t \in \mathbb{R}_{\geq t_0} \backslash \mathcal{T}, \\ x(t) = h_{\sigma(t)}(x(t^-), u(t^-)), \quad t \in \mathcal{T}, \end{cases} \tag{1}$$

where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^{n_x}$ is the system state, $u(t) \in \mathcal{U} \subseteq \mathcal{L}_\infty^{n_u}$ is the external input, $B(t) \in \mathbb{R}^{n_w}$ is a Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P})$. Denote $x(t^-) := \lim_{s\nearrow t} x(s)$. A strictly increasing and infinite set $\mathcal{T} := \{t_1, t_2, \ldots\}$ is called an impulsive switching time sequence, in which the switching and impulses occur synchronously. The switching signal $\sigma : \mathbb{R}_{\geq t_0} \to \mathcal{M} = \{1, \ldots, M\}$ is a piecewise right-continuous Markov chain on the probability space taking values in a finite state space $\mathcal{M}$ with the generator $\Pi = [\pi_{ij}]$, $i, j \in \mathcal{M}$, given by

$$\mathbb{P}\{\sigma(t+\Delta t) = j|\sigma(t) = i\} = \begin{cases} \pi_{ij}\Delta t + o(\Delta t), & i \neq j, \\ 1 + \pi_{ii}\Delta t + o(\Delta t), & i = j, \end{cases}$$

where $\Delta t > 0$, $\pi_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ and $\pi_{ii} = -\sum_{j\neq i} \pi_{ij}$. For each $i \in \mathcal{M}$, $f_i : \mathbb{R}_{\geq t_0} \times \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ and $g_i : \mathbb{R}_{\geq t_0} \times \mathcal{X} \times \mathcal{U} \to \mathbb{R}^{n_x\times n_w}$ are assumed to be continuous with respect to t, x, u and uniformly locally Lipschitz with respect to x, u. The function $h_i : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ is assumed to be continuous with respect to x, u. Moreover, assume that $f_i(\cdot, 0, 0) \equiv 0$, $g_i(\cdot, 0, 0) \equiv 0$ and $h_i(0, 0) \equiv 0$ for all $i \in \mathcal{M}$, which implies that the system (1) admits a trivial solution $x(t) \equiv 0$. Assume that given an initial condition, there exists a unique stochastic process satisfying the system (1); see [20], [41], [42].

Definition 1: Given a switching signal σ and an impulsive switching time sequence $\mathcal{T}$, the system (1) is *stochastically input-to-state stable (SISS)*, if for an arbitrary $\varepsilon \in (0, 1)$, there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all $x(t_0) \in \mathcal{X}$, $u \in \mathcal{U}$ and $t \in \mathbb{R}_{\geq t_0}$,

$$\mathbb{P}\{|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma(\|u\|)\} \geq 1 - \varepsilon. \quad (2)$$

Given a set $\mathcal{S}$ of admissible impulsive switching time sequences, if the system (1) is SISS for each $\mathcal{T} \in \mathcal{S}$ and β, γ do not depend on the choice of $\mathcal{T} \in \mathcal{S}$, then the system (1) is *uniformly SISS* over $\mathcal{S}$.

Definition 2: Given a switching signal σ and an impulsive switching time sequence $\mathcal{T}$, the system (1) is *stochastically globally stable (SGS)*, if for an arbitrary $\varepsilon \in (0, 1)$, there exist $\gamma_1, \gamma_2 \in \mathcal{K}_\infty$ such that for all $x(t_0) \in \mathcal{X}$ and $u \in \mathcal{U}$,

$$\mathbb{P}\{|x(t)| \leq \gamma_1(|x(t_0)|) + \gamma_2(\|u\|)\} \geq 1 - \varepsilon, \quad \forall t \in \mathbb{R}_{\geq t_0}. \quad (3)$$

Given a set $\mathcal{S}$ of admissible impulsive switching time sequences, if the system (1) is SGS for each $\mathcal{T} \in \mathcal{S}$ and γ_1, γ_2 do not depend on the choice of $\mathcal{T} \in \mathcal{S}$, then the system (1) is *uniformly SGS* over $\mathcal{S}$.

Definitions 1 and 2 are extensions of those for switched systems [4], [19], impulsive systems [20], [21], [23], [24] and impulsive switched systems [18], [22]. To investigate the stochastic stability properties of the system (1), we choose a family of multiple Lyapunov functions $\{V_i : \mathbb{R}_{\geq t_0} \times \mathcal{X} \to \mathbb{R}_{\geq 0} | V_i \in \mathcal{C}^{1,2}, i \in \mathcal{M}\}$. For each $i \in \mathcal{M}$, a differential operator of the function V_i and an SISS-Lyapunov function are defined as follows.

Definition 3 [41]: Given each $i \in \mathcal{M}$ and any $\mathcal{C}^{1,2}$ function $V_i : \mathbb{R}_{\geq t_0} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$, the *differential operator* $\mathscr{L}$, which is associated with the continuous differential equation in (1), is defined as

$$\begin{aligned}\mathscr{L}V_i(t, x) = &\frac{\partial V_i(t, x)}{\partial t} + \frac{\partial V_i(t, x)}{\partial x} f_i(t, x, u) \\ &+ \frac{1}{2}\operatorname{tr}\left[g_i^\top(t, x, u)\frac{\partial^2 V_i(t, x)}{\partial x^2} g_i(t, x, u)\right] \\ &+ \sum_{j\in\mathcal{M}} \pi_{ij} V_j(t, x).\end{aligned}$$

By mean of the differential operator $\mathscr{L}$ and Itô's formula in [43, Ch. IV. 3], [41, Th. 6.4], it obtains that for all $t \in \mathbb{R}_{\geq t_0}\backslash\mathcal{T}$,

$$dV_i(t, x(t)) = \mathscr{L}V_i(t, x(t))dt + \frac{\partial V_i(t, x(t))}{\partial x} g_i(t, x, u)dB(t),$$

and then it follows from the proofs of Lemma 1 and Theorem 1 in [36] that for all $t \in \mathbb{R}_{\geq t_0}\backslash\mathcal{T}$,

$$d\,\mathbb{E}[V_i(t, x(t))] = \mathbb{E}[\mathscr{L}V_i(t, x(t))]dt. \quad (4)$$

Definition 4: For each $i \in \mathcal{M}$, a $\mathcal{C}^{1,2}$ function $V_i : \mathbb{R}_{\geq t_0} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ is called an *SISS-Lyapunov function*, if there exist $\alpha_{1i}, \alpha_{2i}, \rho_i, \varphi_i \in \mathcal{K}_\infty$, $\psi_i \in \mathcal{P}$ such that for all $x(t_0) \in \mathcal{X}$, $u \in \mathcal{U}$,

$$\alpha_{1i}(|x|) \leq V_i(t, x) \leq \alpha_{2i}(|x|), \quad t \in \mathbb{R}_{\geq t_0}, \quad (5)$$

$$V_i(t, x) \geq \rho_i(\|u\|) \Rightarrow \begin{cases} \mathscr{L}V_i(t, x) \leq -\varphi_i(V_i(t, x)), & t \in \mathbb{R}_{\geq t_0}\backslash\mathcal{T}, \\ V_i(t, h_i(x, u)) \leq \psi_i(V_i(t, x)), & t \in \mathcal{T}. \end{cases} \quad (6)$$

In addition, if $\varphi_i(v) = c_i v$ and $\psi_i(v) = e^{-d_i} v$, where $c_i, d_i \in \mathbb{R}$, then V_i is an *exponential SISS-Lyapunov function*.

Remark 1: Definition 4 generalizes those for stochastic nonlinear systems in [36] and [41], that for stochastic impulsive systems in [21] and those for deterministic impulsive systems in [23] and [24]. It is known from [41], [44], and [45] that the SISS-Lyapunov function is equivalent to the SISS property for general stochastic nonlinear systems. However, this equivalent relationship is invalid for both impulsive systems and switched systems; see [21]–[24], [26], [35]–[38]. To establish SISS of impulsive and/or switched systems, some additional conditions are required, such as average dwell-time (ADT) conditions (e.g., [23], [36]) and fixed dwell-time (FDT) conditions (e.g., [21], [24], [27] and the proceeding section of this paper). □

III. Main Results

In this section, sufficient conditions are established to guarantee the stochastic stability properties of the system (1). Both the case that the continuous dynamics is stable and the case that the discrete dynamics is stable are studied. In the following, we start from the former case.

To this end, three classes of impulsive switching time sequences are introduced. In this paper, we use the notation $\mathcal{S}_{\text{all}}$ to represent the set of all the admissible impulsive switching time sequence $\mathcal{T}$. Moreover, given a constant $\theta > 0$, $\mathcal{S}_1(\theta)$ denotes the set of the impulsive switching time sequences satisfying $\inf_{k\in\mathbb{N}}\{t_{k+1} - t_k\} \geq \theta$ and $\mathcal{S}_2(\theta)$ denotes the set of the impulsive switching time sequences satisfying $\sup_{k\in\mathbb{N}}\{t_{k+1} - t_k\} \leq \theta$. Hence, $\mathcal{S}_1(\theta)$ implies that the impulsive switching intervals have a lower bound, whereas $\mathcal{S}_2(\theta)$ implies that the impulsive switching intervals have a upper bound. In the following, if such lower (or upper) bound satisfies certain condition, that is, FDT (or reverse FDT) condition, then the system (1) is stable.

A. The Case That the Continuous Dynamics Is Stable

In this subsection, we study the case that the continuous dynamics is stable. For both the subcase that φ_i are the same and the subcase that ψ_i are the same, sufficient conditions are derived for the stability of the system (1).

Theorem 1: Consider the system (1). Suppose for all $i \in \mathcal{M}$, $V_i : \mathbb{R}_{\geq t_0} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ is an SISS-Lyapunov function with $\alpha_{1i}, \alpha_{2i}, \rho_i, \varphi_i \in \mathcal{K}_\infty$, $\psi_i \in \mathcal{P}$, where $\varphi_i = \varphi$ is convex and ψ_i is concave. If there exist certain $\theta, \delta > 0$ such that for all $a > 0$,

$$\sup_{i \in \mathcal{M}} \int_a^{\psi_i(a)} \frac{ds}{\varphi(s)} \leq \theta - \delta, \tag{7}$$

and for all $t_{k-1}, t_k \in \mathcal{T}$, $k \in \mathbb{N}_{>0}$,

$$V_{\sigma(t_k)}(t_k, x(t_k)) \leq V_{\sigma(t_{k-1})}(t_k, x(t_k)), \tag{8}$$

then the system (1) is SISS over $\mathcal{S}_1(\theta)$. In addition, if $\psi_i \leq \mathrm{Id}$ for all $i \in \mathcal{M}$, then the system (1) is SISS over $\mathcal{S}_{\mathrm{all}}$.

Proof: Given a switching signal σ and certain $\mathcal{T} \in \mathcal{S}_1(\theta)$, SISS of the system (1) is established by the construction of the functions β and γ satisfying (2). As a result, the proof is divided into two cases, that is, the case $u(t) \equiv 0$ and the case $u(t) \neq 0$.

Case 1: $u(t) \equiv 0$. In this case, (6) is rewritten directly as

$$\begin{aligned} \mathscr{L}V_i(t,x) &\leq -\varphi(V_i(t,x)), \quad t \in \mathbb{R}_{\geq t_0} \backslash \mathcal{T}, \\ V_i(t, h_i(x,u)) &\leq \psi_i(V_i(t,x)), \quad t \in \mathcal{T}. \end{aligned} \tag{9}$$

Since $\varphi_i = \varphi \in \mathcal{P}$ is convex and $\psi_i \in \mathcal{P}$ is concave, it follows from (8) and Jensen's inequality in [43, Ch. II. 18.3] that

$$\mathbb{E}[\mathscr{L}V_i(t,x)] \leq -\varphi(\mathbb{E}[V_i(t,x)]), \quad t \in \mathbb{R}_{\geq t_0} \backslash \mathcal{T}, \tag{10}$$

$$\mathbb{E}[V_i(t, h_i(x,u))] \leq \psi_i(\mathbb{E}[V_i(t,x)]), \quad t \in \mathcal{T}. \tag{11}$$

For the impulsive switching interval $[t_k, t_{k+1})$, $k \in \mathbb{N}$, assume there exists $t' \in [t_k, t_{k+1})$ such that $\mathbb{E}[V_i(t', x(t'))] = 0$. According to the facts that $\mathbb{E}[V_i(t, x(t))] \geq 0$ for all $t > 0$ and that $x(t) = 0$ is an equilibrium point of the system (1), it follows from (10)-(11) that $\mathbb{E}[V_i(t, x(t))] \equiv 0$ for all $t \geq t'$. Next, we only need to study the scenario that $\mathbb{E}[V_i(t, x(t))] > 0$.

According to the FDT condition (7), define the following function

$$F(\varrho) := \int_\nu^\varrho \frac{ds}{\varphi(s)}, \quad \forall \varrho > 0, \tag{12}$$

where $\nu > 0$ is fixed. Observe that $F : \mathbb{R}_{>0} \to \mathbb{R}$ is continuous and strictly increasing, so is its inverse $F^{-1} : \mathbb{R} \to \mathbb{R}_{>0}$. Using (4) and integrating (10) from t_k to any $t \in [t_k, t_{k+1})$ yield that

$$\int_{t_k}^t \frac{\mathbb{E}[\mathscr{L}V_i(s, x(s))]ds}{\varphi(\mathbb{E}[V_i(s, x(s))])} \leq -(t - t_k). \tag{13}$$

According to (4), Itô's formula in [43, Ch. IV. 3], Fubini's Theorem in [43, Ch. II. 12.2] and (13), it follows that

$$\int_{\mathbb{E}[V_i(t_k, x(t_k))]}^{\mathbb{E}[V_i(t, x(t))]} \frac{ds}{\varphi(s)} \leq -(t - t_k), \quad \forall t \in [t_k, t_{k+1}).$$

That is, for all $t \in [t_k, t_{k+1})$,

$$F(\mathbb{E}[V_{\sigma(t_k)}(t, x(t))]) - F(\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))]) \leq -(t - t_k).$$

Because $t_k \in \mathcal{S}_1(\theta)$, letting $t \to t_{k+1}^-$ yields that

$$F(\mathbb{E}[V_{\sigma(t_k)}(t_{k+1}^-, x(t_{k+1}^-))]) - F(\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))]) \leq -\theta. \tag{14}$$

Combining (8), (11) and (14) gives that for any $t_k, t_{k+1} \in \mathcal{T}$ and $\sigma(t_k), \sigma(t_{k+1}) \in \mathcal{M}$,

$$\begin{aligned} &F(\mathbb{E}[V_{\sigma(t_{k+1})}(t_{k+1}, x(t_{k+1}))]) - F(\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))]) \\ &\leq F(\mathbb{E}[V_{\sigma(t_k)}(t_{k+1}, x(t_{k+1}))]) - F(\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))]) \\ &\leq F(\psi_{\sigma(t_k)}(\mathbb{E}[V_{\sigma(t_k)}(t_{k+1}^-, x(t_{k+1}^-))])) \\ &\quad - F(\mathbb{E}[V_{\sigma(t_k)}(t_{k+1}^-, x(t_{k+1}^-))]) \\ &\quad + F(\mathbb{E}[V_{\sigma(t_k)}(t_{k+1}^-, x(t_{k+1}^-))]) - F(\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))]) \\ &\leq \theta - \delta - \theta = -\delta. \end{aligned}$$

Repeating the above mechanism, one has that

$$F(\mathbb{E}[V_{\sigma(t_{k+1})}(t_{k+1}, x(t_{k+1}))]) \leq F(\mathbb{E}[V_{\sigma(t_1)}(t_1, x(t_1))]) - k\delta, \tag{15}$$

which does hold for all $k \in \mathfrak{K}$, where $\mathfrak{K} := \{k \in \mathbb{N}_{>0} | F(\mathbb{E}[V_{\sigma(t_1)}(t_1, x(t_1))]) - k\delta \geq \lim_{\varrho \downarrow 0} F(\varrho)\}$.

Define $r := V_{\sigma(t_0)}(t_0, x(t_0))$ and $k_1 := \max_{k \in \mathfrak{K}} k$ (if not exists, $k_1 := \infty$). Using (15) and the definitions of r, k_1, the class $\mathcal{KL}$ function $\beta_1(r,s)$ is constructed as follows:

$$\begin{aligned} \beta_1(r, t_1 - t_0) &:= \max\Big\{ \max_{\sigma(t_1) \in \mathcal{M}} \mathbb{E}[V_{\sigma(t_1)}(t_1, x(t_1))], \\ &\qquad \max_{\sigma(t_1) \in \mathcal{M}} \psi_{\sigma(t_1)}(\mathbb{E}[V_{\sigma(t_1)}(t_1, x(t_1))]) \Big\}, \\ \beta_1(r, t_{k+1} - t_0) &:= F^{-1}(F(\beta_1(r, t_1 - t_0)) - k\delta), \end{aligned}$$

where $k \in \{1, \ldots, k_1\}$. In the interval $(t_{k+1} - t_0, t_k - t_0)$ and $k \in \{1, \ldots, k_1\}$, $\beta_1(r,s)$ is required to be continuously decreasing and to lie above every solution of (1). Moreover, such requirement could also be satisfied in $[0, t_1 - t_0)$ by the construction. Moreover, if $k_1 < \infty$, then $\beta_1(r,s)$ in $[t_{k_1} - t_0, \infty)$ is defined to be continuous and decreasing to zero as $s \to \infty$.

Therefore, it follows from the construction of $\beta_1(r,s)$ that

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \leq \beta_1(V_{\sigma(t_0)}(t_0, x(t_0)), t - t_0), \quad \forall t \in \mathbb{R}_{\geq t_0},$$

where $\beta_1(r,s)$ is continuous and decreasing with respect to s. If $k_1 < \infty$, $\beta_1(r,s) \to 0$ as $s \to \infty$ by the construction. If $k_1 < \infty$, we need to prove that $\beta_1(r,s) \to 0$ as $s \to \infty$.

Claim 1: If $\beta_1(r, t_k - t_0) \to 0$ as $k \to \infty$, then $\beta_1(r,s) \to 0$ as $s \to \infty$.

Proof of Claim 1: Suppose Claim 1 is not true, then there exists an $\epsilon > 0$ (may be related to r) such that

$\lim_{k\to\infty}\beta_1(r, t_k - t_0) = \epsilon$. Denote $\vartheta := \min_{\epsilon\le\upsilon\le\beta_1(r,0)}\varphi(\upsilon)$. Using the middle-value theorem gives that

$$\begin{aligned}\delta &\le F(\beta_1(r, t_k - t_0)) - F(\beta_1(r, t_{k+1} - t_0))\\ &\le \frac{\beta_1(r, t_k - t_0) - \beta_1(r, t_{k+1} - t_0)}{\vartheta},\end{aligned}$$

which implies that $\beta_1(r, t_k - t_0) - \beta_1(r, t_{k+1} - t_0) \ge \delta\vartheta > 0$. It follows that as $k \to \infty$, $\beta_1(r, t_k - t_0)$ is decreasing to zero, which contradicts with the assumption that $\lim_{k\to\infty}\beta_1(r, t_k - t_0) = \epsilon > 0$. Thus, the claim is true. ■

Define $\beta_2(r, t) := \sup_{0\le b\le r}\beta_1(b, t)$. It is clear that $\beta_2(r, t) \ge \beta_1(r, t)$ for all $r > 0$ and $t \in \mathbb{R}_{\ge t_0}$. Furthermore, define $\beta_3(r, t) := \frac{1}{r}\int_r^{2r}\beta_2(s, t)ds + re^{-t}$. Note that $\beta_3(r, t) \in \mathcal{KL}$ and $\beta_3(r, t) \ge \beta_2(r, t)$ for all $r > 0$ and $t \in \mathbb{R}_{\ge t_0}$. It obtains that for all $t \in \mathbb{R}_{\ge t_0}$,

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \le \beta_3(V_{\sigma(t_0)}(t_0, x(t_0)), t - t_0). \tag{16}$$

Applying Markov's inequality in [43, Ch. II, 18.1] to (16), it obtains that for an arbitrary $\varepsilon_1 \in (0, 1)$, there exists $\beta_4(\upsilon, t) := \beta_3(\upsilon, t)/\varepsilon_1 \in \mathcal{KL}$ such that for all $t \in \mathbb{R}_{\ge t_0}$,

$$\begin{aligned}&\mathbb{P}\{V_{\sigma(t)}(t, x(t)) > \beta_4(V_{\sigma(t_0)}(t_0, x(t_0)), t - t_0)\}\\ &\qquad\le \frac{\mathbb{E}[V_{\sigma(t)}(t, x(t))]}{\beta_4(V_{\sigma(t_0)}(t_0, x(t_0)), t - t_0)} \le \varepsilon_1,\end{aligned}$$

which implies that for all $t \in \mathbb{R}_{\ge t_0}$,

$$\mathbb{P}\{|x(t)| > \beta(|x(t_0)|, t - t_0)\} \le \varepsilon_1, \tag{17}$$

where $\beta(\upsilon, t) := \alpha_1^{-1}(\beta_4(\alpha_2(\upsilon), t)) \in \mathcal{KL}$, $\alpha_1(\upsilon) := \min_{i\in\mathcal{M}}\alpha_{1i}(\upsilon)$ and $\alpha_2(\upsilon) := \max_{i\in\mathcal{M}}\alpha_{2i}(\upsilon)$.

Case 2: $u(t) \neq 0$. In this case, define the set $\mathbf{B}_1 := \{x \in \mathcal{X}|V_i(t, x) \le \rho_i(\|u\|), i \in \mathcal{M}\}$ and $\bar{t} := \inf\{t \in \mathbb{R}_{\ge t_0}|x(t) \in \mathbf{B}_1\}$ (if no exists, $\bar{t} := \infty$), then (16) holds for all $t \le \bar{t}$. For $t > \bar{t}$, if there exists some $\tilde{t} > \bar{t}$ such that $x(\tilde{t}) \in \partial\mathbf{B}_1$, then it obtains from (6) that $x(\tilde{t}) \in \mathbf{B}_1$. Define $\mathbf{B}_2 := \{x \in \mathcal{X}|V_i(t, x) \le \bar{\psi}(\|u\|)\}$, where $\bar{\psi}(\upsilon) := \max_{i\in\mathcal{M}}\{\max\{\max_{0\le v\le\rho_i(\upsilon)}\psi_i(v), \rho_i(\upsilon)\}\}$. Note that $\mathbf{B}_1 \subseteq \mathbf{B}_2$. If there exist $t_k \in \mathcal{T}$, $t_k > \bar{t}$ and $\varepsilon_2 > 0$ such that $x(t_k) \notin \mathbf{B}_1$ and $x(t) \in \mathbf{B}_1$ for $t \in (t_k - \varepsilon_2, t_k)$, then $x(t) \in \mathbf{B}_2$ for $t \in (t_k - \varepsilon_2, t_k)$. If $x(t) \notin \mathbf{B}_1$ for all $t \ge t_k$, then it has been proven that $\mathbb{E}[V_{\sigma(t)}(t, x(t))] < \mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))]$. Therefore, $x(t) \in \mathbf{B}_2$ for all $t > t_k$, which implies that for all $t > \bar{t}$,

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \le \bar{\psi}(\|u\|).$$

Applying Markov's inequality, we have that there exist a $\kappa \in \mathcal{K}_\infty$ and an arbitrary $\varepsilon_3 = \varepsilon_3(\kappa) > 0$ such that for all $t > \bar{t}$,

$$\mathbb{P}\{V_{\sigma(t)}(t, x(t)) > \kappa(\bar{\psi}(\|u\|))\} \le \frac{\mathbb{E}[V_{\sigma(t)}(t, x(t))]}{\kappa(\bar{\psi}(\|u\|))} \le \varepsilon_3. \tag{18}$$

Define $\gamma(\upsilon) := \alpha_1^{-1}(\kappa(\bar{\psi}(\upsilon)))$. Combining (17) and (18) yields that for all $t \in \mathbb{R}_{\ge t_0}$,

$$\mathbb{P}\{|x(t)| \le \beta(|x(t_0)|, t - t_0) + \gamma(\|u\|)\} \ge 1 - \max\{\varepsilon_1, \varepsilon_3\}.$$

That is, the system (1) is SISS over $\mathcal{S}_1(\theta)$.

On the other hand, if $\psi_i \le \mathrm{Id}$ for all $i \in \mathcal{M}$, then the left-hand side of the FDT condition (7) is less than 0, which implies that there are no constraints on θ, that is, there is no requirement on the impulsive switching time sequence. As a result, the system (1) is SISS over $\mathcal{S}_{\text{all}}$ and the proof is completed. ■

Remark 2: The inequality (7) is a type of fixed dwell-time (FDT) condition, which first appeared in [38, Ch. 3, Th. 48] for autonomous impulsive systems. The FDT condition provides the lower bound for impulsive switching intervals, and plays a similar role as the ADT condition [23], [46] in stability analysis. The relationship between the FDT condition and the ADT condition was studied in [21] and [24, Sec. 3.2]. For exponential SISS-Lyapunov functions, the FDT condition (7) is $\theta - \delta \ge \max_{i\in\mathcal{M}}\{-c^{-1}d_i\}$ with $c_i = c > 0$ and $d_i < 0$, which is simpler and similar to the ADT condition. Furthermore, Theorem 1 recovers many previous works [21], [23]–[25], [36] as the special cases. For instance, if there are no switches in the system (1), then Theorem 1 is similar to those in [21] and [24]. If there are no impulses in the system (1), the theorem 1 is similar to the results in [23], [25], and [36]. □

Remark 3: It is obvious from the proof of Theorem 1 that the construction of β depends on the impulsive switching time sequence whereas the construction of γ does not. It implies that the system (1) is not uniformly SISS over $\mathcal{S}_1(\theta)$. However, if the impulsive switching time sequence is periodic, then the construction of β depends only on the period of the impulsive switching time sequence. In this case, uniform SISS of the system (1) is obtained; see [21], [24] for more details. □

Theorem 2: Under the same assumptions as in Theorem 1, if the FDT condition (7) holds for $\delta \equiv 0$, then the system (1) is uniformly SGS over $\mathcal{S}_1(\theta)$. In addition, if $\psi_i \le \mathrm{Id}$, then the system (1) is uniformly SGS over $\mathcal{S}_{\text{all}}$.

Proof: The proof is proceeded along the same fashion as the proof of Theorem 1 until (15) holds for $\delta \equiv 0$. Define $\chi(\upsilon) := \max\{\upsilon, \max_{i\in\mathcal{M}}\psi_i(\upsilon)\}$. Instead of (16), the following inequality holds for all $t \in \mathbb{R}_{\ge t_0}$ in Case 1 and for all $t \le \bar{t}$ in Case 2,

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \le \chi(V_{\sigma(t_0)}(t_0, x(t_0))) \le \chi(\alpha_2(|x(t_0)|)),$$

where $\alpha_2(\upsilon) := \max_{i\in\mathcal{M}}\alpha_{2i}(\upsilon)$. For all $t \ge \bar{t}$ in Case 2, it follows from the proof of Theorem 1 that $\mathbb{E}[V_{\sigma(t)}(t, x(t))] \le \bar{\psi}(\|u\|)$. As a result, we have that for all $t \in \mathbb{R}_{\ge t_0}$,

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \le \chi(\alpha_2(|x(t_0)|)) + \bar{\psi}(\|u\|).$$

Exploiting Markov's inequality, it obtains that for an arbitrary $\varepsilon \in (0, 1)$ and all $t \in \mathbb{R}_{\ge t_0}$,

$$\mathbb{P}\left\{V_{\sigma(t)}(t, x(t)) \le \chi(\alpha_2(|x(t_0)|))/\varepsilon + \bar{\psi}(\|u\|)/\varepsilon\right\} \ge 1 - \varepsilon.$$

That is,

$$\mathbb{P}\{|x(t)| \le \gamma_1(|x(t_0)|) + \gamma_2(\|u\|)\} \ge 1 - \varepsilon,$$

where $\alpha_1(\upsilon) := \min_{i\in\mathcal{M}}\alpha_{1i}(\upsilon)$, $\alpha_2(\upsilon) := \max_{i\in\mathcal{M}}\alpha_{2i}(\upsilon)$, $\gamma_1(\upsilon) := \alpha_1^{-1}(2\chi(\alpha_2(\upsilon))/\varepsilon)$ and $\gamma_2(\upsilon) := \alpha_1^{-1}(2\bar{\psi}(\upsilon)/\varepsilon)$.

According to the construction of χ and ρ, both γ_1 and γ_2 do not depend on the choice of $\mathcal{T}$. Hence, the system (1) is uniformly SGS. ■

In Theorem 1, φ_i are the same for all $i \in \mathcal{M}$, which means that all the estimates of the derivatives of $V_i(t,x)$ are limited with the same bounded function. Another scenario is that ψ_i are the same for all $i \in \mathcal{M}$, that is, all the growths of $V_i(t,x)$ by the jumps are limited with the same bounded function. For this case, the following theorem establishes the stability conditions for the system (1).

Theorem 3: Consider the system (1). Suppose for each $i \in \mathcal{M}$, $V_i : \mathbb{R}_{\geq t_0} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ is an SISS-Lyapunov function and $\alpha_{1i}, \alpha_{2i}, \rho_i, \varphi_i \in \mathcal{K}_\infty$, $\psi_i \in \mathcal{P}$, where φ_i is convex and $\psi_i = \psi$ is concave. If there exists a $\theta > 0$ such that

$$\sup_{i\in\mathcal{M}} \int_a^{\psi(a)} \frac{ds}{\varphi_i(s)} < \theta, \quad \forall a > 0, \tag{19}$$

then the system (1) is SISS over $\mathcal{S}_1(\theta)$.

Proof: Let $\mathcal{N}(t) := V_{\sigma(t_k)}(t, x(t))$ for all $t \in [t_k, t_{k+1})$ and $k \in \mathbb{N}$. Define $\Phi(\upsilon) := \max_{i\in\mathcal{M}}\{\alpha_{2i}(\alpha_{1i}^{-1}(\upsilon))\}$, $\rho(\upsilon) := \max_{i\in\mathcal{M}}\{\rho_i(\upsilon)\}$, $\xi(\upsilon) := \max\{\psi(\rho(\upsilon)), \Phi(\upsilon)\}$ and $\Omega := \xi(\|u\|)$.

Claim 2: If there exists a $T \in \mathbb{R}_{\geq t_0}$ such that $\mathcal{N}(T) \leq \psi^{-1}(\Omega)$, then $\mathcal{N}(t) \leq \Omega$ for all $t \geq T$.

Proof of Claim 2: Suppose Claim 2 is invalid, there exists $t \geq T$ such that $\mathcal{N}(t) > \Omega$. Define $\hat{t} := \inf\{t \geq T | \mathcal{N}(t) > \Omega\}$. Thus, there exists a $k \in \mathbb{N}$ such that $\hat{t} \in [t_k, t_{k+1})$. Assume $\hat{t} \neq t_k$, it obtains that $\mathcal{N}(\hat{t}) = \Omega > \psi^{-1}(\Omega) \geq \rho(\|u\|)$ due to instability of the discrete dynamics. As a result, we can find a sufficiently small $\varpi > 0$ such that $\mathcal{N}(\hat{t}) \geq \rho(\|u\|)$ for $[\hat{t}, \hat{t}+\varpi]$. However, it follows from (6) that $\mathcal{N}(t)$ is decreasing on $[\hat{t}, \hat{t}+\varpi]$, which contradicts with the definition of $\hat{t}$. In the sequel, $\hat{t} = t_k$ and $\mathcal{N}(\hat{t}) \leq \Omega$.

It follows from the definition of $\hat{t}$ that $\mathcal{N}(t) \leq \Omega$ on $t \in [t_{k-1}, t_k)$. For all $t \in [t_{k-1}, t_k)$, if $\mathcal{N}(t) > \psi^{-1}(\Omega) \geq \rho(\|u\|)$, then

$$\mathbb{E}[\mathscr{L}\mathcal{N}(t)] \leq -\varphi_{\sigma(t_{k-1})}(\mathbb{E}[\mathcal{N}(t)]). \tag{20}$$

Since $t_k \in \mathcal{S}_1(\theta)$, integrating (20) from t_{k-1} to $t \in [t_{k-1}, t_k)$ and letting $t \to t_k^-$ yield that

$$\begin{aligned}
\int_{t_k^-}^{t_{k-1}} \frac{\mathbb{E}[\mathscr{L}\mathcal{N}(t)]ds}{\varphi_{\sigma(t_{k-1})}(\mathbb{E}[\mathcal{N}(t)])} &\geq t_k^- - t_{k-1} \geq \theta, \\
\int_{t_k^-}^{t_{k-1}} \frac{\mathbb{E}[\mathscr{L}\mathcal{N}(t)]ds}{\varphi_{\sigma(t_{k-1})}(\mathbb{E}[\mathcal{N}(t)])} &\leq \int_{\psi^{-1}(\Omega)}^{\Omega} \frac{\mathbb{E}[\mathscr{L}\mathcal{N}(t)]ds}{\varphi_{\sigma(t_{k-1})}(\mathbb{E}[\mathcal{N}(t)])} \\
&< \theta,
\end{aligned}$$

which is a contradiction due to (19). If there exists certain $t \in [t_{k-1}, t_k)$ such that $\mathcal{N}(t) < \psi^{-1}(\Omega)$, then define $\check{t} := \sup\{t \in [t_{k-1}, t_k) | \mathcal{N}(t) \leq \psi^{-1}(\Omega)\}$. Suppose $\check{t} \neq t_k$, then it obtains that $\mathcal{N}(t) \geq \psi^{-1}(\Omega) \geq \rho(\|u\|)$ for $t \in [\check{t}, t_k)$, which in turn implies from (5) that $\mathcal{N}(t)$ is decreasing on $[\check{t}, t_k)$. This is a contradiction with the definition of $\check{t}$. As a result, $\check{t} = t_k$ and $\mathcal{N}(\check{t}^-) \leq \psi^{-1}(\Omega)$.

It follows from above analysis that $\check{t} = \hat{t} = t_k$. Furthermore, if $\mathcal{N}(\hat{t}^-) \geq \rho(\|u\|)$, then $\mathcal{N}(\hat{t}) \leq \psi(\mathcal{N}(\hat{t}^-)) \leq \Omega$ from (6). If $\mathcal{N}(\hat{t}^-) < \rho(\|u\|)$, then $\mathcal{N}(\hat{t}) \leq \psi(\rho(\|u\|)) \leq \Omega$ follows from the fact that $\rho(\|u\|) \leq \psi^{-1}(\Omega) < \Omega$. All these two cases contradict with the assumption that $\mathcal{N}(\hat{t}) > \Omega$. Therefore, the claim is valid. ■

Define $\bar{t} := \inf\{t \in \mathbb{R}_{\geq t_0} | \mathcal{N}(t) \leq \Omega\}$. Claim 2 implies that $\mathcal{N}(t) \leq \Omega$ for all $t \geq \bar{t}$, which gives that for all $t \geq \bar{t}$,

$$\mathbb{E}[\mathcal{N}(t)] \leq \xi(\|u\|). \tag{21}$$

It follows from the definition of $\bar{t}$ that $\mathcal{N}(t) \geq \Omega \geq \rho(\|u\|)$ for $t_0 \leq t \leq \bar{t}$. Thus, for $t \in [t_0, \bar{t}]$, (9) and (10) can be rewritten as

$$\begin{aligned}
\mathbb{E}[\mathscr{L}\mathcal{N}(t)] &\leq -\varphi_i(\mathbb{E}[\mathcal{N}(t)]), \quad t \in (t_k, t_{k+1}), \\
\mathbb{E}[\mathcal{N}(t)] &\leq \psi(\mathbb{E}[\mathcal{N}(t^-)]), \quad t = t_k.
\end{aligned}$$

According to (4), the comparison principle for hybrid systems (e.g., [47, Lemma C.1], [20, Lemma 1]) and [22, Lemma 3.2], there exists $\bar{\beta} \in \mathcal{KL}$ such that for all $t \in [t_0, \bar{t}]$,

$$\mathbb{E}[\mathcal{N}(t)] \leq \bar{\beta}(\mathbb{E}[\mathcal{N}(t_0)], t - t_0). \tag{22}$$

Combining (5), (21) with (22) and applying the Markov's inequality yield that for an arbitrary $\varepsilon \in (0,1)$, there exist $\beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty$ such that for all $t \in \mathbb{R}_{\geq t_0}$,

$$\mathbb{P}\{|x(t)| \leq \beta(|x(t_0)|, t - t_0) + \gamma(\xi(\|u\|))\} \geq 1 - \varepsilon,$$

where $\beta(\upsilon, s) := \alpha_1(2\bar{\beta}(\alpha_2(\upsilon), s)/\varepsilon)$, $\gamma(\upsilon) := \alpha_1(2\xi(\upsilon)/\varepsilon)$. As a result, the proof is completed. ■

Remark 4: The inequality (19) is another type of FDT condition, and has the same role as (7) in stability analysis. Furthermore, the FDT condition (19) could rewritten as: for some $\theta, \delta > 0$,

$$\sup_{i\in\mathcal{M}} \int_a^{\psi(a)} \frac{ds}{\varphi_i(s)} \leq \theta - \delta, \quad \forall a > 0. \tag{23}$$

Under the FDT condition (23), the stability analysis could be proceeded along the similar fashion as the proof of Theorem 3 and SISS of the system (1) is obtained. In addition, compared with the conditions given in Theorem 1, the inequality (8) is not needed in Theorem 3 and relaxes the requirements on Lyapunov functions. □

Remark 5: For the case that $\psi_i \equiv \psi$ for all $i \in \mathcal{M}$, similar result could be found in [22, Th. 4.2] for impulsive switched systems. Reference [22, Th. 4.2] is a special case of Theorem 3 if the considered system is deterministic. In addition, Theorem 3 provides a general lower bound for all the impulsive switching intervals, whereas each subsystem had a lower bound for its impulsive switching intervals in [22]. □

Remark 6: The case that $\varphi_i \equiv \varphi$ for all $i \in \mathcal{M}$ is studied in Theorems 1, and the case that $\psi_i \equiv \psi$ for all $i \in \mathcal{M}$ is studied in Theorems 3. Both of such two cases are more general than the previous works [4], [19], [25] where $\varphi_i \equiv \varphi$ and $\psi_i \equiv \psi$. However, because the switching and impulsive effects, multiple general Lyapunov functions cannot be applied directly, which deserves further study. □

B. The Case That the Discrete Dynamics Is Stable

In this subsection, the case that the discrete dynamics is stable is studied. For both the subcase that φ_i are the same and the subcase that ψ_i are the same, sufficient conditions are established for the stability properties of the system (1).

Theorem 4: Consider the system (1). Suppose for each $i \in \mathcal{M}$, $V_i : \mathbb{R}_{\geq t_0} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ is an SISS-Lyapunov function with $\alpha_{1i}, \alpha_{2i}, \rho_i, -\varphi_i \in \mathcal{K}_\infty$, $\psi_i \in \mathcal{P}$, where $-\varphi_i = -\varphi$ and ψ_i are concave. If (8) holds for all $t_{k-1}, t_k \in \mathcal{T}$, $k \in \mathbb{N}_{>0}$, and there exist certain $\theta, \delta > 0$ such that

$$\inf_{i\in\mathcal{M}} \int_{\psi_i(a)}^{a} \frac{ds}{-\varphi(s)} \geq \theta + \delta, \quad \forall a > 0, \tag{24}$$

then the system (1) is SISS over $\mathcal{S}_2(\theta)$.

Proof: Similar to the proof of Theorem 1, the proof is divided into two cases based on the construction of β and γ satisfying (2). Both the case $u(t) \equiv 0$ and the case $u(t) \neq 0$ are considered.

Case 1: $u(t) \equiv 0$. Since $-\varphi, \psi_i$ are concave, it follows from (6) and Jensen's inequality in [43, Ch. II. 18.3] that

$$\begin{aligned} \mathbb{E}[\mathscr{L}V_i(t, x(t))] &\leq -\varphi(\mathbb{E}[V_i(t, x(t))]), \quad t \in \mathbb{R}_{\geq t_0} \backslash \mathcal{T}, \\ \mathbb{E}[V_i(t, h_i(x, u))] &\leq \psi_i(\mathbb{E}[V_i(t, x(t))]), \quad t \in \mathcal{T}. \end{aligned} \tag{25}$$

Integrating (25) yields that for all $t \in [t_k, t_{k+1})$,

$$\int_{t_k}^{t} \frac{\mathbb{E}[\mathscr{L}V_i(s, x(s))]ds}{-\varphi(\mathbb{E}[V_i(s, x(s))])} \leq t - t_k.$$

Following the similar fashion of the proof of Theorem 1, define the functions as follows:

$$F(\varrho) := \int_{\nu}^{\varrho} \frac{ds}{-\varphi(s)},$$

where $\nu > 0$ is fixed and $\varrho > 0$. Thus, $F : \mathbb{R}_{>0} \to \mathbb{R}$ and its inverse $F^{-1} : \mathbb{R} \to \mathbb{R}_{>0}$ are continuous and strictly increasing.

Analogous to the proof of Theorem 1, since $t_k \in \mathcal{S}_1(\theta)$, it follows from (24) that for all $k \in \mathbb{N}$, the inequality (15) holds and for all $t \in [t_k, t_{k+1})$,

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \leq F^{-1}(F(\mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))]) + (t - t_k)). \tag{26}$$

By the similar construction of the function $\beta_1(r, s)$ as in the proof of Theorem 1, it holds that for all $t \in \mathbb{R}_{\geq t_0}$,

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \leq \beta_1(V_{\sigma(t_0)}(t_0, x(t_0)), t - t_0). \tag{27}$$

Following the similar line as the proof of Theorem 1 for the case $u(t) \equiv 0$, it obtains that (16) holds for all $t \in \mathbb{R}_{\geq t_0}$.

Case 2: $u(t) \neq 0$. the set $\mathbf{B}_1$ and $\bar{t}$ are the same as those in the proof of Theorem 1. Thus, the inequality (17) holds for all $t \in [t_0, \bar{t}]$. Define $\tilde{t} := \inf\{t_k \in \mathbb{R}_{\geq t_0} | t_k > \bar{t}\}$. For $t \in [\bar{t}, \tilde{t})$, combining $V_i(x(\bar{t}), \bar{t}) \leq \rho_i(\|u\|)$, (8) and (26) gives that

$$\begin{aligned} \mathbb{E}[V_{\sigma(t)}(t, x(t))] &\leq F^{-1}(F(\mathbb{E}[V_{\sigma(\bar{t})}(\bar{t}, x(\bar{t}))]) + \theta) \\ &\leq F^{-1}(F(\rho(\|u\|)) + \theta), \end{aligned} \tag{28}$$

where $\rho(\upsilon) := \max_{i\in\mathcal{M}}\{\rho_i(\upsilon)\}$. Because $\mathbb{E}[V_{\sigma(\tilde{t})}(\tilde{t}, x(\tilde{t}))] < \mathbb{E}[V_{\sigma(\bar{t})}(\bar{t}, x(\bar{t}))]$ from (15), the inequality (28) also holds for all $t > \bar{t}$. Implementing Markov's inequality to (28), there exist $\kappa \in \mathcal{K}_\infty$ and an arbitrary $\varepsilon_3 = \varepsilon_3(\kappa) \in (0, 1)$ such that for all $t \in \mathbb{R}_{\geq t_0}$,

$$\mathbb{P}\{V_{\sigma(t)}(t, x(t)) > \kappa(\tilde{\psi}(\|u\|))\} \leq \frac{\mathbb{E}[V_{\sigma(t)}(t, x(t))]}{\kappa(\tilde{\psi}(\|u\|))} \leq \varepsilon_3,$$

where $\tilde{\psi}(\upsilon) := F^{-1}(F(\rho(\upsilon)) + \theta)$, which means that

$$\mathbb{P}\{|x(t)| \leq \alpha_1^{-1}(\kappa(\tilde{\psi}(\|u\|)))\} \geq 1 - \varepsilon_3.$$

The remaining is similar to the proof of Theorem 1. As a result, the system (1) is SISS over $\mathcal{S}_2(\theta)$. ■

In Theorem 4, the inequality (24) is a type of reverse FDT (RFDT) condition, which is a counterpart to the reverse average dwell-time (RADT) condition in [23]. Different from the RADT condition bounding the impulsive switching intervals on average, the RFDT condition gives an upper bound of the impulsive switching intervals.

Remark 7: If $\varphi(\upsilon) \equiv 0$ for all $\upsilon \geq 0$, then the continuous dynamics is neutral for SISS of the system (1). It follows from (25) that there exists a constant K such that for all $t \in [t_k, t_{k+1})$,

$$\mathbb{E}[V_{\sigma(t_k)}(t, x(t))]) - \mathbb{E}[V_{\sigma(t_k)}(t_k, x(t_k))] \leq K. \tag{29}$$

Since the discrete dynamics is stable, we have that for an arbitrary $\varepsilon \in (0, 1)$, there exist $\tilde{\beta} \in \mathcal{KL}$ and $\tilde{\gamma} \in \mathcal{K}_\infty$ such that for all $t_k \in \mathcal{T}$,

$$\mathbb{P}\{|x(t_k)| \leq \tilde{\beta}(|x(t_0)|, t_k - t_0) + \tilde{\gamma}(\|u\|)\} \geq 1 - \varepsilon. \tag{30}$$

As a result, it obtains from (8), (29) and (30) that for all $t \geq t_0$,

$$\mathbb{P}\{|x(t)| \leq \tilde{\beta}(|x(t_0)|, t - t_0) + \tilde{\gamma}(\|u\|) + K\} \geq 1 - \varepsilon,$$

which implies that the system (1) is stochastically input-to-state practically stable; see [1], [2, Ch. 2.9]. Especially, SISS of the system (1) is guaranteed in the case that $K \leq 0$. □

Theorem 5: Under the same assumptions as in Theorem 4, if the reverse fixed dwell-time condition (24) holds for $\delta \equiv 0$, then the system (1) is uniformly SGS over $\mathcal{S}_2(\theta)$.

Proof: Since all the assumptions in Theorem 4 hold for $\delta \equiv 0$, following the same fashion as the proof of Theorem 4, (15) holds for $\delta \equiv 0$. Thus, define $\chi(\upsilon) := \max\{\upsilon, \max_{i\in\mathcal{M}}\{\psi_i(\upsilon)\}\}$. Instead of (26), it obtains that for all $t \in [t_0, \bar{t}]$,

$$\mathbb{E}[V_{\sigma(t)}(t, x(t))] \leq \chi(V_{\sigma(t_0)}(t_0, x(t_0))) \leq \chi(\alpha_2(|x(t_0)|)).$$

The remaining is a combination of the proofs of Theorem 2 and Theorem 4. Hence, the system (1) is uniformly SGS over $\mathcal{S}_2(\theta)$. ■

As a counterpart of Theorem 3, if ψ_i are the same for all $i \in \mathcal{M}$, then the following theorem provides the sufficient conditions for SISS of the system (1).

Theorem 6: Consider the system (1). Suppose for each $i \in \mathcal{M}$, $V_i : \mathbb{R}_{\geq t_0} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ is an SISS-Lyapunov function and $\alpha_{1i}, \alpha_{2i}, \rho_i, -\varphi_i \in \mathcal{K}_\infty$, $\psi_i \in \mathcal{P}$, where $-\varphi_i$ and $\psi_i = \psi$ are concave. If there exists a $\theta > 0$ such that for all $a > 0$,

$$\inf_{i\in\mathcal{M}} \int_{\psi(a)}^{a} \frac{ds}{-\varphi_i(s)} > \theta, \tag{31}$$

then the system (1) is SISS over $\mathcal{S}_2(\theta)$.

Proof: Let $\mathcal{N}(t) := V_{\sigma(t_k)}(t, x(t))$ for all $t \in [t_k, t_{k+1})$ and $k \in \mathbb{N}_{>0}$. Define $\rho(\upsilon) := \max_{i\in\mathcal{M}}\{\rho_i(\upsilon)\}$, $\xi(\upsilon) := \max\{\psi^{-1}(\rho(\upsilon)), \Phi(\upsilon)\}$ and $\Omega := \xi(\|u\|)$.

Claim 3: If there exists a $T \in \mathbb{R}_{\geq t_0}$ such that $\mathcal{N}(T) \leq \psi^{-1}(\Omega)$, then $\mathcal{N}(t) \leq \Omega$ for all $t \geq T$.

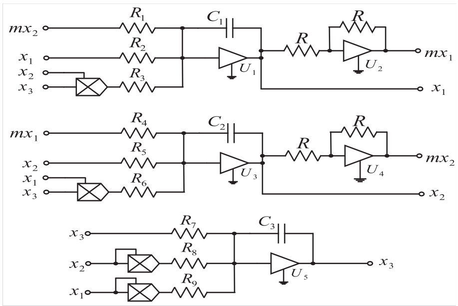

Fig. 2. Circuit diagram of the 3-D novel chaotic system (33) without external input and disturbances.

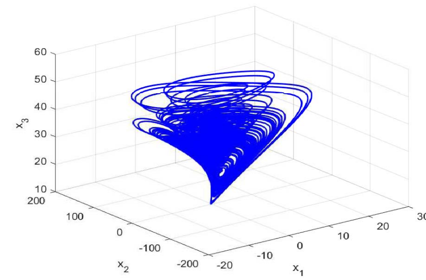

Fig. 3. The state trajectory of the 3-D chaotic system (33) with $a_1 = 22$, $b_1 = 600$, $c_1 = 3$, $p_1 = 11$.

Proof of Claim 3: Suppose Claim 3 is invalid, then there exists a $t \geq T$ such that $\mathcal{N}(t) > \Omega$. Define $\hat{t} := \inf\{t \geq T | \mathcal{N}(t) > \Omega\}$. Thus, there exists a $k \in \mathbb{N}_{>0}$ such that $\hat{t} \in [t_k, t_{k+1})$. Define $\check{t} := \sup\{t \in [t_k, \hat{t}) | \mathcal{N}(t) \leq \psi(\Omega)\}$. It obtains that $\hat{t} \geq \check{t} \geq t_k$. If $\hat{t} = \check{t}$, then $\hat{t} = \check{t} = t_k$. Moreover, if $\mathcal{N}(t_k^-) \geq \rho(\|u\|)$, then it follows from (6) and the stability of the discrete dynamics that $\mathcal{N}(t_k) \leq \psi(\mathcal{N}(t_k^-)) \leq \psi(\Omega) \leq \Omega$. If $\mathcal{N}(t_k^-) < \rho(\|u\|)$, then $\mathcal{N}(t_k) \leq \alpha_{2\sigma(t_k)}(\alpha_{1\sigma(t_k^-)}(\rho(\|u\|))) \leq \Omega$. Both of these two cases contradict with the definition of $\hat{t}$. In the sequel, $\hat{t} > \check{t}$, $\mathcal{N}(\hat{t}) = \Omega$ and $\mathcal{N}(\check{t}) = \psi(\Omega)$. For $t \in [\check{t}, \hat{t}] \subseteq [t_k, t_{k+1})$, it holds that $\psi(\Omega) \leq \mathcal{N}(t) \leq \Omega$, which in turn gives that $\mathcal{N}(t) \geq \rho(\|u\|)$. As a result, for $t \in [\check{t}, \hat{t}]$,

$$\mathbb{E}[\mathscr{L}\mathcal{N}(t)] \leq -\varphi_{\sigma(t_k)}(\mathbb{E}[\mathcal{N}(t)]). \tag{32}$$

Integrating (32) from $\check{t}$ to $\hat{t}$ yields that

$$\int_{\check{t}}^{\hat{t}} \frac{\mathbb{E}[\mathscr{L}\mathcal{N}(s)]ds}{-\varphi_{\sigma(t_k)}(\mathbb{E}[\mathcal{N}(s)])} < t_{k+1}^- - t_k \leq \theta,$$

$$\int_{\check{t}}^{\hat{t}} \frac{\mathbb{E}[\mathscr{L}\mathcal{N}(s)]ds}{-\varphi_{\sigma(t_k)}(\mathbb{E}[\mathcal{N}(s)])} \geq \int_{\psi(\Omega)}^{\Omega} \frac{\mathbb{E}[\mathscr{L}\mathcal{N}(s)]ds}{-\varphi_{\sigma(t_k)}(\mathbb{E}[\mathcal{N}(s)])} > \theta,$$

which is a contradiction. Therefore, the claim is valid. ■

The remaining of the proof is similar to the proof of Theorem 3. Hence, SISS of the system (1) is established. ■

Remark 8: The inequality (31) is another type of RFDT condition. Similar to Remark 4, the RFDT condition (31) could be rewritten as: for some $\theta, \delta > 0$,

$$\inf_{i \in \mathcal{M}} \int_{\psi(a)}^{a} \frac{ds}{-\varphi_i(s)} \geq \theta + \delta, \quad \forall a > 0,$$

and then SISS of the system (1) is still established. □

IV. Illustrative Examples

In this section, two numerical examples are presented to demonstrate the validity of the developed results.

Example 1. As stated in [3], many electrical circuits with switching capacitors usually have the impulses at the switches. Consider the following stochastic switched 3-D novel chaotic circuit system [15] with parameter white-noise excitation:

$$dx(t) = [A_{\sigma(t)}x(t) + f_{\sigma(t)}(x(t)) + u_{\sigma(t)}(t)]dt + g_{\sigma(t)}(t, x(t))dB(t), \tag{33}$$

where $x(t) \in \mathbb{R}^3$ is the system state. $u(t) \in \mathbb{R}^3$ and $B(t) \in \mathbb{R}^3$ are the external disturbance and random noise, respectively. The generator of the Markovian switching and the matrices A_i are given as follows:

$$\Pi = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}, \quad A_i = \begin{bmatrix} -a_i & a_i & 0 \\ b_i & -1 & 0 \\ 0 & 0 & -c_i \end{bmatrix}.$$

In addition, $f_i(x(t)) = (x_2(t)x_3(t), x_1(t)x_3(t), p_i x_2^2(t) - p_i x_1^2(t))^\top$. The circuit realization and the state trajectory of such chaotic system without external input and disturbances are presented in Fig. 2 and Fig. 3, respectively; see [15] for more details. To synchronize the system (33), the stochastic impulsive switched controlled slave system is given by (see e.g., [29], [32])

$$dz(t) = [A_{\sigma(t)}z(t) + f_{\sigma(t)}(z(t)) + U_{\sigma(t)}(t)]dt + g_{\sigma(t)}(t, z(t))dB(t), \tag{34}$$

where $U_i(t) = \sum_{k=1}^{\infty} E_{ik}(z(t) - x(t))\delta(t - t_k)$ is an impulsive controller, $E_{ik} \in \mathbb{R}^{3\times 3}$ are constant matrices, and $\delta(t - t_k)$ is the Dirac impulse function with discontinuous time sequence $\mathcal{T}$. That is, the control input is implemented in the impulsive switching time sequence. In addition, let $E_{ik} = E_i$.

Define the synchronization error as $e(t) := z(t) - x(t)$. According to (33)-(34), the synchronization error dynamics is obtained as the following stochastic impulsive switched system:

$$\begin{cases} de(t) = [A_{\sigma(t)}e(t) + H_{\sigma(t)}(e(t)) - u_{\sigma(t)}(t)]dt \\ \qquad + G_{\sigma(t)}(t, e(t))dB(t), \quad t \notin \mathcal{T}, \\ e(t) = (I + E_{\sigma(t)})e(t^-), \quad t \in \mathcal{T}, \end{cases} \tag{35}$$

where $H_i(e(t)) := f_i(z(t)) - f_i(x(t))$ and $G_i(e(t)) := g_i(t, z(t)) - g_i(t, x(t))$.

Assume that $\mathcal{M} = \{1, 2\}$ and choose the Lyapunov functions $V_1(t, x(t)) = e^\top(t)e(t)$ and $V_2(t, e(t)) = |e(t)|$. Thus, the condition (5) holds with $\alpha_{1i}(v) := \min\{v, v^2\}$ and $\alpha_{2i}(v) := \max\{v, v^2\}$. In addition, set $E_1 = E_2 = -0.5I$, $g_i(t, x(t)) = g_i(t, z(t)) = \exp(-0.4t)(1, 1, 1)^\top$ and $\rho_1(v) = \rho_2(v) = 3v^2$. In the sequel, it obtains from computation that for all $t \notin \mathcal{T}$,

$$\mathscr{L}V_1(t, e(t)) \leq \Omega_1 V_1(t, e(t)) + p_1 V_1^2(t, e(t)),$$
$$\mathscr{L}V_2(t, e(t)) \leq \Omega_2 V_2(t, e(t)) + (p_2 + 1)V_1^2(t, e(t)),$$

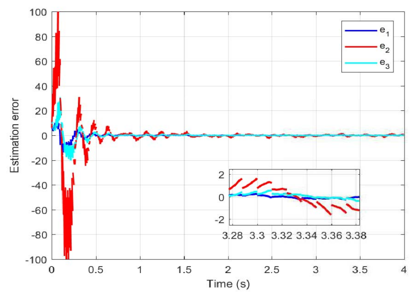

Fig. 4. The state response of the synchronization error dynamics (35) with a periodic impulsive switching time sequence and $\theta = 0.012$.

where $\Omega_1 := a_1 + b_1 - 1$ and $\Omega_2 := \max\{a_2 - 1, b_2 - a_2\} + 1$, and that for all $t \in \mathcal{T}$,

$$V_1(t, e(t)) \le \lambda^2_{\max}(I + E_1)V_1(t^-, e(t^-)),$$
$$V_2(t, e(t)) \le \lambda^2_{\max}(I + E_2)V_1(t^-, e(t^-)).$$

As a result, the Lyapunov functions are not exponential, and then the ADT condition (see [19], [36]) is not available here.

Integrating the left-hand side of the RFDT condition (24) yields that

$$\mathfrak{I}_1(\upsilon) = \int_{J_1\upsilon}^{\upsilon} \frac{ds}{\Omega_1 s + p_1 s^2} = \frac{p_1}{\Omega_1} \ln\left(\frac{p_1 J_1 \upsilon + \Omega_1}{J_1 p_1 \upsilon + J_1 \Omega_1}\right),$$
$$\mathfrak{I}_2(\upsilon) = \int_{J_2\upsilon}^{\upsilon} \frac{ds}{\Omega_2 s + (p_2+1)s^2} = \frac{(p_2+1)}{\Omega_2} \ln\left(\frac{(p_2+1)J_2\upsilon + \Omega_2}{J_2(p_2+1)\upsilon + J_2\Omega_2}\right),$$

where $J_i := \lambda^2_{\max}(I + E_i)$ and $i \in \mathcal{M}$. It follows from Theorem 4 that if $\min\{\mathfrak{I}_1(\upsilon), \mathfrak{I}_2(\upsilon)\} \ge \theta + \delta$ for some $\theta, \delta > 0$, then the system (35) is SISS. Let $a_1 = 22$, $b_1 = 600, c_1 = 3, p_1 = 11$ and $a_2 = 20, b_2 = 4$, $c_2 = 5, p_2 = 1$. Thus, $\theta + \delta \le \min\{0.0285, 0.0265\}$. As a result, we choose $\theta = 0.012$. Given the initial state $e(0) = (5, 6, 10)^\top$, $u_1(t) = 0.2\sin(0.2t)(1, 1, 1)^\top$, $u_2(t) = 0.5\cos(0.5t)(1, 1, 1)^\top$, the Gaussian white noise $B(t)$ with zero-mean and variance of 40, and the periodic impulsive switching time sequence, the state trajectory of the system (35) is shown in Fig. 4.

Example 2. Consider the following networked control system presented in [22] with stochastic disturbances:

$$\begin{cases} dx(t) = [A_i x(t) + f_i(x(t)) + D_i u(t)]dt \\ \qquad +[E_i x(t) + g_i(x(t))]dB(t), \\ y(t) = Cx(t) + \omega(t), \end{cases} \tag{36}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the disturbance input, $y(t) \in \mathbb{R}^p$ is the state measurement and $\omega(t) \in \mathbb{R}^n$ is the measurement noise. The Markovian switching function $\sigma(t) : \mathbb{R}_{\ge 0} \to \mathcal{M} = \{1, \ldots, M\}$ is piecewise continuous and the generator is Π. $f_i(x(t))$ and $g_i(x(t))$ are the nonlinear perturbations of the state and $i \in \mathcal{M}$. The matrices in (36) are of appropriate dimensions.

To generate the remote state estimate, one measurement is allowed to be sent at the time instance in $\mathcal{T} := \{t_1, t_2, \ldots\}$. The estimation mechanism is presented in Fig. 5. Assume that the network is free of transmission errors and time delay. The estimator of the system (36) is given by

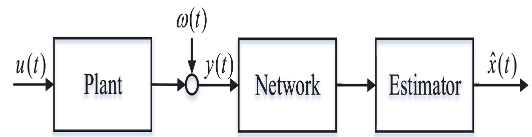

Fig. 5. State estimation via network.

$$\begin{cases} d\hat{x}(t) = [A_i\hat{x}(t) + \hat{f}_i(\hat{x}(t))]dt \\ \qquad +[E_i\hat{x}(t) + \hat{g}_i(\hat{x}(t))]dB(t), \quad t \notin \mathcal{T}, \\ \hat{y}(t) = C\hat{x}(t), \quad t \notin \mathcal{T}, \\ \hat{y}_l(t) = \begin{cases} y_{j_k}(t^-), & l = j_k, \\ \hat{y}_l(t^-), & l \ne j_k, \end{cases} \quad t \in \mathcal{T}, \\ \hat{x}(t) = C^\top(CC^\top)^{-1}\hat{y}(t), \quad t \in \mathcal{T}, \end{cases} \tag{37}$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the state estimate and $j_k \in \mathbb{N}, l \in \{1, \ldots, n\}$. $\hat{f}_i(\hat{x}(t))$ and $\hat{g}_i(\hat{x}(t))$ are the estimates of $f_i(x(t))$ and $g_i(x(t))$, respectively. In (37), if $t \in (t_k, t_{k+1})$, then the estimate $\hat{x}(t)$ evolves continuously; if $t = t_k$, then a measurement $y(t_k)$ is sent to the remote estimator and update the output estimate $\hat{y}(t_k)$. Assume that the matrix C is of full rank, and then $C^\top(CC^\top)^{-1}$ is the right-hand inverse of C.

According to [23, Sec. 8.2], a try-once-discard (TOD) protocol is used to determine that the index $j_k \in \{1, \ldots, n\}$, that is, the index corresponding to the largest $|\hat{y}_j(t_k^-) - y_j(t_k^-)|$, where $j \in \{1, \ldots, n\}$. Based on the system (36)-(37), the dynamics of the estimation error $e(t) := \hat{x}(t) - x(t)$ is derived as follows and denoted by $\mathcal{E}$:

$$\begin{cases} de(t) = [A_i e(t) + \hat{f}_i(\hat{x}(t)) - f_i(x(t)) - D_i u(t)]dt \\ \qquad +[E_i e(t) + \hat{g}_i(\hat{x}(t)) - g_i(x(t))]dB(t), \\ e_l(t) = \begin{cases} \omega_{j_k}(t^-), & l = j_k, \\ e_l(t^-), & l \ne j_k, \end{cases} \quad t \in \mathcal{T}. \end{cases}$$

Our objective is to analyze the SISS property of the estimate error dynamics $\mathcal{E}$. To this end, we first present the following assumption.

Assumption 1: For all $x, \hat{x} \in \mathbb{R}^n$ and $i \in \mathcal{M}$, there exist constants $\lambda_i, \eta_i, L_i, K_i, H_i \ge 0$ such that

$$E_i^\top C^\top C E_i \le \eta_i C^\top C,$$
$$C^\top C A_i + A_i^\top C^\top C \le \lambda_i C^\top C,$$
$$e^\top C^\top C[\hat{f}_i(\hat{x}) - f_i(x)] \le L_i e^\top C^\top C e,$$
$$e^\top E_i^\top C^\top C[\hat{g}_i(\hat{x}) - g_i(x)] \le K_i e^\top C^\top C e,$$
$$[\hat{g}_i(\hat{x}) - g_i(x)]^\top C^\top C[\hat{g}_i(\hat{x}) - g_i(x)] \le H_i e^\top C^\top C e.$$

For all $i \in \mathcal{M}$, choose the Lyapunov functions $V_i(t, e(t)) := e^\top(t)C^\top C e(t)$. It obtains from [23, Sec. 8.2] and [47, Proposition 2.6] that for each $i \in \mathcal{M}$, there exists a constant $\phi \in ((n-1)/n, 1)$ and a function $\rho \in \mathcal{K}_\infty$ such that, if $V_i(t, e(t)) \ge \rho(|u(t)|)$, then

$$V_i(t, e(t)) \le \phi V_i(t^-, e(t^-)). \tag{38}$$

Moreover, based on Assumption 1, it obtains that for all $t \in (t_k, t_{k+1})$, if $V_i(t, e(t)) \ge \lambda_{\max}(D_i D_i^\top)|u(t)|^2/\varepsilon$, then the

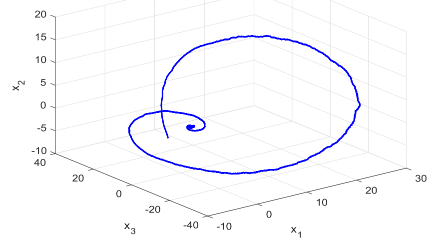

Fig. 6. State trajectory of the first subsystem, which also could be treated as a chaotic attractor in Chua's circuit.

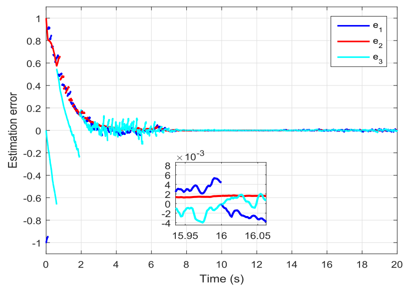

Fig. 7. The state response of the system $\mathcal{E}$ with the vanishing input, a periodic impulsive switching time sequence and $\theta = 0.1$.

differential of $V_i(t, e(t))$ is given by

$$\begin{aligned}\mathscr{L}V_i(t,e(t)) = & \; 2e^\top(t)C^\top C[A_i e(t) + \hat{f}_i(\hat{x}(t)) - f_i(x(t)) \\ & -D_i u(t)] + [E_i e(t) + \hat{g}_i(\hat{x}(t)) - g_i(x(t))]^\top \\ & \times C^\top C[E_i e(t) + \hat{g}_i(\hat{x}(t)) - g_i(x(t))] \\ & + \sum_{j\in\mathcal{M}} \pi_{ij} V_j(t,e(t)) \\ \le & \; \Lambda_i V_i(t,e(t)),\end{aligned}$$

where $\Lambda_i := 2\lambda_i + \eta_i + 2L_i + K_i + H_i + \varepsilon - 1 + \sum_{j\in\mathcal{M}} \pi_{ij}$ and $\varepsilon > 0$ could be arbitrarily small. Thus, the RFDT condition (24) in Theorem 4 is $\inf_{i\in\mathcal{M}} \Im_i(r) = \inf_{i\in\mathcal{M}} \int_{\phi r}^{r} \frac{ds}{\Lambda_i r} \ge \theta + \delta$, then the estimate error dynamics $\mathcal{E}$ is SISS.

In the following, Pick $n = 3$ and C is an identity matrix. That is, the system state $x(t) \in \mathbb{R}^3$ is fully observable. Let $\varepsilon = 1$, $f_i(x) = \text{sat}(x_1)(\alpha_i(a-b), 0, 0)^\top$, $\hat{f}_i(\hat{x}) = \text{sat}(\hat{x}_1)(\alpha_i(a-b), 0, 0)^\top$, $D_i = \alpha_i(1, 0.1, 0)^\top$, $g_i(x) = \text{sat}(x_1)(0.2, 0.1, 0.05)^\top$, $\hat{g}_i(\hat{x}) = \text{sat}(\hat{x}_1)(0.2, 0.1, 0.05)^\top$, $E_i = \text{diag}(0.1\alpha_i, 0.2\alpha_i, 0.3\alpha_i)$, and

$$\Pi = \begin{bmatrix} 0.3 & 0.7 \\ 0.55 & 0.45 \end{bmatrix}, \quad A_i = \begin{bmatrix} -\alpha_i(1-b) & \alpha_i & 0 \\ 1 & -1 & 1 \\ 0 & -\beta_i & 0 \end{bmatrix},$$

where $\text{sat}(\upsilon) = (|\upsilon + 1| - |\upsilon - 1|)/2$, and $\alpha_1 = 9/10, \beta_1 = 10/7, \alpha_2 = 1, \beta_1 = 8/5, a = 8/7, b = 5/7$. As a result, the state system (34) could be viewed as a stochastic hybrid system switching between two stochastic Chua's circuit; see Fig. 6. It obtains from computation that $\Lambda_1 = 3.4295, \Lambda_2 = 3.5873$ and $\Im_1(\upsilon) = 0.1182, \Im_2(\upsilon) = 0.1130$. Thus, we choose $\theta = 0.1$.

If $\omega(t) = e^{-t}(1,1,1)^\top$ and $u(t) = 0.4e^{-0.4t}$, i.e., the external input is vanishing, then under the initial state $e(0) = (-1, 1, 0)^\top$, the Gaussian white noise with zero-mean and variance of 1 and the periodic impulsive switching time sequence, the state response of the estimate error system $\mathcal{E}$ is given in Fig. 7. If $\omega(t) = 0.2\cos(t)(1,1,1)^\top$ and $u(t) = \sin(4t)$, i.e., the external input is periodic, then under the initial state $e(0) = (-1, 1, 0)^\top$, the Gaussian white noise with zero-mean and variance of 1, and the periodic impulsive switching time sequence, the state response of the estimate error system $\mathcal{E}$ is given in Fig. 8.

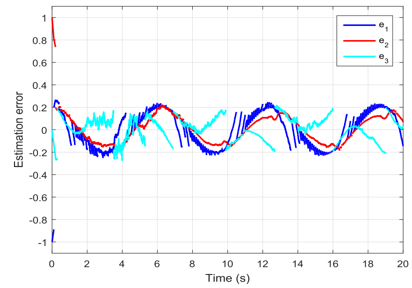

Fig. 8. The state response of the system $\mathcal{E}$ with the periodic input, a periodic impulsive switching time sequence and $\theta = 0.1$.

V. Conclusion

In this paper, we studied the stability properties of stochastic impulsive switched systems. Both the stable continuous dynamics case and the stable discrete dynamics case were studied. Using general Lyapunov functions and fixed dwell-time, stability conditions were derived. Future research could be directed to stability analysis of stochastic impulsive switched time-delay systems.

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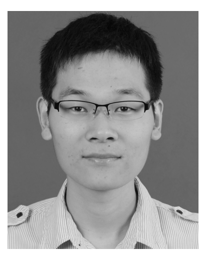

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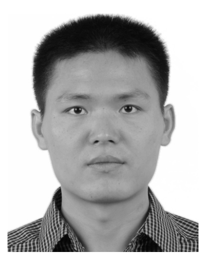

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