H_∞ Model Reduction for Interval Frequency Negative Imaginary Systems

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***Abstract*— This paper studies an H_∞ model reduction problem for interval frequency negative imaginary (IFNI) systems. For a given IFNI system, our goal is to find a reduced-order IFNI system satisfying a pre-specified H_∞ approximation error bound over the finite-frequency interval. Necessary and sufficient conditions in terms of matrix inequalities are derived for the existence and construction of an H_∞ reduced-order IFNI system. An improved iterative algorithm is provided to solve the matrix inequalities and to minimize the H_∞ approximation error. The proposed method is further clarified via the application to the electrical circuits, such as high-order Sallen–Key low-pass filter, piezoelectric tube scanner, and *RLC* circuit. The simulation results on these electrical circuits are compared with the finite-frequency interval Gramians-based model reduction method both in the frequency domain and time domain.**

***Index Terms*— H_∞ model reduction, interval frequency, negative imaginary systems, iterative algorithm, electrical circuits.**

I. Introduction

IN MANY practical engineering fields, such as integrated circuits, there are many complex systems described by differential equations. Mathematical modeling of such systems are often high order transfer functions, which poses serious difficulties to be simulated on the computer [1]. Model reduction has been recognized as an effective way to solve these problems and is still a topic of active research. Over the past decades, some classical model reduction methods, such as the balanced truncation method [2], Hankel-norm approximation [3] and Krylov-subspace model reduction method [4], have been shown to be effective in reducing the order of general linear systems. In recent years, model reduction methods based on convex optimization technology have drawn considerable attention [5]–[9]. For instance, H_∞ model reduction methods based on linear matrix inequalities (LMIs) have been widely used to deal with the structure preserving model reduction problems for different classes of systems, including switched systems [10], positive systems [11], Markovian jump systems [12], T-S fuzzy systems [9], and negative imaginary (NI) systems [7], [13]. The advantages of the H_∞ model reduction methods are twofold. Firstly, the stability of the reduced-order system can be guaranteed. Secondly, the H_∞ approximation error can be minimized.

For many applications, the approximation error only needs to be minimized over finite frequency interval. For instance, the aircraft mathematical models derived from the physical dynamics are only valid within a specific frequency interval [14], the elliptic low pass filter in [15] has the pass band edge frequencies, the swing-arm positioning system in [16] satisfies the positive real property over a finite frequency interval. In such cases, it is desirable to achieve lower approximation error over a finite frequency interval. However, the classical model reduction methods that approximate the original system over the entire frequency are not applicable. Hence, model reduction methods over the finite frequency interval have drawn profound interest and have been widely used to solve the structure preserving model reduction problems [6], [15], [17]–[20]. Model reduction problem over a finite frequency interval has been investigated for continuous-time systems [17] and discrete-time linear systems [18]. Furthermore, H_∞ model reduction based on LMIs has been applied to the frequency-limited model reduction for positive systems [6]. Optimal model reduction method has been proposed for bilinear systems [19] on a given frequency interval. A more complicated situation to find a reduced-order 2-D digital filter system over a finite frequency interval has been handled in [15] and [20].

NI system is an important class of dynamical system with equal number of inputs and outputs, having a real-rational, proper transfer function matrix $G(s)$ that satisfies the frequency domain condition $j[G(j\omega) - G^*(j\omega)] \geq 0$ for all $\omega \in (0, \infty)$ except values of ω where $j\omega$ is a pole of $G(j\omega)$ [21]. For the single-input single-output case, the imaginary part of the frequency response is non-negative, that is, $\Im[G(j\omega)] \leq 0$. NI system arises in many practical engineering systems, for example, lightly damped structures [21], [22], RLC circuit networks [23], atomic force microscopes [24], [25], robotic manipulator arms [26]. NI theory is related to positive real theory. Under certain technical assumptions, NI transfer functions can be transformed into positive real transfer functions by multiplying the NI transfer functions with sI or $-(\frac{1}{s})I$ and vice versa, see [27]. An overview of NI systems theory and applications is referred to the survey paper [28]. In practice, many dynamical systems satisfy the NI property over the finite frequency interval, for instance,

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the capacitance subsystem of piezoelectric tube scanners [29], a Sallen-key low pass filter that cascaded with a gain multiplier circuit [30]. In [31], the concept of finite frequency NI systems [32] has been extended to the interval frequency negative imaginary (IFNI) case. For model reduction of IFNI systems, it is desirable that the reduced-order system preserves the IFNI structure and gives better approximations over the frequency interval. However, the results on model reduction for NI systems [7], [13], [33], [34] over the entire positive frequency may not suitable. Moreover, the existing frequency interval Gramian-based methods [35], [36] can not guarantee the IFNI structure for the reduced-order systems. Therefore, it is necessary to develop new approaches to solve the model reduction problem for IFNI systems.

In this paper, we investigate the H_∞ model reduction problem for IFNI systems, including low frequency negative imaginary (LFNI), middle frequency negative imaginary (MFNI) and high frequency negative imaginary (HFNI) cases. For a given IFNI system $G(s)$, the objective of the paper is to find a reduced-order IFNI system $G_{\mathrm{r}}(s)$ such that $\|G(s)-G_{\mathrm{r}}(s)\|_\infty^\Omega < \gamma$, where γ is a prescribed approximation error bound, Ω is the given frequency interval. We split the original IFNI system into an asymptotically stable subsystem and a semi-stable subsystem. Then, finite frequency H_∞ model reduction is implemented on the asymptotically stable subsystem. By using the projection lemma, necessary and sufficient conditions in terms of matrix inequalities are derived for the existence and construction of the desired reduced-order IFNI system. In these new conditions, the reduced-order IFNI system matrices are decoupled with the matrix variables induced by the IFNI lemma and GKYP lemma. An improved algorithm including two stages is developed to search for the reduced-order IFNI system and to minimize the H_∞ approximation error over the finite frequency interval. The proposed model reduction method is illustrated through examples from a Sallen-key filter, a piezoeletric tube scanner and an RLC circuit. Compared with the finite frequency interval Gramians-based model reduction method [35], the proposed model reduction method guarantees the IFNI structure and achieves lower approximation error. The limitation is that the proposed model reduction method is not applicable for IFNI systems with non-minimal realizations. Moreover, only a sub-optimal reduced-order IFNI system can be obtained by the proposed algorithm.

Notation: All the matrices are assumed to be compatible dimensions and the symbol $\star$ within a square matrix represents the symmetric part. $\mathbb{R}^{n\times n}$ denotes the set of $n\times n$ real matrices. $0_{m\times n}$ denotes an $m\times n$ zero matrix and I_n represents identity matrix of order n. $\mathcal{H}$ denotes the set of matrices defined by $\mathcal{H} \triangleq \{U : U = [0_{(r_2+m)\times n_2}\ \Lambda]\}$, $\mathcal{H}_1$ denotes the set of matrices defined by $\mathcal{H}_1 \triangleq \{U_1 : U_1 = [0_{(r_2+m)\times n_1}\ \Lambda]\}$, where $\Lambda \in \mathbb{R}^{(r_2+m)\times r_2}$. Let A^* denotes the complex conjugate transpose of a complex matrix A. The real part and imaginary part of a complex number s, respectively, are denoted by $\Re[s]$ and $\Im[s]$. The notation $P > 0$ (≥ 0) means that matrix P is positive definite (semi-definite). For a matrix $X \in \mathbb{C}^{n\times n}$, $\mathrm{Her}(X)$ indicates $X^* + X$. The set of square, real-rational, proper transfer functions is denoted by $\mathbf{G}$.

II. Problem Statement

Consider an IFNI system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t),\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$. The transfer function of system (1) is given by

$$G(s) = C(sI - A)^{-1}B.$$

A reduced-order system for (1) is

$$\begin{aligned}\dot{x}_{\mathrm{r}}(t) &= A_{\mathrm{r}}x_{\mathrm{r}}(t) + B_{\mathrm{r}}u(t),\\ y_{\mathrm{r}}(t) &= C_{\mathrm{r}}x_{\mathrm{r}}(t),\end{aligned} \tag{2}$$

where $x_{\mathrm{r}}(t) \in \mathbb{R}^r$, $y_{\mathrm{r}}(t) \in \mathbb{R}^m$, $A_{\mathrm{r}} \in \mathbb{R}^{r\times r}$, $B_{\mathrm{r}} \in \mathbb{R}^{r\times m}$, $C_{\mathrm{r}} \in \mathbb{R}^{m\times r}$ with $1 \leq r < n$.

From (1) and (2), the approximation error system is given by

$$\begin{aligned}\dot{x}_{\mathrm{e}}(t) &= A_{\mathrm{e}}x_{\mathrm{e}}(t) + B_{\mathrm{e}}u(t),\\ e(t) &= C_{\mathrm{e}}x_{\mathrm{e}}(t),\end{aligned} \tag{3}$$

where $x_{\mathrm{e}}(t) = \left[x^{\mathsf{T}}(t)\ x_{\mathrm{r}}^{\mathsf{T}}(t)\right]^{\mathsf{T}}$ is the augmented state vector, $e(t) = y(t) - y_{\mathrm{r}}(t)$ is the output error and

$$A_{\mathrm{e}} = \begin{bmatrix} A & 0\\ 0 & A_{\mathrm{r}}\end{bmatrix},\quad B_{\mathrm{e}} = \begin{bmatrix} B\\ B_{\mathrm{r}}\end{bmatrix},\quad C_{\mathrm{e}} = \begin{bmatrix} C & -C_{\mathrm{r}}\end{bmatrix}.$$

The transfer function of system (3) is given by

$$G_{\mathrm{e}}(s) = C_{\mathrm{e}}(sI - A_{\mathrm{e}})^{-1}B_{\mathrm{e}}.$$

Define the frequency interval set as

$$\Omega \triangleq \Omega_L \bigcup \Big(\bigcup_{\hat{l}=1}^{N} \Omega_{M_{\hat{l}}}\Big) \bigcup \Omega_H, \tag{4}$$

where Ω_L, $\Omega_{M_{\hat{l}}}$, Ω_H stand for the low, middle and high frequency ranges,

$$\begin{aligned}\Omega_L &= \{\omega \in \mathbb{R} : 0 < \omega \leq \bar{\omega}_0\},\\ \Omega_{M_{\hat{l}}} &= \{\omega \in \mathbb{R} : \underline{\omega}_{\hat{l}} \leq \omega \leq \bar{\omega}_{\hat{l}}, \bar{\omega}_{\hat{l}} > \underline{\omega}_{\hat{l}} > \bar{\omega}_{\hat{l}-1}\},\\ \Omega_H &= \left\{\omega \in \mathbb{R} : \underline{\omega}_h \leq \omega, \underline{\omega}_h > \bar{\omega}_N\right\},\quad \hat{l} = 1, \ldots, N.\end{aligned}$$

The H_∞ model reduction problem for IFNI systems can be formulated as follows.

Problem 1: Given a frequency interval Ω, $\gamma > 0$ and r $(1 \leq r < n)$. The H_∞ model reduction problem for IFNI system (1) is to find a reduced-order system (2), such that

1) reduced-order system (2) is IFNI over the frequency interval Ω;
2) system (3) satisfies $\|G_{\mathrm{e}}(s)\|_\infty^\Omega < \gamma$.

Some preliminaries are presented. Firstly, the definition of IFNI system is given as follows.

Definition 1 [31]: Given a transfer function matrix $G(s) \in \mathbf{G}$. $G(s)$ is said to be IFNI in the frequency interval Ω if

1) $G(s)$ has no poles in $\Re[s] > 0$;
2) $j[G(j\omega) - G^*(j\omega)] \geq 0$ for all $\omega \in \Omega$ where $j\omega$ is not a pole of $R(s)$;

TABLE I
Ψ_l DEFINED FOR EACH FREQUENCY INTERVAL

	Ω_L	$\Omega_{M_{\hat{l}}}$	Ω_H
l	0	$1,\ldots,N$	$N+1$
Ψ_l	$\begin{bmatrix}-1 & 0\\ 0 & \bar{\omega}_0^2\end{bmatrix}$	$\begin{bmatrix}-1 & j\omega_{cl}\\ -j\omega_{cl} & -\underline{\omega}_{\hat{l}}\bar{\omega}_{\hat{l}}\end{bmatrix}$	$\begin{bmatrix}1 & 0\\ 0 & -\underline{\omega}_h^2\end{bmatrix}$

3) if $j\omega_0, \omega_0 \in \mathbf{\Omega}$, is a pole of $G(s)$, then it is a simple pole and the corresponding residue matrix of $jG(s)$ is positive semidefinite Hermitian;
4) if $G(s)$ has a pole at $s = 0$, then $\lim_{s\to 0} s^2G(s)$ is positive semidefinite Hermitian and $\lim_{s\to 0} s^kG(s) = 0$ for $k \geq 3$;
5) $G(\infty) = G^{\mathsf{T}}(\infty)$.

Remark 1: In Definition 1, if $\mathbf{\Omega} = \mathbf{\Omega}_L$, then $G(s)$ is said to be LFNI, which could be considered as an extension of the definition of finite frequency NI [32] by allowing poles at the origin. If $\mathbf{\Omega} = \mathbf{\Omega}_{M_{\hat{l}}}$, then $G(s)$ is said to be MFNI. If $\mathbf{\Omega} = \mathbf{\Omega}_H$, $G(s)$ is said to be HFNI. In Problem 1, the statement 1) means that we need to find a reduced-order system satisfies the NI structure for the given $\mathbf{\Omega}$, that is, to find a reduced-order system satisfies the conditions given in Definition 1.

Let $\omega_{cl} = (\underline{\omega}_{\hat{l}}+\bar{\omega}_{\hat{l}})/2$ be the middle point of each frequency interval $\mathbf{\Omega}_{M_{\hat{l}}}$. We now define $\Phi = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, and Ψ_l, shown in Table I, for each frequency interval.

The IFNI lemma is given as follows.

Lemma 1 [31]: Consider a transfer function matrix $G(s) \in \mathbf{G}$ with minimal state-pace realization (A, B, C, D). Suppose that $G(s)$ has no poles in the open right-half of the complex plane and that the pure imaginary poles of $G(s)$, if any, are simple, and the zero pole, if any, are either a simple or double pole. When A has eigenvalues $j\omega_i$ $(i \in \{1,\ldots,p\})$ such that $\omega_i \in \mathbf{\Omega}\bigcup\{0\}$, the residue matrix of $A(sI-A)^{-1}$ at $s = j\omega_i$ is given by $\Phi_i = \lim_{s\to j\omega_i}(s-j\omega_i)A(sI-A)^{-1}$. Then $G(s)$ is IFNI if and only if $D = D^{\mathsf{T}}$,

1) $C\Phi_iB = (C\Phi_iB)^* \geq 0$ for all $i \in \{1,\ldots,p\}$ if $j\omega_i$ is an eigenvalue of A;
2) there exist real symmetric matrices P_0, $Q_0 \geq 0$, P_{N+1}, $Q_{N+1} \geq 0$ and Hermitian matrices $P_{\hat{l}} = P_{\hat{l}}^*$, $Q_{\hat{l}} = Q_{\hat{l}}^* \geq 0$, $\hat{l} = 1,\ldots,N$, such that

$$\begin{bmatrix}A & B\\ I_n & 0_{n\times m}\end{bmatrix}^{\mathsf{T}}(\Phi\otimes P_l+\Psi_l\otimes Q_l)\begin{bmatrix}A & B\\ I_n & 0_{n\times m}\end{bmatrix} - \begin{bmatrix}0_{n\times n} & A^{\mathsf{T}}C^{\mathsf{T}}\\ CA & CB+B^{\mathsf{T}}C^{\mathsf{T}}\end{bmatrix} \leq 0, \quad (5)$$

for all $l = 0, \hat{l}, N+1$.

Remark 2: Note that when the system (A, B, C, D) is asymptotically stable, the condition 1) in Lemma 1 can be removed. That is, the asymptotically stable system (A, B, C, D) is IFNI if and only if (5) and $D = D^{\mathsf{T}}$ hold.

The following lemma is known as the GKYP lemma.

Lemma 2 [37]: Suppose that the approximation error system (3) is asymptotically stable, then $\|G_e(s)\|_\infty^{\Omega} < \gamma$ if and only if there exist real symmetric matrices P_{el}, $Q_{el} > 0$, such that

$$\begin{bmatrix}A_e & B_e\\ I_{n+r} & 0\end{bmatrix}^{\mathsf{T}}(\Phi\otimes P_{el}+\Psi_l\otimes Q_{el})\begin{bmatrix}A_e & B_e\\ I_{n+r} & 0\end{bmatrix} + \begin{bmatrix}C_e & 0\\ 0 & I_m\end{bmatrix}^{\mathsf{T}}\begin{bmatrix}I_m & 0\\ 0 & -\gamma^2I_m\end{bmatrix}\begin{bmatrix}C_e & 0\\ 0 & I_m\end{bmatrix} < 0, \quad (6)$$

for all $l = 0, \hat{l}, N+1$.

The following lemma is known as the projection lemma.

Lemma 3 [38]: Given a symmetric matrix Σ of dimension $n\times n$ and two matrices Ξ_1, Ξ_2 of column dimension n, there exists an unstructured matrix W that satisfies

$$\Xi_1^{\mathsf{T}}W\Xi_2 + \Xi_2^{\mathsf{T}}W^{\mathsf{T}}\Xi_1 + \Sigma < 0,$$

if and only if the following inequalities are satisfied:

$$N_{\Xi_1}^{\mathsf{T}}WN_{\Xi_1} < 0, \quad N_{\Xi_2}^{\mathsf{T}}WN_{\Xi_2} < 0,$$

where N_{Ξ_1} and N_{Ξ_2} are arbitrary matrices whose columns form a basis of the nullspaces of Ξ_1 and Ξ_2.

III. MAIN RESULTS

In this section, the main results of this paper are presented. We first split the IFNI system into a semistable subsystem and an asymptotically stable subsystem. H_∞ model reduction method is proposed for the asymptotically stable subsystem. Necessary and sufficient conditions are derived for the existence and construction of the H_∞ reduced-order IFNI system. These conditions show that the desired reduced-order system can be found by solving LMIs. Moreover, iterative algorithm is provided to solve the LMIs and to minimize the H_∞ approximation error.

A. Separation of IFNI System

Given an IFNI system (1), the minimal state-space realization can be transformed into the following block diagonal form [26]

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}B_1\\ B_2\end{bmatrix}u(t),$$
$$y(t) = \begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}, \quad (7)$$

where $A_1 \in \mathbb{R}^{n_1\times n_1}$, $A_2 \in \mathbb{R}^{n_2\times n_2}$, $B_1 \in \mathbb{R}^{n_1\times m}$, $B_2 \in \mathbb{R}^{n_2\times m}$, $C_1 \in \mathbb{R}^{m\times n_1}$, $C_2 \in \mathbb{R}^{m\times n_2}$, $n_1+n_2 = n$. A_1 is a diagonalizable matrix with purely imaginary eigenvalues, the eigenvalues of A_2 have strictly negative real parts. Thus, the transfer function of (1) can be rewritten as

$$G(s) = G_1(s) + G_2(s),$$

where

$$G_1(s) = C_1(sI-A_1)^{-1}B_1, \quad G_2(s) = C_2(sI-A_2)^{-1}B_2.$$

Here, the subsystem $G_1(s)$ is semi-stable, the subsystem $G_2(s)$ is asymptotically stable.

Remark 3: According to the proof of [26, Lemma 7], for the given NI system with the minimal state-space realization (1), there always exist a nonsingular transformation T such that $(T^{-1}AT, T^{-1}B, CT)$ be the real Jordan canoncial form. Moreover, the transformation T can be chosen such that the real Jordan blocks of $T^{-1}AT$ are ordered according to the eigenvalues of the matrix A. Thus, we have that any NI system can be transformed to the block diagonal form (7) with the minimality assumption.

A reduced-order system approximate system (7) at the given frequency interval Ω is given by

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_{r_2}(t)\end{bmatrix} = \begin{bmatrix}A_1 & 0\\ 0 & A_{r_2}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_{r_2}(t)\end{bmatrix} + \begin{bmatrix}B_1\\ B_{r_2}\end{bmatrix}u(t),$$
$$y_r(t) = \begin{bmatrix}C_1 & C_{r_2}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_{r_2}(t)\end{bmatrix}, \tag{8}$$

where $n_1 + r_2 = r$, $A_{r_2} \in \mathbb{R}^{r_2\times r_2}$, $B_{r_2} \in \mathbb{R}^{r_2\times m}$, $C_{r_2} \in \mathbb{R}^{m\times r_2}$. The eigenvalues of A_{r_2} have strictly negative real parts. Here, the subsystem $(A_{r_2}, B_{r_2}, C_{r_2})$ is the reduced-order asymptotically stable subsystem.

Define

$$\bar{F}_1 = \begin{bmatrix}0_{n_1\times r_2} & 0\\ I_{r_2} & 0_{r_2\times m}\end{bmatrix}, \quad \bar{A}_1 = \begin{bmatrix}A_1 & 0\\ 0 & 0_{r_2\times r_2}\end{bmatrix},$$
$$\bar{B}_1 = \begin{bmatrix}B_1\\ 0_{r_2\times m}\end{bmatrix}, \quad \bar{C}_1 = \begin{bmatrix}C_1 & 0_{m\times r_2}\end{bmatrix}, \ \bar{H} = \begin{bmatrix}0_{m\times r_2} & I_m\end{bmatrix}.$$

The system matrices of reduced-order system (8) can be rewritten as

$$A_r = \bar{A}_1 + U_1^\top \bar{F}_1^\top, \quad B_r = \bar{B}_1 + U_1^\top \bar{H}^\top, \ C_r = \bar{C}_1 + V_1, \tag{9}$$

where $U_1 = \begin{bmatrix}0_{r_2\times n_1} & A_{r_2}^\top\\ 0_{m\times n_1} & B_{r_2}^\top\end{bmatrix} \in \mathcal{H}_1$, $V_1 = \begin{bmatrix}0_{m\times n_1} & C_{r_2}\end{bmatrix}$.

Note that the transfer function of system (8) is

$$G_r(s) = G_1(s) + G_{r_2}(s), \quad G_{r_2}(s) = C_{r_2}(sI - A_{r_2})^{-1}B_{r_2}.$$

Thus, the transfer function of the approximation error system (3) can be rewritten as

$$G_e(s) = G(s) - G_r(s) = G_2(s) - G_{r_2}(s).$$

A minimal state-space realization of $G_e(s)$ is given by

$$A_e = \begin{bmatrix}A_2 & 0\\ 0 & A_{r_2}\end{bmatrix}, \quad B_e = \begin{bmatrix}B_2\\ B_{r_2}\end{bmatrix}, \quad C_e = \begin{bmatrix}C_2 & -C_{r_2}\end{bmatrix}. \tag{10}$$

Let

$$\bar{F} = \begin{bmatrix}0_{n_2\times r_2} & 0\\ I_{r_2} & 0_{r_2\times m}\end{bmatrix}, \quad \bar{A} = \begin{bmatrix}A_2 & 0\\ 0 & 0_{r_2\times r_2}\end{bmatrix},$$
$$\bar{B} = \begin{bmatrix}B_2\\ 0_{r_2\times m}\end{bmatrix}, \quad \bar{C} = \begin{bmatrix}C_2 & 0_{m\times r_2}\end{bmatrix}.$$

The system matrices in (10) can be rewritten as

$$A_e = \bar{A} + U^\top \bar{F}^\top, \quad B_e = \bar{B} + U^\top \bar{H}^\top, \quad C_e = \bar{C} + V, \tag{11}$$

where

$$U = \begin{bmatrix}0_{r_2\times n_2} & A_{r_2}^\top\\ 0_{m\times n_2} & B_{r_2}^\top\end{bmatrix} \in \mathcal{H}, \quad V = \begin{bmatrix}0_{m\times n_2} & -C_{r_2}\end{bmatrix}.$$

In terms of the block diagonal form (7), a new condition is proposed to test the IFNI structure.

Proposition 1: Consider a transfer function matrix $G(s) \in \mathbf{G}$ with a minimal state-pace realization (7). Suppose that the nonzero eigenvalues of A_1 are simple and the zero eigenvalues of A_1, if any, are either simple or double. Let $j\omega_i$ ($i \in \{1,\ldots,p\}$) be the eigenvalues of A_1, $\omega_i \in \Omega\bigcup\{0\}$, the residue matrix of $A_1(sI - A_1)^{-1}$ at $s = j\omega_i$ is given by $\hat{\Phi}_i = \lim_{s\to j\omega_i}(s - j\omega_i)A_1(sI - A_1)^{-1}$. Then $G(s)$ is IFNI if and only if

1) $C_1\hat{\Phi}_i B_1 = (C_1\hat{\Phi}_i B_1)^* \geq 0$ for all $i \in \{1,\ldots,p\}$ if $j\omega_i$ is an eigenvalue of A_1;
2) there exist real symmetric matrices P_0, $Q_0 \geq 0$, P_{N+1}, $Q_{N+1} \geq 0$ and Hermitian matrices $P_{\hat{l}} = P_{\hat{l}}^*$, $Q_{\hat{l}} = Q_{\hat{l}}^* \geq 0$, $\hat{l} = 1,\ldots,N$, such that (5) holds.

Proof: In view of Lemma 1, $G(s)$ is IFNI if and only if (5) and $C\Phi_i B = (C\Phi_i B)^* \geq 0$ hold. For the block diagonal form (7), the residue matrix Φ_i can be calculated as

$$\begin{aligned}\Phi_i &= \lim_{s\to j\omega_i}(s - j\omega_i)\begin{bmatrix}A_1 & 0\\ 0 & A_2\end{bmatrix}\begin{bmatrix}(sI - A_1) & 0\\ 0 & (sI - A_2)\end{bmatrix}^{-1}\\ &= \lim_{s\to j\omega_i}(s - j\omega_i)\begin{bmatrix}A_1(sI - A_1)^{-1} & 0\\ 0 & A_2(sI - A_2)^{-1}\end{bmatrix}\\ &= \begin{bmatrix}\lim_{s\to j\omega_i}(s - j\omega_i)A_1(sI - A_1)^{-1} & 0\\ 0 & 0\end{bmatrix} = \begin{bmatrix}\hat{\Phi}_i & 0\\ 0 & 0\end{bmatrix}.\end{aligned}$$

It follows that

$$C\Phi_i B = \begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}\hat{\Phi}_i & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}B_1\\ B_2\end{bmatrix} = C_1\hat{\Phi}_i B_1.$$

Thus, $C\Phi_i B = (C\Phi_i B)^* \geq 0$ and $C_1\hat{\Phi}_i B_1 = (C_1\hat{\Phi}_i B_1)^* \geq 0$ are equivalent. ■

Remark 4: For a given IFNI system (7), Problem 1 is solvable if and only if there exists a reduced-order system (8) satisfies (5) and $\|G_e(s)\|_\infty^\Omega < \gamma$. However, the inequalities (5) and $\|G_e(s)\|_\infty^\Omega < \gamma$ are non-convex, which makes it difficult to construct the reduced-order system. Thus, the main challenge of this problem is how to transform these inequalities into convex problems.

B. H_∞ Model Reduction: IFNI

In this subsection, we first investigate the H_∞ model reduction problem for LFNI systems. Necessary and sufficient conditions in terms of matrix inequalities are proposed for the existence and construction of the H_∞ reduced-order LFNI system. Then, the proposed results are extended to the MFNI and HFNI cases.

Firstly, the following necessary and sufficient conditions are proposed for the existence and construction of the H_∞ reduced-order system over the low frequency range.

Lemma 4: Given $\gamma > 0$, r_2 ($1 \leq r_2 < n_2$), $n_1 + r_2 = r$, $\Omega_L = \{\omega \in \mathbb{R} : 0 < \omega \leq \bar{\omega}_0\}$, LFNI system (1). There exists a reduced-order system (2) such that the approximation error system (3) is asymptotically stable and satisfies $\|G_e(s)\|_\infty^{\Omega_L} < \gamma$ if and only if there exist matrices $Q \in \mathbb{R}^{(n_2+r_2)\times(n_2+r_2)}$, $Q > 0$, $P_{e_0} \in \mathbb{R}^{(n_2+r_2)\times(n_2+r_2)}$, $Q_{e_0} \in \mathbb{R}^{(n_2+r_2)\times(n_2+r_2)}$, $Q_{e_0} > 0$, $X_1 \in \mathbb{R}^{(n_2+r_2)\times(n_2+r_2)}$, $X_2 \in \mathbb{R}^{(n_2+r_2)\times(n_2+r_2)}$, $U \in \mathcal{H}$, V, such that the following matrix inequalities hold:

$$\mathcal{A} = \mathrm{Her}((\bar{A}^\top + \bar{F}U)Q) < 0, \tag{12}$$

$$\Upsilon = \begin{bmatrix}\Upsilon_{11} & \Upsilon_{12} & \Upsilon_{13} & 0_{(n_2+r_2)\times m}\\ \star & \Upsilon_{22} & \Upsilon_{23} & \bar{C}^\top + V^\top\\ \star & \star & -\gamma^2 I_m & 0_{m\times m}\\ \star & \star & \star & -I_m\end{bmatrix} < 0, \tag{13}$$

where

$$\begin{aligned}\Upsilon_{11} &= -Q_{e_0} - \mathrm{Her}(X_1),\\ \Upsilon_{12} &= P_{e_0} - X_2 + X_1^\top \bar{A} + X_1^\top U^\top \bar{F}^\top,\\ \Upsilon_{13} &= X_1^\top \bar{B} + X_1^\top U^\top \bar{H}^\top,\\ \Upsilon_{22} &= \bar{\omega}_0^2 Q_{e_0} + \mathrm{Her}(X_2^\top \bar{A} + X_2^\top U^\top \bar{F}^\top),\\ \Upsilon_{23} &= X_2^\top \bar{B} + X_2^\top U^\top \bar{H}^\top.\end{aligned}$$

Proof: Note that $G_e(s)$ is asymptotically stable if and only if there exists a matrix $Q \in \mathbb{R}^{(n_2+r_2)\times(n_2+r_2)}$, $Q > 0$, such that $\mathrm{Her}(A_e^\top Q) < 0$. Substituting $A_e = \bar{A} + U^\top \bar{F}^\top$ into $\mathrm{Her}(A_e^\top Q) < 0$, we arrive at (12). That is, $\mathrm{Her}(A_e^\top Q) < 0$ and (12) are equivalent.

In view of Lemma 2 [37], the asymptotically stable system $G_e(s)$ satisfies $\|G_e(s)\|_\infty^{\Omega_L} < \gamma$ if and only if there exist matrices $P_{e_0} \in \mathbb{R}^{(n_2+r_2)\times(n_2+r_2)}$, $Q_{e_0} \in \mathbb{R}^{(n_2+r_2)\times(n_2+r_2)}$, $Q_{e_0} > 0$, such that (6). Now, we prove that (6) and (13) are equivalent.

Note that

$$N_{\Xi_1}^\top \Sigma N_{\Xi_1} = -\gamma^2 I_m < 0, \tag{14}$$

where $N_{\Xi_1} = \begin{bmatrix} 0_{m\times(n_2+r_2)} & 0_{m\times(n_2+r_2)} & I_m \end{bmatrix}^\top$,

$$\Sigma = \begin{bmatrix} -Q_{e_0} & P_{e_0} & 0_{(n_2+r_2)\times m} \\ \star & \bar{\omega}_0^2 Q_{e_0} + C_e^\top C_e & 0_{(n_2+r_2)\times m} \\ \star & \star & -\gamma^2 I_m \end{bmatrix}.$$

The inequality (6) can be rewritten as

$$N_{\Xi_2}^\top \Sigma N_{\Xi_2} < 0, \tag{15}$$

where

$$N_{\Xi_2} = \begin{bmatrix} A_e^\top & I_{n_2+r_2} & 0 \\ B_e^\top & 0 & I_m \end{bmatrix}^\top.$$

According to Lemma 3 [38], (14) and (15) are equivalent to

$$\begin{aligned}&\Sigma + \Xi_1^\top X^\top \Xi_2 + \Xi_2^\top X \Xi_1 \\ &= \begin{bmatrix} \Upsilon_{11} & P_{e_0} - X_2 + X_1^\top A_e & X_1^\top B_e \\ \star & \bar{\omega}_0^2 Q_{e_0} + C_e^\top C_e + \mathrm{Her}(X_2^\top A_e) & X_2^\top B_e \\ \star & \star & -\gamma^2 I_m \end{bmatrix} < 0,\end{aligned} \tag{16}$$

where Υ_{11} is defined in (13), $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$,

$$\begin{aligned}\Xi_1 &= \begin{bmatrix} I_{n_2+r_2} & 0 & 0 \\ 0 & I_{n_2+r_2} & 0_{(n_2+r_2)\times m} \end{bmatrix},\\ \Xi_2 &= \begin{bmatrix} -I_{n_2+r_2} & A_e & B_e \end{bmatrix},\end{aligned}$$

$X_1 \in \mathbb{R}^{(n_2+r_2)\times(n_2+r_2)}$, $X_2 \in \mathbb{R}^{(n_2+r_2)\times(n_2+r_2)}$. Substituting (11) into (16) and using the Schur complement, we have that (13). Thus, we obtain that the inequalities (6) and (13) are equivalent. The proof is completed. ■

Remark 5: For finite frequency H_∞ model reduction problems, Lemma 3 [38] has been widely used to deal with GKYP lemma, such as in [5], [39], and [40]. Note that, the obtained results in [39, Th. 3] are not applicable for the high frequency case. However, the inequalities (12), (13) in Lemma 4 are also applicable for the middle frequency and high frequency cases.

Remark 6: For the middle frequency case, the matrices Υ_{11}, Υ_{12}, Υ_{22} in (13) are defined as

$$\begin{aligned}\Upsilon_{11} &= -Q_{e_{\hat{l}}} - \mathrm{Her}(X_1),\\ \Upsilon_{12} &= P_{e_{\hat{l}}} + j\omega_{cl} Q_{e_{\hat{l}}} - X_2 + X_1^\top \bar{A} + X_1^\top U^\top \bar{F}^\top,\\ \Upsilon_{22} &= -\omega_{\hat{l}}\bar{\omega}_{\hat{l}} Q_{e_{\hat{l}}} + \mathrm{Her}(X_2^\top \bar{A} + X_2^\top U^\top \bar{F}^\top),\end{aligned}$$

for all $\hat{l} = 1, \ldots, N$. Similarly, by replacing the matrices Υ_{11}, Υ_{12}, Υ_{22} in (13) with

$$\begin{aligned}\Upsilon_{11} &= Q_{e_{N+1}} - \mathrm{Her}(X_1),\\ \Upsilon_{12} &= P_{e_{N+1}} - X_2 + X_1^\top \bar{A} + X_1^\top U^\top \bar{F}^\top,\\ \Upsilon_{22} &= -\omega_h^2 Q_{e_{N+1}} + \mathrm{Her}(X_2^\top \bar{A} + X_2^\top U^\top \bar{F}^\top),\end{aligned}$$

the necessary and sufficient conditions in Lemma 4 are obtained for the existence and construction of the H_∞ reduced-order system over the high frequency range.

Based on Lemma 4, the following necessary and sufficient conditions are derived for the existence and construction of the H_∞ reduced-order LFNI system.

Theorem 1: Given $\gamma > 0$, r_2 $(1 \le r_2 < n_2)$, $n_1 + r_2 = r$, LFNI system (7) and $\Omega_L = \{\omega \in \mathbb{R} : 0 < \omega \le \bar{\omega}_0\}$. Problem 1 is solvable if and only if there exist matrices $Q \in \mathbb{R}^{(n_2+r_2)\times(n_2+r_2)}$, $Q > 0$, $P_{e_0} \in \mathbb{R}^{(n_2+r_2)\times(n_2+r_2)}$, $Q_{e_0} \in \mathbb{R}^{(n_2+r_2)\times(n_2+r_2)}$, $Q_{e_0} > 0$, $P_{r_0} \in \mathbb{R}^{r\times r}$, $Q_{r_0} \in \mathbb{R}^{r\times r}$, $Q_{r_0} \ge 0$, matrices $X_1 \in \mathbb{R}^{(n_2+r_2)\times(n_2+r_2)}$, $X_2 \in \mathbb{R}^{(n_2+r_2)\times(n_2+r_2)}$, $Y_1 \in \mathbb{R}^{r\times r}$, $Y_2 \in \mathbb{R}^{r\times r}$, $\tilde{U} \in \mathcal{H}$, $M \in \mathcal{H}$, $\tilde{U}_1 \in \mathcal{H}_1$, $M_1 \in \mathcal{H}_1$, V, V_1, block diagonal matrices $S = \mathrm{diag}\{S_2, S_1\} > 0$, $S_2 \in \mathbb{R}^{|n_1-n_2|\times|n_1-n_2|}$, diagonal matrix $S_1 \in \mathbb{R}^{r\times r}$, $S_1 > 0$, such that the following matrix inequalities

$$\mathcal{W} = \begin{bmatrix} \mathcal{W}_{11} & -Q - \bar{F}M \\ \star & -S \end{bmatrix} < 0, \tag{17}$$

$$L = \begin{bmatrix} L_{11} & L_{12} & X_1^\top \bar{B} & 0_{(n_2+r_2)\times m} & X_1^\top \\ \star & L_{22} & L_{23} & \bar{C}^\top + V^\top & X_2^\top + \bar{F}M \\ \star & \star & L_{33} & 0_{m\times m} & \bar{H}M \\ \star & \star & \star & -I_m & 0_{m\times(n_2+r_2)} \\ \star & \star & \star & \star & -S \end{bmatrix} < 0, \tag{18}$$

$$Z = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & 0_{r\times r} & Y_1^\top \\ \star & Z_{22} & Z_{23} & \varepsilon I_r & Y_2^\top + \bar{F}_1 M_1 \\ \star & \star & Z_{33} & 0_{m\times r} & \bar{H}M_1 \\ \star & \star & \star & -\varepsilon I_r & 0_{r\times r} \\ \star & \star & \star & \star & -S_1 \end{bmatrix} < 0, \tag{19}$$

hold for all $\varepsilon > 0$, where

$$\begin{aligned}\mathcal{W}_{11} &= \mathrm{Her}(\bar{A}^\top Q - \bar{F}\tilde{U}M^\top \bar{F}^\top) + \bar{F}\tilde{U}S\tilde{U}^\top \bar{F}^\top,\\ L_{11} &= -Q_{e_0} - \mathrm{Her}(X_1), \quad L_{12} = P_{e_0} - X_2 + X_1^\top \bar{A},\\ L_{22} &= \bar{\omega}_0^2 Q_{e_0} + \mathrm{Her}(X_2^\top \bar{A} - \bar{F}\tilde{U}M^\top \bar{F}^\top) + \bar{F}\tilde{U}S\tilde{U}^\top \bar{F}^\top,\\ L_{23} &= X_2^\top \bar{B} - \bar{F}\tilde{U}M^\top \bar{H}^\top - \bar{F}M\tilde{U}^\top \bar{H}^\top + \bar{F}\tilde{U}S\tilde{U}^\top \bar{H}^\top,\\ L_{33} &= -\gamma^2 I_m - \mathrm{Her}(\bar{H}\tilde{U}M^\top \bar{H}^\top) + \bar{H}\tilde{U}S\tilde{U}^\top \bar{H}^\top,\\ Z_{11} &= -Q_{r_0} - \mathrm{Her}(Y_1), \quad Z_{12} = P_{r_0} - Y_2 + Y_1^\top \bar{A}_1,\\ Z_{13} &= -\bar{C}_1^\top - V_1^\top + Y_1^\top \bar{B}_1,\\ Z_{22} &= \bar{\omega}_0^2 Q_{r_0} - 2\varepsilon I_r + \mathrm{Her}(Y_2^\top \bar{A}_1 - \bar{F}_1\tilde{U}_1 M_1^\top \bar{F}_1^\top)\\ &\quad + \bar{F}_1\tilde{U}_1 S_1 \tilde{U}_1^\top \bar{F}_1^\top,\end{aligned}$$

$$Z_{23} = Y_2^\top \bar{B}_1 - \bar{F}_1\tilde{U}_1 M_1^\top \bar{H}^\top - \bar{F}_1 M_1 \tilde{U}_1^\top \bar{H}^\top + \bar{F}_1\tilde{U}_1 S_1 \tilde{U}_1^\top \bar{H}^\top,$$
$$Z_{33} = -\varepsilon I_m - \mathrm{Her}(\bar{H}\tilde{U}_1 M_1^\top \bar{H}^\top) + \bar{H}\tilde{U}_1 S_1 \tilde{U}_1^\top \bar{H}^\top.$$

Proof: According to Proposition 1 and Lemma 4, Problem 1 is solvable if and only if there exist symmetric matrices $Q > 0$, P_{e_0}, $Q_{e_0} > 0$, P_{r_0}, $Q_{r_0} \geq 0$, such that (12), (13) and the following matrix inequality hold,

$$\begin{bmatrix} A_\mathrm{r} & B_\mathrm{r} \\ I_r & 0 \end{bmatrix}^\top \begin{bmatrix} -Q_{r_0} & P_{r_0} \\ P_{r_0} & \bar{\omega}_0^2 Q_{r_0} \end{bmatrix} \begin{bmatrix} A_\mathrm{r} & B_\mathrm{r} \\ I_r & 0 \end{bmatrix} - \begin{bmatrix} 0_{r\times r} & A_\mathrm{r}^\top C_\mathrm{r}^\top \\ C_\mathrm{r} A_\mathrm{r} & C_\mathrm{r} B_\mathrm{r} + B_\mathrm{r}^\top C_\mathrm{r}^\top \end{bmatrix} < 0. \quad (20)$$

Now, we prove that (12), (13) and (20) are equivalent to (17), (18) and (19).

($\Rightarrow$) The inequality (20) can be rewritten as

$$\hat{N}_{\hat{\Xi}_2}^\top \hat{\Sigma}_\mathrm{r} \hat{N}_{\hat{\Xi}_2} \leq 0,$$

where

$$\hat{N}_{\hat{\Xi}_2} = \begin{bmatrix} A_\mathrm{r} & B_\mathrm{r} \\ I_r & 0 \\ 0 & I_m \end{bmatrix}, \quad \hat{\Sigma}_\mathrm{r} = \begin{bmatrix} -Q_{r_0} & P_{r_0} & -C_\mathrm{r}^\top \\ P_{r_0} & \bar{\omega}_0^2 Q_{r_0} & 0_{r\times m} \\ -C_\mathrm{r} & 0_{m\times r} & 0_{m\times m} \end{bmatrix}.$$

The above inequality is equivalent to

$$\hat{N}_{\hat{\Xi}_2}^\top \hat{\Sigma}_\mathrm{r} \hat{N}_{\hat{\Xi}_2} - \hat{N}_{\hat{\Xi}_2}^\top \mathrm{diag}\{0_{r\times r}, \varepsilon I_r, \varepsilon I_m\} \hat{N}_{\hat{\Xi}_2} < 0, \quad (21)$$

for all $\varepsilon > 0$.

Note that

$$\hat{N}_{\hat{\Xi}_1}^\top (\hat{\Sigma}_\mathrm{r} - \mathrm{diag}\{0_{r\times r}, \varepsilon I_r, \varepsilon I_m\}) \hat{N}_{\hat{\Xi}_1} = -\varepsilon I_m < 0, \quad (22)$$

where $\hat{N}_{\hat{\Xi}_1} = \begin{bmatrix} 0_{m\times r} & 0_{m\times r} & I_m \end{bmatrix}^\top$. According to Lemma 3 [38], (21) and (22) are equivalent to

$$\begin{aligned} &\hat{\Sigma}_\mathrm{r} - \mathrm{diag}\{0_{r\times r}, \varepsilon I_r, \varepsilon I_m\} + \hat{\Xi}_1^\top Y^\top \hat{\Xi}_2 + \hat{\Xi}_2^\top Y \hat{\Xi}_1 \\ &= \begin{bmatrix} Z_{11} & P_{r_0} - Y_2 + Y_1^\top A_\mathrm{r} & -C_\mathrm{r}^\top + Y_1^\top B_\mathrm{r} \\ \star & \bar{\omega}_0^2 Q_{r_0} - \varepsilon I_r + \mathrm{Her}(A_\mathrm{r}^\top Y_2) & Y_2^\top B_\mathrm{r} \\ \star & \star & -\varepsilon I_m \end{bmatrix} < 0, \end{aligned} \quad (23)$$

where Z_{11} is defined in (19), $Y = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix}$,

$$\hat{\Xi}_1 = \begin{bmatrix} I_r & 0 & 0_{r\times m} \\ 0 & I_r & 0_{r\times m} \end{bmatrix}, \quad \hat{\Xi}_2 = \begin{bmatrix} -I_r & A_\mathrm{r} & B_\mathrm{r} \end{bmatrix},$$

$Y_1 \in \mathbb{R}^{r\times r}$, $Y_2 \in \mathbb{R}^{r\times r}$. Substituting (9) into (23) and using the Schur complement, we have that

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \Pi_{13} & 0_{r\times r} \\ \star & \Pi_{22} & \Pi_{23} & \varepsilon I_r \\ \star & \star & -\varepsilon I_m & 0_{m\times r} \\ \star & \star & \star & -\varepsilon I_r \end{bmatrix} < 0, \quad (24)$$

where

$$\begin{aligned} \Pi_{11} &= -Q_{r_0} - \mathrm{Her}(Y_1), \\ \Pi_{12} &= P_{r_0} - Y_2 + Y_1^\top \bar{A}_1 + Y_1^\top U_1^\top \bar{F}_1^\top, \\ \Pi_{13} &= -\bar{C}_1^\top - V_1^\top + Y_1^\top \bar{B}_1 + Y_1^\top U_1^\top \bar{H}^\top, \\ \Pi_{22} &= \bar{\omega}_0^2 Q_{r_0} - 2\varepsilon I_r + \mathrm{Her}(\bar{A}_1^\top Y_2 + \bar{F}_1 U_1 Y_2), \\ \Pi_{23} &= Y_2^\top \bar{B}_1 + Y_2^\top U_1^\top \bar{H}^\top. \end{aligned}$$

There always exist a diagonal matrix $S_1 > 0$ and a block diagonal matrix $S = \mathrm{diag}\{S_2, S_1\} > 0$, such that

$$-S - Q\mathcal{A}^{-1}Q < 0, \quad -S - \phi\Upsilon^{-1}\phi^\top < 0,$$
$$-S_1 - \phi_1 \Pi^{-1} \phi_1^\top < 0,$$

where $S_2 \in \mathbb{R}^{|n_1-n_2|\times|n_1-n_2|}$, $S_1 \in \mathbb{R}^{r\times r}$,

$$\begin{aligned} \phi &= \begin{bmatrix} X_1 & X_2 & 0_{(n_2+r_2)\times m} & 0_{(n_2+r_2)\times(n_2+r_2)} \end{bmatrix}, \\ \phi_1 &= \begin{bmatrix} Y_1 & Y_2 & 0_{r\times m} & 0_{r\times r} \end{bmatrix}. \end{aligned}$$

Using the Schur complement, the above inequalities are equivalent to

$$\begin{bmatrix} \mathcal{A} & -Q \\ \star & -S \end{bmatrix} < 0, \quad (25)$$

$$\begin{bmatrix} \Upsilon & \phi^\top \\ \star & -S \end{bmatrix} < 0, \quad (26)$$

$$\begin{bmatrix} \Pi & \phi_1^\top \\ \star & -S_1 \end{bmatrix} < 0. \quad (27)$$

Let

$$T_0 = \begin{bmatrix} I_{n_2+r_2} & 0 \\ U^\top \bar{F}^\top & I_{n_2+r_2} \end{bmatrix},$$

$$T_1 = \begin{bmatrix} I_r & 0 & 0 & 0 & 0 \\ 0 & I_r & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & I_r & 0 \\ 0 & -U_1^\top \bar{F}_1^\top & -U_1^\top \bar{H}^\top & 0 & I_r \end{bmatrix}.$$

Multiplying (25) to the right by T_0 and to the left by $T_0^\top$, (27) to the right by T_1 and to the left by $T_1^\top$, one obtains

$$\begin{bmatrix} \mathrm{Her}(\bar{A}^\top Q) - \bar{F}USU^\top \bar{F}^\top & -Q - \bar{F}US \\ \star & -S \end{bmatrix} < 0, \quad (28)$$

$$\begin{bmatrix} \Pi_{11} & \tilde{\Pi}_{12} & \tilde{\Pi}_{13} & 0_{r\times r} & Y_1^\top \\ \star & \tilde{\Pi}_{22} & \tilde{\Pi}_{23} & \varepsilon I_r & Y_2^\top + \bar{F}_1 U_1 S_1 \\ \star & \star & \tilde{\Pi}_{33} & 0_{m\times r} & \bar{H} U_1 S_1 \\ \star & \star & \star & -\varepsilon I_r & 0_{r\times r} \\ \star & \star & \star & \star & -S_1 \end{bmatrix} < 0, \quad (29)$$

where Π_{11} is defined in (24),

$$\begin{aligned} &\tilde{\Pi}_{12} = P_{r_0} - Y_2 + Y_1^\top \bar{A}_1, \quad \tilde{\Pi}_{13} = -\bar{C}_1^\top - V_1^\top + Y_1^\top \bar{B}_1, \\ &\tilde{\Pi}_{22} = \bar{\omega}_0^2 Q_{r_0} - 2\varepsilon I_r + \mathrm{Her}(Y_2^\top \bar{A}_1) - \bar{F}_1 U_1 S_1 U_1^\top \bar{F}_1^\top, \\ &\tilde{\Pi}_{23} = Y_2^\top \bar{B}_1 - \bar{F}_1 U_1 S_1 U_1^\top \bar{H}^\top, \\ &\tilde{\Pi}_{33} = -\varepsilon I_m - \bar{H} U_1 S_1 U_1^\top \bar{H}^\top. \end{aligned}$$

Let $\tilde{U} = U$, $M = US$, $\tilde{U}_1 = U_1$, $M_1 = U_1 S_1$, we arrive at (17), (19).

Let

$$T_2 = \begin{bmatrix} I_{n_2+r_2} & 0 & 0 & 0 & 0 \\ 0 & I_{n_2+r_2} & 0 & 0 & 0 \\ 0 & 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & I_m & 0 \\ 0 & -U^\top \bar{F}^\top & -U^\top \bar{H}^\top & 0 & I_{n_2+r_2} \end{bmatrix}.$$

Similarly, multiplying (26) to the right by T_2 and to the left by $T_2^\top$, (18) can be obtained.

($\Leftarrow$) Suppose that there exist matrices Q, P_{e_0}, Q_{e_0}, P_{r_0}, Q_{r_0}, X_1, X_2, Y_1, Y_2, $\tilde{U}$, M, $\tilde{U}_1$, M_1, V, V_1, S, S_1, such that (17), (18), (19) hold for all $\varepsilon > 0$. Using the Schur complement, (17), (18), (19) are equivalent to

$$\mathcal{W}_{11} - (-Q - \bar{F}M)S^{-1}(-Q - \bar{F}M)^\top < 0, \quad (30)$$

$$\mathcal{Z}_a - J_1 \operatorname{diag}\{-\frac{1}{\varepsilon}I_r, -S_1^{-1}\}J_1^\top < 0, \quad (31)$$

$$\mathcal{L}_a - J_2 \operatorname{diag}\{-I_m, -S^{-1}\}J_2^\top < 0, \quad (32)$$

where

$$\mathcal{Z}_a = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ \star & Z_{22} & Z_{23} \\ \star & \star & Z_{33} \end{bmatrix}, \ \mathcal{L}_a = \begin{bmatrix} L_{11} & L_{12} & X_1^\top \bar{B} \\ \star & L_{22} & L_{23} \\ \star & \star & L_{33} \end{bmatrix},$$

$$J_1 = \begin{bmatrix} 0_{r\times r} & Y_1^\top \\ \varepsilon I_r & Y_2^\top + \bar{F}_1 M_1 \\ 0_{m\times r} & 0_{m\times r} \end{bmatrix},$$

$$J_2 = \begin{bmatrix} 0_{(n_2+r_2)\times m} & X_1^\top \\ \bar{C}^\top + V^\top & X_2^\top + \bar{F}M \\ 0_{m\times m} & \bar{H}M \end{bmatrix},$$

$\mathcal{W}_{11}$ is defined in (17), Z_{11}, Z_{12}, Z_{13}, Z_{22}, Z_{23}, Z_{33} are defined in (18), L_{11}, L_{12}, L_{22}, L_{23}, L_{33} are defined in (19).

It follows from

$$\psi_1(M_1 - \tilde{U}_1 S_1)S_1^{-1}(M_1 - \tilde{U}_1 S_1)^\top \psi_1^\top \geq 0,$$
$$\psi_2(M - \tilde{U}S)S^{-1}(M - \tilde{U}S)^\top \psi_2^\top \geq 0,$$

that

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \Upsilon_{22} & \Upsilon_{23} \\ \star & \star & \Upsilon_{33} \end{bmatrix} \leq 0, \quad (33)$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & E_{22} & E_{23} \\ \star & \star & E_{33} \end{bmatrix} \leq 0, \quad (34)$$

where

$$\psi_1 = \begin{bmatrix} 0_{(m+r_2)\times r} & \bar{F}_1^\top & \bar{H}^\top \end{bmatrix}^\top,$$
$$\psi_2 = \begin{bmatrix} 0_{(m+r_2)\times(n_2+r_2)} & \bar{F}^\top & \bar{H}^\top \end{bmatrix}^\top,$$
$$\Upsilon_{22} = -\bar{F}_1 M_1 S_1^{-1} M_1^\top \bar{F}_1^\top - \bar{F}_1 \tilde{U}_1 S_1 \tilde{U}_1^\top \bar{F}_1^\top + \operatorname{Her}(\bar{F}_1 M_1 \tilde{U}_1^\top \bar{F}_1^\top),$$
$$\Upsilon_{23} = -\bar{F}_1 M_1 S_1^{-1} M_1^\top \bar{H}^\top - \bar{F}_1 \tilde{U}_1 S_1 \tilde{U}_1^\top \bar{H}^\top + \bar{F}_1 M_1 \tilde{U}_1^\top \bar{H}^\top + \bar{F}_1 \tilde{U}_1 M_1^\top \bar{H}^\top,$$
$$\Upsilon_{33} = -\bar{H} M_1 S_1^{-1} M_1^\top \bar{H}^\top - \bar{H} \tilde{U}_1 S_1 \tilde{U}_1^\top \bar{H}^\top + \operatorname{Her}(\bar{H} M_1 \tilde{U}_1^\top \bar{H}^\top),$$
$$E_{22} = -\bar{F}MS^{-1}M^\top \bar{F}^\top - \bar{F}\tilde{U}S\tilde{U}^\top \bar{F}^\top + \operatorname{Her}(\bar{F}M\tilde{U}^\top \bar{F}^\top),$$
$$E_{23} = -\bar{F}MS^{-1}M^\top \bar{H}^\top - \bar{F}\tilde{U}S\tilde{U}^\top \bar{H}^\top + \bar{F}M\tilde{U}^\top \bar{H}^\top + \bar{F}\tilde{U}M^\top \bar{H}^\top,$$
$$E_{33} = -\bar{H}MS^{-1}M^\top \bar{H}^\top - \bar{H}\tilde{U}S\tilde{U}^\top \bar{H}^\top + \operatorname{Her}(\bar{H}M\tilde{U}^\top \bar{H}^\top).$$

Combing (31) with (33) and (30), (32) with (34), we have that

$$\tilde{\mathcal{W}}_{11} - (-Q - \bar{F}M)S^{-1}(-Q - \bar{F}M)^\top < 0, \quad (35)$$

$$\tilde{\mathcal{Z}}_a - J_1 \operatorname{diag}\{-\frac{1}{\varepsilon}I_r, -S_1^{-1}\}J_1^\top < 0, \quad (36)$$

$$\tilde{\mathcal{L}}_a - J_2 \operatorname{diag}\{-I_m, -S^{-1}\}J_2^\top < 0, \quad (37)$$

where $\tilde{\mathcal{W}}_{11} = \operatorname{Her}(\bar{A}^\top Q) - \bar{F}MS^{-1}M^\top \bar{F}^\top$,

$$\tilde{\mathcal{Z}}_a = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ \star & Z_{22} + \Upsilon_{22} & Z_{23} + \Upsilon_{23} \\ \star & \star & Z_{33} + \Upsilon_{33} \end{bmatrix},$$

$$\tilde{\mathcal{L}}_a = \begin{bmatrix} L_{11} & L_{12} & X_1^\top \bar{B} \\ \star & L_{22} + E_{22} & L_{23} + E_{23} \\ \star & \star & L_{33} + E_{33} \end{bmatrix}.$$

Let $U_1 = M_1 S_1^{-1}$, $U = MS^{-1}$. Using the Schur complement, the inequalities (35), (36), (37) are equivalent to (28), (29) and

$$\begin{bmatrix} L_{11} & \tilde{\mathcal{L}}_{12} & X_1^\top \bar{B} & 0_{(n_2+r_2)\times m} & X_1^\top \\ \star & \tilde{\mathcal{L}}_{22} & \tilde{\mathcal{L}}_{23} & \bar{C}^\top + V^\top & X_2^\top + \bar{F}US \\ \star & \star & \tilde{\mathcal{L}}_{33} & 0_{m\times m} & \bar{H}US \\ \star & \star & \star & -I_m & 0_{m\times(n_2+r_2)} \\ \star & \star & \star & \star & -S \end{bmatrix} < 0, \quad (38)$$

where

$$\tilde{\mathcal{L}}_{12} = P_{e_0} - X_2 + X_1^\top \bar{A},$$
$$\tilde{\mathcal{L}}_{22} = \bar{\omega}_0^2 Q_{e_0} + \operatorname{Her}(X_2^\top \bar{A}) - \bar{F}USU^\top \bar{F}^\top,$$
$$\tilde{\mathcal{L}}_{23} = X_2^\top \bar{B} - \bar{F}USU^\top \bar{H}^\top, \quad \tilde{\mathcal{L}}_{33} = -\gamma^2 I_m - \bar{H}USU^\top \bar{H}^\top.$$

Multiplying (28) to the right by T_0^{-1} and to the left by $(T_0^\top)^{-1}$, multiplying (38) to the right by T_2^{-1} and to the left by $(T_2^\top)^{-1}$ and multiplying (29) to the right by T_1^{-1} and to the left by $(T_1^\top)^{-1}$, one obtains (25), (26), (27).

Substituting (11) into (25), (26), using the Schur complement, one obtains (12), (13). Substituting (9) into (27), using the Schur complement, we arrive at (23). According to Lemma 3 [38], (23) is equivalent to (21) and (22), which implies that (20). Thus, (12), (13) and (20) are equivalent to (17), (18) and (19). The proof is completed. ∎

Remark 7: Compared with the conditions in Proposition 1 and Lemma 4, the reduced-order system matrices are decoupled with the matrix variables Q, P_{e_0}, Q_{e_0}, P_{r_0}, Q_{r_0} in Theorem 1. Note that (17), (18), (19) are still bilinear with respect to the matrix variables Q, P_{e_0}, Q_{e_0}, P_{r_0}, Q_{r_0}, X_1, X_2, Y_1, Y_2, $\tilde{U}$, M, $\tilde{U}_1$, M_1, V, V_1, S_1, S. However, these inequalities become LMIs when $\tilde{U}_1$, $\tilde{U}$ are fixed. This implies that these matrix inequalities can be solved efficiently by available numerical algorithm, see subsection III-C.

Remark 8: The obtained results in Theorem 1 are also applicable for HFNI systems. When the matrices L_{11}, L_{12}, L_{22} in (18), the matrices Z_{11}, Z_{12}, Z_{22} in (19) are replaced by

$$L_{11} = Q_{e_{N+1}} - \operatorname{Her}(X_1), \quad L_{12} = P_{e_{N+1}} - X_2 + X_1^\top \bar{A},$$
$$L_{22} = -\underline{\omega}_h^2 Q_{e_{N+1}} + \operatorname{Her}(X_2^\top \bar{A} - \bar{F}\tilde{U}M^\top \bar{F}^\top) + \bar{F}\tilde{U}S\tilde{U}^\top \bar{F}^\top,$$
$$Z_{11} = -Q_{r_{N+1}} - \operatorname{Her}(Y_1), \quad Z_{12} = P_{r_{N+1}} - Y_2 + Y_1^\top \bar{A}_1,$$
$$Z_{22} = -\underline{\omega}_h^2 Q_{r_{N+1}} - 2\varepsilon I_r + \operatorname{Her}(Y_2^\top \bar{A}_1 - \bar{F}_1 \tilde{U}_1 M_1^\top \bar{F}_1^\top) + \bar{F}_1 \tilde{U}_1 S_1 \tilde{U}_1^\top \bar{F}_1^\top,$$

Theorem 1 provides the necessary and sufficient conditions for the existence and the construction of the reduced-order HFNI systems.

Remark 9: Similarly, Theorem 1 can be extended to the MFNI systems by using the complex version Lemma 3 [38].

Algorithm 1 Algorithm for IFNI Model Reduction

Input: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times m}$, Ω_L, $\gamma > 0$, $1 \le r_2 < n_2$, $\mu \le 0$, $\Delta_\gamma > 0$, σ_1, σ_2 are sufficiently small positive numbers.

Output: A_r^*, B_r^*, C_r^*, γ^*.

Initilization

Use Jordan transform to get the Jordan block form (7).

Set $j, k = 0$, $\gamma^{(k)} = \gamma$. Choose some initial values $\tilde{U}_1^{(i)}$, $\tilde{U}^{(i)}$ for matrices $\tilde{U}_1$, $\tilde{U}$. Let $\tilde{U}_1^{(j)} = \tilde{U}_1^{(i)}$, $\tilde{U}^{(j)} = \tilde{U}^{(i)}$, $\varepsilon^{(-1)} = 1$.

repeat

Solve the following optimization problem for matrices Q, Q_{e_0}, P_{e_0}, Q_{r_0}, P_{r_0}, X_1, X_2, V, M, S, Y_1, Y_2, V_1, M_1, S_1, ε:

$$\begin{aligned} &\min\ \varepsilon \\ &\text{s.t. } \mathcal{W} - \mathrm{diag}\{\mu I_{n_2+r_2}, 0\} < 0, \quad Z < 0, \\ &\qquad L - \mathrm{diag}\{0, \mu I_{n_2+r_2}, \mu I_m, 0\} < 0, \end{aligned} \tag{39}$$

where $\mathcal{W}$, L and Z are defined in (17), (18) and (19).

if $\varepsilon \le \sigma_1$ **then**

Denote the obtained $\tilde{U}_1^{(j)}$, $\tilde{U}^{(j)}$, $V_1^{(j)}$ as $\tilde{U}_1^{(k)}$, $\tilde{U}^{(k)}$, $V_1^{(k)}$. Set $\gamma^{(k+1)} = \gamma^{(k)} - \Delta_\gamma$, $k = k + 1$.

else

Fix $\varepsilon^{(i)} = \varepsilon$, solve the following optimization problem for matrices Q, Q_{e_0}, P_{e_0}, Q_{r_0}, P_{r_0}, X_1, X_2, V, M, S, Y_1, Y_2, V_1, M_1, S_1,

$$\begin{aligned} &\min\ \mathrm{trace}(S_1) \\ &\text{s.t. } \mathcal{W} - \mathrm{diag}\{\mu I_{n_2+r_2}, 0\} < 0, \ Z < 0, \\ &\qquad L - \mathrm{diag}\{0, \mu I_{n_2+r_2}, \mu I_m, 0\} < 0, \end{aligned} \tag{40}$$

Update $\tilde{U}_1^{(j+1)}$, $\tilde{U}^{(j+1)}$ according to

$$\tilde{U}_1^{(j+1)} = M_1 S_1^{-1}, \quad \tilde{U}^{(j+1)} = M S^{-1}.$$

Set $j = j + 1$.

end if

until $|\varepsilon^{(j)} - \varepsilon^{(j-1)}| \le \sigma_2$, output $\gamma^* = \gamma^{(k-1)}$,

$$\begin{aligned} A_r^* &= \begin{bmatrix} A_1 & 0 \\ 0 & A_{r_2}^{(k-1)} \end{bmatrix}, \quad B_r^* = \begin{bmatrix} B_1 \\ B_{r_2}^{(k-1)} \end{bmatrix}, \\ C_r^* &= \begin{bmatrix} C_1 & C_{r_2}^{(k-1)} \end{bmatrix}, \end{aligned} \tag{41}$$

where

$$A_{r_2}^{(k-1)} = N_1 (\tilde{U}_1^{(k-1)})^\mathsf{T} N_2, \quad B_{r_2}^{(k-1)} = N_1 (\tilde{U}_1^{(k-1)})^\mathsf{T} N_3,$$
$$C_{r_2}^{(k-1)} = V_1^{(k-1)} N_1^\mathsf{T},$$

Algorithm 2 Algorithm for Initial $\tilde{U}_1$, $\tilde{U}$, μ

Input: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times m}$, Ω_L, $1 \le r_2 < n_2$, γ to be a sufficiently large number, δ to be a sufficiently small positive number.

Output: $\tilde{U}_1$, $\tilde{U}$, μ.

Initilization

Use Jordan transform to get the Jordan block form (7).

Use the existing methods, such as balanced truncation method, to get a reduced-order system $(\mathcal{A}_{r_2}, \mathcal{B}_{r_2}, \mathcal{C}_{r_2})$.

Set $i = 0$, Choose $A_{r_2}^{(i)} = \mathcal{A}_{r_2}$, $B_{r_2}^{(i)} = \mathcal{B}_{r_2}$, $\mu^{(-1)} = 1$.

repeat

Let

$$\tilde{U}_1^{(i)} = \begin{bmatrix} 0 & (A_{r_2}^{(i)})^\mathsf{T} \\ 0 & (B_{r_2}^{(i)})^\mathsf{T} \end{bmatrix}, \quad \tilde{U}^{(i)} = \begin{bmatrix} 0 & (A_{r_2}^{(i)})^\mathsf{T} \\ 0 & (B_{r_2}^{(i)})^\mathsf{T} \end{bmatrix},$$

solve the following optimization problem for matrices Q, Q_{e_0}, P_{e_0}, Q_{r_0}, P_{r_0}, X_1, X_2, V, M, S, Y_1, Y_2, V_1, M_1, S_1, ε, μ:

$$\begin{aligned} &\min\ \mu \\ &\text{s.t. } \mathcal{W} - \mathrm{diag}\{\mu I_{n_2+r_2}, 0\} < 0, \quad Z < 0, \\ &\qquad L - \mathrm{diag}\{0, \mu I_{n_2+r_2}, \mu I_m, 0\} < 0, \end{aligned} \tag{42}$$

where $\mathcal{W}$, L and Z are defined in (17), (18), (19).

if $\mu \le 0$ **then**

Stop. Output $\tilde{U}_1^{(i)}$, $\tilde{U}^{(i)}$ and $\mu^{(i)}$.

else

Fix $\mu^{(i)} = \mu$, solve the following optimization problem for matrices Q, Q_{e_0}, P_{e_0}, Q_{r_0}, P_{r_0}, X_1, X_2, V, M, S, Y_1, Y_2, V_1, M_1, S_1, ε,

$$\begin{aligned} &\min\ \mathrm{trace}(S_1) \\ &\text{s.t. } \mathcal{W} - \mathrm{diag}\{\mu^{(i)} I_{n_2+r_2}, 0\} < 0, \quad Z < 0, \\ &\qquad L - \mathrm{diag}\{0, \mu^{(i)} I_{n_2+r_2}, \mu^{(i)} I_m, 0\} < 0, \end{aligned} \tag{43}$$

Update $\tilde{U}_1^{(i+1)}$, $\tilde{U}^{(i+1)}$ according to

$$\tilde{U}_1^{(i+1)} = M_1 S_1^{-1}, \quad \tilde{U}^{(i+1)} = M S^{-1}.$$

Set $i = i + 1$.

end if

until $|\mu^{(i)} - \mu^{(i-1)}| \le \delta$.

Algorithm 3 Improved Algorithm for IFNI Model Reduction

Apply Algorithm 2 to find the initial values $\tilde{U}_1^{(i)}$, $\tilde{U}^{(i)}$, $\mu^{(i)}$.

Fix $\mu = \mu^{(i)}$. Set $j, k = 0$ and choose the initial values as $\tilde{U}_1^{(j)} = \tilde{U}_1^{(i)}$, $\tilde{U}^{(j)} = \tilde{U}^{(i)}$.

Apply Algorithm 1 to find a sub-optimal reduced-order IFNI system.

Define the matrices L_{11}, L_{12}, L_{22} in (18), the matrices Z_{11}, Z_{12}, Z_{22} in (19) as

$$L_{11} = -Q_{e_{\hat{l}}} - \mathrm{Her}(X_1), \quad L_{12} = P_{e_{\hat{l}}} + j\omega_{cl} Q_{e_{\hat{l}}} - X_2 + X_1^\mathsf{T} \bar{A},$$
$$L_{22} = -\underline{\omega}_{\hat{l}} \bar{\omega}_{\hat{l}} Q_{e_{\hat{l}}} + \mathrm{Her}(X_2^\mathsf{T} \bar{A} - \bar{F} \tilde{U} M^\mathsf{T} \bar{F}^\mathsf{T}) + \bar{F} \tilde{U} S \tilde{U}^\mathsf{T} \bar{F}^\mathsf{T},$$
$$Z_{11} = -Q_{r_{\hat{l}}} - \mathrm{Her}(Y_1), \quad Z_{12} = P_{r_{\hat{l}}} + j\omega_{cl} Q_{r_{\hat{l}}} - Y_2 + Y_1^\mathsf{T} \bar{A}_1,$$

$$\begin{aligned} Z_{22} = &-\underline{\omega}_{\hat{l}} \bar{\omega}_{\hat{l}} Q_{r_{\hat{l}}} - 2\varepsilon I_r + \mathrm{Her}(Y_2^\mathsf{T} \bar{A}_1 - \bar{F}_1 \tilde{U}_1 M_1^\mathsf{T} \bar{F}_1^\mathsf{T}) \\ &+ \bar{F}_1 \tilde{U}_1 S_1 \tilde{U}_1^\mathsf{T} \bar{F}_1^\mathsf{T}, \end{aligned}$$

where symmetric matrices $P_{e_{\hat{l}}} \in \mathbb{C}^{(n_2+r_2)\times(n_2+r_2)}$, $Q_{e_{\hat{l}}} \in \mathbb{C}^{(n_2+r_2)\times(n_2+r_2)}$, $Q_{e_{\hat{l}}} > 0$, $P_{r_l} \in \mathbb{C}^{r\times r}$, $Q_{r_{\hat{l}}} \in \mathbb{C}^{r\times r}$, $Q_{r_{\hat{l}}} \ge 0$, $\hat{l} = 1, \ldots, N$. Then, Theorem 1 provides the necessary and sufficient conditions for the existence and the construction of the reduced-order MFNI systems.

C. Iterative Algorithm for LFNI Model Reduction

In this subsection, an iterative algorithm is developed to find the desired reduced-order LFNI system and to minimize the H_∞ approximation error. Similarly, the iterative algorithm can also be developed for the MFNI and HFNI cases by considering Remarks 7 and 8.

Motivated by Shen and Lam [39], the iterative Algorithm 1 is provided to find a solution of Problem 1 and minimize the H_∞ approximation error bound γ.

The matrices N_1, N_2, N_3 in (41) are defined as

$$N_1 = \begin{bmatrix} 0_{n_1 \times r_2} \\ I_{r_2} \end{bmatrix}^{\mathsf{T}}, \ N_2 = \begin{bmatrix} I_{r_2} \\ 0_{m \times r_2} \end{bmatrix}, \ N_3 = \begin{bmatrix} 0_{r_2 \times m} \\ I_m \end{bmatrix}. \quad (44)$$

For the fixed $\tilde{U}_1^{(i)}$, $\tilde{U}^{(i)}$, the optimization problems (39), (40) are LMI problems and can be solved effectively. Note that σ_1 is a sufficiently small positive number. If a solution $\varepsilon \leq \sigma_1$ to (39) is found, then we can conclude that the inequalities in (39) hold for almost all $\varepsilon > 0$. This implies that the resulting reduced-order system satisfies the inequalities in Theorem 1 for almost all $\varepsilon > 0$. Thus, the reduced-order system obtained by Algorithm 1 is a solution to Problem 1. With the increasing iteration numbers k, γ is guaranteed to be reduced, that is, it is expected to reduce the H_∞ approximation error as the iteration number increases.

Note that the initial choice of $\tilde{U}_1$, $\tilde{U}$ and μ will affect the feasibility of Algorithm 1. This is because a solution $\varepsilon \leq \sigma_1$ to (39) can not to be guaranteed for arbitrarily chosen initial values $\tilde{U}_1$, $\tilde{U}$ and μ. Moreover, if inappropriate initial values are selected, the optimization problems in Algorithm 1 may be infeasible. Thus, it is of importance to select a good initial values $\tilde{U}_1$, $\tilde{U}$ and μ. To optimizing the initial $\tilde{U}_1$, $\tilde{U}$ and μ, the iterative Algorithm 2 is provided.

If a solution $\mu \leq 0$ to (42) can not be found, then we can conclude that Problem 1 may not have solutions with the given γ. If a solution $\mu \leq 0$ to (42) is found, then output $\tilde{U}_1^{(i)}$, $\tilde{U}^{(i)}$, $\mu^{(i)}$ as the initial values. Thus, an improved iterative algorithm including two stages is developed to find a solution of Problem 1 and to minimize the H_∞ approximation error.

IV. Application to Electronic Circuits

In this section, the proposed model reduction method is applied to the Sallen-key filter, the Piezoeletric tube scanner and the RLC circuit. The optimization problems are solved by SeDuMi toolbox [41]. The performance of the proposed method is compared with the finite frequency interval Gramians-based model reduction method [35]. It is shown that the proposed model reduction method guarantees the IFNI structure and achieve lower approximation error both in frequency-domain and time-domain.

A. Sallen-Key Filter

Consider the fifth-order Sallen-key filter in Figure 1. G_1 is a third-order Sallen-key filter cascaded with a gain multiplier circuit, G_2 is a second-order Sallen-key filter. The transfer function from V_i to V_o is given by $G(s) = G_1(s)G_2(s)$, where

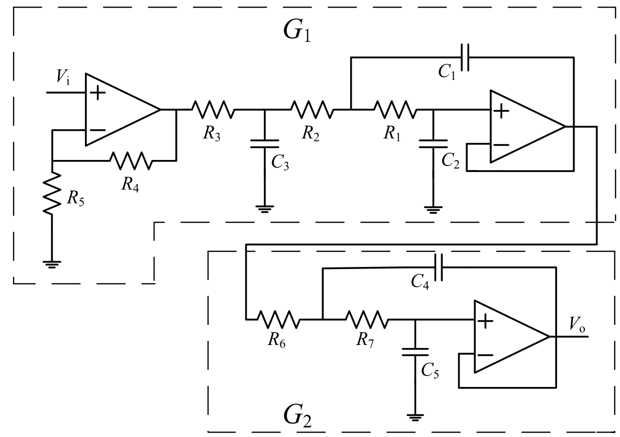

Fig. 1. Sallen-key filter.

$$\begin{aligned} G_1(s) &= \frac{1 + \frac{R_4}{R_5}}{R_1R_2R_3C_1C_2C_3s^3 + g_1s^2 + g_2s + 1}, \\ G_2(s) &= \frac{1}{R_6R_7C_4C_5s^2 + (R_6C_5 + R_7C_5)s + 1}, \\ g_1 &= R_1R_2C_1C_2 + R_1R_3C_1C_2 + R_1R_3C_2C_3 \\ &\quad + R_2R_3C_2C_3, \\ g_2 &= R_1C_2 + R_3C_2 + R_3C_3 + R_2C_2, \end{aligned}$$

Let $R_i = 100\Omega$, $C_j = 0.01$F, $i = 1, \ldots, 7$, $j = 1, \ldots, 5$. It can be verified that the system is LFNI by either Definition 1 or Lemma 1 [31]. Be letting the imaginary part of the transfer function $G(j\omega)$ be zero, we found that $\Im[G(j\omega)] \leq 0$ for $\omega \in (0, 0.7]$. Let $\bar{\omega}_0 = 0.7$rad/s. A solution to (5) can be found and given by

$$P_0 = \begin{bmatrix} 0.1024 & -0.0604 & -0.1312 & 0.0860 & -0.0184 \\ -0.0604 & 0.4640 & 0.1916 & -0.0662 & -0.0235 \\ -0.1312 & 0.1916 & -0.6371 & 0.2680 & 0.0211 \\ 0.0860 & -0.0662 & 0.2680 & -1.3837 & -0.2022 \\ -0.0184 & -0.0235 & 0.0211 & -0.2022 & -2.4895 \end{bmatrix},$$

$$Q_0 = \begin{bmatrix} 0.0604 & 0.2004 & 0.2760 & -0.1165 & 0.0175 \\ 0.2004 & 0.7421 & 0.7532 & -0.4061 & 0.0687 \\ 0.2760 & 0.7532 & 1.6976 & -0.1584 & 0.0064 \\ -0.1165 & -0.4061 & -0.1584 & 1.5228 & 0.0354 \\ 0.0175 & 0.0687 & 0.0064 & 0.0354 & 1.6808 \end{bmatrix} \geq 0.$$

Here, we are interested in finding a second-order LFNI system with the H_∞ approximation error minimized over the low frequency, $\omega \in (0, 0.7]$. By Algorithm 3 with $\delta = 1 \times 10^{-5}$, $\sigma_1 = 0.01$, $\sigma_2 = 1 \times 10^{-5}$, the following second-order system is obtained

$$A_r = \begin{bmatrix} -0.1992 & -0.07064 \\ -0.1872 & -0.5158 \end{bmatrix}, \quad B_r = \begin{bmatrix} -0.6732 \\ 0.7267 \end{bmatrix}, \\ C_r = \begin{bmatrix} -1.05 & -0.8491 \end{bmatrix}. \quad (45)$$

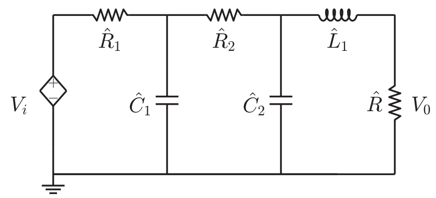

Fig. 2. Reduced-order RLC circuit.

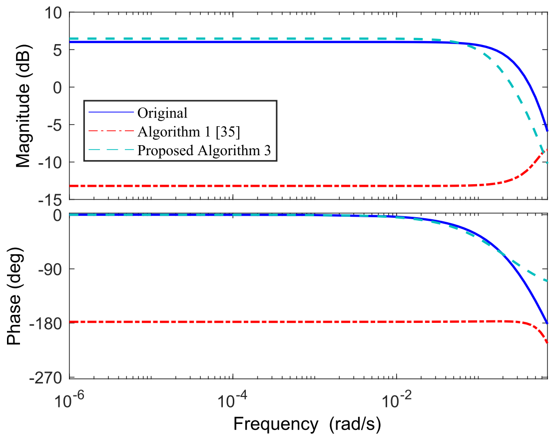

Fig. 3. Bode plots of the original and the reduced-order systems for frequency interval, $\omega \in (0, 0.7]$.

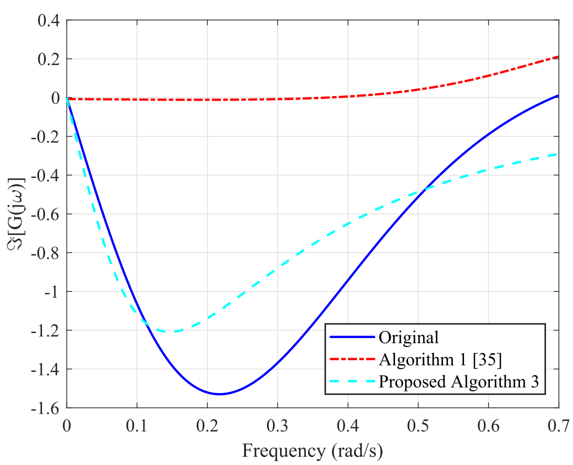

Fig. 4. $\Im[G(j\omega)]$, $\omega \in (0, 0.7]$ for the original and the reduced-order systems.

Moreover, a realization in terms of RLC circuit is shown in Figure 2 with $\hat{R}_1 = 36.2\Omega$, $\hat{R}_2 = 123.6\Omega$, $\hat{R} = 2.11\Omega$, $\hat{C}_1 = \hat{C}_2 = 0.05$F, $\hat{L}_1 = 1$H. The sub-optimal H_∞ approximation error is $\gamma^* = 0.3581$, which is smaller than 2.223 obtained by Algorithm 1 [35].

The bode plots of the original and the reduced-order systems obtained by Algorithm 1 [35], the proposed Algorithm 3 are shown in Figure 3. Moreover, Figure 4 shows the imaginary part of the frequency response $\Im[G(j\omega)]$ for the original and the reduced-order systems. It can be seen from Figure 3 that the reduced-order system (45) approximate the original system well over the frequency interval, $\omega \in (0, 0.7]$. From Figure 4, we can see that the reduced-order (45) satisfies $\Im[G_\mathrm{r}(j\omega)] \leq 0$ for $\omega \in (0, 0.7]$. This means that $j[G_\mathrm{r}(j\omega) - G_\mathrm{r}^*(j\omega)] \geq 0$. In addition, the obtained reduced-order system (45) is stable. Thus, according to Definition 1, the reduced-order system obtained by the proposed Algorithm 3 is LFNI. However, the reduced-order system obtained by Algorithm 1 [35] does not satisfy the LFNI structure.

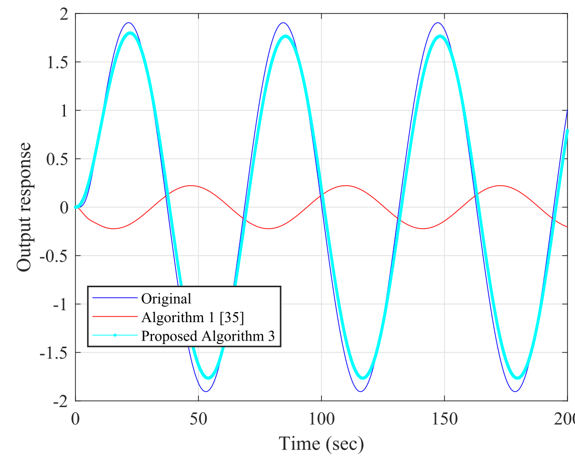

Fig. 5. Output responses for $u(t) = \sin(0.1t)$.

Moreover, the time-domain simulation is also provided to illustrate the effectiveness of the proposed model reduction method. The corresponding output responses of the original and the reduced-order system for input $u(t) = \sin(0.1t)$ are given in Figure 5. It can be seen that the reduced-order system obtained by the proposed model reduction method follow the original output accurately. From these results, it can be obtained that the proposed model reduction method yields good reduced-order systems to approximate the original IFNI system both in the frequency-domain and time-domain.

B. Piezoelectric Tube Scanner

Consider the piezoelectric tube scanner in [42]. The minimal state-space realization of the piezoelectric tube scanner from the voltage input V_{x+} to the displacement output d_y is given by the first equation at the bottom of the next page.
It can be verified that the system is MFNI by either Definition 1 or Lemma 1 [31]. Assuming that the imaginary part of the transfer function $G(j\omega)$ be zero, we found that $\Im[G(j\omega)] \leq 0$ for $\omega \in [6500, 9500]$. Let $\underline{\omega}_{\hat{l}} = 6500$rad/s, $\bar{\omega}_{\hat{l}} = 9500$rad/s, a solution to (5) can be found and given by the second equation at the bottom of the next page.

The sixth-order system is reduced to second-order system over the frequency interval, $\omega \in [6500, 9500]$. By Algorithm 3, the following second-order system is obtained

$$A_\mathrm{r} = \begin{bmatrix} -2.011 & -6103 \\ 6103 & -18.35 \end{bmatrix}, \quad B_\mathrm{r} = \begin{bmatrix} -4.508 \\ 11.58 \end{bmatrix},$$
$$C_\mathrm{r} = \begin{bmatrix} 12.354 & 15.654 \end{bmatrix}. \tag{46}$$

The sub-optimal H_∞ approximation error is $\gamma^* = 1.96$, which is smaller than 3.9 obtained by Algorithm 1 [35].

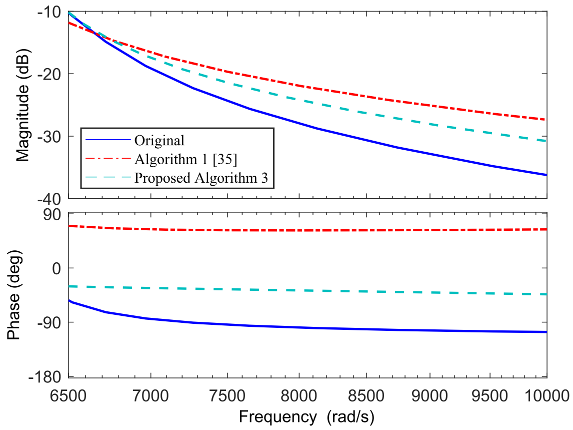

Fig. 6. Bode plots of the original and the reduced-order systems for frequency interval, $\omega \in [6500, 9500]$.

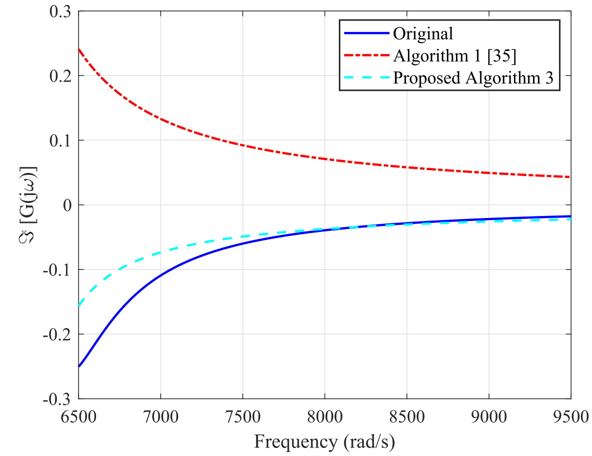

Fig. 7. $\Im[G(j\omega)]$, $\omega \in [6500, 9500]$ for the original and the reduced-order systems.

The bode plots of the original and the reduced-order systems obtained by Algorithm 1 [35], the proposed Algorithm 3 are shown in Figure 6. Moreover, Figure 7 shows the imaginary part of the frequency response $\Im[G(j\omega)]$ for the original and the reduced-order systems. From Figure 7, we can see that the reduced-order (46) satisfies $\Im[G_{\mathrm{r}}(j\omega)] \leq 0$ for $\omega \in [6500, 9500]$. This means that $j[G_{\mathrm{r}}(j\omega) - G_{\mathrm{r}}^{*}(j\omega)] \geq 0$. In addition, the obtained reduced-order system (46) is stable. Thus, we have that the reduced-order system obtained by the proposed Algorithm 3 preserves the MFNI structure. However, the reduced-order system obtained by Algorithm 1 [35] does not satisfy the MFNI structure.

Furthermore, the corresponding output responses of the original and the reduced-order system for input $u(t) = \sin(0.1t)$ are given in Figure 8. It can be seen that the reduced-order system obtained by the proposed model reduction method approximate the original output well. Thus, the proposed model reduction method yields good approximation performance both in the frequency-domain and time-domain. However, only a sub-optimal reduced-order system can be obtained by our proposed method. In practice, it is desirable to find the global optimal reduced-order system. How to extend the results of this paper to find the optimal reduced-order system is worth future research.

C. RLC Circuit

Consider the n-stage RLC circuit in Figure 9. The input is the voltage $V(t)$ and the output is the charge on the first

$$A = \begin{bmatrix} -498.07 & -6156.1 & 519.03 & 283.81 & -1091.7 & -796.8 \\ 5623.9 & -8.762 & -376.93 & 752.44 & -989.36 & -863.58 \\ -554.96 & 34.261 & -466.65 & 5998.3 & 849.04 & -1578.9 \\ -192.83 & -798.48 & -6500.9 & -89.24 & 378.8 & -661.32 \\ -311.39 & 109.74 & 85.414 & 163.26 & -1350.9 & -51.35 \\ -220.68 & -129.77 & -440.52 & -422.72 & 14.29 & -1843.3 \end{bmatrix},$$
$$B = \begin{bmatrix} 8.3647 & 1.5217 & -6.4335 & -7.1282 & -16.034 & 0.029016 \end{bmatrix}^{\mathsf{T}},$$
$$C = \begin{bmatrix} -18.296 & -15.838 & -33.085 & -16.047 & 6.5911 & -59.707 \end{bmatrix}.$$

$$P_{\hat{l}} = \begin{bmatrix} 12738 & 3501 + 4817j & 22671 - 1311j & 11316 - 2506j & -2004 - 119j & -3783 + 2264j \\ \star & 15975 & 16548 - 2333j & -23848 + 1499j & -1388 - 167j & 8388 + 2556j \\ \star & \star & 34572 & 348 + 2965j & -5752 - 243j & 1072 + 182j \\ \star & \star & \star & 37823 & -1902 + 207j & -12873 - 3790j \\ \star & \star & \star & \star & 1874 & -3973 + 961j \\ \star & \star & \star & \star & \star & 17229 \end{bmatrix},$$
$$Q_{\hat{l}} = \begin{bmatrix} 0.7676 & -0.1555 + 0.7039j & -0.4673 - 0.026j & 0.0127 + 0.4393j & -0.0991 - 0.073j & 0.045 - 0.0077j \\ \star & 0.6769 & 0.0709 + 0.4338j & 0.4002 - 0.1006j & -0.0468 + 0.1056j & -0.0162 - 0.0397j \\ \star & \star & 0.2854 & -0.0226 - 0.267j & 0.0628 + 0.0411j & -0.0271 + 0.0062j \\ \star & \star & \star & 0.2516 & -0.0434 + 0.0555j & -0.0037 - 0.0259j \\ \star & \star & \star & \star & 0.0197 & -0.0051 + 0.0053j \\ \star & \star & \star & \star & \star & 0.0027 \end{bmatrix}$$
$$\geq 0.$$

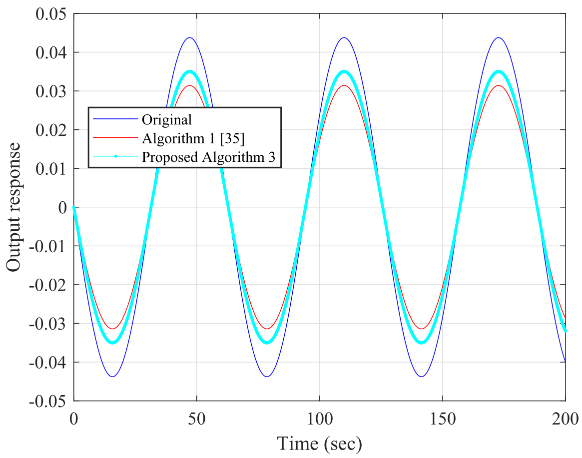

Fig. 8. Output responses for $u(t) = \sin(0.1t)$.

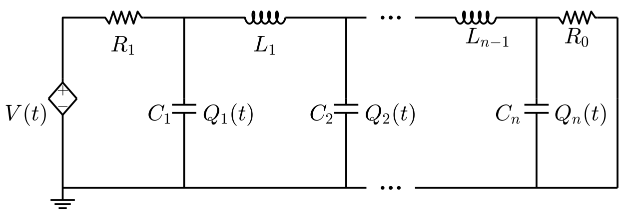

Fig. 9. RLC circuit.

capacitor $Q_1(t)$. The input-output relationship from $u(t) = V(t)$ to $y(t) = Q_1(t)$ is given by

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t),$$

where

$$x(t) = \begin{bmatrix} u_1(t) & i_{L_1}(t) & u_2(t) & \cdots & i_{L_{n-1}}(t) & u_n(t) \end{bmatrix}^{\mathsf{T}},$$

$u_k(t)$ is the voltage across capacitor C_k and $i_{L_k}(t)$ represent the current through inductor L_k,

$$A = \begin{bmatrix} -\frac{1}{C_1R_1} & -\frac{1}{C_1} & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{L_1} & 0 & -\frac{1}{L_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{C_2} & 0 & -\frac{1}{C_2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{L_2} & 0 & -\frac{1}{L_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{L_{n-1}} & 0 & -\frac{1}{L_{n-1}} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{C_n} & -\frac{1}{R_0C_n} \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{1}{C_1R_1} & 0 & \cdots & 0 \end{bmatrix}^{\mathsf{T}}, \quad C = \begin{bmatrix} C_1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Here, we consider a fifth-order RLC network with $C_1 = C_2 = 1F$, $L_1 = L_2 = 1H$, $R_1 = R_0 = 0.5\Omega$. It can be verified that the system is LFNI by either Definition 1 or Lemma 1. By letting the imaginary part of the transfer function $G(j\omega)$

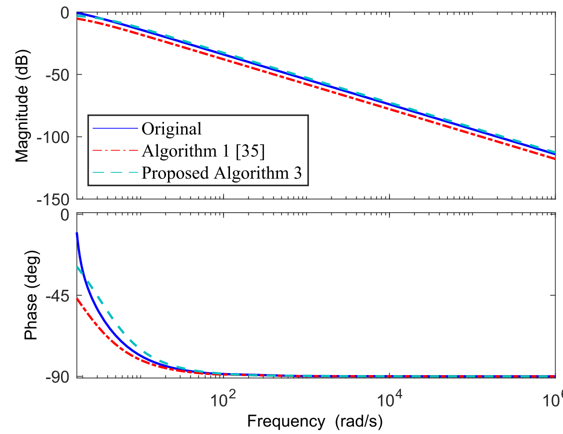

Fig. 10. Bode plots of the original and the reduced-order systems for frequency interval $\omega \in [1.63, \infty)$.

be zero, we can find that $\Im[G(j\omega)] \le 0$ for $\omega \in [1.63, \infty)$. Let $\underline{\omega}_h = 1.63$rad/s. A solution to (6) can be found and given by

$$P_{N+1} = \begin{bmatrix} 0.78 & 0.08 & -0.37 & 0.113 & -0.014 \\ 0.08 & -0.078 & 0.09 & -0.816 & 0.281 \\ -0.37 & 0.09 & 0.34 & 0.13 & -0.34 \\ 0.113 & -0.816 & 0.13 & -0.247 & -0.126 \\ -0.014 & 0.281 & -0.34 & -0.126 & 0.708 \end{bmatrix},$$

$$Q_{N+1} = \begin{bmatrix} 0.79 & -0.107 & 0.207 & 0.05 & -0.028 \\ -0.107 & 0.542 & -0.119 & 0.18 & 0.06 \\ 0.207 & -0.119 & 0.5691 & -0.11 & 0.058 \\ 0.05 & 0.18 & -0.11 & 0.72 & -0.157 \\ -0.028 & 0.06 & 0.0577 & -0.157 & 1.04 \end{bmatrix} \ge 0.$$

The goal of this example is to find the second-order HFNI system with the H_∞ approximation error minimized over the frequency interval, $\omega \in [1.63, \infty)$. By Algorithm 3, the following second-order system is obtained

$$A_{\mathrm{r}} = \begin{bmatrix} -4.497 & -0.5437 \\ -1.257 & -4.03 \end{bmatrix}, \quad B_{\mathrm{r}} = \begin{bmatrix} 1.946 \\ 0.5013 \end{bmatrix},$$
$$C_{\mathrm{r}} = \begin{bmatrix} 1.791 & -2.307 \end{bmatrix}. \tag{47}$$

Moreover, a realization in terms of RLC circuit is shown in Figure 2 with $\hat{R}_1 = 19\Omega$, $\hat{R}_2 = 30\Omega$, $\hat{R} = 0.8\Omega$, $\hat{C}_1 = \hat{C}_2 = 0.01$F, $\hat{L}_1 = 0.134$H. The sub-optimal H_∞ approximation error is $\gamma^* = 0.135$, which is smaller than 0.3267 obtained by Algorithm 1 [35].

The bode plots of the original and the reduced-order systems obtained by Algorithm 1 [35], the proposed Algorithm 3 are shown in Figure 10. Moreover, Figure 11 shows the imaginary part of the frequency response $\Im[G(j\omega)]$ for the original and the reduced-order systems over the frequency interval, $\omega \in [1.63, \infty)$. It can be seen from Figure 10 that the reduced-order system (47) approximate the original system well over the frequency interval, $\omega \in [1.63, \infty)$. From Figure 11, we can see that the reduced-order (47) satisfies $\Im[G_{\mathrm{r}}(s)] \le 0$ for $\omega \in [1.63, \infty)$. This means $j[G_{\mathrm{r}}(j\omega) - G_{\mathrm{r}}^*(j\omega)] \ge 0$. In addition, the reduced-order system (47) is stable. Thus, we have that the

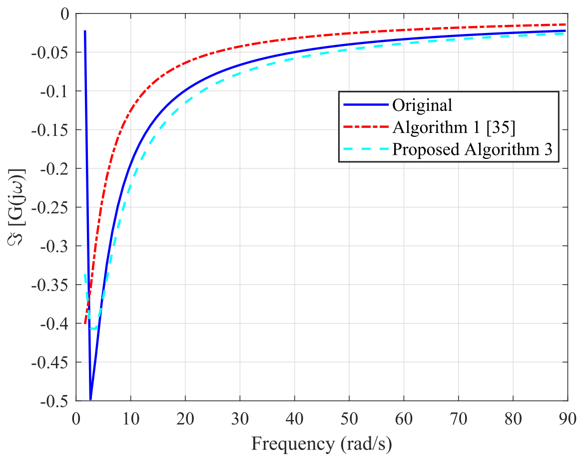

Fig. 11. $\Im[G(j\omega)]$, $\omega \in [1.63, \infty)$ for the original and the reduced-order systems.

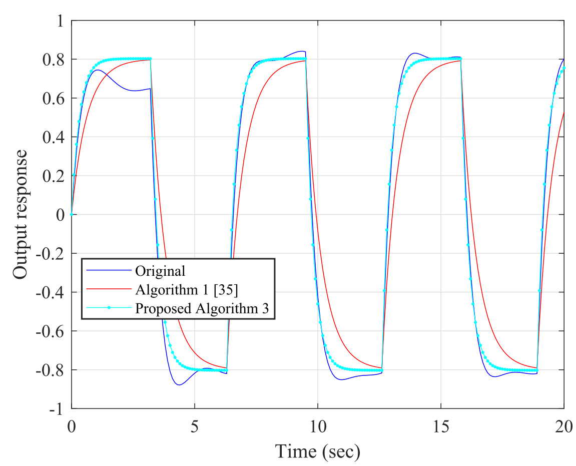

Fig. 12. Output responses for $u(t) = \sqrt{t}$.

reduced-order system obtained by the proposed Algorithm 3 preserves the HFNI structure. Moreover, the corresponding output responses of the original and the reduced-order system for input $u(t) = \sqrt{t}$ are given in Figure 12. It can be seen that the reduced-order system obtained by the proposed model reduction method follow the original output accurately.

From the simulation results, we can conclude that the reduced-order systems obtained by the proposed Algorithm 3 give better approximations of the original systems both in the frequency-domain and time-domain. And the proposed Algorithm 3 result in lower H_∞ approximation errors. The conservatism is that only a sub-optimal reduced-order IFNI system can be obtained. Question concerning the optimal structure preserving model reduction for IFNI systems is under investigation.

V. Conclusions

The H_∞ model reduction problem for interval frequency negative imaginary systems has been studied in this paper. Necessary and sufficient conditions in terms of matrix inequalities have been established for the existence and construction of the reduced-order interval frequency negative imaginary system. It has shown that a desired reduced-order system can be obtained by solving these matrix inequalities. Moreover, an improved iterative algorithm including two stages has been provided to find the sub-optimal reduced-order interval frequency negative imaginary system. The proposed model reduction method has been applied to solve the model reduction problems for electrical circuits over a finite frequency interval. Simulation results have been provided to show the effectiveness and advantages of the proposed model reduction method.

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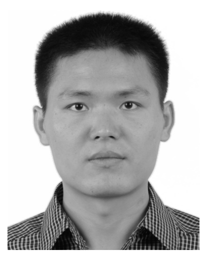

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