A dynamic lattice method for elastic wave modeling and migration in TTI media
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Summary
A dynamic lattice method (DLM) is developed to simulate seismic wave propagation in transversely isotropic (TTI) media with tilted symmetry axis (TTI media). Basing on a particle lattice model, this method calculates the micro-mechanical interactions between particles in the lattice instead of solving the wave equation. Seismic waves in continua are approximated by the dynamics of these particles and the elastic properties of the continuum are represented by properties of the particle lattice. Our research reveals the theoretical connections between the TI medium and the particle lattice model. We have applied this method to elastic wave simulation in TI media with free surface topography and reverse-time migration on a TI model to test its usefulness in elastic wave simulation for TI media.

Introduction
A variety of numerical methods such as finite difference methods (FDM) (Virieux, 1984, 1986) and finite element methods (FEM) (Lysmer and Drake, 1972) are developed to offer approximate solutions to the elastic wave equations. Based on continuum mechanics, these methods are governed by the spatially continuous differential equation where special calculations of particular boundary conditions at discontinuities are required. Consequently, for these methods, despite the naturally discontinuous features of the geological materials, such as fractures and faults, models are smoothed to satisfy the continuous requirement.

Apart from these elastic wave equation based methods, another type of particle based method has been quite successful in dealing with discontinuities. Originated from solid-state physics models of crystalline materials (Hoover et al., 1974), these methods treat geological materials as assemblages of numerous interconnected discrete particles and focus on “microscopic” interactions between these particles. In terms of elastic wave simulation, Toomey and Bean (2000) discuss a “discrete particle scheme” for elastic wave simulation in isotropic media. Instead of solving the elastic wave equations, their method considers the medium to be composed of interconnected particles closely packed in a hexagonal lattice and uses the dynamics of the discrete particles to simulate elastic waves. Del Valle-García and Sánchez-Sesma (2003) improve Toomey’s method by including a bond-bending force term to get rid of the restriction on Poisson ratio. O’Brien and Bean (2004) extend this particle lattice method to the 3-dimensional case by using the elastic lattice method.

Basing on their studies, we make use of the orientations of the bonds in the lattice to create anisotropy. By comparing the elastic energy density stored in the anisotropic lattice with TI media we find the connections between the lattice model and TI media. We apply our dynamic lattice method (DLM) to elastic wave simulation in TI media with the presence of free surface topography and reverse-time migration on a TI model. Numerical tests show that this method can accurately deal with the elastodynamic and anisotropic feature of TI media.

Dynamic Lattice Method
We employ an anisotropic particle lattice model to represent TI media. Figure 1a shows a basic unit of the particle lattice where the central particle is bonded with 8 neighbors.

These bonds can be stretched or bended when external forces are applied to the particles. As Figure 1b shows, micro-mechanical interactions between these particles are achieved through the stretching and bending of the bonds. Coefficients $k_1$ and $b_{ij}$ are introduced to calibrate the stiffness of these bonds when they are stretched and bended respectively. The bigger these coefficients are the harder it is to stretch or bend the bonds. In order to incorporate anisotropy, we set the stiffness coefficients in different orientations separately. According to Monette and Anderson (1994), the elastic energy in a distorted lattice unit is given by

$$E = \frac{1}{2} \sum_{i} \left( k_i |r_{ii}' - r_{ii}|^2 + \frac{1}{2} \sum_{j \neq i} b_{ij} (\cos \theta_{ij} - \cos \theta_{ji}) \right),$$

(1)

in which $E$ is the elastic energy, $r_{ii}'$ and $r_{ii}$ are the vectors for bonds that connect particle 0 and i in the distorted and undistorted lattices respectively. Similarly, $\theta_{ij}$ and $\theta_{ji}$ are the angles between bonds 0-i and 0-j in the distorted and undistorted lattices respectively. mod(i, 8) represents the...
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In order to improve the CPU time efficiency of the DLM, we apply a linear modification to simplify the calculation of equations 4 and 5. The micromechanical interaction force acting upon particle 0 can be written as the linear combination of \( \mathbf{u}_{0i} \) given by

\[
\mathbf{u}_{0i} = \mathbf{u}_i - \mathbf{u}_0 \quad (i = 1, 2, \ldots 8).
\]

where \( \mathbf{u}_i \) is the displacement of particle \( i \) which is the neighbor of particle 0. Therefore, the interaction force acting on particle 0 can be written as

\[
F = S \mathbf{U},
\]

In which \( \mathbf{U} = (\mathbf{u}_{01} \mathbf{u}_{02} \ldots \mathbf{u}_{08})^T \) and \( S \) denotes the linear combination, we apply the Velocity-Verlet accumulation (Allen and Tildesley, 1987), to solve the movements of the particles.

Dispersion and Stability

According to Saenger and Bohlen (2004), we perform a plane wave analysis to illustrate the dispersion property of DLM. For any given wavenumber \( k \), the phase velocity \( v(k) \) is given by

\[
v(k) = \frac{1}{2\mu a^2 k^2} (g_{11} + g_{22} \pm \sqrt{(g_{11} - g_{22})^2 + 4g_{12}g_{22}}).
\]

where

\[
\begin{align*}
g_{11} &= \begin{pmatrix} h_2 + h_3 + h_4 + k_1 \\ h_2 h_3 + h_2 + 1 + k_2 \\ h_2 h_3 + h_2 + 1 + k_2 \\ h_2 h_3 + h_2 + 1 + k_2 \\ h_2 h_3 + h_2 + 1 + k_2 \\ h_2 h_3 + h_2 + 1 + k_2 \\ h_2 h_3 + h_2 + 1 + k_2 \\ h_2 h_3 + h_2 + 1 + k_2 
\end{pmatrix}^T, \\
g_{12} &= \begin{pmatrix} h_2 h_3 + h_2 + 1 + k_2 \\ h_2 h_3 + h_2 + 1 + k_2 \\ h_2 h_3 + h_2 + 1 + k_2 \\ h_2 h_3 + h_2 + 1 + k_2 \\ h_2 h_3 + h_2 + 1 + k_2 \\ h_2 h_3 + h_2 + 1 + k_2 \\ h_2 h_3 + h_2 + 1 + k_2 \\ h_2 h_3 + h_2 + 1 + k_2 
\end{pmatrix}^T,
\end{align*}
\]

We calculate the relative errors of phase velocity in different media. Figure 2 shows relative errors of phase velocity with respect to different wavenumber i.e. \( k_x \) and \( k_z \) ranging from \(-2\pi/15\alpha\) to \(2\pi/15\alpha\). The elastic tensors of these media are given by table 1. As Figure 2 demonstrates, when \(|k| < 2\pi/15\alpha\) i.e. the lattice spacing \( a < \lambda/15\) where \( \lambda \) is the wavelength, the dispersion is suppressed. Note that our DLM only uses a 3x3 lattice unit which is the first order in terms of spatial interaction; it is potential to extend our method to

To conduct seismic modeling, it is necessary to find the relationship between micro-mechanical stiffness coefficients and the macroscopic parameters such as elasticity tensor and Thomsen moduli (1986). One way to find this connection is through the comparison between the elastic energy of a lattice unit given by equation 1 and the elastic energy density for TI media in continuum theory. For TTI media, the number of independent stiffness coefficients can be reduced to 6, as \( k_1, k_3, k_2, k_4, b_{12} = b_{34} = b_{56} = b_{78} \) and \( b_{23} = b_{45} = b_{67} = b_{83} \). The relationship between stiffness coefficients and the elasticity tensor is given by

\[
\mathbf{w}(\theta)(k_1, k_3, k_2, k_4, h_1, h_2)^T = (C_{11}, C_{13}, C_{15}, 0, 0)^T,
\]

where \( C_{11}, C_{33}, C_{13} \) and \( C_{44} \) are the elements in the elasticity tensor of the corresponding TI medium when the symmetry axis is in vertical direction, \( \theta \) is the rotated angle for the TTI medium, and \( \mathbf{W}(\theta) \) is a 6 x 6 matrix whose determinant is not zero, indicating that the transformation from elasticity tensors to stiffness coefficients is unique. Therefore, we can always find the corresponding stiffness coefficients for any TI medium as long as it can be described by elasticity tensor. According to our research, this matrix is written as

\[
\mathbf{W}(\theta) = \begin{pmatrix}
\cos^2 \theta & 2\cos \theta \sin \theta & -\sin^2 \theta & -\cos^2 \theta & -2\cos \theta \sin \theta & \sin^2 \theta \\
\sin^2 \theta & 2\cos \theta \sin \theta & \cos^2 \theta & -2\cos \theta \sin \theta & \sin^2 \theta & -\cos^2 \theta \\
\cos^2 \theta & -2\cos \theta \sin \theta & \sin^2 \theta & -2\cos \theta \sin \theta & \cos^2 \theta & -\sin^2 \theta \\
-\sin^2 \theta & 2\cos \theta \sin \theta & -\cos^2 \theta & \sin^2 \theta & -2\cos \theta \sin \theta & \cos^2 \theta \\
\cos^2 \theta & -2\cos \theta \sin \theta & -\sin^2 \theta & \cos^2 \theta & -2\cos \theta \sin \theta & \sin^2 \theta \\
-\sin^2 \theta & 2\cos \theta \sin \theta & \cos^2 \theta & -\sin^2 \theta & \cos^2 \theta & -2\cos \theta \sin \theta
\end{pmatrix}
\]

Elastic wave propagation are simulated by the dynamics of the particles. We apply the Lagrange’s equations (Landau and Lifshitz, 1976) to calculate the complex dynamics of the particles. According to the Lagrange’s equations, the dynamics of the particles in a lattice unit are governed by

\[
\frac{d}{dt} \frac{\partial L}{\partial \mathbf{u}^*_i} = \frac{\partial L}{\partial \mathbf{u}^*_i} \quad (i = 0, 1, 2, \ldots 8)
\]

and

\[
\frac{d}{dt} \frac{\partial L}{\partial \mathbf{u}^*_i} = \frac{\partial L}{\partial \mathbf{u}^*_i} \quad (i = 0, 1, 2, \ldots 8),
\]

in which \( \mathbf{u}^*_i \) and \( \mathbf{u}^*_i \) are the x- and z-components of the displacement for particle \( i \), respectively, whereas \( \mathbf{u}^*_y \) and \( \mathbf{u}^*_y \) are the x- and z-components of the velocity for particle \( i \), respectively; \( L \) is the Lagrange’s function for the particle system, which is given by

\[
L = T - V,
\]

where

\[
T = \sum_{i=1}^{8} \frac{1}{2} m_i \mathbf{u}^*_i^2
\]

and the potential energy \( V \) equals to the elastic energy given by equation 1.
higher orders by considering larger lattice units, e.g., a 4×4 lattice unit.

Table 1. Elastic properties of the TI models.

<table>
<thead>
<tr>
<th>Model</th>
<th>(C_{11}(\text{GPa}))</th>
<th>(C_{22}(\text{GPa}))</th>
<th>(C_{33}(\text{GPa}))</th>
<th>(C_{44}(\text{GPa}))</th>
<th>(\rho(10^3 \text{ kg/m}^3))</th>
<th>(\theta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>18.0</td>
<td>18.0</td>
<td>4.0</td>
<td>7.0</td>
<td>2.0</td>
<td>0°</td>
</tr>
<tr>
<td>2</td>
<td>25.5</td>
<td>18.4</td>
<td>18.5</td>
<td>5.6</td>
<td>2.0</td>
<td>10°</td>
</tr>
</tbody>
</table>

Figure 2. Relative errors of phase velocity for model 1 in the upper panel and model 2 in the lower panel. \(a\) is the lattice spacing. \(k_x\) and \(k_z\) are the \(x\)- and \(z\)- components of wave number \(k\) respectively.

We analyze the stability condition of the DLM according to Von Neumann’s method (Press et al., 1986). The stability condition of DLM is given by

\[
0 \leq \frac{\Delta t^2}{\rho a^2}\left(\frac{h_{12} + h_{13}}{a} + k_i\right) \leq 1 \quad (l = 1, 3) \quad (14)
\]

and

\[
0 \leq \frac{\Delta t^2}{\rho a^2}\left(\frac{h_{14} + h_{24}}{a} + k_i + k_z \sqrt{k_1^2 + \left(\frac{h_{12} - h_{13}}{a} + k_1 - k_z\right)^2} \right) \leq 1. \quad (15)
\]

For VTI media the stability condition for DLM is given by

\[
C_{11} \leq C_{33}, \quad (16)
\]

and

\[
\frac{\Delta t^2}{a^2} \leq \frac{\rho}{C_{11}} = \frac{1}{(1+2\varepsilon)V_p^2}. \quad (17)
\]

Equation 16 defines the scope of application for DLM with regard to VTI media which indicates that the DLM will face a stability problem when anisotropy gets too high. This limitation is probably caused by the shape of the lattice structure. It is possible to eliminate this restriction by changing the square lattice into a hexagonal lattice.

**Numerical Examples**

To demonstrate the effect of the dynamic lattice method in elastic wave simulation in TTI media. We compare the seismograms from our method with those from a high quality elastic wave equation based DRP/opt MacCormack finite difference method (Zhu et al., 2009; Zhang et al., 2012). This FDM is a popular and well tested non-staggered scheme. It solves the first-order partial differential velocity-stress equations with low dispersion or dissipation.

A homogeneous TTI model showed in Figure 3 is used to conduct the comparison. For dynamic lattice method, the lattice spacing is 1m which is the same as the grid spacing of the finite difference method. The time stepping for both methods is 0.2 ms. We place a single force source in vertical direction at \(x = 500\) m and \(z = 300\) m. The wavelet of the source is a Ricker wavelet with 20 Hz dominant frequency. The seismograms generated by both methods are plotted in Figure 4.

Figure 4. Comparison between the seismograms (displacements) from the DLM and those from the FDM. Seismograms of \(x\)- and \(z\)- components are plotted on the left and right columns respectively.

Despite that the DLM does not solve the elastic wave equation, seismograms of both methods are all most identical. This indicates our particle lattice model can truly reflect the elastic
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properties of the corresponding continuum. In terms of CPU time, it takes 145 seconds for our DLM on a single CPU to calculate the movements of 800×800 particles for 1200 time steps, while the FDM we used for comparison takes 391 seconds to calculate the same problem under the same condition. We conduct a seismic wave simulation using our method on a free surface topography model (Figure 5) in which the lattice spacing is 1 m, the time stepping is 0.2 ms and the dominant frequency of the source is 30 Hz. In order to implement the free surface condition in the presence of topography, we remove the particles that are above the surface from the lattice network. Through the interactions between the particles near the surface, the free surface condition is automatically implemented. As the snapshots in Figure 6 show, the surface waves generated by the free surface indicate that our method is capable in dealing with free surface topography.

Figure 5. A TTI model with free surface topography. Single force source is placed at the surface where \( x = 1200 \) m.

Figure 6. Snapshots of x- and z-component of the displacement field.

We also implement the DLM in reverse time migration on a 2D TTI overthrust model (Fei et al., 1998) displayed in Figure 7a. The TI thrust is composed of four parts with different symmetry axis. All shots and receivers have \( z = 0 \) km. First shot has \( x = 0.2 \) km, last shot has \( x = 5.8 \) km with shot interval of 0.06 km. Receiver offsets from -2 km to 2 km with an interval of 5 m. The dominant frequency of the source is 20 Hz. The sampling rate is 1000 Hz. The lattice spacing is 5 m and time stepping is 1ms. Absorbing boundary condition is applied at all boundaries including the surface. Image in Figure 7b is obtained by applying the cross correlation imaging condition on z components of the shot wave field and receiver wave field. Reflectors are imaged in correct places. We can also see the interface of the thrust block. We believe this is caused by the divergence of the symmetry axis.

Figure 7. (a) 2D Overthrust model. Reflectors are placed at \( z = 1.94 \) km and \( z = 0.28 \) km. (b) Pre-stack image of z component of the 2D TTI overthrust model. Note the shallow interface between the thrust blocks is imaged.

Conclusion

By varying the micro-mechanical interaction with bond orientation, we create an anisotropic lattice network. Our research indicates that the macroscopic parameters, which define a TTI medium such as elasticity tensor and Thomsen moduli, can be transformed into microscopic parameters that calibrates the anisotropic elastic properties of the lattice model through a linear transformation. Our numerical experiments show that the dynamic lattice method is capable of simulating elastic waves in TTI media. The particle feature of this method makes it convenient for our method to implement free surface condition in the presence of topography.

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