A new finite-difference scheme for frequency-domain seismic modeling on non-uniform grids
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Summary

In this paper we introduce a new finite-difference scheme for 2D frequency-domain acoustic wave modeling on irregular grids. It is developed from the average-derivative optimal nine-point scheme (ADM). The ADM scheme breaks the limitation on directional sampling intervals but can’t be implemented on non-uniform grids. Our new scheme overcomes this restriction and is more generalized than the ADM scheme. It can be proved that the ADM scheme is a particular case of our new scheme. A simple two-layer model is tested for verifying the feasibility and efficiency of our new scheme.

Introduction

Seismic wave forward modeling is an important part for full waveform inversion (FWI) and migration and also an important tool to understand wave propagation phenomenon in realistic underground media. Besides time-domain forward modeling, frequency-domain forward modeling is widely used, especially for multi-scale FWI, because it is convenient to manipulate single frequency wavefield. Frequency-domain operators have stronger stability than time-domain operators as it is not limited by time-marching step and no accumulative error occurs. Therefore, it’s a reasonable choice to employ frequency-domain operators to perform numerical modeling when dealing with frequency-dependent physical problems with limited frequency bandwidth.

Nevertheless, a crucial disadvantage of frequency-domain operators is that it would cost memory and time prohibitively when we handle large model areas. Different from the time-domain operators, every single frequency is computed by solving equations independently in frequency-domain operators. A huge sparse impedance matrix needs to be conducted when we solve the system of linear equations. Although influenced by the solver we used, the computational cost increases with the grid number exponentially no matter which solver is chosen (Li et al., 2015). Hence, to reduce the memory and time cost, less grid nodes are used in the numerical computation.

In order to reduce the number of grid nodes on the premise of ensuring the accuracy of modeling, many kinds of finite-difference scheme are proposed in recent years. Pratt and Worthington (1990) propose conventional second-order central finite-difference scheme. For this scheme, thirteen grid nodes per wave-length is needed in order to suppress the numerical dispersion and achieve the satisfying result. A rotated optimal nine-point scheme is presented by Jo et al. (1996) which is more efficient than conventional second-order finite-difference scheme. Four grid nodes per wavelength is enough to obtain the accurate modeling result. An average-derivative optimal nine-point scheme (ADM) proposed by Chen (2012) improves the rotated nine-point scheme. Same to the former, four grid nodes per wavelength is necessary for this scheme to suppress the numerical dispersion greatly. This scheme can overcome the restriction on directional sampling intervals to which the scheme of Jo et al. (1996) is limited.

In addition to these schemes which are aimed to improve the finite-difference accuracy, other approaches are employing non-uniform grids spatially to decrease the overall grid number. In case of complex underground media, using non-uniform grids is a reasonable strategy. Quite a few papers about non-uniform grids have been published in recent years (Oprsal and Zahradnik, 1999; Tessmer, 2000; Adriano and Oliveira, 2003; Zhang and Chen, 2006; Liu et al., 2011; Liu and Sen, 2011; Chu and Stoffa, 2012; Zhang et al., 2013). However, almost all studies focus on the time-domain operators and frequency-domain operators are hardly investigated.

In this paper, we propose a new finite-difference scheme for frequency-domain acoustic wave propagator on non-uniform grids. The mathematical principle is deduced and the numerical dispersion is analyzed. A simple numerical example is presented to demonstrate the validity of this new scheme.

Theory

Consider the 2D acoustic wave equation in frequency domain

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) p(x, z, \omega) = S(x_0, z_0, \omega),
\]

where \( p \) denotes frequency-domain wavefield, and \( S \) denotes the source and \( (x_0, z_0) \) is the source position. We introduce a new finite-difference scheme for equation 1.
$$\Delta x_1 \left( \hat{p}_{m+1,n} - \hat{p}_{m,n} \right) - \Delta x_2 \left( \hat{p}_{n,n} - \hat{p}_{m-1,n} \right)$$

\[
\frac{\Delta x_2 (\Delta x_1 + \Delta x_2)}{2} + \Delta x_1 \left( \hat{p}_{m+1,n} - \hat{p}_{m,n} \right) - \Delta x_2 \left( \hat{p}_{n,n} - \hat{p}_{m-1,n} \right)
\]

\[+ \frac{\alpha^2}{v_m n} \left( \sum_{i,j=1}^{m+1} c_{i,j} p_{m+i,n+j} \right) = S,
\]

(2)

where

\[
\hat{p}_{m+1,n} = \frac{1}{2} \hat{p}_{m+1,n+1} + \frac{1}{2} \hat{p}_{m+1,n-1} + \frac{1}{2} \hat{p}_{m,n},
\]

\[
\hat{p}_{m,n} = \frac{1}{2} \hat{p}_{m+1,n} + \frac{1}{2} \hat{p}_{m-1,n} + \frac{1}{2} \hat{p}_{m,n+1} + \frac{1}{2} \hat{p}_{m,n-1}.
\]

It can be testified that the ADM scheme is a special case of this scheme when the following constraints are fulfilled,

\[
\Delta x_1 = \Delta x_2, \Delta z_1 = \Delta z_2,
\]

\[
\alpha_1 = \alpha_2, \beta_1 = \beta_2,
\]

\[
c_{1,-1} = c_{1,1},
\]

\[
c_{1,0} = c_{0,1},
\]

(3)

Therefore, we call our new scheme as the generalized ADM (GADM) for convenience.

![Figure 1: Schematic of the GADM. Weights are shown by rows and columns.](image)

The coefficients in equation 2 are still to be determined. We obtain the optimal weighted coefficients by minimizing the numerical dispersion. We perform a plane wave analysis to illustrate the numerical dispersion properties. A plane wave that propagates in the mesh as Figure 1 shows can be written as

\[
p_{m,n} = p_0 \exp \left( -i(k_x x + k_z z) \right),
\]

\[
p_{m,n+1} = p_0 \exp \left( -i(k_x (x + \Delta x_1) + k_z z) \right),
\]

\[
p_{m+1,n} = p_0 \exp \left( -i(k_x x + \Delta x_2 + k_z z) \right),
\]

\[
p_{m-1,n} = p_0 \exp \left( -i(k_x (x - \Delta x_1) + k_z z) \right),
\]

\[
p_{m+1,n-1} = p_0 \exp \left( -i(k_x (x - \Delta x_1) + k_z (z + \Delta z_2)) \right),
\]

\[
p_{m,n-1} = p_0 \exp \left( -i(k_x x + k_z (z - \Delta z_1)) \right),
\]

\[
p_{m+1,n+1} = p_0 \exp \left( -i(k_x (x + \Delta x_2) + k_z (z + \Delta z_2)) \right),
\]

\[
p_{m-1,n+1} = p_0 \exp \left( -i(k_x (x - \Delta x_1) + k_z (z + \Delta z_2)) \right),
\]

\[
p_{m+1,n+1} = p_0 \exp \left( -i(k_x (x + \Delta x_2) + k_z (z + \Delta z_2)) \right),
\]

(5)

where \(k_x\) is the horizontal wavenumber, \(k_z\) is the vertical wavenumber and \(p_0\) is the wave amplitude.

Let

\[
\theta = \arcsin(k_z/k_x),
\]

\[
r_1 = \frac{D_{\max}}{\Delta x_1}, r_2 = \frac{D_{\max}}{\Delta x_2}, r_3 = \frac{D_{\max}}{\Delta z_1}, r_4 = \frac{D_{\max}}{\Delta z_2},
\]

\[
G = \frac{2\pi}{k_D \delta_{\max}}
\]

(6)

where \(\theta\) is the angle between wavenumber direction and vertical axis, \(D_{\max}\) is the maximal grid space in non-uniform mesh and \(G\) is the number of grid points (respect to maximal space) per wavelength.

Substituting equation 5 into equation 2 and supposing the constant velocity \(v\), we achieve the normalized phase velocity as

\[
\frac{v_p}{v} = G \sqrt{\sum_{i,j=-1}^{1} b_{i,j} a_{i,j}}
\]

(7)

where

\[
\begin{bmatrix}
\alpha_{i-1} & a_{i-1} & a_{i-1} \\
\alpha_{i+1} & a_{i+1} & a_{i+1} \\
1 - \alpha_i & a_{i} & a_{i} \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\cos(2\sin\theta) + 2\cos\theta & \cos(2\sin\theta) & \cos(2\sin\theta) - 2\cos\theta \\
\cos(2\sin\theta) & \cos(2\sin\theta) + 1 & \cos(2\sin\theta) - 2\cos\theta \\
\cos(2\sin\theta) & \cos(2\sin\theta) & \cos(2\sin\theta) + 2\cos\theta \\
\end{bmatrix}
\]

(8)

\[
\begin{bmatrix}
b_{i-1} & b_{i-1} & b_{i-1} \\
b_{i+1} & b_{i+1} & b_{i+1} \\
b_{1,0} & b_{1,0} & b_{1,0} \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{1}{r_1} - \frac{1}{r_2} & \frac{1}{r_1} & \frac{1}{r_1} \\
\frac{1}{r_1} & \frac{1}{r_1} - \frac{1}{r_2} & \frac{1}{r_1} \\
\frac{1}{r_1} & \frac{1}{r_1} & \frac{1}{r_1} \\
\end{bmatrix}
\]

(9)

Consequently, the weighted coefficients are determined by minimizing the phase error

\[
E(\alpha_1, \alpha_2, \beta_1, \beta_2, c_{i,j}) = \int \left( 1 - \frac{v_p}{v} \right)^2 d\theta
\]

(10)
where

$$\bar{k} = \frac{1}{G}$$

(11)

\(\bar{k}\) varies from 0 to 0.25 and \(\theta\) varies from 0 to \(\pi/2\), respectively.

We employ a constrained nonlinear optimization approach to achieve the optimal coefficients. Different grid space ratios (i.e., different \(r_1, r_2, r_3\) and \(r_4\)) lead to different optimal coefficients. In this study, \(r_i (i = 1, 2, 3, 4)\) varies from 1 to 5 with an interval of 0.5. Therefore, a “dictionary” composed by \(9 \times 9 \times 9 \times 9\) groups of optimal coefficients is obtained. For every discrete point, we can select the corresponding coefficients from the “dictionary” according to the ratio of maximal grid space and four grids spaces around it.

Here we choose four groups of optimal coefficients among the huge “dictionary” randomly. The normalized phase velocity curves are illustrated in Figure 2 and their corresponding grid spaces ratios is shown in Table 1. Due to the limited space, values of the four groups of optimal coefficients are not shown. From Figure 2 and Table 1, we can observe that the phase error is influenced slightly by different grid spaces ratios. Nevertheless, for all grid spaces ratios, the phase errors in different directions are within \(\pm 0.5\%\) even if the number of grid points is small as four.

Table 1: Grid spaces ratios corresponding to four groups of optimal coefficients chosen randomly.

<table>
<thead>
<tr>
<th>(r_1)</th>
<th>(r_2)</th>
<th>(r_3)</th>
<th>(r_4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>1.5</td>
<td>1.5</td>
<td>2.0</td>
</tr>
<tr>
<td>b)</td>
<td>5.0</td>
<td>1.5</td>
<td>5.0</td>
</tr>
<tr>
<td>c)</td>
<td>3.5</td>
<td>4.0</td>
<td>1.5</td>
</tr>
<tr>
<td>d)</td>
<td>3.0</td>
<td>5.0</td>
<td>2.0</td>
</tr>
</tbody>
</table>

**Numerical examples**

We verify the GADM scheme by a simple two-layer model (Figure 3). The size of this model is 3000m in \(x\) direction and 1600m in \(z\) direction. The shot position is at 1500m on the surface. The source is Ricker wavelet of which the dominant frequency is 20Hz and the frequency bandwidth is \([5\text{Hz}, 60\text{Hz}]\). The hybrid absorbing boundary conditions introduced by Moreira et al. (2014) is appended around the model area.

![Figure 3: The two-layer velocity model. The upper layer is 2000m/s and the below layer is 3000m/s.](image)

![Figure 2: Normalized phase velocity curves defined by four groups of optimal coefficients chosen randomly. Numerical phase velocity \(v_{ph}\) is normalized with respect to the true velocity \(v\) and plotted versus \(1/G\). Different colors of curves denote different directions of wavenumber.](image)
The grid space in the GADM scheme is non-uniform as Figure 4 shows. The grid space in z direction (dz) varies with depth and the grid space in x direction (dx) varies with distance, respectively. dz is 6.25m in the upper area and 7.5m in the lower area. dz increases gradually from 6.25m to 7.5m with an increment of 0.1m in Area 1. dx is 6.25m in global model area except in Area 2. In Area 2, dx increases from 6.25m to 7.5m with an increment of 0.1m and then decreases from 7.5m to 6.25m with an increment of -0.1m. In Area 3, dx varies in the same way as in Area 2 and dz varies in the same way as in Area 1. The snapshot at 0.5s is illustrated in Figure 5a.

In order to test validity of the GADM scheme, an experiment with the ADM is performed simultaneously. The grid space in whole model area in this experiment is dx = dz = 6.25m. Except that meshes are different, all the other parameters in two experiments are coincident. The snapshot at 0.5s is presented in Figure 5b. Compared to modeling result of the ADM, the result of the GADM scheme is satisfactory.

We compare the memory and time cost in two experiments and the results are shown in Table 2. From Table 2, we can realize that the time and memory cost in the GADM scheme is less than in the ADM scheme. The GADM scheme is more efficient than the ADM scheme.

Table 2: Cost comparison between two methods

<table>
<thead>
<tr>
<th></th>
<th>The GADM scheme</th>
<th>The ADM scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td>Memory cost/G</td>
<td>0.54</td>
<td>0.65</td>
</tr>
<tr>
<td>Time cost/s</td>
<td>227</td>
<td>281</td>
</tr>
</tbody>
</table>

Conclusions

We propose a new finite-difference scheme for frequency-domain propagator on non-uniform grids. Its validity and efficiency have been certified through a simple numerical example. Although complex model tests have not been implemented so far, it’s promising to be applied on more non-uniform problems. The method can be easily extended to higher order and higher dimension form.

Acknowledgments

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EDITED REFERENCES
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REFERENCES