Unsteady fluid-dynamic force solely in terms of control-surface integral

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In experimental aerodynamics (and hydrodynamics) it is well known that, if the flow past a solid body is steady, then the total force on the body can be conveniently estimated by the measured flow data on an appropriate control surface alone. We now show that, for the first time, the steady-flow condition can be removed provided that the flow is incompressible: two innovative formulas for the total force acting on any solid body that moves and deforms arbitrarily in a viscous incompressible fluid, solely in terms of control-surface integrals, are derived based on derivative-moment transformations. The formulas are verified by a numerical test for flow over a two-dimensional fishlike swimming body. © *2005 American Institute of Physics*. [DOI: 10.1063/1.2055528]

As an applied branch of fluid dynamics, a main concern of external aerodynamics (and hydrodynamics) is the forces experienced by a solid body moving through a viscous fluid, of which the experimental determination has been the ultimate basis of all relevant studies. In engineering applications, balances are widely employed in force measurement, but its accuracy is limited by support interference. It is especially difficult to measure the very small drag component of streamlined bodies. Most of all, the relation between the force and local flow quantities, which is of crucial importance for flow diagnosis, is missing. Information on this relationship can only be obtained by using integral-type force formulas, which as a fundamental requirement should be expressed in a way that permits convenient experimental measurement of a relevant integrand with high accuracy. Unfortunately, despite developments made over the past century, theoretical fluid dynamics has never provided any general integral formulas satisfying this requirement.

Consider an incompressible flow with uniform density ρ over a body as an example. The force acting on the body takes the following alternative standard forms based on (a) the direct integral of surface stresses over the body surface and (b) the rate of change of total momentum in a generic control volume V_f:

$$\boldsymbol{F} = -\int_{\partial B} (-p\boldsymbol{n} + \boldsymbol{\tau})dS \tag{1a}$$

$$= -\rho\frac{d}{dt}\int_{V_f} \boldsymbol{u}\, dV + \int_{\Sigma} [-p\boldsymbol{n} + \boldsymbol{\tau} - \rho\boldsymbol{u}(u_n - v_n)]dS. \tag{1b}$$

Here, $\boldsymbol{u}$ is the fluid velocity, p is the pressure, V_f is a fluid domain bounded externally by an arbitrary control surface Σ and internally by the material body surface ∂B, with the unit normal vector $\boldsymbol{n}$ pointing out of V_f, and $\boldsymbol{\tau}=\mu\boldsymbol{\omega}\times\boldsymbol{n}$ is the shear stress with μ being the dynamic viscosity and $\boldsymbol{\omega}=\nabla\times\boldsymbol{u}$ the vorticity (see Fig. 1). For generality we allow the body surface ∂B to have a specified velocity distribution $\boldsymbol{u}=\boldsymbol{b}(\boldsymbol{x},t)$ and the control surface Σ to have an arbitrary velocity $\boldsymbol{v}(\boldsymbol{x},t)$. u_n and v_n in (1b) are the normal components of $\boldsymbol{u}(\boldsymbol{x},t)$ and $\boldsymbol{v}(\boldsymbol{x},t)$ on Σ, respectively. For completeness, we give the proof of (1b) at the end of this Brief Communication.

While these standard formulas can be used in a numerical simulation after the flow field is solved, subjected to the numerical accuracy of course, none of them are convenient in experimentally determining the forces with a high accuracy, which, however, is supposed to serve as the test bed of all numerical results. For example, owing to the difficulty in measuring the distributed shear stress $\boldsymbol{\tau}$, it is not easy to use (1a). On the other hand, when the flow is steady in V_f, viewed in a frame of reference fixed to a rigid body, the first term of (1b) vanishes, so that the total force can conveniently follow from a survey of the flow data over a fixed control surface Σ (with $\boldsymbol{v}=\boldsymbol{0}$) only:

$$\boldsymbol{F} = \int_{\Sigma} (-p\boldsymbol{n} + \boldsymbol{\tau} - \rho\boldsymbol{u}\boldsymbol{u}\cdot\boldsymbol{n})dS. \tag{2}$$

This method has long been adopted as a complement to balance measurement, in particular when the front and side boundaries of V_f are sufficiently far from the body so that the flow thereon can be assumed approximately uniform at large Reynolds numbers and only a wake-plane survey is needed. Unfortunately, the flow steadiness is a very severe limitation. It excludes the general applicability of the control-surface survey method to many complex flows of significant interest, such as vehicle maneuvering, fish swimming, insect flight, dynamically deformable smart wings, fluid-solid coupling, and active flow control by unsteady excitation, among others. Today, with the rapid development of noninterference

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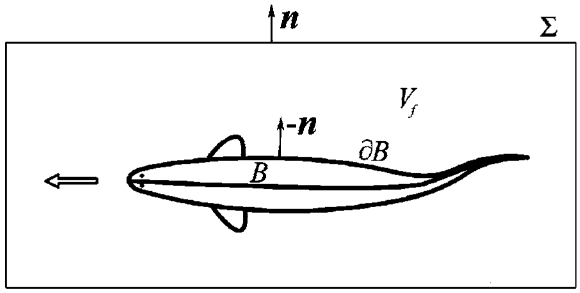

FIG. 1. Sketch of domain and notations.

instantaneous flow-field survey techniques such as particle image velocimetry (PIV), it is in principle possible to obtain the flow data in V_f and their time dependence, and hence to estimate the volume integral in (1b). However, it is still not easy to survey the flow field in a finite V_f all the way down to the body surface. Therefore, a pure control-surface integral is still highly desired.

Although this desired goal can by no means be achieved by any standard force formulas, we now show that it can be done by a transformation of (1b) to a nonstandard form by the familiar identity

$$\int_V \boldsymbol{f}\, dV = -\int_V \boldsymbol{x}(\nabla\cdot\boldsymbol{f})dV + \int_{\partial V} \boldsymbol{x}(\boldsymbol{f}\cdot\boldsymbol{n})dS \tag{3}$$

for any piecewise differentiable vector field $\boldsymbol{f}$, with $\boldsymbol{x}$ being the position vector measured from any fixed origin. We call these kinds of integral identities, which express the integral of a vector field by certain moments of its spatial derivatives, the "derivative-moment transformations" (DMT for short). One may recall that (3) was the basis of the classic Föppl total vorticity theorem

$$\int_V \boldsymbol{\omega}\, dV = \boldsymbol{0} \text{ if } \omega_n = 0 \text{ at } \partial V.$$

Therefore, owing to the continuity equation $\nabla\cdot\boldsymbol{u}=0$, we see immediately that (1b) can be cast to

$$\boldsymbol{F} = -\rho\frac{d}{dt}\left(\int_{\partial B} \boldsymbol{x} b_n dS + \int_\Sigma \boldsymbol{x} u_n dS\right) + \int_\Sigma [-p\boldsymbol{n} + \boldsymbol{\tau} - \rho\boldsymbol{u}(u_n - v_n)]dS, \tag{4}$$

where b_n is the specified normal component of the body-surface velocity.

The pressure term in (4) can be replaced by an acceleration term, through another DMT identity for the integral of a normal vector $\phi\boldsymbol{n}$ over a closed surface S, with ϕ being any tangentially piecewise differentiable scalar:

$$\int_S \phi\boldsymbol{n}\, dS = -\frac{1}{k}\int_S \boldsymbol{x}\times(\boldsymbol{n}\times\nabla\phi)dS, \tag{5}$$

where $k=n-1$ and $n=2,3$ is the spatial dimensionality. Thus, setting $\phi=-p$ and using the local momentum balance

$$\rho\boldsymbol{a} = -\nabla p - \mu\nabla\times\boldsymbol{\omega}, \tag{6}$$

where $\boldsymbol{a}=D\boldsymbol{u}/Dt$ is the material acceleration, we obtain

$$-\int_\Sigma p\boldsymbol{n}\, dS = -\frac{\rho}{k}\int_\Sigma \boldsymbol{x}\times(\boldsymbol{n}\times\boldsymbol{a})dS - \frac{\mu}{k}\int_\Sigma \boldsymbol{x}\times[\boldsymbol{n}\times(\nabla\times\boldsymbol{\omega})]dS.$$

Therefore, an alternative to (4), we have

$$\boldsymbol{F} = -\rho\frac{d}{dt}\left(\int_{\partial B} \boldsymbol{x} b_n dS + \int_\Sigma \boldsymbol{x} u_n dS\right) - \frac{\rho}{k}\int_\Sigma \boldsymbol{x}\times(\boldsymbol{n}\times\boldsymbol{a})dS + \boldsymbol{F}_\Sigma, \tag{7}$$

where

$$\boldsymbol{F}_\Sigma \equiv -\frac{\mu}{k}\int_\Sigma \boldsymbol{x}\times[\boldsymbol{n}\times(\nabla\times\boldsymbol{\omega})]dS + \int_\Sigma \boldsymbol{\tau}\, dS \tag{8}$$

collects all viscous vortical effects on the force.

It should be stressed that the above force formulas in terms of control-surface integrals alone exist for incompressible flow only. In an unsteady compressible flow disturbances propagate with finite speed, and hence a volume integral over V_f has to be involved.

Two more remarks on DMT-based formulas are in order here. First, for two-dimensional flow, identity (3) will become trivial if $\boldsymbol{f}$ is perpendicular to the flow plane (as is the case for the vorticity). But it is well applicable if $\boldsymbol{f}$ is on the plane, as is the present case for the velocity $\boldsymbol{u}$. Second, each of the identities (3) and (5) belongs to one type of derivative-moment transformation: the "inner-product" type and the "cross-product" type, of which a more systematic analysis has been given by Wu and Wu.[1] The cross-product type may lead to a fruitful set of nonstandard force formulas able to reveal the local dynamic mechanisms responsible for the force and moment, for which a preliminary report of a complete theory is given by Wu, Lu, and Zhuang.[2]

We now proceed to verify the new formulas (4) and (7) by calculating a two-dimensional viscous flow, governed by (6) and the continuity equation, over a fishlike swimming body. In dimensionless form, the Reynolds number is defined as $Re=UL/\nu$, where U is the free-stream velocity, L the length of the body, and ν the kinematic viscosity. A NACA0012 airfoil is used as the contour of the body at an equilibrium position of undulating motion. The midline of the body is making a transversal oscillation in the form of a wave traveling in the streamwise direction, described by

$$y_m = A_m(x)\cos[2\pi(x-ct)], \quad 0 \leqslant x \leqslant 1, \tag{9}$$

where A_m and c are the amplitude and phase speed of the traveling wave. To model reasonably the lateral motion of the backbone undulation of fish swimming, $A_m(x)$ is approximated by a quadratic polynomial

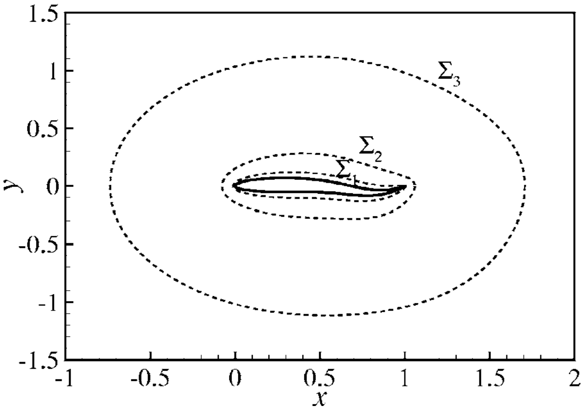

FIG. 2. Three control domains with the control surfaces $\Sigma_i(i=1,2,3)$ at an instant $t/T=0$ with T being the period of wave travelling in (9).

$$A_m(x) = C_0 + C_1 x + C_2 x^2, \tag{10}$$

where the coefficients C_0, C_1, and C_2 are solved from the kinematic data of a steadily swimming saithe,[3] which gives

$$A_m(0) = 0.02, \quad A_m(0.2) = 0.01, \quad A_m(1.0) = 0.10. \tag{11}$$

To set the kinematic conditions on the deformable body, as used and confirmed previously by Wassersug and Hoff[4] and Liu *et al.*,[5] we assume that the body length is unchanged during swimming and its undulation is purely a lateral compressive motion.

The governing equations are solved using the finite volume method (FVM) in a time-accurate manner proposed by Liu and Kawachi.[6] A third-order upwind scheme is employed to compute the convective term in an ultimate conservative scheme.[7] The viscous term is evaluated by a Gauss integration in FVM. The discretized formulation was described in detail in Ref. 6. Since our goal is to compute flow around the undulating body, a method of regenerating O-type grids fitting the deforming body surface at each time step is employed, with the outside boundary of computational domain being fixed. The basic code used here was provided by Liu and Kawachi[6] and has been validated extensively, especially for the hydrodynamics and undulating propulsion of tadpoles with Re up to 10^5.[5,6]

As the phase speed of traveling wave in (9) is a key parameter for the propulsion of the undulating body,[8–10] we calculate three typical cases with $c=0.5$, 1.0, and 1.5 at $Re=10^4$. To verify (4) and (7), three different control domains with control loops $\Sigma_i(i=1,2,3)$ coinciding with three circumferential gridlines shown in Fig. 2 are tested. Note that since the integration domain V_f is a subset of the computational domain, Σ_i vary as the regeneration of the deformable grids.

The time dependence of the lift and drag coefficients, C_L and C_D, during one cycle is shown in Fig. 3 after periodic results are reached through a few cycles. The results are calculated by (4) and (7) with different control domains as well as by the stress-integral formula (1a). The relative errors

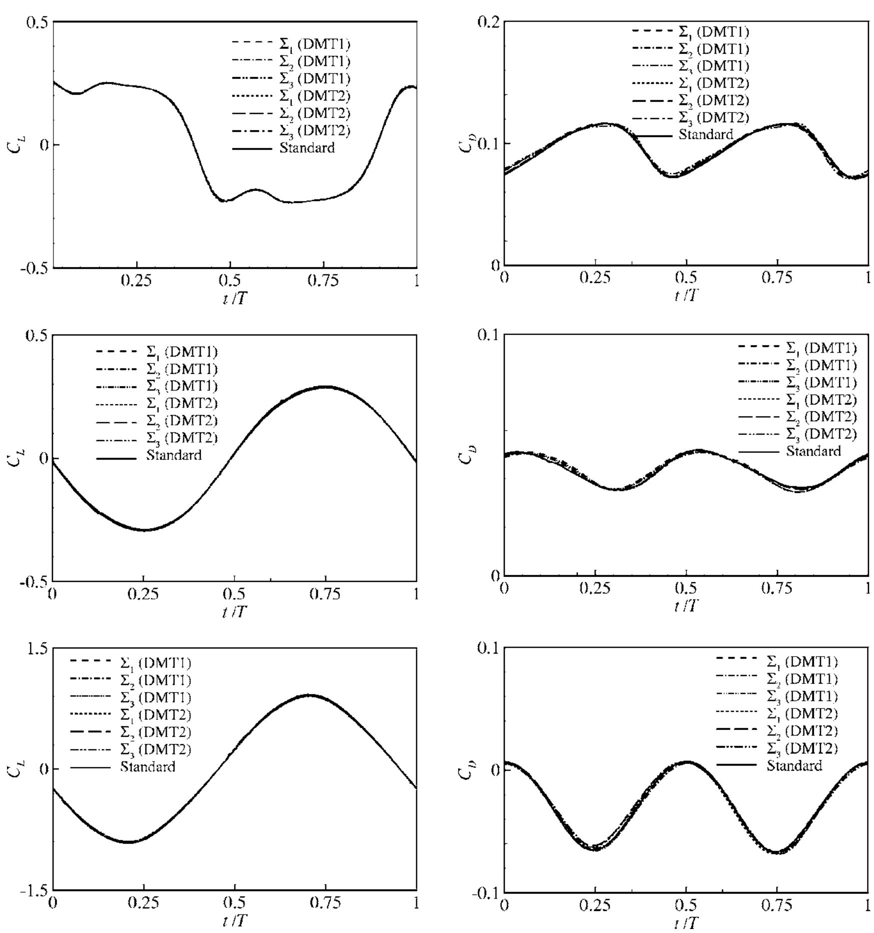

FIG. 3. Time-dependent lift (C_L, left column) and drag (C_D, right column) coefficients during one cycle for $c=0.5$ (top panels), 1.0 (middle panels), and 1.5 (bottom panels). Here, "DMT1" and "DMT2" denote the results calculated by (4) and (7) with different control domains, respectively, and "standard" represents the results calculated by the conventional stress-integral formula (1a).

of the time-averaged C_L and C_D predicted by (4) and (7), with respect to the values by (1a), are found to be less than 1%. The independence of computed forces of the control domain is well confirmed as it should.

The above results suggest that, with high-accuracy flow data gained by the PIV in a neighborhood of Σ, (4) or (7) may easily lead to a force estimate with an accuracy higher than those estimates based on any conventional experimental means. In this regard, an important issue is that the formulas should be robust, i.e., insensitive to the inevitable measurement inaccuracy. Thanks to the comment and suggestion of a referee, a numerical test of this robustness has been carried out to mimic the effect of measurement inaccuracy in PIV data (typically 5%) on the overall quality of the force prediction. In the test, artificial noise of 5% free stream velocity was added to the data computed by both (4) and (7). Since the influence of the noise is quickly weakened by the modulation of the body's transversal motion, the noise was introduced 20 times evenly in a period. The error response was observed to be less than 1% or smaller (figure not shown).

Note that with a given velocity field along and near Σ measured by PIV, the relative merit of (4) and (7) in application depends on the accuracy of data processing methods for inferring pressure and acceleration. For example, the pressure can be inferred by the averaged omnidirectional integration algorithm (Liu and Katz[11]), which can filter the noise of the acceleration estimate. Thus, it is expected that by this algorithm the force predicted by (4) may be more accurate than that by (7).

Finally, we prove (1b). For any tensor field $\mathcal{F}(\boldsymbol{x},t)$ defined on a deformable and movable control volume V_f, with $\boldsymbol{v}(\boldsymbol{x},t)$ being the velocity of ∂V_f, the Reynolds transport theorem reads

$$\frac{d}{dt}\int_{V_f} \mathcal{F}\, dV = \int_{V_f} \frac{\partial \mathcal{F}}{\partial t} dV + \int_{\partial V_f} \mathcal{F}\boldsymbol{n}\cdot\boldsymbol{v}\, dS. \tag{12}$$

We may continue $\boldsymbol{v}(\boldsymbol{x},t)$ smoothly into the interior of V_f, so that

$$\begin{aligned}\int_{\partial V_f} \mathcal{F}\boldsymbol{n}\cdot\boldsymbol{v}\, dS &= \int_{V_f} \nabla\cdot(\boldsymbol{v}\mathcal{F})dV \\ &= \int_{V_f} \nabla\cdot[\boldsymbol{u}\mathcal{F}-(\boldsymbol{u}-\boldsymbol{v})\mathcal{F}]dV,\end{aligned}$$

where $\boldsymbol{u}$ is the fluid velocity. Then (12) becomes

$$\begin{aligned}\frac{d}{dt}\int_{V_f} \mathcal{F}\, dV &= \int_{V_f} \left(\frac{D\mathcal{F}}{Dt} + \mathcal{F}\nabla\cdot\boldsymbol{u}\right)dV \\ &\quad - \int_{\partial V_f} (u_n - v_n)\mathcal{F}\, dS.\end{aligned} \tag{13}$$

Thus, for incompressible flow we have

$$\int_{V_f} \boldsymbol{a}\, dV = \frac{d}{dt}\int_{V_f} \boldsymbol{u}\, dV + \int_{\partial V_f} (u_n - v_n)\boldsymbol{u}\, dS. \tag{14}$$

But integrating (6) over V_f yields

$$\boldsymbol{f} = -\rho\int_{V_f} \boldsymbol{a}\, dV + \int_{\Sigma} (-p\boldsymbol{n} + \boldsymbol{\tau})dS, \tag{15}$$

of which a combination with (14) proves (1b).

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