AN INEQUALITY RELATING THE ORDER, MAXIMUM DEGREE, DIAMETER AND CONNECTIVITY OF A STRONGLY CONNECTED DIGRAPH*†

XU JUNMING (徐俊明)

(*University of Science and Technology of China, Hefei 230026, China*)

Abstract

We prove that if there is a strongly connected digraph of order n, maximum degree d, diameter k and connectivity c, then $n \le c\frac{d^k-d}{d-1} + d + 1$. It improves the previous known results, and it, in fact, is the best possible for several interesting cases. A similar result for arc connectivity is also established.

§1. Introduction

To realize highly reliable and efficient communication, means to find networks (i.e., digraphs) with smaller diameter and larger connectivity for a given order and a maximum degree (see [1]). Consequently, the following problem is raised naturally: If there is a strongly connected digraph of order n, maximum degree d, diameter k and connectivity c, how are these parameters related one to another? In this paper, we prove an inequality which improves previous results and show that our result is sharp for several interesting cases.

A digraph G consists of a finite set of vertices $V(G)$ and a set of arcs $A(G)$ which is a subset of all ordered pairs (u, v), u, $v \in V(G)$. G may have loops but no multiple arcs. The order of G is the number of its vertices, the degree $d(v)$ of a vertex $v \in V(G)$ is the maximum value of its out-degree and in-degree, the maximum degree of G is $\max\{d(v) : v \in V(G)\}$. A vertex cut set C of G is a proper subset of $V(G)$ such that $G - C$ is either not strongly connected or a single vertex. The (vertex) connectivity of G, denoted by $c(G)$, is the minimum cardinality of all cut sets of G. The diameter of G denoted by $D(G)$, is the maximum distance between any pair of vertices. From now on, a strongly connected digraph of order n, maximum degree d, diameter k and connectivity c is denoted as an (n, d, k, c)-digraph.

*Received May 13, 1989. Revised October 16, 1990.

†This project is supported by the National Natural Science Foundation of China.

It is easy to see that if there is an (n, d, k, c)-digraph G, then

$$n \leq 1 + d + d^2 + \cdots + d^k = \frac{d^{k+1} - 1}{d - 1}, \tag{1}$$

and the equality holds if G is a complete digraph of order $d + 1$ (and $k = 1$) or a directed cycle of order $k + 1$ (and $d = 1$). For $d \geq 2$ and $k \geq 2$, Flensnik and Znam ([2]) showed that the upper bound given in (1) can be improved as

$$n \leq d\frac{d^k - 1}{d - 1}, \tag{2}$$

which was rediscovered later by Bridge and Toueg too ([3]). In this paper, we illustrate that the inequality (2) is the best possible for $k = 2$ and $d \geq 2$.

A further relation among the parameters of an (n, d, k, c)-digraph for $1 \leq c \leq d - 1$ was established by Imase, Soneoka and Okada ([4]). Namely, they proved that

$$n \leq c\left(\frac{d^k - 1}{d - 1} + d\right). \tag{3}$$

In this paper, the inequality (3) is improved as follows:

$$n \leq c\frac{d^k - d}{d - 1} + d + 1. \tag{4}$$

We also illustrate that the inequality (4) is the best possible for $c = d - 1$ and some other interesting cases. A similar result for arc connectivity is also established.

In the following section, a class of digraphs $G_I(n, d)$ is considered. In Section 3, our main result is derived, namely, the inequality (4) is proved. In Section 4, several examples are given to show that our result is the best possible and an analogy of (4) for arc connectivity is considered in Section 5.

§2. A Class of Digraphs

In this section we consider a class of digraphs $G_I(n, d)$ proposed by Imase and Itoh ([5]), where n, d are given integers, $1 < d < n - 1$. The vertex set of $G_I(n, d)$ is $V(G_I(n, d)) = \{0, 1, \cdots, n - 1\}$ and $(i, j) \in A(G_I(n, d))$ if and only if

$$j \equiv d(n - 1 - i) + r (\mathrm{mod}\, n) \quad \text{for} \quad 0 \leq i \leq n - 1, \quad 0 \leq r \leq d - 1.$$

It is clear that $G_I(n, d)$ is a d-regular digraph. In [5] it is proved that the diameter of $G_I(n, d)$ satisfies $D(G_I(n, d)) \leq \lceil \log_d n \rceil$, where $\lceil x \rceil$ is the minimum integer not smaller than x. In particular, if $n = d^k + d^{k-q}$ (where k is an arbitrary integer and q is an odd number less than or equal to k), $D(G_I(n, d)) = \lceil \log_d n \rceil - 1 = k$. A d-regular digraph with diameter k and order $n (= d^k + d^{k-1})$ was first constructed by Reddy et al. ([6]).

Later, Imase, Soneoka and Okada ([4]) proved that $c(G_I(n, d)) \geq d - 1$. Recently, Du and Hwang ([7]) have shown that $c(G_I(n, d))$ is either $d - 1$ if d and n are relatively prime or d if both d and $d + 1$ divide n. Thus $c(G_I(d^k + 1, d)) = d - 1$ if $k (\geq 3)$ is an odd number and $c(G_I(d^k + d^{k-1}, d)) = d$, where k is the diameter.

Noting that $G_I(n, d)$ may have loops, Du and Hwang ([7]) have made some study on the number of loops of $G_I(n, d)$. The proof of the following result is easy and is left to reader.

Theorem 1. $G_I(n, d)$ has no loops when $d + 1$ divides n.

§3. Main Results

Theorem 2. If there is an (n,d,k,c)-digraph, $c \le d-1, d \ge 2$ and $k \ge 2$, then

$$n \le c\frac{d^k - d}{d-1} + d + 1. \tag{4}$$

Proof. Let G be an (n,d,k,c)-digraph. By the definition of (vertex) connectivity, there is a cut vertex set V_0 such that $|V_0| = c$. The set $V(G)\backslash V_0$ can be partitioned into two disjoint nonempty subsets V_1 and V_2 such that $G - V_0$ has no arcs from V_1 to V_2. Let $n_1 = |V_1|$ and $n_2 = |V_2|$, and $V_0 = \{z_1, z_2, \cdots, z_c\}$.

Let

$$m = \max_{v\in V_1}\Big\{d(v,V_0)\}, \ l = \max_{u\in V_2}\{d(V_0,u)\}, \ m^* = \max_{v\in V_1}\{d^*(v,V_0)\Big\}, \tag{5}$$

where $d(v,V_0)$ (or $d^*(v,V_0)$) is the minimum (or maximum) distance from v to any vertex in V_0, while $d(V_0,u)$ is the minimum distance from any vertex in V_0 to u. It is easy to see that $m^* \ge m > 0$ and $l > 0$ since $V_1 \cap V_0 = \emptyset$ and $V_2 \cap V_0 = \emptyset$.

Let v be a vertex in V_1 such that $d(v,V_0) = m$ and let u be a vertex in V_2 such that $d(V_0,u) = l$. Since any directed path from v to u goes through V_0 and the diameter of G is k, there exists a vertex z in V_0 such that $d(v,z) + d(z,u) = d(v,u) \le k$. By our choice of $v, d(v,z) \ge m$. Therefore,

$$l = d(V_0,u) \le d(z,u) \le k - d(v,z) \le k - m. \tag{6}$$

If $m = k-1, l = 1$, we consider the transpose G' of G which is the digraph with $V(G') = V(G)$ and $(u,v) \in A(G') \Leftrightarrow (v,u) \in A(G)$. It is easy to reduce the case $m = k-1$ in G to the case $m = 1$ in G'. So it suffices to consider the case $1 \le m \le k-2$.

Let

$$Q_i = \{v \in V_1 : d(v,V_0) = i\}, \qquad 1 \le i \le m.$$

Then

$$n_1 = |V_1| \le \sum_{i=1}^{m} |Q_i| \le c\sum_{i=1}^{m} d^i = c\frac{d^{m+1} - d}{d-1}. \tag{7}$$

Also let

$$P_j = \{u \in V_2 : d(V_0,u) = j\}, \qquad 1 \le j \le l.$$

Then

$$n_2 = |V_2| \le \sum_{j=1}^{l} d^j = c\frac{d^{l+1} - d}{d-1}.$$

Since $l \le k - m$ by (6),

$$n_2 \le c\frac{d^{k-m+1} - d}{d-1}. \tag{8}$$

We first show the following lemma which is stronger than Theorem 2.

Lemma. For $m \ge 2$ or $m = 1$ and $m^* \ge 2$,

$$n \le c\frac{d^k - 1}{d-1}. \tag{9}$$

In fact, the function $f(m) = d^{m+1} + d^{k-m+1}$ is concave upward on the interval $[2, k-2]$ and

$$f(m) \le f(2) = f(k-2) = d^3 + d^{k-1}. \tag{10}$$

It follows that, if $2 \le m \le k-2$, then from (7), (8) and (10),

$$\begin{aligned} n =& n_1 + c + n_2 \le c\frac{d^{m+1} - d}{d-1} + c + c\frac{d^{k-m+1} - d}{d-1} \\ \le& c\frac{d^3 + d^{k-1} - d - 1}{d-1} < c\frac{d^k - 1}{d-1}. \end{aligned}$$

The reason why the last inequality is valid is that $d^k - d^{k-1} - d^3 + d = d(d-1)(d^{k-2} - d - 1) > 0$ because $d \ge 2$ and $k \ge 4$.

For $m = 1$ and $m^* \ge 2$ we consider the two cases $k = 2$ and $k \ge 3$, separately. As for case $k = 2$, the proof is easy and is left to the reader. To complete the proof of Lemma for $k \ge 3$, it suffices to prove that

$$n_2 \le c\frac{d^k - d}{d-1} - n_1. \tag{11}$$

If there is some $j\,(1 \le j \le k-3)$ such that $|P_j| < cd^j$, then

$$\begin{aligned} n_2 \le& c\frac{d^k - d}{d-1} - (1 + d + d^2) \le c\frac{d^k - d}{d-1} - (1 + d + cd) \\ \le& c\frac{d^k - d}{d-1} - (1 + d + n_1) < c\frac{d^k - d}{d-1} - n_1, \end{aligned}$$

and the inequality (11) holds. Next, let $P_0 = V_0$ and assume that

$$|P_j| = cd^j, \qquad 0 \le j \le k-3. \tag{12}$$

Let $P_0(z_i) = \{z_i\}$, $i = 1, 2, \cdots, c$, and let

$$P_j(z_i) = \{u \in V_2 : d(z_i, u) = j\}, \qquad j = 1, 2, \cdots, k-1, \quad i = 1, 2, \cdots, c.$$

Then from (12), we have

$$\begin{aligned} &|P_j(z_i)| = d^j, && 1 \le i \le c, && 0 \le j \le k-3, && (13) \\ &P_j(z_i) \cap P_j(z_m) = \emptyset, && 1 \le i \ne m \le c, && 0 \le j \le k-3, && (14) \\ &|P_{j+1}(z_i)| = |P_j(z_i)|d, && 1 \le i \le c, && 0 \le j \le k-4. && (15) \end{aligned}$$

Let $N^+(v) = \{u \in V(G-v) : (v, u) \in A(G)\}$ for a vertex v in $V(G)$. By the assumption $m^* \ge 2$, there exist some vertices, say $v \in V_1$ and $z_1 \in V_0$ such that $z_1 \notin N^+(v)$. Let

$$\overline{P_{k-2}}(z_1) = P_{k-2}(z_1) \cap P_{k-2}.$$

Then

$$|\overline{P_{k-2}}(z_1)| \le |P_{k-2}(z_1)| \le d^{k-2}.$$

Let

$$|\overline{P_{k-2}}(z_1)| = d^{k-2} - x. \tag{16}$$

And let

$$\begin{aligned} S &= \overline{P_{k-2}}(z_1) \bigcap \Big(\bigcup_{i=2}^{c} P_{k-2}(Z_i) \Big), \qquad s = |S|, \\ W &= \overline{P_{k-2}}(z_1) \backslash S, \\ U &= \{x \in N^+(w) \cap P_{k-1} : w \in W\}. \end{aligned}$$

Then. from (16),

$$|W| = |\overline{P_{k-2}}(z_1)| - s = d^{k-2} - x - s, \tag{17}$$

$$|U| = |W|d - t = (d^{k-2} - x - s)d - t. \tag{18}$$

Taking any one of the vertices in U, say u, we consider the shortest directed path R from v to u. The length of R is $d(v,u)$, which is the number of arcs occurring in R. Let the successor of v on R is some vertex $z_{i_0} \in V_0$. Then $z_{i_0} \neq z_1$ since $z_1 \notin N^+(v)$. From (14) the jth successor of z_{i_0} on R is not in $P_j(z_1) \cap P_j$ for $0 \le j \le k-3$ and $W \cap P_{k-2}(z_{i_0}) = \emptyset$. Hence $d(v,u) = k$, and so the $(k-1)$th successor of z_{i_0} on R is u, i.e., $u \in P_{k-1}(z_1) \cap P_{k-1}(z_{i_0})$. By an arbitrary choice of the vertex $u \in U$, it follows that

$$\Big| P_{k-1}(z_1) \bigcap \Big(\bigcup_{i=2}^{c} P_{k-1}(z_i) \Big) \bigcap P_{k-1} \Big| \ge |U|,$$

$$|P_{k-1}| \le cd^{k-1} - |U| - (x+s)d - t.$$

Thus by (18), $k \ge 3$ and $c \le d-1$, we have

$$\begin{aligned} & x + s + |U| + t + (x+s)d = x + s + (d^{k-2} - x - s)d - t + t + (x+s)d \\ = & d^{k-1} + x + s \ge dc \ge n_1. \end{aligned}$$

Therefore

$$n_2 \le c\frac{d^k - d}{d-1} - (x + s + |U| + t + (x+s)d) \le c\frac{d^k - d}{d-1} - n_1.$$

The inequality (11) is proved and Lemma is established.

To prove Theorem 2, we only want to consider the case $m^* = m = 1$ by Lemma.

In this case, $n_1 \le d$ and $n_2 \le c\frac{d^k - d}{d-1}$ from (8). Thus if $c = 1$, then the theorem holds obviously. Next, we suppose $c \ge 2$.

If there is a j $(1 \le j \le k-2)$ such that $|P_j| \le cd^j - 1$, we have

$$\begin{aligned} n_2 \le & |P_1| + |P_2| + \cdots + |P_{j-1}| + |P_j| + \cdots + |P_{k-1}| \\ \le & cd + cd^2 + \cdots + cd^{j-1} + (cd^j - 1)d + \cdots + (cd^j - 1)d^{k-1-j} \\ = & c\frac{d^k - d}{d-1} - \frac{d^{k-j} - d}{d-1}, \end{aligned}$$

and the inequality (4) holds immediately. So we suppose that

$$|P_j| = cd^j, \qquad 1 \le j \le k-2. \tag{19}$$

Let $z_1 \in V_0$ and let

$$\begin{aligned} S' &= \{u \in P_{k-1} : (u, z_1) \in A(G)\}, & s' &= |S'|, \\ T' &= \{u \in P_{k-2} : (u, v) \in A(G) \text{ for some } v \in V_1\}, & t' &= |T'|, \\ T^* &= \{u \in P_{k-2} : (u, z_1) \in A(G)\}, & t^* &= |T^*|, \\ T^{**} &= \{u \in P_{k-2} : (u, w) \in A(G) \text{ for some } w \in P_{k-2}, w \neq u\}, & t^{**} &= |T^{**}|. \end{aligned}$$

Thus

$$|P_{k-1}| \le |P_{k-2}|d - (t' + t^* + t^{**}),$$
$$n_2 \le c\frac{d^k - d}{d-1} - (t' + t^* + t^{**}) \tag{20}$$

and

$$s' + t^* = d - n_1. \tag{21}$$

Take $z_i \in V_0$, $i = 2, 3, \cdots, c$. Let R_i denote the shortest directed (z_i, z_1) path. From (19) and $d(z_i, z_1) \le k$, every R_i must contain either a vertex in T' or a vertex in $S' \cup T^* \cup T^{**}$. Therefore,

$$c - 1 \le t' + t^* + t^{**} + s'. \tag{22}$$

From (21) and (22) we have

$$n_1 \le t' + t^{**} + d - c + 1. \tag{23}$$

Thus from (20) and (23), the inequality (4) is proved.

The proof of Theorem 2 is completed.

From Theorem 2 we immediately have the following result.

Corollary. For every (n, d, k, c)-digraph with $d \ge 2$ and $k \ge 2$, the (vertex) connectivity

$$c \ge \frac{(n-d-1)(d-1)}{d^k - d}.$$

Particularly,

$$c = \begin{cases} = d, & \text{if } n > d^k + 1; \\ \ge d - 1, & \text{if } n \ge (d-2)\dfrac{d^k - 1}{d-1} + 4. \end{cases}$$

Remark. From the corollary it is directly obtained that the connectivity of $G_I(n, d)$ is d if $n = d^k + d^{k-q}$ and q is an odd number less than k, though it can also been obtained from Du' and Hwang's result (see Theorem 3.2 in [7]). In particular, it is obtained that the connectivity of the digraph with diameter k and order $n(= d^k + d^{k-1})$ constructed by Reddy et al. ([6], also see [5]) is d.

§4. The Sharpness of Results

In this section we illustrate our results to be the best possible for several interesting cases.

Example 1. the right side of the inequality (4) is $d^k + 1$ for $c = d - 1$. We have already seen in Section 2 that the loopless digraph $G_I(d^k + 1, d)$ is a $(d^k + 1, d, k, d-1)$-digraph if k is an odd number. This example shows that the uppper bound of n given in (4) is the best possible for $c = d - 1$ and an odd number $k(\ge 3)$. Meanwhile this example also shows that the lower bound of n such that c is d in the corollary is the best possible.

Example 2. The right side of the inequality (4) is $2d + 1$ for $c = 1$, $k = 2$ and $d \ge 2$. Let $K_d^{(j)}$ be a complete digraph with the vertex set $V(K_d^{(j)}) = \{x_1^{(j)}, x_2^{(j)}, \cdots, x_d^{(j)}\}$, $j = 1, 2$ and K_1 be a digraph consisting of a single vertex x. Let

$$\begin{aligned} G =& \big(K_d^{(1)} \cup K_d^{(2)} \cup K_1\big) \bigcup \{(x_i^{(1)}, x) : i = 1, 2, \cdots, d\} \\ & \bigcup \{(x, x_i^{(2)}) : i = 1, 2, \cdots, d\} \bigcup \{(x_i^{(2)}, x_i^{(1)}) : i = 1, 2, \cdots, d\}. \end{aligned}$$

It is easily verified that G is a $(2d+1, d, 2, 1)$-digraph. This shows that the upper bound of n given in (4) is the best possible for $c = 1$, $k = 2$ and $d \geq 2$.

Example 3. The right side of the inequality (2) is $d^2 + d$ for $k = 2$. We have also seen in Section 2 (or from Corollary) that the loopless digraph $G_I(d^2+d, d)$ is a $(d^2+d, d, 2, d)$-digraph, which shows that the upper bound given in (2) is the best possible for $k = 2$ and $d \geq 2$.

§5. Arc Connectivity

Considering arc rather than vertex connectivity in this section, we can prove an analogy of Theorem 2 by a similar argument. An arc cut set E of the strongly connected digraph G is a subset of $A(G)$ such that $G - E$ is not strongly connected. The arc connectivity of G, denoted by $c'(G)$, is the minimum cardinality of all arc cut sets of G. A strongly connected digraph of order n, maximum degree d, diameter k and arc connectivity c' is denoted as an (n, d, k, c')-digraph.

Theorem 3. If there is an (n, d, k, c')-digraph, $c' \leq d-1$, $d \geq 2$ and $k \geq 4$, then

$$n \leq c'\left(\frac{d^{k-1}-1}{d-1} + 1\right) + d. \tag{24}$$

Proof. Let G be an (n, d, k, c')-digraph. By the definition of arc connectivity, there exists an arc cut set E_0 such that $|E_0| = c'$. The set $V(G - E_0)$ can be partitioned into two disjoint nonempty subsets U_1 and U_2 such that $G - E_0$ has no arc from U_1 to U_2. Let U_0 be a set of the initial vertices of the arcs in E_0 and U_0' be a set of the terminal vertices of the arcs in E_0. It is clear that $U_0 \subset U_1$, $U_0' \subset U_2$ and $|U_0|, |U_0'| \leq c'$. Let $n_1 = |U_1|$ and $n_2 = |U_2|$. Then $n = n_1 + n_2$.

The following notations are defined in the same manner as in the proof of Theorem 2. Let

$$m = \max_{v \in U_1}\{d(v, U_0)\}, \qquad l = \max_{v \in U_2}\{d(U_0', u)\}.$$

It was proved by Imase et al. (see [4]) that $1 \leq m \leq k-2$ and $1 \leq l \leq k-2$. (Note that the assumption $c' < d$ was used in their proof.)

Let $v \in U_1 \backslash U_0$ such that $d(v, U_0) = m$ and let $u \in U_2 \backslash U_0'$ such that $d(U_0', u) = l$. Since any directed path from v to u contains some arc $(u_0, u_0') \in E_0$, $u_0 \in U_0, u_0' \in U_0'$, then $d(v, u_0)) + d(u_0', u) = d(v, u) - 1 \leq k - 1$. Thus

$$l = d(U_0', u) \leq d(u_0', u) \leq k - 1 - d(v, U_0) \leq k - 1 - m. \tag{25}$$

Let

$$Q_i = \{v \in U_1 \backslash U_0 : \ d(v, U_0) = i\}, \qquad 1 \leq i \leq m.$$

Then

$$n_1 = |U_1| \leq |U_0| + \sum_{i=1}^{m} |Q_i| \leq |U_0|\left(1 + \frac{d^{m+1} - d}{d-1}\right).$$

Noting that $|U_0| \leq c'$, therefore,

$$n_1 \leq c'\frac{d^{m+1}-1}{d-1}. \tag{26}$$

Let

$$P_j = \{u \in U_2 \backslash U_0' : d(U_0', u) = j\}, \qquad 1 \leq j \leq l.$$

Then

$$n_2 = |U_2| \le |U_0'| + \sum_{j=1}^{l} |P_j| \le |U_0'| \frac{d^{l+1}-1}{d-1}.$$

Therefore, from (25) and noting $|U_0'| \le c'$,

$$n_2 \le |U_0'| \frac{d^{k-m}-1}{d-1} \le c' \frac{d^{k-m}-1}{d-1}. \tag{27}$$

The function $f(m) = d^{m+1} + d^{k-m}$ is concave upward on the interval $[2, k-3]$, and

$$f(m) = f(2) = f(k-3) = d^3 + d^{k-2}. \tag{28}$$

Therefore, if $2 \le m \le k-3$, then from (26), (27) and (28), the inequality (24) is valid.

If $m = k-2$, we consider the transpose G' of G and reduce the case $m = k-2$ in G to the case $m = 1$ in G'. Hence, to complete the proof of the inequality (24), it suffices to consider the case $m = 1$. (Note $l = k-2$ in this case.)

If $U_0 \subseteq N^+(v)$ for every $v \in U_1 \backslash U_0$, then $n_1 \le d + |U_0| \le d + c'$. Hence from (27), the inequality (24) is valid.

In the following, we suppose that there exists a vertex v in $U_1 \backslash U_0$ such that $|N^+(v) \cap U_0| < |U_0|$. Then $n_1 \le c'(d+1)$. If $|U_0'| < c'$, then, since $l = k-2 \ge 2$,

$$n_2 \le c' \frac{d^{k-1}-1}{d-1} - (1 + d + d^2), \tag{29}$$

and so, also from (26), (29) and $c' < d$, we can derive the inequality (24).

If $|U_0'| = c'$, in the same way as in proving the inequality (11), it can be proved that

$$n_2 - c' < c' \frac{d^{k-1}-1}{d-1} - n_1.$$

The details are left to the reader and so the proof of Theorem 3 is completed.

Corollary. For every (n, d, k, c')-digraph with $d \ge 2$, $k \ge 4$ and $0 < c' < d$, the arc connectivity

$$c' \ge \frac{(n-d)(d-1)}{d^{k-1} + d - 2}.$$

Particularly,

$$c' = \begin{cases} = d, & \text{if } n > d^{k-1} + 2d - 2; \\ \ge d-1, & \text{if } n > (d-2)\left(\dfrac{d^{k-1}-1}{d-1} + 1\right) + d. \end{cases}$$

Acknowledgement. We wish to thank Prof. Li Qiao for bringing the problem to our attention, and also for some useful discussions on the problem.

References

[1] Wilkov, R.S., Analysis and Design of Reliable Computer Networks, *IEEE Trans. Commum.*, COM-1 **20**(1972), 660–678.

[2] Plesnil, J. and Znam, S., Strong Geodetic Directed Graphs, *Acta Fcta Rerum Natur. Univ. Camenian Math.*, **29**(1974), 29–34.

[3] Bridge, W. C. and Toueg, S., On the Impossibility of Ditrected Moore Graphs, *J. Combin. Theory*, Ser. B, **29**(1980), 339–341.

[4] Imase, M., Soneoka, T. and Okada, K., Connectivity of Regular Directed Graphs with Small Diameters, *IEEE Trans. Comput.*, C-**34**(1985), 267–273.

[5] Imase, M. and Itoh, M., A Design for Directed Graph with Minimum Diameter, *IEEE Trans. Comput.*, C-**32**(1983), 782–784.

[6] Reddy, S.M., Kuhl, J. G., Hosseini, S.H. and Lee, H., On digraph with Minimum Dimeter and Maximum Connectivity, Proceedings of Allerton Conf. On Communications, Control and Computers, Illinois, 1018–1026, 1982.

[7] Du, D.Z. and Hwang, F.K., Generalized de Bruijn Digraphs, *Networks*, **18**(1988), 27–38.